UNIVERSITAT DE BARCELONA

Essays on political economy and matching markets

Gerard Domènech i Gironell

PhD in Economics

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PhD candidate:

Gerard Domènech i Gironell

Advisors:

Marina Núñez Oliva

Oriol Tejada Pinyol

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Als meus pares, Jaume i Isabel

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Contents

1. Introduction

As Douglass North writes in the very first lines of the first part of his extremely influential book "Insitutions, Institutional Change and Economic Performance", published in 1990 (North 1990):

> Institutions are the rules of the game in a society or, more formally, are the humanly devised constraints that shape human interaction. In consequence they structure incentives in human exchange, whether political, social, or economic.

Given that institutions are meant to " structure incentives", it is of paramount importance to understand precisely how, given a set of rules, incentives are shaped. Here is where game theory, which aims to investigate strategic interactions between rational agents, comes into play. Even if game theory as a field of its own was mostly developed in the 20th century, some of its core ideas can be traced back to works in the 18th century, such as in the Rousseau's 1755 seminal "Discourse on the Origin of Inequality" (Rousseau 2004) [1]. Game theory studies which outcomes might arise when several strategic agents interact, focusing on the incentives of each agent depending on the behavior of the others. In this thesis I will make use of the tools that game theory provides to study how the incentives certain institutions induce might generate outcomes that are not in line with the initial purpose of the institution itself. The three papers that make up the three chapters that follow after this introduction, aim to investigate this issue formally in a variety of settings: from a job market to an election.

In the second chapter, that is coauthored with Marina Núñez and published in *Games and Economic Behavior* (Domènech and Núñez 2022), we focus on a job market, in which firms and workers are to be matched with each other. Each firm has a certain productivity with each worker, and both workers and firms have a maximum number of partners they can be matched to. If a firm and a worker are matched with each other, their joint productivity has to be shared among them. This

[1] Fudenberg and Tirole analyze as examples of games some reasoning paths and "rational analysis" of Rousseau in Fudenberg and Tirole (1991), probably the most widely used game theory textbook.

model, known as multiple-partners assignment game, was firstly introduced in Sotomayor (1992), and is a generalization of the simple assignment game, introduced in the seminal Shapley and Shubik (1972). The model can also be read in terms of buyers and sellers, where workers are the sellers and firms the buyers. In this framework, there is a set of solutions of particular interest, the stable rules, which involve a firms-workers matching and an allocation of the productivities such that no firm and worker that are not matched together would prefer to drop one of their partners in order to associate. In our work, we provide an axiomatization of two especially relevant stable rules: the firms-optimal and the workers-optimal stable rules. The axioms that we use allow us to discuss manipulability aspects of these rules, which is a relevant topic in the matching literature. Focusing on the side of the firms (it works similarly for workers), we know from Pérez-Castrillo and Sotomayor (2017) that the firms-optimal stable rules are manipulable by any firm that is able to hire more than one worker. However, in our work we show that manipulation is not profitable if a firm can only over-report all productivities by the same amount. Hence, we show that a firm must be able to asymmetrically manipulate the productivities with the different workers in the market in order to gain from manipulation. This result lets a planner better foresee, given the particularities of the market, when the mentioned relevant stable rules can be expected to produce the desired outcome and when they might lead to manipulation.

In the third chapter, that is product of a collaboration with Dimitrios Xefteris, we study how an interest group might affect the outcome of an election. We consider the case of a committee, which might be viewed as a Parliament, that must choose whether or not to implement a *socially desirable* reform. The decision is taken via an election and votes are aggregated by means of simple majority. We assume that all the voters find the reform desirable and that there exists an interest group willing to bribe voters in order to stop the reform from passing. The setting is in line with Dal Bò (2007), that shows that, even if the interest group might not have to spend a lot to buy the election, it needs a potentially huge budget in order to make credible bribes to the voters. We inquire what an interest group with a fixed and limited budget can do to block the reform, and we do so by considering a simple bribing scheme: the interest group commits to giving to each voter who votes against the reform an equal part of the budget. We show that, if voters value the election outcome enough, two symmetric completely mixed-strategy Nash equilibria always coexist, one in which the reform is more likely to pass and one in which it is less likely. Furthermore, we prove that when the size of the committee grows asymptotically large, the reform is implement almost for sure, or almost surely not, depending on which of the two equilibria is played. The existence of these equilibria relies on voters' being aware their vote might not be decisive, and hence inferring how likely they are to be pivotal (individually). Our results are consistent and a potential explanation of the "Tullock paradox", which states that some interest groups seem to obtain disproportionately large political favors (Tullock, 1967; Tullock, 1997). In addition, our work also poses doubts on the efficiency of vote trading, allowing for a critique, in the spirit of Neeman (1999), based on the fact that the prices of votes might not reflect its real value.

In the fourth chapter, which is the fruit of joint work with Caio Lorecchio and Oriol Tejada, we investigate how social networks, or information groups (groups of people that share information), might affect the incentives to acquire information ahead of an election. Anthony Downs, in his seminal book "An Economic Theory of Democracy" (Downs, 1957), points out that voters might not find profitable to acquire costly information, since their vote is likely to not change the outcome of the election; this is what he called "rational ignorance". Martinelli (2006) studies this issue formally in the context of an election in which citizens must choose between two options but are initially unaware of which one is better for society (they have common preferences). However, prior to voting (the decision is taken by simple majority), citizens can individually acquire a costly signal about the right alternative to implement, the accuracy of which is increasing in the cost incurred. That is: if a citizen wants to be more certain about which alternative should be implemented, s/he has to incur a higher cost. This cost can be viewed as an effort that citizens must undertake in order to obtain and/or process information. In this setting, Martinelli (2006) shows that there is only one symmetric equilibrium; i.e. in which all citizens acquire the same information level. Furthermore, he shows that, even if the individual level of information goes to zero as the size of the electorate grows large, the probability that the society as a whole chooses the right alternative does not, and in fact, in some (not extreme) cases, it goes to one. In our work we study what happens if we allow the possibility of communication between voters. To be precise, we allow for voters to communicate to (some) other voters, truthfully or not, how much information they have individually purchased and which is the alternative they deem more likely to be the right one to implement. Hence, we assume the existence of an underlying communication network. We focus on the case in which all voters can communicate between them, and prove that in this case, truthful reporting is consistent with voters *believing* the messages they receive [2]. Intuitively, the existence of such a communication network might have two (opposed) effects compared to the benchmark case without communication: *i*) a voter might find beneficial to free-ride on the information acquired by the other voters, and *ii*) a voter might want to acquire more information, given that it can be enjoyed

[2] We also show that this is not the case when the underlying network is not complete.

by everyone. We prove that both these effects translate into a *dictator equilibrium* always existing, which is an equilibrium in which only one voter acquires information, and the rest rely on it. We show that such equilibrium, which never exists without communication, might provide a higher or a lower probability of choosing the right alternative compared to the symmetric equilibrium of case of no communication. In addition, we show that if information is costly enough, the symmetric equilibrium of the case without communication carries over to the case with communication. In fact we prove that this is the only possible symmetric equilibrium of the game and, importantly, that, whenever it exists, it provides the exact same probability of choosing the right alternative whether there is communication or not.

After these three chapters, in Chapter 5, I address some concluding remarks. These are meant not only to highlight the main contributions and findings of the central chapters, but also to identify main directions for further research.

2. Axioms for the optimal stable rules and fair-division rules in a multiple-partners job market.[1]

Abstract

In the multiple-partners job market, introduced in Sotomayor (1992), each firm can hire several workers and each worker can be hired by several firms, up to a given quota. We show that, in contrast to what happens in the simple assignment game, in this extension, the firms-optimal stable rules are neither valuation monotonic nor pairwise monotonic. However, we show that the firms-optimal stable rules satisfy a weaker property, what we call *firm-covariance*, and that this property characterizes these rules among all stable rules. This property allows us to shed some light on how firms can (and cannot) manipulate the firms-optimal stable rules. In particular, we show that firms cannot manipulate them by constantly over-reporting their valuations. Analogous results hold when focusing on the workers. Finally, we extend to the multiple-partners market a known characterization of the fair-division rules on the domain of simple assignment games.

Keywords: assignment game; stable rules; fair division.

JEL Codes: C78, D78.

2.1. Introduction

The aim of this paper is to study some allocation rules in a two-sided job market with firms on one side and workers on the other side. Each agent has a quota that

[1]This chapter was published as a paper coauthored with Marina Núñez in *Games and Economic Behavior* (Domènech and Núñez 2022).

determines in how many partnerships with agents of the opposite side this agent can enter. Each potential partnership has a value and a rule determines a matching and an allocation of the value of each partnership between the partners. This model is an extension of the well-known Shapley and Shubik assignment game.

The assignment game was introduced in Shapley and Shubik (1972) as a coalitional game model for a two-sided market, formed by buyers and sellers or firms and workers, where each agent on one side is to be matched to at most one agent on the opposite side. The objective is to propose a matching and an allocation of the worth of each matched pair among the partners in such a way that no buyer-seller pair (or firm-worker pair) blocks the proposed matching because they can get a higher payoff by being matched together.

Shapley and Shubik prove that, for such markets, stable outcomes always exist and form a complete lattice, which guarantees the existence of an optimal stable outcome for each side of the market. They also prove the coincidence between the core, the set of stable payoff vectors and the set of competitive equilibria payoff vectors.

Many extensions of the Shapley and Shubik assignment game, that we will call the simple assignment game, have been studied since then. The first ones allow agents to be matched to more than one partner. Kaneko (1976) assumes that buyers can only buy one good from one seller while each seller can sell to more than one buyer. The core is also non-empty but (strictly) contains the set of competitive equilibrium payoff vectors. Thompson (1981) allows that both buyers and sellers can take part in multiple partnerships, up to a given quota exogenously determined for each agent. This extension was also studied in Sánchez-Soriano et al. (2001) with the name of transportation game and in Sotomayor (2002). It turns out that the core, that also contains the set of competitive equilibrium payoff vectors, is non-empty but has no longer a lattice structure. In this model, existence of optimal core allocations for each side of the market is still an open question.

A different extension of the simple assignment game is introduced in Sotomayor (1992) with the name of multiple-partners assignment game, and this is the model that better fits with our initial job market situation. In the multiple-partners assignment game, each agent can also take part in multiple partnerships, as many as the agent's quota allows, but can trade at most one unit with each possible partner. Utilities are assumed to be additively separable. Again, an outcome consists of a matching and an allocation of the worth of each partnership between the two partners. In this setting, a notion of (pairwise) stable outcome is similarly defined. Sotomayor (1992) shows that the set of stable payoffs is non-empty and a subset of the core, and that it can be strictly smaller than the core. Sotomayor (1999) adds that the set of stable payoffs is endowed with a complete lattice structure under two convenient partial order relations. Although these partial orders are not defined by the preferences of the agents, all agents on the same side of the market agree on the best stable payoff for them. The relationship with the set of competitive equilibrium payoffs is analysed in Sotomayor (2007) and a mechanism that yields the buyers-optimal competitive equilibrium payoff, which coincides with the buyers-optimal stable payoff, is obtained in Sotomayor (2009). Pérez-Castrillo and Sotomayor (2019) analyses how the optimal stable and competitive solutions react to the introduction of a new agent to the market, depending on whether it is a buyer or a seller.

The aim of our paper is to study stable allocation rules, that is, rules that given the values of all possible partnerships select a stable outcome. In particular we will focus on the two optimal stable rules. We first generalize some monotonicity properties: pairwise monotonicity and firm-valuation monotonicity (or worker-valuation monotonicity), that are satisfied by optimal stable rules in the simple assignment game. Firm-valuation monotonicity states that if the values of a firm weakly decrease but this does not modify the partners of this firm given by the rule, then this firm should not receive a higher payoff in any of its partnerships. In the simple assignment game this property characterizes the firms-optimal stable rules among all stable rules (van den Brink et al., 2021).

We show that the optimal stable rules for the multiple-partners assignment game do not satisfy the aforementioned monotonicity properties: the firms-optimal stable rules are neither firm-valuation monotonic nor pairwise monotonic. However, the firms-optimal stable rules satisfy a weaker form of valuation monotonicity. We strengthen the conditions under which a decrease of the valuations of the firm should imply a decrease in that firm's payoffs: we only require monotonicity when all its valuations are decreased by the same amount. This weak firm-valuation monotonicity is a consequence of what we call firm-covariance. Roughly speaking, a rule is firm-covariant if when all valuations of a firm decrease in a constant amount and all optimal matchings of the initial market still remain optimal, then the payoff this firm obtains in each partnership decreases in exactly that constant amount. We prove that firm covariance characterizes the firms-optimal stable rules among all stable rules, and worker covariance characterizes the workers-optimal stable rules among all stable rules.

Secondly, we focus on how agents can misrepresent their preferences to manipulate a stable rule in the multiple-partners assignment game. Pérez-Castrillo and Sotomayor (2017) analyse the manipulability of competitive equilibrium rules for this market game (with buyers instead of firms and sellers instead of workers). They show that (i) any agent who does not receive her/his optimal competitive equilibrium payoff under a competitive rule can profitably misrepresent her/his valuations, assuming the others tell the truth; (ii) if the buyers-optimal (respectively, sellers-optimal) competitive equilibrium rule is used in a market with more than one vector of equilibrium prices, then there is a seller (respectively, buyer) who can profitably misrepresent his (respectively, her) valuations and (iii) an agent with a quota of one cannot manipulate a rule in a market if and only if the rule gives to this agent her/his most preferred equilibrium payoff.

Since in multiple-partners assignment games the payoff vector of the buyers-optimal stable rule coincides with that of the buyers-optimal competitive equilibrium rule (Sotomayor, 2007), only the buyers with capacity one cannot manipulate the rule. However, we show that these stable rules that are optimal for one side of the market have a weaker non-manipulability property: on the domain of multiple-partner job markets where all firm-worker pairs are acceptable, no firm can manipulate a firms-optimal stable rule by constantly over-reporting its valuations. Similarly, no worker can manipulate the workers-optimal stable rule by under-reporting his/her valuation.

There is some experimental evidence that bidders tend to over-report valuations in some auctions. See for instance Kagel and Levin (1993) for second price auctions, Kagel and Levin (2009) for the Vickrey multi-unit demand auction, or Kagel et al. (2014) for some combinatorial auctions with package bidding. We see that, although over-reporting may be profitable for firms (or buyers) if a firms-optimal stable rule is implemented in a multiple-partners job market, the least sophisticated form of over-reporting which consists in adding the same constant to all firm's valuations, does not bring any additional profit.

Finally, we consider the fair-division rules. The payoff vector of these rules is the midpoint between the firms-optimal and the workers-optimal payoff vectors. On the domain of simple assignment games, these rules have been characterized in van den Brink et al. (2021) by means of two properties: great valuation fairness and weak derived consistency. We adapt the definition of these two properties to the domain of multiple-partners job markets. Great valuation fairness requires that when the value of all firm-worker pair decreases by a constant amount (up to a given threshold that guarantees that all optimal matchings of the initial market remain optimal) then all players suffer the same reduction in the payoff they receive from the rule. Weak derived consistency only requires consistency of the payoffs when the market is reduced at a firm-worker pair that have the same payoff at any stable outcome. We show that these two properties individualize the fair-division rules among all stable rules.

The structure of the paper is as follows. In Section 2 we introduce the multiple-partners job market, Section 3 contains the characterizations of the two optimal stable rules, Section 4 discusses the manipulability of these rules and Section 5 characterizes the fair-division rules.

2.2. The multiple-partners assignment game

Let $F = \{f_1, f_2, \ldots, f_m\}$ be a finite set of firms and $W = \{w_1, w_2, \ldots, w_n\}$ a finite set of workers. Each firm f_i values in $h_{ij} \geq 0$ being matched to worker w_j. Also, each worker w_j has a reservation value $t_j \geq 0$, that can be interpreted as how much worker w_j values each one of his available slots. If firm f_i hires worker w_j, then a value $a_{ij} = \max\{h_{ij} - t_j, 0\} \geq 0$ is generated that has to be shared by both partners. Sometimes we will normalize reservation values of workers at zero and then $a_{ij} = h_{ij}$ for all $i = 1, 2 \ldots, m$ and $j = 1, 2, \ldots, n$. Each firm f_i can hire at most r_i workers and each worker w_j can work for at most s_j firms.

A *multiple-partners assignment market* or a *multiple-partners job market* is then defined by (F, W, a, r, s) where $a = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$, $r = (r_i)_{i=1,\ldots,m}$ and $s = (s_j)_{j=1,\ldots,n}$. The set of all possible valuation profiles for a set F of firms and a set W of workers is denoted by $\mathscr{A}^{F \times W}$. We add a dummy agent on each side of the market, f_0 and w_0, such that $a_{00} = a_{i0} = a_{0j} = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. As for the quotas, a dummy player may form as many partnerships as needed to fill up the quotas of the non-dummy players. We write $F_0 = F \cup \{f_0\}$ and $W_0 = W \cup \{w_0\}$. A dummy player may be matched to more than one player of the opposite side and more than once to the same player. When all non-dummy agents have quota 1, this model coincides with the one in Shapley and Shubik (1972) and we will say it is a *simple assignment game*.

A *matching* μ is a subset of $F_0 \times W_0$, that does not violate the quotas of the players, that is, each $f_i \in F$ appears in exactly r_i pairs of μ and each $w_j \in W$ appears in exactly s_j pairs of μ, since a firm that does not fill some of its positions is assumed to be matched to the dummy worker w_0 (and similarly for workers with unfilled positions). When necessary, we denote by μ_{f_i} the set of partners of firm $f_i \in F$ in matching μ, that is $\mu_{f_i} = \{j \in W_0 \mid (f_i, w_j) \in \mu\}$. Similarly, for all $w_j \in W$, $\mu_{w_j} = \{i \in F_0 \mid (f_i, w_j) \in \mu\}$.

The set of all matchings is $\mathscr{M}(F, W, r, s)$. A matching μ is *optimal* if, for any other $\mu' \in \mathscr{M}(F, W, r, s)$, $\sum_{(f_i, w_j) \in \mu} a_{ij} \geq \sum_{(f_i, w_j) \in \mu'} a_{ij}$. The set of optimal matchings is $\mathscr{M}_a(F, W, r, s)$.

From this market situation, a coalitional game $(F \cup W, w_a)$, the *multiple-partners assignment game* is defined with set of agents $F \cup W$ and coalitional function

$$w_a(T) = \max_{\mu \in \mathscr{M}(F \cap T, W \cap T, r, s)} \sum_{(f_i, w_j) \in \mu} a_{ij}$$

for all $T \subseteq F \cup W$ with $T \cap F \neq 0$ and $T \cap W \neq 0$, and $w_a(T) = 0$ otherwise.

An outcome for the market (F,W,a,r,s) consists of a matching and the payoffs that each agent obtains from each of the partnerships he/she establishes in this matching. That is, if firm f_i hires worker w_j at a salary v_{ij}, this firm receives $u_{ij} = a_{ij} - v_{ij}$.

Definition 2.1. *Let (F,W,a,r,s) be a multiple-partner job market. A feasible outcome is $(u,v;\mu)$ where $\mu \in \mathscr{M}(F,W,r,s)$ and for each $(f_i,w_j) \in \mu$,*

1. $u_{ij} + v_{ij} = a_{ij}$,
2. $u_{ij} \geq a_{i0}$ *and* $v_{ij} \geq a_{0j}$,
3. *if* $f_i = f_0$, *then* $v_{0j} = a_{0j}$,
4. *if* $w_j = w_0$, *then* $u_{i0} = a_{i0}$.

Notice that u_{ij} and v_{ij} are only defined if $(f_i,w_j) \in \mu$. Hence, u and v contain a list of dissagregated payoffs for each agent, one for each partnership established by μ. Also, as a consequence of the above definition, the payoff of the dummy agents is always zero.

For these markets, a notion of stability, sometimes called pairwise stability, is defined in Sotomayor (1992).

Definition 2.2. *Let (F,W,a,r,s) be a multiple-partner job market. A stable outcome is a feasible outcome $(u,v;\mu)$ such that for all $(f_i,w_j) \notin \mu$,*

$$u_{ik} + v_{lj} \geq a_{ij} \text{ for all } (f_i,w_k) \in \mu \text{ and } (f_l,w_j) \in \mu. \tag{2.1}$$

Notice that if there existed (f_i,w_k) and (f_l,w_j) in μ such that $u_{ik} + v_{lj} < a_{ij}$, then f_i and w_j might break their current partnerships with w_k and f_l, respectively, and form a new one together, because this could give to each of them a higher payoff.

It is shown in Sotomayor (1992) that if $(u,v;\mu)$ is a stable outcome, then μ is an optimal matching.

For the multiple-partners assignment game, stable outcomes always exist. This is proved in Sotomayor (1992) in two different ways: one of them uses linear programming and the second one, that we comment on below, is based on a replication of the players and a convenient way of defining the valuation matrix.

Given any multiple partners-assignment game (F,W,a,r,s) we can define a *related simple assignment game* $(\tilde{F},\tilde{W},\tilde{a})$ in the following way. Each firm f_i with quota r_i is replicated r_i times and each worker w_j with quota s_j is replicated s_j

times:

$$\tilde{F} = \{f_{ik} \mid i = 1,\ldots,m;\, k = 1,\ldots,r_i\} \text{ and } \tilde{W} = \{w_{kj} \mid j = 1,\ldots,n;\, k = 1,\ldots,s_j\}$$

with unitary quotas, $\tilde{r}_{ik} = 1$ for all $i = 1,\ldots,m$ and $k = 1,\ldots,r_i$, and $\tilde{s}_{kj} = 1$ for all $j = 1,\ldots,n$ and $k = 1\ldots,s_j$. Moreover, given $\mu \in \mathscr{M}_a(F,W,r,s)$, we define a one-to-one matching $\tilde{\mu}$ between $\tilde{F}$ and $\tilde{W}$ in this way: (i) if $(f_{ik}, w_{lj}) \in \tilde{\mu}$, then $(f_i, w_j) \in \mu$ and (ii) if $(f_i, w_j) \in \mu$, there exist one and only one $k = 1,\ldots,r_i$ and one and only one $l = 1\ldots,s_j$ such that $(f_{ik}, w_{lj}) \in \tilde{\mu}$. This means that if f_i hires w_j under μ, then one copy of f_i hires one copy of w_j under $\tilde{\mu}$ and that no other copies of them are matched. After defining $\tilde{a}$, it can be shown that $\tilde{\mu}$ is optimal for $(\tilde{F}, \tilde{W}, \tilde{a})$.

Then, given $\tilde{\mu}$ as defined above, the valuation matrix $\tilde{a}$ of this related simple assignment game $(\tilde{F}, \tilde{W}, \tilde{\alpha})$ is defined by

$$\tilde{a}_{ik,lj} = \begin{cases} 0 & \text{if } (f_i, w_j) \in \mu \text{ and } (f_{ik}, w_{lj}) \notin \tilde{\mu}, \\ a_{ij} & \text{otherwise.} \end{cases} \tag{2.2}$$

Now, if $(u', v'; \tilde{\mu})$ is a feasible outcome for the simple assignment game $(\tilde{F}, \tilde{W}, \tilde{a})$, we can built a feasible outcome $(u, v; \mu)$ for the multiple-partners assignment game (F, W, a, r, s) in the following way:

$$\begin{aligned} &\text{if } (f_{ik}, w_{lj}) \in \tilde{\mu}, \text{ then define } u_{ij} = u'_{ik},\ v_{ij} = v'_{lj}, \text{ and} \\ &u_{i0} = v_{0j} = 0, \text{ whenever } i \text{ or } j \text{ are matched to a dummy partner.} \end{aligned} \tag{2.3}$$

Proposition 2 in Sotomayor (1992) shows that $(u', v'; \tilde{\mu})$ is stable for $(\tilde{F}, \tilde{W}, \tilde{a})$ if and only if $(u, v; \mu)$ is stable for (F, W, a, r, s).

Since it is well-known that stable outcomes always exist for the simple assignment game, the above result guarantees also existence for the multiple-partners assignment game. Moreover, Sotomayor (1999) proves that the payoff vectors of the set of stable outcomes form a convex and compact lattice and, as a consequence, there exists a unique optimal stable payoff vector for each side of the market. To this end, the problem that u_{ij} and v_{ij} are indexed according to the current matching, that may differ from one stable matching to another, has to be solved. However, it is also proved in Theorem 1 in Sotomayor (1999) that in every stable outcome a player gets the same payoff in any nonessential partnership (those partnerships that occur in some but not all optimal matchings). Because of that, given a stable outcome $(u, v; \mu)$ and another optimal matching μ', we can reindex u_{ij} and v_{ij} according to μ' and still get a stable outcome compatible with μ'.

As a consequence of all that, to obtain the firms-optimal stable outcome in the multiple-partners assignment game we only need to obtain the firms-optimal stable payoff vector in the related simple assignment game. In the simple assignment game, the maximum stable payoff of a firm $f_{ik} \in \tilde{F}$ is its marginal contribution, $\overline{u}_{ik}(\tilde{a}) = w_{\tilde{a}}(\tilde{N}) - w_{\tilde{a}}(\tilde{N} \setminus \{f_{ik}\})$, and similarly for the workers, $\overline{v}_{lj}(\tilde{a}) = w_{\tilde{a}}(\tilde{N}) - w_{\tilde{a}}(\tilde{N} \setminus \{w_{lj}\})$ (see Demange (1982) and Leonard (1983)). Hence, $(\overline{u}(a), \underline{v}(a))$ defined from $(\overline{u}(\tilde{a}), \underline{v}(\tilde{a}))$ as in 2.3 are the optimal stable payoff vectors in the multiple-partners assignment game, and the maximum total stable payoff of an agent in the multiple-partners assignment game is

$$\overline{U}_i(a) = \sum_{(f_i,w_k)\in\mu} \overline{u}_{ik}(a) \text{ for all } f_i \in F \quad \text{and} \quad \overline{V}_j(a) = \sum_{(k,j)\in\mu} \overline{v}_{kj}(a) \text{ for all } w_j \in W,$$

given any optimal matching μ. Notice that, for all $f_i \in F$,

$$\begin{aligned} \overline{U}_i(a) = \sum_{(f_i,w_k)\in\mu} \tilde{w}_a(\tilde{N}) - \tilde{w}_a(\tilde{N} \setminus \{f_{ik}\}) \quad &\leq \quad \tilde{w}_a(\tilde{N}) - \tilde{w}_a(\tilde{N} \setminus \{f_{i1}, f_{i2}, \ldots, f_{ir_i}\}) \\ &= \quad w_a(N) - w_a(N \setminus \{f_i\}), \end{aligned}$$

where, in contrast to the simple assignment game, the inequality $\overline{U}_i(a) \leq w_a(N) - w_a(N \setminus \{f_i\})$ may be strict.

The set of total payoffs (U, V) to the agents in the multiple-partners assignment game has been studied in Fagebaume et al. (2010), where it is proved that the maximum of any pair of stable (total) payoffs for the firms is stable but the minimum need not be, even if we restrict the multiplicity of partnerships to one of the sides.

The aim of the present paper is to study the properties of stable allocation rules. An allocation rule selects a feasible outcome for each multiple-partners job market.

Definition 2.3. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule φ consists of maps $(u, v; \mu)$ from valuation profiles $a \in \mathscr{A}^{F\times W}$ to feasible outcomes $(u(a), v(a); \mu(a))$. That is, for each $a \in \mathscr{A}^{F\times W}$, $\varphi(a) \equiv (u(a), v(a); \mu(a))$ is a feasible outcome for (F, W, a, r, s).*

An allocation rule is a stable rule if it always selects a stable outcome.

Definition 2.4. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule $\varphi \equiv (u, v; \mu)$ is a stable rule if for each valuation profile $a \in \mathscr{A}^{F\times W}$, $\varphi(a) \equiv (u(a), v(a); \mu(a))$ is a stable outcome for (F, W, a, r, s).*

In the next section we study some outstanding stable rules: the firms-optimal stable rules, that for each valuation profile select the firms-optimal stable payoffs together with a compatible matching, and the workers-optimal stable rules, that select the workers-optimal stable payoffs with a compatible matching. Notice from the above discussion of the literature, that for each of these two type of rules the associated payoff vector is uniquely determined, although the compatible matching may not be unique.

2.3. Valuation monotonicity properties

We begin by considering some monotonicity properties that are satisfied by the optimal stable rules in the simple assignment game and we see whether they are satisfied by the corresponding rules in the multiple-partners assignment game. The first one is *pairwise monotonicity*. A rule for the simple assignment game is pairwise monotonic if whenever a single valuation of the market weakly increases and the remaining ones do not change, then the rule does not decrease the payoff of neither the firm nor the worker related to that valuation. It turns out that both optimal stable rules, and also the fair division rules, are pairwise monotonic (Núñez and Rafels, 2002). If we want to discriminate between these stable rules, we need to consider different changes in the valuation profile. In van den Brink et al. (2021), a rule for the simple assignment game is said to be *firm-valuation monotonic*[2] if whenever all valuations of a single firm weakly decrease but this does not change which worker is hired by this firm, then the payoff to this firm cannot increase. It turns out that firms-optimal stable rules are the only stable rules for the simple assignment game that are firm-valuation monotonic. Of course, parallel definitions and results follow for the workers-optimal stable rule.

Let us now generalize the definition of the above monotonicity properties to the multiple-partners assignment game. Notice in the next definition that we can easily compare the payoffs a firm receives in different matchings since we requiere that the firm keeps the same partners after decreasing the valuations.

Definition 2.5. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule $\varphi \equiv (u, v, ; \mu)$ satisfies* ***firm-valuation monotonicity (FVM)*** *if for all $a, a' \in \mathscr{A}^{F\times W}$ such that there is a firm $f_t \in F$ such that $a'_{ij} = a_{ij}$ for all $f_i \in F \setminus \{f_t\}$ and all $w_j \in W$ and $a'_{tj} \leq a_{tj}$ for all $w_j \in W$, then*

$$\mu_{f_t}(a) = \mu_{f_t}(a') \Rightarrow u_{tk}(a') \leq u_{tk}(a) \text{ for all } k \in \mu_{f_t}(a).$$

FVM means that if all valuations of a firm weakly decrease but this does not modify which workers it is assigned to, then the rule cannot give this firm a higher payoff in any of its partnerships.

[2] The market considered in van den Brink et al. (2021) is formed by buyers and sellers and hence this property is called there *buyer-valuation monotonicity*

When defining a pairwise monotonicity property for the multiple-partners assignment game, in order to be able to compare the payoffs before and after the change of a value, we also need to require that the two agents related to the value that has increased or decreased keep the same partners after this change. Hence, when applied to rules for the simple assignment game, this property is weaker than the usual pairwise monotonicity for these games.

Definition 2.6. *Fix a set F of firms with quotas r and a set W of workers with quotas s. An allocation rule $\varphi \equiv (u,v,;\mu)$ satisfies* ***pairwise monotonicity (PM)*** *if for all $a,a' \in \mathscr{A}^{F\times W}$ such that there is a firm-worker pair $(f_t,w_k) \in F\times W$ such that $a'_{ij} = a_{ij}$ if $(f_i,w_j) \neq (f_t,w_k)$ and $a'_{tk} \leq a_{tk}$, then $\mu_{f_t}(a) = \mu_{f_t}(a')$ and $\mu_{w_k}(a) = \mu_{w_k}(a')$ imply*

$$u_{tj}(a') \leq u_{tj}(a) \text{ for all } j \in \mu_{f_t}(a) \text{ and } v_{ik}(a') \leq v_{ik}(a) \text{ for all } i \in \mu_{w_k}(a).$$

The next example shows that firms-optimal stable rules do not satisfy any of the above monotonicity properties.

Example 2.1. *Let be a multiple partner assignment game with three firms, $F = \{f_1,f_2,f_3\}$ and three workers, $W = \{w_1,w_2,w_3\}$, all agents with quota 2, and valuation matrix*

$$a = \begin{pmatrix} 4.5 & 20 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

There is only one optimal matching, given by

$$\mu = \{(f_1,w_2),(f_1,w_3),(f_2,w_1),(f_2,w_2),(f_3,w_1),(f_3,w_3)\},$$

and hence any stable rule must select this matching μ. The worth of the grand coalition is $w_a(N) = 36$.

Assume the value a_{11} increases in 0.1, that is $a'_{11} = 4.6$ and the other values remain unchanged. Hence, the new valuation matrix is

$$a' = \begin{pmatrix} 4.6 & 20 & 4 \\ 5 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

and notice that μ is also the only optimal matching for (F,W,a',r,s).

To compute the payoffs in the firms-optimal stable rule, we obtain a related simple assignment game as in Sotomayor (1992): $\tilde{F} = \{f_{11},f_{12},f_{21},f_{22},f_{31},f_{32}\}$, $\tilde{W} =$

$\{w_{11}, w_{21}, w_{12}, w_{22}, w_{13}, w_{23}\}$ *and*

$$\tilde{a} = \left(\begin{array}{cc|cc|cc} 4.5 & 4.5 & 20 & 0 & 0 & 0 \\ 4.5 & 4.5 & 0 & 0 & 4 & 0 \\ \hline 5 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 & 1 & 1 \\ \hline 0 & 2 & 3 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 2 \end{array} \right).$$

Here, $\overline{u}_{11}(\tilde{a}) = 36 - 17.5 = 18.5 = \overline{u}_{12}(a)$*. If we replace in* $\tilde{a}$ *the 4.5 entries with 4.6, the resulting matrix* $\tilde{a}'$ *is a simple assignment game related to the valuation matrix* a' *and we can easily check that*

$$\overline{u}_{11}(\tilde{a}') = 36 - 17.6 = 18.4 = \overline{u}_{12}(a') < \overline{u}_{12}(a).$$

As a consequence, firms-optimal stable rules are neither firm-valuation monotonic nor pairwise monotonic.

One may also ask about the behaviour of the total payoff of a firm in front of these changes. But in this example it is easy to check that $\overline{u}_{12}(\tilde{a}') = \overline{u}_{12}(\tilde{a}) = 36 - 32 = 4 = \overline{u}_{13}(a') = \overline{u}_{13}(a)$*. Hence,* $\overline{U}_1(a') = 22.4 < 22.5 = \overline{U}_1(a)$*.*

Taking into account the example above, we strengthen the requirement of firm-monotonicity by assuming that all valuations of a given firm are decreased by the same constant amount. Analogously, we will study a new worker-monotonicity property assuming the valuations of all firms with respect to a given worker decrease by the same constant amount. We will see how the two optimal stable rules react to these changes in the valuations.

2.3.1. Firm-covariance of the firms-optimal stable rules

We consider a multiple-partners assignment game (F, W, a, r, s) that is "balanced", in the sense that $\sum_{i \in F} r_i = \sum_{j \in W} s_j$. This assumption is without loss of generality since we could always add a fake dummy agent with the necessary quota. We analyse the behaviour of an allocation rule when the valuations of a firm f_{i_0} decrease by the same amount $c \geq 0$, under the assumption that values that become negative are truncated at zero: $a^c_{i_0 j} = \max\{0, a_{i_0 j} - c\}$ for all $w_j \in W$. These values are allowed to decrease in this way as long as no optimal matching of the initial problem becomes non-optimal. We then say that a rule for the multiple-partners assignment game is *firm-covariant* if the firm pays this cost c in each of its partnerships.

This property can be interpreted by saying that if a constant fee c is applied to some firm whenever it hires a worker, then this fee is completely paid by the firm, and not shared with the workers that it hires.

Definition 2.7. *A rule $\varphi \equiv (u,v;\mu)$ is* ***firm-covariant (FC)*** *if for all (F,W,a,r,s), all $f_{i_0} \in F$ and all $c \geq 0$ such that*

(i) $a^c_{i_0 j} = \max\{0, a_{i_0 j} - c\}$ *for all $w_j \in W$ and $a^c_{ij} = a_{ij}$ for all $f_i \in F \setminus \{f_{i_0}\}$,*

(ii) $c \leq a_{i_0 j}$ *for all $(f_{i_0}, w_j) \in \mu$ and $\mu \in \mathscr{M}_a(F,W)$ and*

(iii) $\mathscr{M}_a(F,W) \subseteq \mathscr{M}_{a^c}(F,W)$,

then,

$$\begin{array}{rcll} u_{i_0 j}(a^c) & = & u_{i_0 j}(a) - c & \text{for all } (f_{i_0}, w_j) \in \mu, \text{ and} \\ u_{ij}(a^c) & = & u_{ij}(a), & \text{for all } f_i \in F \setminus \{f_{i_0}\} \text{ and } (f_i, w_j) \in \mu. \end{array}$$

Notice that conditions (ii) and (iii) together imply that the worth of the grand coalition is still attained at the original optimal matchings.

As we remark after Definition 2.15 in the Appendix, requiring that c satisfies conditions (ii) and (iii) in Definition 2.7 is equivalent to requiring $c \leq c^*$ where this threshold c^*, as defined in (2.5), is the minimum $c \geq 0$ such that there is an optimal matching of (F,W,a^c,r,s) with a zero entry.

When we analyse if the firms-optimal stable rules satisfy this property, we may consider the firms-optimal stable rules of the related simple assignment game and study there how the payoff of such a rule changes when all the copies of a given firm decrease their valuations by the same amount $c \geq 0$. To this end, in the Appendix we introduce the property of strong firm-covariance for stable rules of the simple assignment game, by requiring that several firms decrease their valuations in a given constant, and we provide an axiomatic characterization of their firms-optimal stable rules making use of this property. This strong firm-covariance can be defined analogously for the multiple-partners assignment game.

An analogous covariance property can be defined when all the valuations of a given worker are decreased by a constant amount.

Definition 2.8. *A rule $\varphi \equiv (u,v;\mu)$ is* ***worker-covariant (WC)*** *if for all (F,W,a,r,s), all $w_{j_0} \in W$ and all $c \geq 0$ such that*

(i) $a^c_{ij_0} = \max\{0, a_{ij_0} - c\}$ *for all $f_i \in F$ and $a^c_{ij} = a_{ij}$ for all $w_j \in W \setminus \{w_{j_0}\}$,*

(ii) $c \leq a_{ij_0}$ *for all $(f_i, w_{j_0}) \in \mu$ and $\mu \in \mathscr{M}_a(F,W)$ and*

(iii) $\mathscr{M}_a(F,W) \subseteq \mathscr{M}_{a^c}(F,W)$, *then,*

$$\begin{aligned} u_{ij_0}(a^c) &= v_{ij_0}(a) - c \quad \text{for all } (f_i, w_{j_0}) \in \mu, \text{ and} \\ u_{ij}(a^c) &= u_{ij}(a), \qquad \text{for all } w_j \in W \setminus \{w_{j_0}\} \text{ and } (f_i, w_j) \in \mu. \end{aligned}$$

The next characterization of the firms-optimal stable rules of the multiple-partners assignment game follows from the results on the simple assignment game developed in the Appendix. We could also state this result replacing firm-covariance (worker-covariance) with strong firm-covariance (strong worker-covariance), since in any case it relies on the strong covariance of the optimal stable rules of the simple assignment game.

Theorem 2.1. 1. *The firms-optimal stable rules are the only stable rules for the multiple-partners assignment game that are firm-covariant.*

2. *The workers-optimal stable rules are the only stable rules for the multiple-partners assignment game that are worker-covariant.*

Proof. Let (F,W,a,r,s) be a multiple-partners assignment game. Let $f_{i_0} \in F$ and $c \geq 0$ that satisfies the conditions in Definition 2.7. Take some $\mu \in \mathcal{M}_a(F,W,r,s)$ and let $(\tilde{F},\tilde{W},\tilde{a})$ be a related simple assignment game where firms and workers have been replicated according their capacity and the valuations are as described in (2.2), given that $\mu \in \mathcal{M}_a(F,W,r,s)$. Let (F,W,a^c,r,s) be the multiple-partners assignment game with a^c as in Definition 2.7. Notice that when we replicate this market we obtain $(\tilde{F},\tilde{W},\tilde{a^c})$ and the valuations satisfy $\widetilde{a^c} = \tilde{a}^{c,I}$, as in Definition 2.15 in the Appendix, where I consists of the r_{i_0} copies of firm f_{i_0}.

As a consequence, if $\overline{u}_{ik}(\tilde{a})$ and $\overline{u}_{ik}(\tilde{a^c})$ are the maximum stable payoffs of the k copy of firm f_i in $(\tilde{F},\tilde{W},\tilde{a})$ and $(\tilde{F},\tilde{W},\tilde{a^c})$, respectively, then from Proposition 2.5 in the Appendix,

$$\overline{u}_{i_0k}(\tilde{a^c}) = \overline{u}_{i_0k}(\tilde{a}) - c \quad \text{and} \quad \overline{u}_{ik}(\tilde{a^c}) = \overline{u}_{ik}(\tilde{a}) \text{ if } i \neq i_0.$$

Hence, if $(f_{i_0}, w_j) \in \mu$ and $(f_{i_0k}, w_{lj}) \in \tilde{\mu}$,

$$\overline{u}_{i_0j}(a^c) = \overline{u}_{i_0k}(\tilde{a^c}) = \overline{u}_{i_0k}(\tilde{a}) - c = \overline{u}_{i_0j}(a) - c.$$

Similarly, if $f_i \in F \setminus \{i_0\}$, $(f_i, w_j) \in \mu$ and $(f_{ik}, w_{lj}) \in \tilde{\mu}$, then

$$\overline{u}_{ij}(a^c) = \overline{u}_{ik}(\tilde{a^c}) = \overline{u}_{ik}(\tilde{a}) = \overline{u}_{i_0j}(a),$$

which shows that the firms-optimal stable rules of the multiple-partners assignment game are firm-covariant.

The converse implication is straightforward since any stable rule for the multiple-partners assignment game that is FC induces a stable rule for the simple assignment game that is strong firm-covariant, and by Theorem 2.3 in the Appendix we know this can only be a firms-optimal stable rule. ■

If we are interested in the maximum total payoff of the firms in a stable outcome, then we have

$$\overline{U}_{i_0}(a^c)=\sum_{k=1}^{r_{i_0}}\overline{u}_{i_0k}(\tilde{a^c})=\sum_{k=1}^{r_{i_0}}(\overline{u}_{i_0k}(\tilde{a})-c)=\overline{U}_{i_0}(a)-r_{i_0}c$$

and for all $f_i\in F\setminus\{f_{i_0}\}$

$$\overline{U}_i(a^c)=\sum_{k=1}^{r_i}\overline{u}_{ik}(\tilde{a^c})=\sum_{k=1}^{r_i}\overline{u}_{ik}(\tilde{a})=\overline{U}_i(a),$$

Notice now that, as a consequence of Theorem 2.1, we deduce that the firms-optimal stable rules of the multiple partners assignment game satisfy a weaker form of valuation monotonicity. We strengthen the conditions under which a decrease of the valuations of the firm should imply a decrease in that firm's payoffs: we only require monotonicity when all valuations are decreased by the same amount.

Definition 2.9. *A rule $\varphi\equiv(U,V;\mu)$ is* ***weak firm-valuation monotonic (WFVM)*** *if for all (F,W,a,r,s), all $f_{i_0}\in F$ and all $c\geq 0$ such that*

(i) $a^c_{i_0j}=\max\{0,a_{i_0j}-c\}$ *for all* $w_j\in W$ *and* $a^c_{ij}=a_{ij}$ *for all* $f_i\in F\setminus\{f_{i_0}\}$,

(ii) $c\leq a_{i_0j}$ *for all* $(f_{i_0},w_j)\in\mu$ *and* $\mu\in\mathscr{M}_a(F,W)$ *and*

(iii) $\mathscr{M}_a(F,W)\subseteq\mathscr{M}_{a^c}(F,W)$,

then,

$$u_{i_0j}(a^c)\quad\leq\quad u_{i_0j}(a)\qquad \textit{for all } (f_{i_0},w_j)\in\mu.$$

Since the firms-optimal stable rules are firm-covariant, they trivially satisfy weak firm-valuation monotonicity.

Corollary 2.1. *On the domain of multiple-partners assignment game, the firms-optimal stable rules satisfy weak firm-valuation monotonicity.*

We can analogously define weak worker-valuation monotonicity (WWVM). A rule for the multiple-partners assignment game is weak worker-valuation monotonic if whenever the values all firms obtain with a given worker decrease by the same amount (with truncation to avoid negative valuations), in such a way that all optimal matchings of the initial market still remain optimal, then the payoff this worker obtains in each partnership does not increase. It is then obtained that the workers-optimal stable rules are weak worker-valuation monotonic.

2.4. Non-manipulability properties

Given a multiple-partners job market, when an allocation rule is to be adopted, then firms and workers are required to report their valuations and this induces a strategic game. Recall that each firm f_i values in $h_{ij} \geq 0$ the possibility of hiring worker w_j and each worker w_j has a reservation value $t_j \geq 0$ and will not accept being hired with a salary below his/her reservation value. Once agents report their valuations, an allocation rule $\varphi(h,t)$ selects a matching $\mu \in \mathscr{M}(F,W,r,s)$ and determines how to split the net profit $a_{ij} = \max\{h_{ij} - t_j, 0\}$ of each partnership $(f_i, w_j) \in \mu$. We may assume the rule simply determines the salary m_{ij} that firm f_i pays to worker w_j if they are matched. Then, in the partnership (f_i, w_j), the payoff of the firm is $u_{ij} = h_{ij} - m_{ij}$ and the payoff of the worker is $v_{ij} = m_{ij} - t_j$.

The question is whether a firm (or a worker) has incentives not to report its true valuations, once known which allocation rule will be applied. In particular, we want to study whether firms (workers) have incentives to manipulate the firms-optimal (workers-optimal) stable rule, since it is something they cannot do in the simple assignment game.

For this strategic analysis, and since population will not change, we may consider the sets of firms and workers, F and W, and their capacities fixed. Then, for any reported valuations (h,t), a firms-optimal stable rule selects an optimal matching μ and for all $(f_i, w_j) \in \mu$ determines a salary $\underline{m}_{ij}$ such that $\underline{v}(h,t)_{ij} = \underline{m}_{ij} - t_j$, where $(\overline{u}(h,t), \underline{v}(h,t))$ is the firms-optimal stable payoff vector. Similarly, the workers-optimal stable rule selects an optimal matching μ and for all $(f_i, w_j) \in \mu$ determines a salary $\overline{m}_{ij}$ such that $\overline{v}(a)_{ij} = \overline{m}_{ij} - t_j$, where $(\underline{u}(h,t), \overline{v}(h,t))$ is the workers-optimal stable payoff vector according the reported valuations.

From Pérez-Castrillo and Sotomayor (2017), that studies the manipulability of the optimal competitive equilibrium rules of the multiple-partners assignment game, and taking into account that every firms-optimal stable rule coincides with a firms-optimal competitive equilibrium rule, we deduce that these rules are manipulable by any firm with capacity greater than one. However, in the example provided in Pérez-Castrillo and Sotomayor (2017), the firm that manipulates the firms-optimal stable rule increases its valuations in a non-constant way, that is, it increases some valuations but not all of them by the same amount.

We may think that "naive" firms, when trying to manipulate an allocation rule, only consider whether to increase or decrease all its valuations by the same constant amount. This idea leads to a weaker non-manipulability property.

Definition 2.10. *Let F be a set of firms with capacities $r=(r_i)_{i\in F}$ and W a set of workers with capacities $s=(s_j)_{j\in W}$. A firm $f_{i_0}\in F$* ***manipulates*** *a rule $\varphi\equiv(m;\mu)$ in a multiple-partners job market (F,W,h,t,r,s)* ***by constantly over-reporting its valuations*** *if there exists $c>0$ such that f_{i_0} gets a higher payoff at $(v(h',t);\mu(h',t))$ than at $(v(h,t);\mu(h,t))$, where $h'_{i_0j}=h_{i_0j}+c$ for all $w_j\in W$ and $h'_{ij}=h_{ij}$ for all $f_i\in F\setminus\{f_{i_0}\}$ and all $w_j\in W$.*

We intend to make use of the firm-covariance property that we introduced in the previous section. But notice that the fact that $h'_{i_0j}=h_{i_0j}+c$ for all $w_j\in W$ does not imply $a'_{i_0j}=\max\{h'_{i_0j}-t_j,0\}=a_{i_0j}+c$, since for some $w_j\in W$ it may happen that $h_{i_0j}-t_j<0$. Because of that, we will restrict the study to the domain of multiple-partners job market where all firm-worker pairs are *mutually acceptable*, that is, $h_{ij}-t_j\geq 0$ for all $(f_i,w_j)\in F\times W$.

Proposition 2.1. *On the domain of multiple-partners job market where all firm-worker pairs are mutually acceptable, no firm can manipulate a firms-optimal stable rule by constantly over-reporting its valuations.*

Proof. Let (F,W,h,t,r,s) be a multiple-partners job market such that $h_{ij}-t_j\geq 0$ for all $(f_i,w_j)\in F\times W$. If firm f_{i_0} reports $h'_{i_0j}=h_{i_0j}+c$ for some $c>0$, then $a'_{i_0j}=\max\{h'_{i_0j}-t_j,0\}=a_{i_0j}+c$ and both markets have the same set of optimal matchings. From Theorem 2.1 and the proof of Proposition 2.5 the salaries m'_{ij} determined by $\varphi(h',t)$, where φ is the firms-optimal stable rule, are the same as the salaries m_{ij} determined by $\varphi(h,t)$, since $\underline{v}_{ij}(a')=\underline{v}_{ij}(a)$, for each (f_i,w_j) in an optimal matching $\mu\in\mathscr{M}_a(F,W,r,s)$. Then, for all $(f_{i_0},w_j)\in\mu$

$$h_{i_0j}-m'_{i_0j}=h_{i_0j}-(\underline{v}_{i_0j}(a')+t_j)=h_{i_0j}-(\underline{v}_{i_0j}(a)+t_j)=h_{i_0j}-m_{i_0j}.$$

Hence, the total payoff of firm f_{i_0} does not improve when reporting h'_{i_0}:

$$\overline{U}_{i_0}(a')=\sum_{(i_0,j)\in\mu}h_{i_0j}-m'_{i_0j}=\sum_{(i_0,j)\in\mu}h_{i_0j}-m_{i_0j}=\overline{U}_{i_0}(a).$$

∎

Notice that, because each firm may value differently each worker in the market, firms may have more sophisticated strategies than the constant over-reporting of Definition 2.10. Take for instance Example 4.2 in Pérez-Castrillo and Sotomayor

(2017) that consists in a market with three workers with capacity one and null reservation value and two firms, the first of them with capacity two, with valuations $h_1 = (7,6,4)$ and $h_2 = (8,6,3)$. Notice that all firm-worker pairs are acceptable. Since there is only one optimal matching, this is the matching selected by any stable rule, $\mu = \{(f_1,w_2),(f_1,w_3),(f_2,w_1)\}$. In the firms-optimal stable rule, f_1 pays salaries $\underline{v}_{12}(a) = 1$ and $\underline{v}_{13}(a) = 0$, with a net profit of $\overline{U}_1(a) = 9$. If f_1 reports $h'_1 = (8,7,7)$, which is a non-constant over-report of its valuations, then the optimal matching does not change but now the salaries paid by f_1 in the firms-optimal stable rule are $\underline{v}_{12}(a') = \underline{v}_{13}(a') = 0$ and the payoff of f_1, taking into account its true valuations, is 10.

Instead, the reservation value of a worker does not depend on which firm he/she is matched to. Hence, when a worker under-reports his reservation value, his/her net valuations with all firms increase by the same amount, provided all firm-worker pairs are acceptable.

Definition 2.11. *Let F be a set of firms with capacities $r = (r_i)_{i\in F}$ and W a set of workers with capacities $s = (s_j)_{j\in W}$. A worker $w_{j_0} \in W$* ***manipulates*** *a rule $\varphi \equiv (m;\mu)$ in a multiple-partners job market (F,W,h,t,r,s)* ***by under-reporting his/her reservation value*** *if there exists $0 \le c \le t_{j_0}$ such that w_{j_0} gets a higher payoff at $(v(h,t');\mu(h,t'))$ than at $(v(h,t);\mu(h,t))$, where $t'_{j_0} = t_{j_0} - c$ and $t'_j = t_j$ for all $w_j \in W \setminus \{w_{j_0}\}$.*

Notice that given a multiple-partners job market where all firm-worker pairs are acceptable, then all firm-worker pair in the market that results when some worker under-reports his/her reservation value are also acceptable.

Proposition 2.2. *On the domain of multiple-partners job market where all firm-worker pairs are mutually acceptable, no worker can manipulate the workers-optimal stable rule by under-reporting his/her reservation value.*

Proof. Let (F,W,h,t,r,s) be a multiple-partners job market such that $h_{ij} - t_j \ge 0$ for all $(f_i,w_j) \in F \times W$. If worker w_{j_0} reports $t'_{j_0} = t_{j_0} - c$ for some $c > 0$, then $a'_{ij_0} = \max\{h'_{ij_0} - t_{j_0}, 0\} = a_{ij_0} + c$ and both markets have the same set of optimal matchings. From Theorem 2.1 and the proof of Proposition 2.5 the salary m'_{ij_0} determined by $\varphi(h,t')$, where φ is the workers-optimal stable rule is the same as the salary m_{ij_0} determined by $\varphi(h,t)$:

$$m'_{ij_0} = \overline{v}_{ij_0}(h,t') + t'_{j_0} = \overline{v}_{ij_0}(h,t) + c + t_{j_0} - c = m_{ij_0}.$$

Hence, given any $\mu \in \mathscr{M}_a(F,W,r,s)$, the payoff to worker w_{j_0} in each partnership $(f_i,w_{j_0}) \in \mu$ is $m'_{ij_0} - t_{j_0} = m_{ij_0} - t_{j_0}$ and w_{j_0} has no incentives to report t'_{j_0} instead of t_j. ∎

This may not be the case when a worker over-reports his/her reservation value. Example 4.1 in Pérez-Castrillo and Sotomayor (2017) shows that in that case such a worker may manipulate the workers-optimal stable rule.

However, the above non-manipulability properties do not characterize neither the firms-optimal stable rules nor the workers-optimal stable rules on the domain where all firm-worker pairs are acceptable. Notice for instance that the workers-optimal stable rule is also non-manipulable by constant over-reporting of one firm's valuations. Take (F,W,a,r,s) where all pairs are acceptable, a firm $f_{i_0} \in F$ and an optimal matching $\mu \in \mathscr{M}_a(F,W)$. Assume $h'_{i_0 j} = h_{i_0 j} + c$ for some $c > 0$, while $h'_{ij} = h_{ij}$ for $i \in F \setminus \{f_{i_0}\}$ and $j \in W$. This implies $a'_{i_0 j} = a_{i_0 j} + c$ for all $j \in W$, while $a'_{ij} = a_{ij}$ otherwise. Consider the related simple assignment game $(\tilde{F}, \tilde{W}, \tilde{a})$ and the corresponding optimal matching $\tilde{\mu}$. If $w_{j_0} \in W$ is such that $(f_{i_0 k}, w_{l j_0}) \in \tilde{\mu}$, where $f_{i_0 k}$ and $w_{l j_0}$ are copies of f_{i_0} and w_{j_0} respectively, then $\overline{v}_{i_0 j_0}(a') = \overline{v}_{l j_0}(\tilde{a}') = w_{\tilde{a}'}(\tilde{F} \cup \tilde{W}) - w_{\tilde{a}'}(\tilde{F} \cup (\tilde{W} \setminus \{w_{l j_0}\}))$ is either $\overline{v}_{i_0 j_0}(a) + c$ or $\overline{v}_{i_0 j_0}(a)$, since $\mu' \in \mathscr{M}_{\tilde{a}'}(\tilde{F}, \tilde{W} \setminus \{w_{l j_0}\})$ if and only if $\mu' \in \mathscr{M}_{\tilde{a}}(\tilde{F}, \tilde{W} \setminus \{w_{l j_0}\})$. This means $\overline{v}_{i_0 j_0}(a') \geq \overline{v}_{i_0 j_0}(a)$ and hence $h_{i_0 j_0} - \overline{v}_{i_0 j_0}(a') \leq h_{i_0 j_0} - \overline{v}_{i_0 j_0}(a)$ and i_0 has no incentives to constantly over-report its valuations.

2.5. The fair division rules

In some situations, specially in two-sided markets without money, it is usual to implement an allocation rule that favours one side of the market. Take for instance the allocation of students to colleges or resident doctors to hospitals. But in a job market, and also in a market with buyers and sellers, it makes sense to assume that matched agents enter a negotiation and agree on a price or salary that is in between those that are optimal for each side.

We extend to the multiple-partners job market the notion of fair division payoff vector that was introduced by Thompson (1981) for the simple assignment game as the midpoint between the two optimal stable payoff vectors. That is, given a set F of firms with quotas r and a set of workers W with quotas s, and a valuation profile $a \in \mathscr{A}^{F_0 \times W_0}$, a fair division rule is $\varphi^\tau \equiv (u^\tau(a), v^\tau(a); \mu)$ where

$$u^\tau_{ij}(a) = \frac{1}{2}\overline{u}_{ij}(a) + \frac{1}{2}\underline{u}_{ij}(a) \text{ and } v^\tau_{ij}(a) = \frac{1}{2}\overline{v}_{ij}(a) + \frac{1}{2}\underline{v}_{ij}(a) \text{ for all } (f_i, w_j) \in \mu$$

and μ is a compatible matching. Notice that there may be several compatible matchings but the payoff vector is uniquely defined.

In van den Brink et al. (2021), and for the simple assignment game, the fair division rule is characterized as the only stable rule that satisfies grand valuation fairness and weak derived consistency. Our aim is to extend these two properties to the multiple-partners job market and see whether they also individualize the fair division rules. To extend the notion of (derived) consistency, that reflects how a solution behaves when some agents leave the market, we will allow in this section for positive values a_{i0} and a_{0j}, for all $f_i \in F$ and $w_j \in W$. Hence, now a valuation profile is $(a_{ij})_{\substack{i\in F_0\\ j\in W_0}}$ with $a_{00} = 0$, and we denote by $\mathscr{A}^{F_0\times W_0}$ the set of all valuation profiles. We assume again that all firm-worker pairs are mutually acceptable, which in the notation just introduced translates to saying that for all $(f_i, w_j) \in F^0 \times W^0$, $a_{ij} \geq a_{i0} + a_{0j}$.

Roughly speaking, grand valuation fairness requires that if all firm-worker valuations decrease by a same amount $c \geq 0$, as long as all optimal matchings of the initial market remain optimal, the payoff all agents receive from each partnership decreases equally.

Definition 2.12. *A rule $\varphi \equiv (u, v; \mu)$ satisfies great valuation fairness (GVF) if for all (F,W,a,r,s) and all $c \geq 0$ such that*

(i) $a^c_{ij} = \max\{0, a_{ij} - c\}$ *for all $f_i \in F$ and $w_j \in W$,*

(ii) $c \leq a_{ij}$ *for all $(f_i, w_j) \in \mu$ and $\mu \in \mathscr{M}_a(F,W,r,s)$ and*

(iii) $\mathscr{M}_a(F,W) \subseteq \mathscr{M}_{a^c}(F,W,r,s)$,

then,

$$u_{ij}(a^c) - u_{ij}(a) = v_{ij}(a^c) - v_{ij}(a) \text{ for all } (f_i, w_j) \in \mu. \tag{2.4}$$

From firm-covariance and worker-covariance of the two optimal stable rules, it follows quite straightforwardly that the fair division rules satisfy GVF.

Proposition 2.3. *On the domain of multiple-partners assignment markets, the fair division rules satisfy GVF.*

Proof. Recall that the minimum c satisfying (i), (ii) and (iii) is the c^* defined in (2.5). As a consequence, if a multiple-partners job market (F,W,a,r,s) is "unbalanced", in the sense that $\sum_{f_i\in F} r_i \neq \sum_{w_j\in W} s_j$, then $c^* = 0$ and GVF does not impose any restriction. Hence, we may focus on markets where the sum of capacities of firms equals those of workers.

Let (F,W,a,r,s) be a multiple-parters job market with $\sum_{f_i\in F} r_i = \sum_{w_j\in W} s_j$, μ an optimal matching and $c \geq 0$ satisfying (i), (ii), and (iii) in Definition 2.12. By strong firm-covariance of the firms-optimal stable rules, taking $I = F$, that is, assuming all firms decrease their valuations in c, we have that $\overline{u}_{ij}(a^c) = \overline{u}_{ij}(a) - c$ and $\underline{v}_{ij}(a^c) = \underline{u}_{ij}(a)$ for all $(f_i, w_j) \in \mu$. Similarly, from worker-covariance of the workers-optimal stable rule, taking $I=W$, we have $\overline{v}_{ij}(a^c)=\overline{v}_{ij}(a)-c$ and $\underline{u}_{ij}(a^c)=\underline{v}_{ij}(a)$ for all $(f_i,w_j)\in\mu$. As a consequence,

$$u^{\tau}_{ij}(a^c)=u^{\tau}_{ij}(a)-\frac{c}{2} \text{ and } v^{\tau}_{ij}(a^c)=v^{\tau}_{ij}(a)-\frac{c}{2} \text{ for all } (f_i,w_j)\in\mu$$

which implies $u^{\tau}_{ij}(a^c)-u^{\tau}_{ij}(a)=\frac{c}{2}=v^{\tau}_{ij}(a^c)-v^{\tau}_{ij}(a)$ and GVF holds. ■

The idea now is to proceed as in the simple assignment game (van den Brink et al., 2021). In that case, we decrease all firm-worker values until reaching the threshold c^*; at this point there is a firm-worker pair (f_i,w_j) whose payoff is fixed, and equal to their individual values a_{i0} and a_{0j}, at any stable outcome. Then, these two agents leave the market with their fixed payoff and we must define the reduced game in such a way that the fair division rule is consistent with respect to this reduction.

For the simple assignment game, a notion of reduced game is introduced in Owen (1992) with the name of derived game. Several solution concepts, such as the core, the optimal stable rules for any side of the market or the nucleolus (Llerena et al., 2015), are derived consistent, that meaning that when we restrict the solution payoff vector to the agents that remain in the derived market game, we get a solution payoff vector of the derived market game. The fair-division rules are not derived consistent, unless the agents that leave the market have a unique stable payoff.

We now propose how to reduce a multiple-partners job market when an individual or a firm-worker pair have a unique stable payoff. Since agents may have capacities that allow for multiple partnership, the firm and worker in that pair may not leave the market but simply reduce their capacities in one unit.

Definition 2.13. *Let (F,W,a,r,s) be a multiple-partner assignment market, μ an optimal matching, $T=\{f_i,w_j\}$ with $(f_i,w_j)\in\mu$ such that $a_{ij}=a_{i0}+a_{0j}$ and $z=(u,v;\mu)$ a stable outcome.*

*The **derived assignment market** relative to T at z is $(F^T,W^T,a^{T,z},r^T,s^T)$ where*

$$F^T=\begin{cases} F\setminus\{f_i\} & \text{if } f_i\in F,\ r_i=1,\\ F & \text{otherwise}\end{cases} \quad \text{and}$$

$$W^T=\begin{cases} W\setminus\{w_j\} & \text{if } w_j\in W,\ s_j=1,\\ W & \text{otherwise}\end{cases},$$

$a^{T,z}_{kl}=a_{kl}$ for all $f_k\in F^T$, $w_l\in W^T$,

$$\begin{aligned}(i)\ a^{T,z}_{k0} &= \max\left\{a_{k0},a_{kj}-v_{ij}\right\}, \text{ for all } f_k\in F^T,\\ (ii)\ a^{T,z}_{0k} &= \max\left\{a_{0k},a_{ik}-u_{ij}\right\}, \text{ for all } w_k\in W^T,\end{aligned}$$

and $r^T_k=r_k-1$ if $f_k\in T\cap F^T$, $r^T_k=r_k$ if $f_k\in F^T\setminus T$,
$s^T_k=s_k-1$ if $w_k\in T\cap W^T$, $s^T_k=s_k$ if $w_k\in W^T\setminus T$.

In the derived assignment market relative to a coalition $T = \{f_i, w_j\}$ such that $a_{ij} = a_{i0} + a_{0j}$, (i) values for the firm-worker pairs 'that are still in the market' are the same as in the original market, (ii) the individual values are modified taking into account the possibilities to trade with agents outside the derived market and (iii) the capacity of each agent in T decreases in one unit.

Notice that the above definition allows for T to contain a dummy agent, like $T = \{f_i, w_0\}$ if $(f_i, w_0) \in \mu$.

Weak derived consistency means that in a derived market at a coalition $T = \{f_i, w_j\}$ such that $a_{ij} = a_{i0} + a_{0j}$, the payoffs for the firms and workers that remain in the market do not change.

Definition 2.14. *On the domain of multiple-partners job markets, a stable allocation rule φ is* ***weak derived consistent (WDC)*** *if for all market (F, W, a, r, s) and all coalition $T = \{f_i, w_j\}$ with $(f_i, w_j) \in \mu$ and $a_{ij} = a_{i0} + a_{0j}$, it holds*

$$\begin{array}{lll} (i) & \mu' = \mu \setminus \{(f_i, w_j)\} \text{ is optimal for } (F^T, W^T, a^{T,(u,v)}, r^T, s^T), & \\ (ii) & u_{kl}(F^T, W^T, a^{T,(u,v)}, r^T, s^T) = u_{kl} & \text{for all } (f_k, w_l) \in \mu' \text{ and} \\ (iii) & v_{kl}(F^T, W^T, a^{T,(u,v)}, r^T, s^T) = v_{kl} & \text{for all } (f_k, w_l) \in \mu' \end{array}$$

where $\varphi(F, W, a, r, s) = (u, v; \mu)$.

Let us first argue in the next lemma that condition (i) in the above definition always holds, since it is necessary to guarantee that u_{kl} and v_{kl} are well-defined.

Lemma 2.1. *Let $(F^T, W^T, a^{T,z}, r^T, s^T)$ be the derived game at $T = \{f_i, w_j\}$ with $(f_i, w_j) \in \mu$ such that $a_{ij} = a_{i0} + a_{0j}$ and $z = (u, v; \mu)$ is a stable outcome. Then,*

1. *$a_{k0}^{T,(u,v)} = a_{k0}$ if $(f_k, w_0) \in \mu$ (and $a_{0l}^{T,(u,v)} = a_{0l}$ if $(f_0, w_l) \in \mu$),*
2. *$(u', v'; \mu')$ is a stable outcome for $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$, where $\mu' = \mu \setminus \{(f_i, w_j)\}$, $u'_{kl} = u_{kl}$ and $v'_{kl} = v_{kl}$ for all $(f_k, w_l) \in \mu'$.*
3. *$\mu' = \mu \setminus \{(f_i, w_j)\}$ is optimal for $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$.*

Proof. Notice that $(f_k, w_0) \in \mu$, implies $a_{k0} = u_{k0} \geq a_{kj} - v_{ij}$ because of the stability of (u, v). In the same way, $(f_0, w_l) \in \mu$ implies $a_{0l} = v_{0l} \geq a_{il} - u_{ij}$, and statement 1) is proved.

To prove 2) notice that for all $(f_k, w_l) \in \mu'$, it holds $(f_k, w_l) \in \mu$ and hence $u'_{kl} + v'_{kl} = u_{kl} + v_{kl} = a_{kl} = a_{kl}^{T,(u,v)}$. This includes the case where $(f_k, w_0) \in \mu'$ and then $u'_{k0} = a_{k0} = a_{k0}^{T,(u,v)}$, and similarly for $(f_0, w_l) \in \mu$. This means that (u', v') is feasible with respect to μ'. Now, if $(f_k, w_l) \in F^T \times W^T$ with $(f_k, w_l) \notin \mu'$ then either $(f_k, w_l) \notin \mu$ and in this case $u'_{kq} + v'_{pl} = u_{kq} + v_{pl} \geq a_{kl} = a_{kl}^{T,(u,v)}$ for $(f_k, w_q) \in \mu'$ and

, $(f_p, w_l) \in \mu'$, or $(f_k, w_l) = (f_i, w_j)$. In this second case, if $(f_i, w_q), (f_p, w_l) \in \mu'$, we have $u'_{iq} + v'_{pj} = u_{iq} + v_{pj} \geq a_{i0} + v_{0j} = a_{ij}$, where the last equality follows from the assumption. Finally, if $(f_k, w_l) \in \mu'$, then $u_{kl} \geq \max\{a_{k0}, a_{kj} - v_{ij}\} = a_{k0}^{T,(u,v)}$, and similarly $v_{lk} \geq a_{0k}^{T,(u,v)}$ for all $(f_l, w_k) \in \mu'$. As a consequence, $(u', v'; \mu')$ satisfies all stability constraints.

It follows from Sotomayor (1992) that μ' is optimal for this market, since we know that $(u', v'; \mu')$ is a stable outcome for the market $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$. ■

Notice that statement 2) in Lemma 2.1 crucially depends on the fact that $a_{ij} = a_{i0} + a_{0j}$. This condition is also important to notice that for all stable payoff vectors it holds $u_{ij} = a_{i0}$ and $v_{ij} = a_{0j}$. Then, a sort of converse of statement 2) holds: under the assumptions of Lemma 2.1, if (u'', v'') is a stable payoff vector for $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$, then, by completing it with the payoffs u_{ij} and v_{ij} we obtain a payoff vector of the initial market (F, W, a, r, s). As an immediate consequence, if (u, v) is a stable payoff vector of (F, W, a, r, s), then the set of stable payoff vectors of $(F^T, W^T, a^{T,(u,v)}, r^T, s^T)$ is precisely the restriction of the set of stable payoff vectors of (F, W, a, r, s) to $F^T \cup W^T$. In particular, the restrictions of $(\overline{u}(a), \underline{v}(a))$, $(\underline{u}(a), \overline{v}(a))$ and $(u^\tau(a), v^\tau(a))$ to $F^T \cup W^T$ are, respectively, the firms-optimal stable payoff vector, the workers-optimal stable payoff vector and the fair division payoff vector of the derived game $(F^T, W^T, a^T, r^T, s^T)$. This proves the next proposition.

Proposition 2.4. *On the domain of multiple-partners job markets, the firms-optimal stable rules, the workers-optimal stable rules and the fair division rules are weak derived consistent.*

We have seen until now that the fair division rules satisfy GVF and WDC. It only remains to see that these two properties characterize these rules among all stable rules. We only sketch the proof, since it is very similar to the one for the simple assignment game in van den Brink et al. (2021).

Theorem 2.2. *On the domain of multiple-partners job markets, the fair division rules are the only stable rules that satisfy GVF and WDC.*

Proof. Let φ be a stable rule that satisfies GVF and WDC, and take a multiple-partners assignment game (F, W, a, r, s). Let $c_1 \geq 0$ be the maximum $c \geq 0$ such that $\mathscr{M}_a(F, W, r, s) \subseteq \mathscr{M}_{a^c}(F, W, r, s)$, and $c \leq a_{ij}$ (and thus $a^c_{ij} = a_{ij} - c$) for all $(f_i, w_j) \in \mu$ for some $\mu \in \mathscr{M}_a(F, W, r, s)$. Then, there is some (f_{i_1}, w_{j_1}) in an optimal matching of (F, W, a^{c_1}, r, s) such that $a^{c_1}_{i_1 j_1} = a_{i_1 0} + a_{0 j_1}$, which means that

$u_{i_1j_1} = a_{i_10}$ and $v_{i_1j_1} = a_{0j_1}$ for each stable payoff vector (u,v) of (F,W,a^{c_1},r,s). Define $T_1 = \{f_{i_1}, w_{j_1}\}$.

Let $z^\varphi(a) = (u^\varphi(a), v^\varphi(a))$ and $z^\varphi(a^{c_1}) = (u^\varphi(a^{c_1}), v^\varphi(a^{c_1}))$ be the payoff vectors selected by the rule φ when applied to (F,W,a,r,s) and (F,W,a^{c_1},r,s) respectively, and $z^\tau(a) = (u^\tau(a), v^\tau(a))$ and $z^\tau(a^{c_1}) = (u^\tau(a^{c_1}), v^\tau(a^{c_1}))$ the payoff vectors selected by any fair division rule in these markets. Trivially, because of the selection of T_1, $u^\varphi_{i_1j_1}(a^{c_1}) = u^\tau_{i_1j_1}(a^{c_1})$ and $v^\varphi_{i_1j_1}(a^{c_1}) = v^\tau_{i_1j_1}(a^{c_1})$. And by GVF of both φ and τ,

$$\begin{aligned} u^\varphi_{i_1j_1}(a) &= u^\varphi_{i_1j_1}(a^{c_1}) + \frac{c_1}{2} = u^\tau_{i_1j_1}(a^{c_1}) + \frac{c_1}{2} = u^\tau_{i_1j_1}(a) \\ v^\varphi_{i_1j_1}(a) &= v^\varphi_{i_1j_1}(a^{c_1}) + \frac{c_1}{2} = v^\tau_{i_1j_1}(a^{c_1}) + \frac{c_1}{2} = v^\tau_{i_1j_1}(a) \end{aligned}$$

Consider now

$$F_1 = \begin{cases} F \setminus \{f_{i_1}\} & \text{if } f_{i_1} \in F \text{ and } r_{i_1} = 1, \\ F & \text{otherwise} \end{cases} \quad \text{and ;}$$

$$W_1 = \begin{cases} W \setminus \{w_{j_1}\} & \text{if } w_{j_1} \in W \text{ and } s_{j_1} = 1, \\ W & \text{otherwise} \end{cases},$$

and the derived market at T_1 and $z^\varphi(a)$. That is, $a_1 = a^{T_1, z^\varphi(a)}$, $r^1 = r^{T_1}$ and $s^1 = s^{T_1}$, as in Definition 2.13. By WDC of φ, $u^\varphi_{ij}(a_1) = u^\varphi_{ij}(a^{c_1})$ and $v^\varphi_{ij}(a_1) = v^\varphi_{ij}(a^{c_1})$ for all $(f_i, w_j) \in \mu_1 = \mu \setminus \{(f_{i_1}, w_{j_1})\}$.

We now repeat the procedure, that is, given $(F_1, W_1, a_1, r^1, s^1)$ we take $c_2 \geq 0$ the maximum $c \geq 0$ such that $\mathscr{M}_{a_1}(F^1, W^1, r^1, s^1) \subseteq \mathscr{M}_{a_1^c}(F^1, W^1, r^1, s^1)$, and $c \leq (a_1)_{ij}$ (and thus $(a_1)^c_{ij} = (a_1)_{ij} - c$) for all $(f_i, w_j) \in \mu$ for some $\mu \in \mathscr{M}_{a_1}(F^1, W^1, r^1, s^1)$. Then, there is some (f_{i_2}, w_{j_2}) in an optimal matching μ_1 of $(F^1, W^1, a_1^{c_2}, r^1, s^1)$ such that $(a_1)^{c_2}_{i_2j_2} = (a_1)_{i_20} + (a_1)_{0j_2}$, which means that $u_{i_2j_2} = a_{i_20}$ and $v_{i_2j_2} = a_{0j_2}$ for each stable payoff vector (u,v) of $(F^1, W^1, (a_1)^{c_2}, r^1, s^1)$. And we define $T_2 = \{f_{i_2}, w_{j_2}\}$. Notice at this point that, from Sotomayor (1999), the components of any stable payoff vector can be reindexed according to the new optimal matching μ_1.

Since at each step the aggregated capacity strictly decreases, we can guarantee that the procedure is finite. Moreover, at each step, some payoffs u_{ij} and v_{ij}, for (f_i, w_j) optimally matched, are proved to coincide in φ and in τ, and by GVF and WDC they also coincide in the initial market. ■

The above theorem shows that the known axiomatic characterization of the fair division rules in the simple assignment game extends to the multiple-partners assignment games. As a consequence, it also follows the logical independence of the two axioms.

2.6. Concluding remarks

The axiomatic characterizations given in this paper for the two-optimal stable rules and the fair-division rules, on the domain of multiple-partners job markets, have in common that all of them rely on the behaviour of the rules when some firm-worker valuations decrease in a constant amount. This provides a unifying approach to all these stable rules.

Furthermore, from the discussion at the end of Section 4, it follows that on the domain where all firm-worker pairs are acceptable, the fair-division rules are also non-manipulable by constant over-reporting of one firm's valuations. Although this is a weak non-manipulability property, we find it interesting since it rules out a kind of manipulation that is frequently observed in experiments.

A. Appendix: Strong firm-covariance for stable rules in the simple assignment game

We analyse the behaviour of an allocation rule for the simple assignment game when the valuations of an arbitrary set I of firms decrease by the same amount $c \geq 0$, under the assumption that values that become negative are truncated at zero: $a^c_{ij} = \max\{0, a_{ij} - c\}$ for all $(f_i, w_j) \in I \times W$. These row values are allowed to decrease in this way as long as no optimal matching of the initial problem becomes non-optimal. A rule is covariant with respect this change if all firms that have seen their values decreased in c, also see their payoff decreased in c.

Definition 2.15. *A rule $\varphi \equiv (u, v; \mu)$ for the simple assignment game is **strong firm-covariant (SFC)** if for all (F, W, a), all $I \subseteq F$ and all $c \geq 0$ such that*

(i) $a^{c,I}_{ij} = \max\{0, a_{ij} - c\}$ *for all* $(f_i, w_j) \in I \times W$ *and* $a^c_{ij} = a_{ij}$ *for all* $(f_i, w_j) \in (F \setminus I) \times W$,

(ii) $c \leq a_{ij}$ *for all* $f_i \in I$, $(f_i, w_j) \in \mu$ *and* $\mu \in \mathscr{M}_a(F, W)$ *and*

(iii) $\mathscr{M}_a(F, W) \subseteq \mathscr{M}_{a^{c,I}}(F, W)$,

then,

$$\begin{aligned} u_i(a^{c,I}) &= u_i(a) - c, \text{ for all } f_i \in I \text{ and} \\ u_i(a^{c,I}) &= u_i(a), \qquad \text{for all } f_i \in F \setminus I. \end{aligned}$$

We can give a threshold for those $c \geq 0$ on the conditions of the above definition.

Lemma 2.2. *Conditions (ii) and (iii) in Definition 2.15 are equivalent to considering $c \le c^*$, where*

$$c^* = \min\{c \ge 0 \mid \exists \mu \in \mathcal{M}_{a^{c,I}}(F,W) \text{ and } (f_i,w_j) \in \mu \text{ with } f_i \in I \text{ and } a_{ij} = 0\}. \quad (2.5)$$

Proof. Let us define $m_a^I = \min\{a_{ij} \mid (f_i,w_j) \in \mu$ for some $\mu \in \mathcal{M}_a(F,W)$ and $f_i \in I\}$. It is quite clear that $c^* \le m_a^I$. Otherwise, if $m_a^I < c^*$, taking $c = m_a^I$, by definition of c^*, we have that for any $\mu \in \mathcal{M}_{a^{c,I}}(F,W)$ it holds $a_{ij}^{c,I} > 0$ for all $(f_i,w_j) \in \mu$ and $f_i \in I$. This implies that there is an optimal matching μ' of the initial market that is not optimal in $(F,W,a^{c,I})$, since by definition of m_a^I, μ' will have a null entry. But $\sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I} > \sum_{(f_i,w_j)\in\mu'} a_{ij}^{c,I}$ implies $\sum_{(f_i,w_j)\in\mu} a_{ij} > \sum_{(f_i,w_j)\in\mu'} a_{ij}$, and contradicts $\mu' \in \mathcal{M}_a(F,W)$.

We now show that if $0 \le c \le c^*$, then c satisfies (ii) and (iii) in Definition 2.15. First, since $c \le c^* \le m_a^I$, $a_{ij}^{c,I} = a_{ij} - c \ge 0$ for all $(f_i,w_j) \in \mu$, for all $\mu \in \mathcal{M}_a(F,W)$, and (ii) is satisfied. Moreover, since $c \le c^*$, by definition of c^*, we have $a_{ij}^{c,I} = a_{ij} - c \ge 0$ for all $(f_i,w_j) \in \mu \in \mathcal{M}_{a^{c,I}}(F,W)$. This implies that all $\mu \in \mathcal{M}_a(F,W)$ is also optimal for $(F,W,a^{c,I})$. Otherwise, if there exists $\mu' \in \mathcal{M}_{a^{c,I}}(F,W)$ such that $\sum_{(f_i,w_j)\in\mu'} a_{ij}^{c} > \sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I}$, then

$$\sum_{(f_i,w_j)\in\mu'} a_{ij} - |I|c = \sum_{(f_i,w_j)\in\mu'} a_{ij}^{c,I} > \sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I} = \sum_{(f_i,w_j)\in\mu} a_{ij} - |I|c,$$

in contradiction with $\mu \in \mathcal{M}_a(F,W)$.

Conversely, if c satisfies (ii) and (iii), we show that $c \le c^*$. Indeed, (ii) implies that $c \le m_a^I$. To see that $c \le c^*$, if we assume on the contrary that $c^* < c \le m_a^I$, we know that none of the matchings $\mu \in \mathcal{M}_a(F,W)$ has a null entry neither in $(F,W,a^{c^*,I})$ nor in $(F,W,a^{c,I})$. Instead, by definition of c^* there is $\mu' \in \mathcal{M}_{a^{c^*,I}}(F,W)$ with $(f_{i_0},w_{j_0}) \in \mu'$ and $a_{i_0j_0}^{c^*,I} = 0 \ge a_{i_0j_0} - c^*$. Then, since $c^* < c \le m_a^I$, $\sum_{(f_i,w_j)\in\mu'} a_{ij}^{c,I} > \sum_{(f_i,w_j)\in\mu} a_{ij}^{c,I}$ for all $\mu \in \mathcal{M}_a(F,W)$, in contradiction with (iii). ■

If a rule φ satisfies Definition 2.15 for $|I| = 1$, we will say φ is **firm-covariant (FC)**. And notice that this definition coincides with Definition 2.7 when applied to a rule for the simple assignment game. We first prove that the firms-optimal stable rules are strong firm-covariant.

We could similarly define when an allocation rule is strong worker-covariant and we would obtain, in an analogous way, that the workers-optimal stable rules are strong worker-covariant.

Proposition 2.5. *The firms-optimal stable rules of the simple assignment game are strong firm-covariant.*

Proof. We can assume without loss of generality that there are as many firms as workers (otherwise we only need to add dummy agents with null valuations in the short side of the market). If $a_{ij} = 0$ for some $f_i \in I$ such that $(f_i, w_j) \in \mu$ and $\mu \in \mathscr{M}_a(F,W)$, then only $c = 0$ satisfies the conditions on Definition 2.15, and SFC is trivially satisfied in that case. So, assume $a_{ij} > 0$ for all $(f_i, w_j) \in \mu$ such that $f_i \in I$ and $\mu \in \mathscr{M}_a(F,W)$.

Let $c \geq 0$ be a constant under the conditions of Definition 2.15, that is, $c \leq a_{ij}$ for all $(f_i, w_j) \in \mu$ with $f_i \in I$ and $\mu \in \mathscr{M}_a(F,W)$ (and thus $a^{c,I}_{ij} = a_{ij} - c$), and moreover any matching μ that is optimal for (F,W,a) is also optimal for $(F,W,a^{c,I})$.

From now on, to simplify notation, we will write just a^c instead of $a^{c,I}$.

Consider the two optimal stable payoff vectors, $(\overline{u}(a), \underline{v}(a))$ and $(\underline{u}(a), \overline{v}(a))$, for (F,W,a). Let $(\overline{u}^c(a), \underline{v}(a))$ be given by

$$\overline{u}^c(a) = \overline{u}_i(a) - c \text{ for all } f_i \in I;\ \overline{u}^c(a) = \overline{u}_i(a) \text{ for all } f_i \in F \setminus I.$$

We show that $(\overline{u}^c(a), \underline{v}(a))$ is stable for (F,W,a^c), that is, we show individual rationality for each firm and worker, and the stability requirements for each firm-worker pair.

(i) Individual rationality for the workers (i.e. $\underline{v}_j \geq 0$ for all $w_j \in W$) follows trivially from the stability of $(\overline{u}(a), \underline{v}(a))$.

(ii) The stability requirements for every firm-worker pair (i.e. $\overline{u}^c(a)_i + \underline{v}(a)_j \geq a^c_{ij}$ for all $f_i \in F$ and $w_j \in W$) follows trivially from the stability of $(\overline{u}(a), \underline{v}(a))$ (under the assumption that $\overline{u}^c_i(a) \geq 0$, which we show next under (iii)).

(iii) It only remains to show individual rationality for the firms. It is obvious that $\overline{u}^c_i(a) = \overline{u}_i(a) \geq 0$ for all $f_i \in F \setminus I$, so we only need to prove that $\overline{u}^c_i(a) = \overline{u}_i(a) - c \geq 0$ for all $f_i \in I$. This implies to show that any c on the conditions of Definition 2.15 satisfies $c \leq \min_{i \in I} \overline{u}_i(a)$. Let us denote by $f_{i_1} \in I$ the firm such that $\overline{u}_{i_1}(a) = \min_{i \in I} \overline{u}_i(a)$.

Notice first that trivially if $c' = \overline{u}_{i_1}(a)$, then $(\overline{u}^{c'}(a), \underline{v}(a))$ is stable for $(F,W,a^{c'})$. Let k be the cardinality of I, $\mu \in \mathscr{M}_a(F,W)$ and $\mu' \in \mathscr{M}_a(F \setminus \{f_{i_1}\}, W)$. Then,

$$\overline{u}_{i_1}(a) = \sum_{(f_i,w_j)\in\mu} a_{ij} - \sum_{(f_i,w_j)\in\mu'} a_{ij}. \tag{2.6}$$

Furthermore,

$$\sum_{(f_i,w_j)\in\mu'} a^{c'}_{ij} \geq \sum_{(f_i,w_j)\in\mu'} a_{ij} - (k-1)c'.$$

On the other hand, $\sum_{(f_i,w_j)\in\mu} a_{ij} - \sum_{(f_i,w_j)\in\mu} a_{ij}^{c'} = kc'$, and then

$$\begin{aligned}\sum_{(f_i,w_j)\in\mu} a_{ij}^{c'} &= \sum_{(f_i,w_j)\in\mu} a_{ij} - kc' \\ &= \sum_{(f_i,w_j)\in\mu'} a_{ij} + c' - kc' = \sum_{(f_i,w_j)\in\mu'} a_{ij} - (k-1)c' \leq \sum_{(f_i,w_j)\in\mu'} a_{ij}^{c'},\end{aligned}$$

where the second equality follows from (2.6). This implies that μ' is also optimal for $(F,W,a^{c'})$ and, as a consequence, if $w_{j_2} \in W$ is the worker unmatched by μ', then $a_{i_1 j_2}^{c'} = 0$. Otherwise, $a_{i_1 j_2}^{c'} > 0$ would contradict the optimality of μ in $(F,W,a^{c'})$.

We finally show that $c \leq \overline{u}_{i_1}(a)$. On the contrary, suppose that $c > c' = \overline{u}_{i_1}(a)$. Since $a_{i_1 j_2}^{c'} = 0$, we have $a_{i_1 j_2} - c < 0$. Then, $\{f_{i_1}, w_{j_2}\}$ belonging to an optimal matching of $(F,W,a^{c'})$ and $c > c'$ implies that the optimal matchings of (F,W,a), which entries have not been truncated, are no longer optimal in (F,W,a^c). This contradicts that c satisfies the conditions of Definition 2.15. So, we have proved that $c \leq \overline{u}_{i_1}(a)$, and as a consequence individual rationality for the firms is satisfied.

Since we showed individual rationality for the firms and the workers ((i) and (iii) above), and the stability requirements for all firm-worker pairs ((ii) above), we have that $(\overline{u}^c(a), \underline{v}(a))$ is a stable payoff vector for (F,W,a^c), for all c under the conditions of Definition 2.15. Analogously, it can be shown that $(\underline{u}(a), \overline{v}^c(a))$ is a stable payoff vector for (F,W,a^c).

Notice that, by $(\overline{u}^c(a), \underline{v}(a))$ being a stable payoff vector of (F,W,a^c), it is the optimal stable payoff vector of (F,W,a^c). Otherwise, one can derive a contradiction with $(\overline{u}(a), \underline{v}(a))$ being the optimal stable payoff vector of (F,W,a). This completes the proof of SFC for the firms-optimal stable rule. ■

The converse implication also holds. In fact, it is even stronger. Any stable rule that satisfies Definition 2.15 for $|I| = 1$ (any single row) must be the firms-optimal stable rule. We state the result for both optimal stable rules but only prove it for the firms-optimal one.

Theorem 2.3. 1. *The firms-optimal stable rules are the only stable rules for the simple assignment game that are firm-covariant.*

2. *The workers-optimal stable rules are the only stable rules for the simple assignment game that are worker-covariant.*

Proof. It has already been proved in Proposition 2.5 that any firms-optimal stable rule is SFC. We need to prove the converse implication. Let $\varphi \equiv (u^\varphi, v^\varphi; \mu)$ be a stable rule that satisfies FC. If φ is not the firms-optimal stable rule, there exists $f_{i_0} \in F$ and a simple assignment game (F,W,a) such that $0 \leq u_{i_0}^\varphi(a) < \overline{u}_{i_0}(a)$. Take then

$I = \{f_{i_0}\} \subseteq F$ and $c^* = \overline{u}_{i_0}(a)$, where c^* as defined in (2.5) satisfies the requirements of Definition 2.15.

Then, by firm-covariance of φ, we get $u^{\varphi}_{i_0}(a^{c^*}) = u^{\varphi}_{i_0}(a) - c^* < \overline{u}_{i_0}(a) - c^* = 0$ which contradicts the stability of φ. ■

The combination of the above results leads to the next straightforward characterization.

Corollary 2.2.

1. *The firms-optimal stable rules are the only stable rules for the simple assignment game that are strong firm-covariant.*
2. *The workers-optimal stable rules are the only stable rules for the simple assignment game that are strong worker-covariant.*

3. Buying elections for peanuts

Abstract

We study an election under the influence of an interest group, assuming that a committee must decide between two options—to implement a reform or to stay with the status quo—and that all its members are aligned and in favor of the reform. The decision is taken via simultaneous voting and simple majority. An interest group that prefers the status quo offers an equal share of a "small" budget to any member that votes against the reform. We demonstrate that even if the available budget is a miniscule fragment of the one required to buy the election for sure (see, e.g., Dal Bò 2007), the interest group can be quite disruptive: there is always a completely mixed equilibrium in which the status quo is the most likely outcome, and the probability of its implementation converges to one as the size of the committee increases. The strategic uncertainty generated by the fact that other equilibria also exist, in which the reform is the most likely winner, seems to be the price that the interest group pays when attempting to buy an election for peanuts. We study the model under different assumptions on how the voting stage proceeds, but concerns on democratic quality do not vanish.

Keywords: elections; interest groups; lobbies; vote-buying; mixed-strategy equilibrium.

JEL Codes: D71, D72.

3.1. Introduction

In democratic systems, the will of the majority is sovereign; however, there are various scenarios in which a minority can wield significant influence over outcomes (see, Leaver and Makris 2006, for comprehensive evidence on such scenarios). An insightful explanation for the existence and natural occurrence of these cases is found in a renowned observation by Pareto in his influential work, the 1927 *Manual of Political Economy* (Pareto 1927)[1]:

> In order to understand how those who champion protection make themselves heard so easily it is necessary to add the consideration which applies to social movements generally[...] If a certain measure A is the case of a loss of one franc to each of a thousand persons, and of a one thousand franc gain to one individual, the latter will expend a great deal of energy, whereas the former will resist weakly; and it is likely that, in the end, the person who is attempting to secure the thousand francs via A will be successful.

Interest groups representing minority concerns can affect collective decision making in various ways. They manipulate electoral outcomes indirectly by "lobbying" decision makers—i.e. by spending resources on advertising and other means of influence—and directly by "bribing" them—i.e. by promising decision makers a payout if they support the interest groups' preferred policy.

In this paper, we abstract from the nuances of indirect influence, and try to understand by the means of a formal model, how expensive it is for an interest group to "buy" an election by directly paying decision makers who support its preferred alternative, when the preferences of the interest group and the decision makers are not aligned. Notice that even such blunt form of intervention cannot be easily discarded merely by proclaiming it illegal. Indeed, as stressed by Levine et al. (2022):

> [B]ribing politicians through campaign contributions is only the tip of the iceberg. Now and historically a simple and effective incentive is to give money to the family or to give money after departing office.[...] If lobbyists take the long view it is hard to legislate against them: Do we pass a law that anyone who has ever worked in government, is likely ever to work in government or who is related to such a person is unemployable?

We consider that there are two alternatives—a reform proposal and a status quo—and that decision makers choose one of the two by means of the simple majority rule; hence, they vote and the alternative with more votes is implemented. To capture the case of maximum tension between the decision makers and the interest group we consider that every decision maker prefers the reform, and that the interest group prefers the status quo. Before the election, the interest group commits to

[1] We quote and cite the first version of the book translated into English, that's why the date of the citation does not correspond to the one of the original work, which is in Italian.

conduct a transfer to each decision maker who will vote for its preferred outcome, and it fulfils its promise upon the publication of the vote record (i.e. ballots are open). We consider simple transfer schemes of the following sort: each decision maker that will vote for the alternative preferred by the interest group, will get an equal share of a fixed monetary amount. Does the interest group need to dedicate huge resources to be able to affect the outcome in its favor, or can it buy an election by promising only a small transfer to the involved decision makers? Answering this question is of paramount importance, as it will allow us to assess the vulnerability of democratic institutions to monetary influence from local elites, or even outsiders.

Our findings draw a rather dim picture of the resilience of electoral institutions to bribing attempts. On top of pure strategy equilibria in which all decision makers vote against their preferred alternative, independently of the bribe amount (such equilibria exist even if the bribe amount is zero), we show that in every symmetric (mixed) equilibrium the alternative preferred by the interest group wins with positive probability, irrespective of the amount that the interest group commits to spend for bribing. What is even worse, there is always a symmetric equilibrium in which the alternative preferred by the interest group is the most likely winner. Moreover, in that equilibrium, given any budget, as the electorate becomes large, we find that the alternative preferred by the interest group wins with a probability that converges to one! That is, an interest group can buy an election for peanuts, especially, when the electorate is large.[2]

The reasoning behind this apparently unintuitive finding is as follows. Conditional on being pivotal, a voter prefers to vote for the reform (since the transfer from the interest group is small), but, conditional on not being pivotal, a voter prefers to vote for the status quo (since the transfer from the interest group is positive). In a completely mixed equilibrium, the expected utilities associated with the two actions should be identical. The expected utility gain from supporting the reform is equal to the probability of being pivotal times the utility gain from the reform, and the expected utility gain from opposing the reform is equal to the expected transfer from the interest group. When each voter supports the reform with a fixed probability, then as the number of voters increases, both the probability of being pivotal and the expected transfer from the interest group converge to zero. However, they converge to zero at a different speed: When the fixed probability of supporting the reform is different than [equal to] one half, then the probability of being pivotal converges faster [slower] to zero than the expected transfer from the interest group. That is,

[2]Our results are a possible explanation, in some settings, for the emergence of a "Tullock paradox" (Tullock 1967 ; Tullock 1997), which states that certain interest groups seem to obtain political favors that are worth significantly more than the resources that these groups have used in order to secure them.

two completely mixed equilibria exist—one in which the reform is the most likely winner, and one in which the status quo prevails with higher probability—and both of them converge to a coin flip (for each individual) between the two actions in the size of the group, for any admissible parametrization. Following Martinelli (2006), we show that both mixed equilibria converge to the even lottery between the two actions "slowly", and hence the probability of the most likely winner prevailing goes to one as the electorate grows.

We are surely not the first to investigate a model of interest group influence on committee elections. Dal Bò (2007) studies a setting similar to ours, but focuses only on pure equilibria of the voting stage and allows for more elaborate transfer schemes (i.e. for any profile of actions, the interest group is allowed to promise any profile of transfers). The difference in predictions that follows from the alternative equilibrium notions and admissible transfer schemes that are used by our paper and Dal Bò (2007) is stark. In that paper, the resources that the interest group needs to commit to buy the election become unboundedly large in the size of the electorate. By allowing more nuanced transfer schemes than equal division, Dal Bò (2007) shows that the interest group does not actually need to spend these resources. However, for its bribing attempt to be convincing, it needs to be able to commit a lot of resources to this cause. By focusing on symmetric mixed equilibria instead, and to simple transfer schemes (equal division), we show that the alternative preferred by the interest group can be implemented in equilibrium with high probability—and almost certainly when the group of voters becomes arbitrarily large—even if the interest group has very few resources to commit to bribing voters. However, it should be noted that other equilibria also exist in which the alternative least preferred by the interest group is the most likely winner. Arguably, the strategic uncertainty generated by the simultaneous existence of multiple equilibrium outcomes is the price that the interest group pays by attempting to buy an election without committing substantial resources.[3] [4]

It is noteworthy, that the, arguably, intuitive transfer technology that we consider in this paper has already received attention by earlier studies, but the corresponding fully mixed equilibria have been deemed "difficult to solve" (Dahm et al. 2014, p. 73). That is, our formal analysis breaks new ground in a model that the literature

[3] This result is also in contrast with Name-Correa and Yildirim (2018), which granting the interest group with less flexibility (it can only make bribes contingent on the result of the election) arrives to the conclusion that larger committees help deter capture.

[4] Chen and Zapal (2022) study a model of sequential vote-buying and see that with upfront payments the interest group might buy the elections at a low cost, provided that agents discount the future. However, in their model there still is a minimum budget that the interest group must be able to spend in order to influence the election.

considers deserving of exploration, but hard to crack.[5] Furthermore, even if during the paper we will mainly talk about bribing, this work shows that the model we study can also explain situations without direct transfers, but with a more subtle lobbying action going on.

Our work is also related to vote trading, in which there is not an external interest group but the trading takes place inside the committee. It is a matter of framing though: the interest group in our model could be a member of the committee, and the main results would just carry over. In this sense, our work poses doubts on the efficiency of vote trading, since the "price of votes" might not reflect their actual value, which is in line with the critique in Neeman (1999). A relevant work in this strand of the literature is Casella and Turban (2014), which is profoundly different from ours but derives a conclusion similar in spirit, also through a mixed-strategy analysis: a minority (inside) the committee can implement a policy with positive probability.

3.2. The model

A committee of N members, where N is an odd number, has to decide between two different options, which we call A (reform) and B (status quo), of which one must be chosen. Each member votes for A or B, then votes are added and the option with more votes is chosen (simple majority).

There is an interest group which strictly prefers option B and is willing to spend a certain budget, M, in order to affect the election. We assume that all committee members strictly prefer A, so there exists a clear conflict between the interest group and the people in charge of the decision. Furthermore, we assume that the committee members are perfectly aligned, hence the results cannot be interpreted as a consequence of internal discrepancies. In other words: we make it as difficult as possible for the interest group.

Hence, we assume that committee members get utility $H > 1$ if A passes and utility 0 if B passes. We normalize the budget of the interest group to be $M = 1$ and we consider that the interest group offers an equal part of the budget to any member that votes for B; if only one member votes for B, he gets the whole budget, if two members do so, they get half of the budget each, and so on. Notice that given that we assume that $H > M$, the interest group cannot bribe a single decision maker ($N = 1$) in any way, hence from now on we assume $N \geq 3$.

[5] There is also a strand of theoretical literature that focuses on the tension that arises when there are several interest groups that have different preferences. Maybe the most relevant example of this strand of literature is Groseclose and Snyder (1996), which presents a model that can account for the existence of supermajorities and other coalition patterns difficult to explain otherwise.

The payoff of each member depends on their personal vote and on the decision taken by the committee. Let N_B^i denote how many of the remaining $N-1$ committee members vote for B when member i is taken out of the pool. If $U_Y^{X,i}$ denotes the utility that member i gets when he votes for X and Y passes, then with the rest of the notation introduced we have:

$$\begin{aligned} U_A^{A,i} &= H, \\ U_A^{B,i} &= H + \frac{1}{N_B^i + 1}, \\ U_B^{A,i} &= 0, \\ U_B^{B,i} &= \frac{1}{N_B^i + 1}. \end{aligned} \tag{3.1}$$

We assume that these utility functions are public. In particular, each member of the committee knows the preferences of the other members.

Notice a couple of things. First of all, we assume that the interest group can see who cast which vote, since he can afterwards perfectly discriminate who has to be paid. This is an assumption we relax in Section 2.3. Secondly, not allowing for abstention is not relevant, since voting for A clearly (weakly) dominates not voting for anyone. At last, given the members' symmetry in preferences, N_B is not really member-specific, but rather strategy-specific, meaning that in general for any two members i and i' that use the same strategy, $N_B^i = N_B^{i'}$.

In the next two subsections we explore the equilibria of this game. We start by looking at pure strategy Nash equilibria, which are rather simple, and then move to symmetric mixed strategy Nash equilibria.

3.3. Pure strategy Nash equilibria

Let's analyze the equilibria of the game when the strategy space of each committee member consists only of voting for A or voting for B. Notice that each member voting for A cannot be part of an equilibrium, since any member would individually deviate and vote for B without affecting the election and taking money from the interest group.

The following proposition characterizes equilibria in this setting.

Proposition 3.1. *All pure strategy Nash equilibria of the game correspond to one of the following scenarios:*

1. *All committee members vote for B.*

2. *Exactly* $(N+1)/2$ *committee members vote for A and the rest vote for B.*

Proof. First of all, it is clear that in any equilibrium in which A is chosen, there have to be exactly $(N+1)/2$ committee members that vote for A. This is the case because: a) if there were more, any agent voting for A would derive strictly more utility by voting for B, since A would still pass; and b) if there were less, A would not pass. Furthermore, any situation in which exactly $(N+1)/2$ members vote for A clearly constitutes an equilibrium of the game, since no member voting for A can deviate without affecting the choice, and since $H > 1$ being worse off, and obviously no member voting for B can do better neither since A is already passing.

Regarding equilibria in which B passes, it is clear that there can only be one, corresponding to the situation in which every member is voting for B. In addition, such a case is obviously an equilibrium, since no one can single-handedly deviate and change the result of the choice. ■

Notice that for the previous proposition to hold, it is irrelevant the budget of the interest group and how it promises to distribute it among voters, as long as it offers a positive quantity to anyone voting for B. And, in fact, the only equilibrium in which B passes looks unlikely and is often dismissed in the literature (since it exists in most settings, as it is difficult to avoid as long as there are no expressive payoffs or majority concerns).

Hence, we have seen that considering pure strategies we can separate equilibria in two classes, only one of which is symmetric, in the sense that all members use the same strategy. In the only symmetric equilibrium in pure strategies everyone votes for B and B passes. This equilibrium will also appear in the next section, where we analyze symmetric mixed strategy Nash equilibria.

3.4. Symmetric mixed strategy Nash equilibria

In this section we study symmetric equilibria of the game, that is: equilibria in which all members choose the same strategy. We consider mixed strategies, hence members assign a probability, which we will note as p, to voting for B, and a probability, $1-p$, to voting for A.

Since we focus on situations in which all members play the same strategy, note that the superscript i in (3.1) is irrelevant. As we already commented on, if all members play the same strategy then N_B^i is the same variable for all of them as well, since clearly taking out any of the members from the group has always the same effect. Hence, from now on we will simply refer to it as N_B.

In any symmetric mixed strategy Nash equilibrium such that $p \in (0,1)$, hence a completely mixed strategy Nash equilibrium, every member has to be indifferent between voting for A and voting for B; otherwise, he would use a pure strategy. So we start by deriving the expected utilities from the point of view of one member when the rest all vote for B with probability p.

A member voting for A only gets utility from A being implemented, and in that case he gets utility H. Hence, the expected utility of a member voting for A when all other members vote for B with probability p is:

$$E(u_m(A)) = H \cdot P\left(N_B \leq \frac{N-1}{2}\right). \tag{3.2}$$

When voting for B instead, a member potentially derives utility both from the interest group and from A passing. So the expected utility of a member voting for B when all other members vote for B with probability p can be written as:

$$\begin{aligned} E(u_m(B)) = (H+1)P(N_B = 0) + \ldots + \left(H + \frac{1}{(N-1)/2}\right) P\left(N_B = \frac{N-3}{2}\right) + \\ + \frac{1}{(N+1)/2} P\left(N_B = \frac{N-1}{2}\right) + \ldots + \frac{1}{N} P\left(N_B = N-1\right) = \\ = H \cdot P\left(N_B \leq \frac{N-3}{2}\right) + E\left(\frac{1}{1+N_B}\right). \end{aligned} \tag{3.3}$$

We know from the previous section that $p = 1$ constitutes a symmetric Nash equilibrium and that $p = 0$ does not. Any other symmetric mixed strategy Nash equilibrium, which will be completley mixed, must make the following equality hold:

$$E(u_m(A)) = E(u_m(B)). \tag{3.4}$$

Simplifying we obtain that Equation 3.4 reduces to:

$$E\left(\frac{1}{1+N_B}\right) = H \binom{N-1}{\frac{N-1}{2}} p^{\frac{N-1}{2}} (1-p)^{\frac{N-1}{2}}. \tag{3.5}$$

Equation (3.5) is very intuitive and it basically states that, in equilibrium, the expected gain in utility that a member gets from voting B, which is the LHS of the equation, should equal the expected gain he gets from voting A, which is its RHS, so that he doesn't strictly prefer one of the two options. Naturally, the equation depends on all parameters of the game. Notice that the equation brings to light something that is fundamental: a member only cares about voting for A when he is pivotal, i. e., when there would be a tie without him; this is precisely what the RHS of (3.5) is capturing.

Just by using the definition of expected value of a random variable we have that:

$$E\left(\frac{1}{1+N_B}\right) = \sum_{i=0}^{N-1} \frac{1}{1+i} \binom{N-1}{i} \frac{p^i(1-p)^{N-1-i}}{i!(N-1-i)!}. \tag{3.6}$$

Before continuing, in order to get a better understanding of the model, let's plot both sides of Equation (3.5) for some parameter values (Figure 1), using p as plotting variable. Intersections of both functions indicate symmetric mixed strategy Nash equilibria of the game.

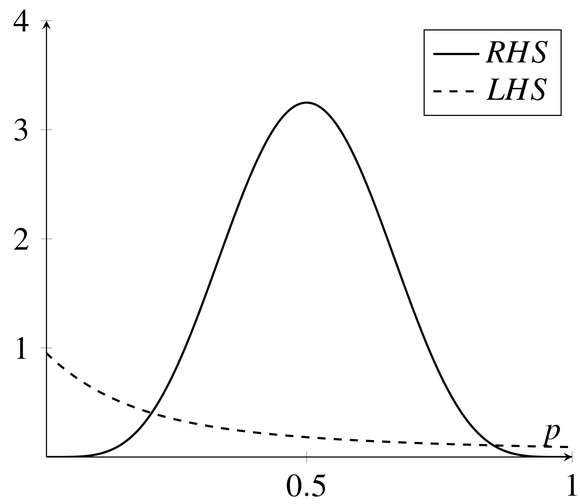

Figure 3.1.: Both sides of Equation 3.5 for $H = 1.2$ and $N = 11$.

Looking at Figure 1, we see that for the chosen parameters there are two symmetric equilibria with $p \in (0,1)$, so in completely mixed strategies. Furthermore, one of them involves a "low" probability of each member voting for B, and the other one a "high" probability of it. We will characterize the equilibria and see that the above feature holds in general.

In the next lemma we derive an alternative expression for (3.6), which is compact and will prove useful.

Lemma 3.1. *For all $p \in (0,1)$ and $N \geq 3$, the following holds* [6]*:*

$$E\left(\frac{1}{1+N_B}\right) = \frac{1-(1-p)^N}{pN}.$$

[6] Notice that

$$\lim_{p\to 0} \frac{1-(1-p)^N}{pN} = 1,$$

so the expression is well-defined at $p = 0$.

Proof. Developing the expression that we know of $E\left(\frac{1}{1+N_B}\right)$ we have that:

$$\begin{aligned}
E\left(\frac{1}{1+N_B}\right) &= \sum_{i=0}^{N-1}\binom{N-1}{i}p^i(1-p)^{N-1-i}\frac{1}{i+1} \\
&= \sum_{i=0}^{N-1}\binom{N}{i+1}p^i(1-p)^{N-1-i}\frac{1}{N} \\
&= \frac{1}{pN}\sum_{i=0}^{N-1}\binom{N}{i+1}p^{i+1}(1-p)^{N-1-i} \\
&= \frac{1}{pN}\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k}-(1-p)^N \\
&= \frac{1-(1-p)^N}{pN},
\end{aligned}$$

where in the last step we have simply used the Newton binomial. ■

With this new expression we have just derived, we can rewrite Equation 3.5, for $p \in (0,1]$, in the following two ways:

$$H\frac{(N-1)!}{\left(\left(\frac{N-1}{2}\right)!\right)^2}p^{\frac{N-1}{2}}(1-p)^{\frac{N-1}{2}} = \frac{1-(1-p)^N}{pN}; \tag{3.7}$$

$$H\frac{N!}{\left(\left(\frac{N-1}{2}\right)!\right)^2}p^{\frac{N+1}{2}}(1-p)^{\frac{N-1}{2}}+(1-p)^N = 1. \tag{3.8}$$

Notice that Equation 3.8 is simply Equation 3.7 slightly rearranged. Both equations will be useful when characterizing symmetric mixed strategy Nash equilibria in the following section.

In order to keep expressions short, we will use the following notation throughout the rest of the paper:

$$\begin{aligned}
f_N(p,H) &= H\frac{(N-1)!}{\left(\left(\frac{N-1}{2}\right)!\right)^2}p^{\frac{N-1}{2}}(1-p)^{\frac{N-1}{2}}; \\
l_N(p,H) &= H\frac{N!}{\left(\left(\frac{N-1}{2}\right)!\right)^2}p^{\frac{N+1}{2}}(1-p)^{\frac{N-1}{2}}+(1-p)^N; \\
u_N(p,H) &= H\frac{N!}{\left(\left(\frac{N-1}{2}\right)!\right)^2}p^{\frac{N+1}{2}}(1-p)^{\frac{N-1}{2}}; \\
g_N(p) &= \frac{1-(1-p)^N}{pN}.
\end{aligned} \tag{3.9}$$

3.4.1. Characterization of symmetric mixed strategy Nash equilibria

In this section we fully describe the symmetric mixed strategy Nash equilibria of the game for the different combinations of parameters, H and N.

First of all, we see that for $N \geq 7$ any member strictly prefers to vote for A if the rest choose $p = 1/2$, for any $H > 1$. In other words, at $p = 1/2$, the LHS of Equation 3.7 is greater than the RHS if $N \geq 7$. This is not true for the cases $N = 3$ and $N = 5$ and they will be discussed at the end of the section.

Proposition 3.2. *If $N \geq 7$, at $p = 1/2$ it holds that:*

$$H \frac{(N-1)!}{((\frac{N-1}{2})!)^2} p^{\frac{N-1}{2}} (1-p)^{\frac{N-1}{2}} > \frac{1-(1-p)^N}{pN}.$$

Proof. What we want to see is clearly equivalent to seeing that the LHS of Equation 3.8 is greater than the RHS under the specified conditions. Substituting $p = 1/2$ into the inequality we get:

$$H \frac{N!}{((\frac{N-1}{2})!)^2} \left(\frac{1}{2}\right)^{\frac{N+1}{2}} \left(\frac{1}{2}\right)^{\frac{N-1}{2}} + \left(\frac{1}{2}\right)^N > 1.$$

It is clearly enough to see that, for $N \geq 7$:

$$\frac{N!}{((\frac{N-1}{2})!)^2} \left(\frac{1}{2}\right)^{\frac{N+1}{2}} \left(\frac{1}{2}\right)^{\frac{N-1}{2}} > 1.$$

We prove it by induction. The case $N = 7$ is easily verified. Then, if we denote as $g(N)$ the LHS of the previous inequality:

$$g(N+2) = \left(\frac{N+2}{N+1}\right) \frac{N!}{((\frac{N-1}{2})!)^2} \left(\frac{1}{2}\right)^{\frac{N+1}{2}} \left(\frac{1}{2}\right)^{\frac{N-1}{2}} = \left(\frac{N+2}{N+1}\right) g(N) > g(N),$$

and the proof is over, since we see that if the inequality holds for N it also holds for $N+2$. ■

Hence, if $N-1$ members of the committee choose $p = 1/2$, the probability of being pivotal of the remaining member is too high to be tempted by the offer of the interest group, given the available budget. In particular, the symmetric profile $p = 1/2$ cannot be an equilibrium of the game.

We are ready to see that, for $N \geq 7$, there is always one, and only one, symmetric mixed strategy Nash equilibrium in which members are more likely to vote for A than for B. The equilibrium will obviously depend on H as well, but its existence in the interval of interest does not.

Proposition 3.3. *For $N \geq 7$, there exists one, and only one, symmetric mixed strategy Nash equilibrium such that $p \in [0, 1/2)$.*

Proof. Let's focus on Equation 3.7. We have to see that it has one, and only one solution in the interval $(0, 1/2)$. Given that we know that at $p = 1/2$ the LHS is greater than the RHS and that both sides are clearly continuous in p, it is enough to see that: i) this relationship is reversed at $p = 0$; ii) the LHS is increasing in $[0, 1/2]$; iii) the RHS is decreasing in $[0, 1/2]$.

Points ii) and iii) are immediate just by differentiating. And i) is immediate to verify, since, with the notation introduced, $f_N(0, H) = 0$ and $g_N(0) = 1$. ■

The next step is to discuss existence of equilibria with $p \in (1/2, 1)$. This case is less clear than the one we have just studied though, since now, in the domain of interest, both sides of (3.5) are decreasing, hence it is not immediate to conclude that there is a unique interior solution. However, we see in the next lines that the same sort of result still holds. We divide the proof in two parts: existence and uniqueness.

Proposition 3.4. *For $N \geq 7$, there exists one symmetric mixed strategy Nash equilibrium such that $p \in (1/2, 1)$.*

Proof. With the notation introduced, it is immediate that $f_N(1, H) = 0$ and $g_N(1) = 1/N$, and we know from Proposition 2.3 that $f_N(1/2, H) > g_N(1/2)$. Since both functions are continuous, they must cross at least once in the interval $(1/2, 1)$, and hence Equation 3.7 must have a solution in this interval. ■

Hence, there is also a symmetric equilibrium in which B is more likely to pass than A. We move on to see that it is also unique in the class of completely mixed Nash equilibria in which this is the case.

Proposition 3.5. *For $N \geq 7$, there exists one, and only one, symmetric mixed strategy Nash equilibrium such that $p \in (1/2, 1)$.*

Proof. Existence comes from the previous proposition. We just need to prove uniqueness. We will see that the LHS of Equation 3.8, which we call $l_N(H, p)$, can only attain once the value 1 in the interval $(0, 1/2)$.

Differentiating we obtain:

$$
\begin{aligned}
l_N'(p,H) &= H\frac{N!}{((N-1)/2)!^2}p^{(N-1)/2}(1-p)^{(N-3)/2}\left(\frac{N+1}{2}-Np\right)-N(1-p)^{N-1}\\
&= (1-p)^{(N-3)/2}\left(H\frac{N!}{((N-1)/2)!^2}p^{(N-1)/2}\left(\frac{N+1}{2}-Np\right)-N(1-p)^{(N+1)/2}\right).
\end{aligned}
$$

It is immediate to check that $l_N(0,H)=1$ and that $l_N'(0,H)<0$, which taking into account that we know that there is one, and only one, solution with $p<1/2$, implies that $l_N(1/2,H)>1$. It is also immediate, just substituting, to see that $l_N'(1/2,H)>0$. Furthermore, if $p\geq\frac{N+1}{2N}$, then $l_N'(p,H)<0$, and we can conclude that $l_N(p,H)$ must have one maximum in the interval $(1/2,\frac{N+1}{2N})$ and that any extreme point of $l_N(p)$ in $[1/2,1]$ has to be in this interval. We prove that there is only one. Notice that, since $l_N'(1/2,H)>0$ and $l_N'\left(\frac{N+1}{2N},H\right)<0$, there cannot be only two extreme points in the interval, if there are more than one, there must be at least three. Hence, seeing that there cannot be three is enough. And in order to see that it is enough to prove that, in the interval of interest, the function

$$
\frac{N!}{H\,((N-1)/2)!^2}p^{(N-1)/2}\left(\frac{N+1}{2}-Np\right)-N(1-p)^{(N+1)/2},
$$

which determines the roots of $l_N(p)$, is strictly concave. This is the case because a strictly concave function has at most two zeroes. It is a simple exercise of derivation to see that for $p>1/2$ the function above is indeed strictly concave, and the proof is concluded. ∎

At this point we have characterized the equilibria of the game for committees with at least 7 members. We have seen that there are three symmetric mixed strategy Nash equilibria, one with $p=1$, in which all voters vote for B, and two that are completely mixed. In one of them, $p<1/2$, and A is more likely to pass, and in the other one, $1/2<p$ and B is more likely to pass. However, we still have to study the cases $N=3$ and $N=5$. We do it in the next lines. We will see, in particular, that there is H^* such that if $H>H^*$ the characterization of the case $N\geq 7$ carries over.

Proposition 3.6. *If $N=3$, there exists $H^*>1$ such that:*

1. *if $H<H^*$, there is no symmetric completely mixed strategy Nash equilibrium;*
2. *if $H=H^*$, there is one, and only one, symmetric completely mixed strategy Nash equilibrium;*
3. *If $H>H^*$, there are exactly two symmetric completely mixed strategy Nash equilibria.*

Furthermore, there exists $H^{**} > H^*$ such that if $H > H^{**}$, the two completely mixed strategy Nash equilibria are at opposite sides of $1/2$.

Proof. We know that $p = 0$ doesn't constitute an equilibrium and $p = 1$ does, hence we only need to assess the solutions of Equation 3.8 in the interval $(0,1)$. If $N = 3$ we can solve the equation algebraically. By doing so, we find that the solutions are:

$$p_1 = 0,\ p_2 = \frac{-\sqrt{3}\sqrt{12H^2 - 12H - 1} + 6H + 3}{2(6H+1)},\ p_3 = \frac{\sqrt{3}\sqrt{12H^2 - 12H - 1} + 6H + 3}{2(6H+1)}.$$

Let H^* be the positive solution to $12H^2 - 12H - 1 = 0$, hence $H^* = \frac{1}{6}\left(2\sqrt{3} + 3\right)$. Notice that $H^* > 1$. If $H < H^*$, it is clear that both p_2 and p_3 are not real. If $H = H^*$, p_2 and p_3 collapse to being the same solution. So there is only left to prove that, if $H > H^*$, both p_2 and p_3 are in the interval $(0,1)$. However, once again the expressions can be solved algebraically (now in H) and it is immediate to see that, in fact, there is no $H > 0$ that makes $p_2 = 0$ or $p_3 = 1$, which with the continuity of the functions in the domain of interest and the fact that their value at H^* is in the interval $(0,1)$, concludes this part of the proof.

The existence of H^{**} is simple with what we already saw. Taking the solution p_2 and derivating it with respect to H, we can see that it is decreasing for $H > 1$. It is straightforward to check that for $H = 2$, $p_2 < 1/2$, which given the continuity of p_2 in the interval of interest guarantees that for $H \geq 2$, $p_2 \in (0, 1/2)$. An analogous argument for p_3 concludes the proof. ■

A similar result can also be proved for $N = 5$, even if now we cannot solve for the equilibria algebraically as in the $N = 3$ case.

Proposition 3.7. *If $N = 5$, there are two, and only two, symmetric mixed strategy Nash equilibria other than $p = 1$. Furthermore, there exists H^{**} such that if $H > H^{**}$, the two completely mixed strategy Nash equilibria are at opposite sides of* $1/2$.

Proof. Let's focus on Equation 3.8. First of all, it is a matter of computation to verify that the LHS is greater than 1 at $p = 0.6$ for any $H > 1$. Now, given that both at $p = 1$ the LHS is clearly smaller than 1 and that at $p = 0$ the derivative is negative, it is enough to prove that it has at most two extreme points in $(0,1)$.

Using our notation, hence denoting by $l(p)$ the LHS of (3.8) (now fixing $N = 5$), and derivating we have that for $p \in (0,1)$:

$$l'(p) = 0 \iff 90Hp^2(1-p) - 60Hp^3 - 5(1-p)^3 = 0. \tag{3.10}$$

Even if, once again, we have a polynomial that we can solve algebraically, now it is not as helpful because the expressions of the solutions are complicated. However, proving what we need is relatively easy. Notice that we just have to see that Equation 3.10 has one negative solution, since any polynomial of degree three has at most three different real roots. Firstly, it is clear that at $p = 0$ it is negative. Secondly, when p tends to $-\infty$ it behaves like $-90Hp^3 - 60Hp^3 + 5p^3$, which for $H > 1$ is clearly positive. And the first part of the proof is concluded, since the polynomial must have a negative root, and as a consequence it can at most have two roots in the interval of interest.

The existence of H^{**} is immediate by checking that we can simply choose it such that:

$$\left(\frac{1}{2}\right)^5 (30H^{**} + 1) = 1,$$

which is Equation 3.8 for $N = 5$ at $p = 1/2$. ■

After the analysis of these two particular cases, we have a complete characterization of the symmetric mixed strategy Nash equilibria of the game. Furthermore, we have seen that there is $H^* > 1$, such that if we only consider $H > H^*$, then no distinction is needed; that is: the study corresponding to $N \geq 7$ carries over to $N \geq 3$. When there is no possible misunderstanding, and in particular it is granted that the two symmetric completely mixed strategy Nash equilibria of the game exist, we will note them as $p_1(H,N)$ and $p_2(H,N)$, with $p_1(H,N) \leq p_2(H,N)$. Notice that for $N \geq 7$, we have in general that $p_1(H,N) \in (0,1/2)$ and $p_2(H,N) \in (1/2,1)$.

It is natural to wonder how the equilibria that we have just characterized react to changes in the parameters of the game. It is especially interesting to study how including more members to the committee might affect these equilibria. We see this next.

3.4.2. Comparative statics

Let's see how an increase on H or N might affect the symmetric mixed equilibria of the game, which we fully described in the previous section. Once again, in order to do so we will mainly focus on Equation 3.7 and Equation 3.8.

Notice, looking at Equation 3.7, that only one side depends on H, hence the result concerning comparative statics on this parameter is trivial. Having in mind the notation we set for the equilibria at the end of the last section:

Proposition 3.8. *If $H' > H > 1$, then the following hold:*

1. *If $N \geq 5$, then $p_1(H',N) < p_1(H,N)$ and $p_2(H,N) < p_2(H',N)$.*
2. *If $N = 3$ and $H \geq H^*$, where H^* is as determined in the proof of Proposition 3.6, $p_1(H',N) < p_1(H,N)$ and $p_2(H,N) < p_2(H',N)$.*

Proof. Take a look at Equation 3.7, which describes symmetric completely mixed strategy Nash equilibria. By taking a greater value for H, we are just moving the LHS upwards. This, along with the fact, which we know, that in the interval $(p_1(H,N), p_2(H,N))$ the LHS is greater than the RHS, proves immediately the result. ■

Hence, increasing H, which represents the utility that each member gets when his most preferred option, A, passes, has a polarizing effect on the two symmetric completely mixed strategy Nash equilibria. Roughly speaking, an increase in H makes pivotality more relevant. From the point of view of one member, the probability with which the others vote for B has to compensate for this increase in the value of pivotality in order to make him indifferent between voting for A and voting for B.

Comparative statics with respect to N are less clear. A quick analysis on both sides of Equation 3.7 is enough to see that the effect of an increase in N is not trivial. In particular, it is not complicated to see that an increase in N causes both sides of the equation to decrease, hence leading to an ambiguous effect at first sight. However, we can establish a result by comparing how much each side decreases (after a certain threshold).

Proposition 3.9. *There exists N^* such that, if $N > N^*$, then*

$$p_1(H,N+2) > p_1(H,N).$$

Proof. First of all, let's write (3.8) for $N+2$, with the notation introduced:

$$4p(1-p)\frac{N+2}{N+1}u_N(p,H) + (1-p)^2(1-p)^N = 1.$$

Notice that there is a threshold $p^*(N) < 1/2$ such that, if $p < p^*(N)$, then it holds that $4p(1-p)\frac{N+2}{N+1} < 1$. This threshold is easy to compute, since it solves a quadratic equation, and is given by:

$$p^*(N) = \frac{1}{2} - \frac{1}{2\sqrt{N+2}}.$$

In order to prove the proposition it is enough to see that there exists N^* such that, if $N > N^*$, then $u_N(p^*(N),H) > 1$, since this clearly guarantees that

$$u_{N+2}(p_1(H,N),H) + (1-p_1(H,N))^{N+2} < u_N(p_1(H,N),H) + (1-p_1(H,N))^N$$

and hence that $p_1(N) < p_1(N+2)$. So let's prove that.

Substituting and grouping terms we have that

$$u_N(p^*(N),H) = H\frac{N!}{((N-1)/2)!^2}\left(\frac{1}{2}-\frac{1}{2\sqrt{N+2}}\right)\left(\frac{1}{4}-\frac{1}{4(N+2)}\right)^{\frac{N-1}{2}}.$$

It can be checked that $u_N(1,p^*(N)) > 1$ holds for $N = 27$, and so it holds for all $H > 1$ as well. Now we prove that if the inequality holds for N it must also hold also for $N+2$. In fact we will see that $u_N(h,p^*(N))$ is increasing in N.

Since $\frac{1}{2}-\frac{1}{2\sqrt{N+2}}$ increases with N, it is enough to see that

$$\left(\frac{1}{4}-\frac{1}{4(N+2)}\right)^{\frac{N-1}{2}} < 4\frac{N+2}{N+1}\left(\frac{1}{4}-\frac{1}{4(N+4)}\right)\left(\frac{1}{4}-\frac{1}{4(N+4)}\right)^{\frac{N-1}{2}}.$$

First of all, it is clear that

$$\left(\frac{1}{4}-\frac{1}{4(N+2)}\right)^{\frac{N-1}{2}} < \left(\frac{1}{4}-\frac{1}{4(N+4)}\right)^{\frac{N-1}{2}},$$

so it is enough to see that

$$1 < 4\frac{N+2}{N+1}\left(\frac{1}{4}-\frac{1}{4(N+4)}\right).$$

Rewriting the RHS of the last inequality, we have that

$$4\frac{N+2}{N+1}\left(\frac{1}{4}-\frac{1}{4(N+4)}\right) = \frac{N+2}{N+1}\cdot\frac{N+3}{N+4} > 1.$$

And we have concluded the proof. ∎

A symmetric result for $p_2(H,N)$ is straightforward with the proof for $p_1(H,N)$.

Proposition 3.10. *There exists N^* such that, if $N > N^*$, then*

$$p_2(H,N+2) < p_2(H,N).$$

Proof. The proof follows just as the one of Proposition 3.9 taking now $p^*(N)$ as the solution of $4p(1-p)\frac{N+2}{N+1} = 1$ with $p > 1/2$. Hence, now $p^*(N) = \frac{1}{2}+\frac{1}{2\sqrt{N+2}}$, which is symmetric to the previous threshold with respect to $1/2$.

The proof is concluded by realizing that for all $x \in (0,1/2)$, $u_N(1/2-x,H) < u_N(1/2+x,H)$. This is immediate because $f_N(p)$ is symmetric around $1/2$ and $u_N(p,H) = H\cdot N\cdot p\cdot f_N(p)$. Hence, $u_N(p^*(N),H) > 1$ trivially holds now as well and the same N^* as before is enough. ∎

Since we know that $p_1(H,N) \in (0,1/2)$ for all $N \geq 7$, Proposition 3.9 tells us that, above a certain N^*, the equilibrium with $p_1(H,N)$ moves monotonically towards $1/2$ as N grows larger. Hence, $\{p_1(H,2k+1)\}_{k\in\mathbb{N}}$ must converge somewhere in $(0,1/2]$. The same argument works for $p_2(H,N)$ in the opposite interval. In the next section we see that both equilibria actually converge to $1/2$.

3.4.3. The limit case: committee size arbitrarily large

In this section we study how the symmetric mixed strategy Nash equilibria of the game behave when N is arbitrarily large. As we have justified, convergence of the equilibria is guaranteed. Notice that the equilibrium with $p = 1$ always exists, hence its convergence is trivial, so we focus on the completely mixed strategy equilibria from now on.

In order to get the main result, we need to study a bit more in depth the RHS of Equation 3.7. We devote the next couple of lemmas to do so.

Lemma 3.2. *For any $p \in [0,1]$ and any $N \geq 3$, $N \geq (1-p)^2(N+2(1-(1-p)^N))$ holds.*

Proof. First of all, we see that the RHS of the inequality is decreasing in p. Derivating and grouping terms we obtain that the derivative of the RHS is $2(1-p)((1-p)^N(N+2)-(N+2))$, which is clearly negative for all $p \in [0,1]$. Checking that the inequality holds for $p = 0$, which is trivial, concludes the proof. ■

Lemma 3.3. *The function $g_{N+2}(p)/g_N(p)$ is decreasing in p.*

Proof. Derivating and simplifying we can see that the derivative is negative if, and only if,

$$N \geq (1-p)^2(N+2(1-(1-p)^N)),$$

which we have just proved in the previous lemma. ■

Since $g_{N+2}(0)/g_N(0) = 1$, it is an immediate consequence of the last result that $g_{N+2}(p) \leq g_N(p)$ for all $p \in [0,1]$. Similarly, since $g_{N+2}(1)/g_N(1) = \frac{N}{N+2}$, we also have that

$$g_{N+2}(p) \geq \frac{N}{N+2} g_N(p) \tag{3.11}$$

for all $p \in [0,1]$.

The function $g_N(p)$, as defined in (3.9), is the expected payment that a member gets from voting B when all others voter for B with probability p. Hence, it is clear that the $g_N(p)$ is decreasing in p: the more likely the others are to vote for B, the less money a members expects to get from voting it. With that in mind, Lemma 3.3, tells us something very intuitive: the reduction in the expected payment from the interest group faced by a single member voting for B when p increases, is more severe as the committee is larger. This is only natural, since the bigger the committee, the more people that are adjusting their voting behavior when we change p.

Now we are ready to prove the main result of this section. As we did in the last section, we will prove the result first for $p_1(H,N)$, and then obtain the one for $p_2(H,N)$ as an immediate consequence.

Proposition 3.11. *Given any $\delta > 0$, there exists $N^*(H)$ such that for all $N > N^*(H)$, there exists $p(N) \in (1/2-\delta, 1/2)$ that constitutes a symmetric mixed strategy Nash equilibrium of the game.*

Proof. Consider any $0 < \delta < 1/2$ and any $N \geq 7$. Let $p_0 = 1/2 - \delta$ throughout this proof.

With the notation introduced up to this point, symmetric completely mixed strategy Nash equilibria are defined by $g_N(p) = f_N(p,H)$. Since $g_N(0) > u_N(0,H)$ and the two functions cross exactly once in the interval $(0,1/2)$, if, for all N, $g_N(p_0) > f_N(p_0,H)$ we are done. Assume that it is not the case for some N.

It is immediate to see that in general $f_{N+2}(p,H) = 4p(1-p)\frac{N}{N+1}f_N(p,H)$. So for all $k \geq 1$, we have that $f_{N+2k}(p_0,H) < (4p_0(1-p_0))^k f_N(p_0,H)$. Furthermore, by applying repeatedly Inequality 3.11, we have that for all $k \geq 1$, $g_{N+2k}(p_0) \geq \frac{N}{N+2k}g_N(p_0)$. Then, for all $k \geq 1$,

$$\frac{f_{N+2k}(p_0,H)}{g_{N+2k}(p_0)} \leq (4p_0(1-p_0))^k (N+2k)\frac{f_N(p_0,H)}{Ng_N(p_0)},$$

and since $4p_0(1-p_0) < 1$ and the exponential decreases faster than any polynomial, it is clear that the RHS of the last inequality goes to 0 as N goes to infinite and we can conclude that

$$\lim_{N\to\infty}\frac{f_N(p_0,H)}{g_N(p_0)} = 0.$$

Hence, using the definition of limit, in particular there exists $N^*(H)$ such that, for all $N \geq N^*(H)$, $f_N(p_0,H)/g_N(p_0) < 1$, or what is the same, such that $f_N(p_0,H) < g_N(p_0)$. ∎

The result concerning the symmetric interval with respect to $1/2$ is proved in exactly the same way. Notice that the expressions we used remain unchanged and that the crucial fact, that is $4p_0(1-p_0) < 1$, holds as well for $p_0 > 1/2$. We state it as a corollary and do not provide a separate proof since the previous one carries over without complications.

Corollary 3.1. *Given any $\delta > 0$, there exists $N^*(H)$ such that for all $N > N^*(H)$, there exists $p(N) \in (1/2, 1/2+\delta)$ that constitutes a symmetric mixed strategy Nash equilibrium of the game.*

Summarizing the results up to this point: for $N \geq 7$, we have seen that the two symmetric completely mixed strategy Nash equilibria of the game are on opposite sides of $1/2$, but they both move towards $1/2$ as N increases (after a certain threshold), and in fact both of them get arbitrarily close to it.

3.4.4. Welfare analysis

We know that both symmetric completely mixed strategy Nash equilibria converge to $1/2$. However how fast they do so is relevant in order to assess how the probability of A passing evolves in each scenario. Broadly speaking, fast convergence of $p_1(H,N)$ to $1/2$ would lead to probabilities of A passing close to $1/2$ in equilibrium for arbitrarily large committees. On the other hand, slow convergence would lead to probabilities of A passing close to 1 in equilibrium for arbitrarily large committees.

If p is the probability with which each of the members of the committee votes for B, then the probability of A passing is:

$$P_A(p) = \sum_{i=0}^{(N-1)/2} \binom{N}{i} p^i (1-p)^{N-i}. \tag{3.12}$$

Next we see that, given a fixed H: i) under $p_1(H,N)$ the probability of A passing converges to 1 as N grows large; and ii) under $p_2(H,N)$ the probability of A passing converges to 0 as N grows large. Hence, in big enough committees, the two completely mixed strategy Nash equilibria provide “completely" different results. We focus on proving the first of these points, since the second one will then be immediate. To prove this result, we draw inspiration from the proof of Theorem 2 of Martinelli (2006), which makes use of the Berry-Esseen theorem.

We start with a technical lemma, which is convenient to prove separately, but that has no value on its own.

Lemma 3.4. *Given $N \geq 5$, it holds that:*

$$\left(\frac{N-2}{N-4}\right)^{\frac{N-1}{2}} (N-2) > \left(\frac{N+2}{N}\right)^{\frac{N-1}{2}} (N+2).$$

Proof. This last inequality is equivalent to:

$$\left(\frac{N-2}{N+2}\right)^{\frac{N+1}{2}}\left(\frac{N}{N-4}\right)^{\frac{N-1}{2}}>1.$$

Taking logarithms on both sides of the inequality we get:

$$\frac{N+1}{2}\left(\log(N-2)-\log(N+2)\right)+\frac{N-1}{2}\left(\log(N)-\log(N-4)\right)>0.$$

It is immediate to see that the LHS goes to 0 as N grows large and that at $N=5$ it is positive. Hence, it is enough to see that the derivative of the LHS is negative. Derivating we obtain:

$$\frac{1}{2}\left(-\frac{8\left(N^2+2\right)}{N\left(N^3-4N^2-4N+16\right)}+\log\left(\frac{N}{N-4}\frac{N-2}{N+2}\right)\right),$$

so we have to see that for $N\geq 5$,

$$-\frac{8\left(N^2+2\right)}{N\left(N^3-4N^2-4N+16\right)}+\log\left(\frac{N}{N-4}\frac{N-2}{N+2}\right)<0.$$

It is once again clear for $N=5$ and the LHS goes to 0 as N grows large, hence it is enough to see that the LHS is increasing. By derivating the LHS we obtain:

$$\frac{48N^4+64N^3-640N^2+128N+256}{(N-4)^2(N-2)^2N^2(N+2)^2},$$

which is positive if, and only if, $48N^4+64N^3-640N^2+128N+256>0$, which is immediate to check for $N\geq 5$ since the polynomial of degree 4 can be solved in general. ■

We follow up by finding sequences of probabilities that we can assess how they behave in terms of convergence of the probability of A passing.

Lemma 3.5. *If committee members of a committee of size N all vote according to the probability $p_k(N)=\frac{1}{2}-\frac{k}{\sqrt{N}}$ for $k<\sqrt{N}$ a natural number, the probability of A passing goes to $\Phi(2k)$ as N grows large.*

Proof. We consider the sequence $p_k(N)=\frac{1}{2}-\frac{k}{\sqrt{N}}$, and hence that each voter votes for A with probability $\frac{1}{2}+\frac{k}{\sqrt{N}}$. This sequence clearly converges to $1/2$ as N grows large. Now consider the random variable X_N^i defined by:

$$X_N^i = \begin{cases} \frac{1}{2} - \frac{k}{\sqrt{N}} & \text{if voter i votes for A} \\ -\frac{1}{2} - \frac{k}{\sqrt{N}} & \text{if voter i votes for B} \end{cases}$$

Then, given N the variables X_N^i are identically distributed, and it is immediate to derive the following:

$$\begin{aligned} E\left(X_N^i\right) &= 0, \\ E\left(\left(X_N^i\right)^2\right) &= \frac{1}{4} - \frac{k^2}{N}, \\ E\left(|X_N^i|^3\right) &= \frac{1}{8} - \frac{2k^4}{N^2}. \end{aligned}$$

A doesn't pass if it gets $(N-1)/2$ votes or fewer. This is equivalent to:

$$\begin{aligned} \sum_{i=1}^{N} X_N^i + N \cdot \left(\frac{1}{2} + \frac{k}{\sqrt{N}}\right) &\leq \frac{N-1}{2} \\ \sum_{i=1}^{N} X_N^i &\leq -\frac{1}{2} - N\left(\frac{k}{\sqrt{N}}\right). \end{aligned}$$

Then, if we let F_N denote the distribution of:

$$\frac{\sum_{i=1}^{N} X_N^i}{\sqrt{N \cdot E\left(\left(X_N^i\right)^2\right)}},$$

the probability of A passing is $1 - F_N(J_N)$ where:

$$J_N = \frac{-\frac{1}{2} - N\left(\frac{k}{\sqrt{N}}\right)}{\sqrt{N \cdot E\left(\left(X_N^i\right)^2\right)}}.$$

By means of the Berry-Esseen theorem we know that, for all w,

$$|F_N(w) - \Phi(w)| \leq \frac{3E\left(|X_N^i|^3\right)}{\left(E\left(\left(X_N^i\right)^2\right)\right)^{3/2}\sqrt{N}}.$$

It is clear that the RHS of the previous inequality goes to 0 as N grows large, so the approximation with the normal distribution is increasingly good. In particular

$$\lim_{N\to\infty} |F_N(J_N) - \Phi(J_N)| = 0,$$

and since in our case $\lim_{N\to\infty} J_N = -2k$, and Φ is continuous and hence

$$\lim_{N\to\infty} |\Phi_N(J_N) - \Phi(-2k)| = 0,$$

we have that the probability of A passing goes to $1-\Phi(-2k) = \Phi(2k)$. ■

Since $\lim_{k\to\infty} \Phi(2k) = 1$, it is sufficient to see the following result, which compares the previous sequences with the one given by $p_1(H,N)$.

Lemma 3.6. *For all k natural, there exists $N^*(k)$ such that if $N \geq N^*(k)$, then*

$$p_1(H,N) < \frac{1}{2} - \frac{k}{\sqrt{N}}.$$

Proof. It is enough to see that there exists N^* such that if $N \geq N^*$ then:

$$u_N(k) = \frac{N!}{\left(\left(\frac{N-1}{2}\right)!\right)^2} (p_k(N))^{\frac{N+1}{2}} (1-p_k(N))^{\frac{N-1}{2}} > 1. \tag{3.13}$$

In order to do so, notice that it is enough to see that there exists N_0 such that $u_{N_0}(k) > 0$ and such that, if $N \geq N_0$, then

$$u_{N+2}(p,k) > \frac{N+2}{N+1} u_N(p,k),$$

since the product $\frac{N+2}{N+1}\frac{N+4}{N+3}\ldots$ diverges for any N. We have that:

$$\begin{aligned}
\frac{u_{N+2}(k)}{u_N(k)} &= \frac{N+2}{N+1} \frac{(N+2-4k^2)^{\frac{N+1}{2}}}{(N-4k^2)^{\frac{N-1}{2}}} \left(\frac{\sqrt{N+2}-2k}{\sqrt{N}-2k}\right) \frac{\sqrt{N}^N}{\sqrt{N+2}^{N+2}} \\
&> \frac{N+2}{N+1} \left(\frac{N+2-4k^2}{N-4k^2}\right)^{\frac{N-1}{2}} \frac{\sqrt{N}^{N-1}}{\sqrt{N+2}^{N+1}} (N+2-4k^2).
\end{aligned}$$

Hence, it is enough to see that:

$$\left(\frac{N+2-4k^2}{N-4k^2}\right)^{\frac{N-1}{2}} (N+2-4k^2) \geq \left(\frac{N+2}{N}\right)^{\frac{N-1}{2}} (N+2).$$

We know from Lemma 3.4 that this inequality holds for the case $k=1$, so it is enough to see that if $a \geq 4$, then for $N \geq a$ the function

$$\left(\frac{N+2-a}{N-a}\right)^{\frac{N-1}{2}}(N+2-a)$$

is increasing in a, which is straightforward derivating and grouping terms properly. ■

Now we are ready to state the main result of this section, which we have already introduced:

Theorem 3.1. *Given $H > 1$ and any $\varepsilon > 0$, there exists N^* such that if $N \geq N^*$, then $P_A(p_1(H,N)) > 1-\varepsilon$ and $P_A(p_2(H,N)) < \varepsilon$.*

Proof. Given $H > 1$ and $\varepsilon > 0$, we know that we can take k^* such that for all $k \geq k^*$, $\Phi(2k) < 1-\varepsilon/2$. Furthermore, we know from Lemma 3.5, that we can choose $N_0 > (k^*)^2$ such that if $N \geq N_0$, then the probability of A passing when $p_k(N) = \frac{1}{2} - \frac{k}{\sqrt{N}}$ is chosen by all members, is such that $\Phi(2k) - \varepsilon/2 < P_A(p_k(N))$. By Lemma 3.6 we know that there exists N_1 such that if $N \geq N_1$, then $p_1(H,N) < \frac{1}{2} - \frac{k}{\sqrt{N}}$, and hence such that if $N \geq N_1$, then $P_A(p_k(N)) < P_A(p_1(H,N))$. It is clear that choosing $N^* = \max\{N_0, N_1\}$ provides the desired result.

The proof concerning $P_A(p_2(H,N))$ is immediate realizing that $1/2 - p_1(H,N) > p_2(H,N) - 1/2$, and hence that the same N^* must work as well, since in general we have: $P_A(p_1(H,N)) < 1 - P_A(p_2(H,N))$. ■

Hence, we have derived both results of interest regarding convergence of the expected outcome of the election when the size of the committee grows large. However, it is also interesting to see numerically how are these outcomes for different values of N, since many committees have a definite size and/or cannot include an arbitrarily large amount of members.

In the following table we report, for different values of N, the probabilities that yield symmetric completely mixed strategy Nash equilibria, and the probability of A passing for each of them. We fix $H = 1.5$ for all cases, since it doesn't play an interesting role. In order to keep notation short, since we stick to $N \geq 7$, we denote simply by p_1 the equilibrium in the interval $(0, 1/2)$, and by p_2 the one in the interval $(1/2, 1)$.

N	p_1	p_2	$P_A(p_1)$	$P_A(p_2)$
7	0.3657299625	0.7555054273	0.7734354867	0.0655875462
15	0.3637414053	0.6977288773	0.8630360795	0.0521503635
31	0.3773898020	0.6532047817	0.9189185447	0.0389922406
95	0.4094768404	0.6007906874	0.9627918559	0.0232716543
215	0.4315274100	0.5730613861	0.9783017149	0.0155159882
617	0.4541302262	0.5474810870	0.9888290885	0.0090122542

We can clearly see the convergence results of Theorem 1 happening, but we can also see that in relatively large committees, such as $N = 15$, the probability of B passing under p_1 is still of almost 14%. Furthermore, it is worth pointing out that in this particular case in order to use the pivotal bribes of Dal Bò (2007) the interest group would need to have a budget of $B = 12$, while we study the extreme case of $B = 1$.

3.5. Alternative assumptions: anonymity, sequentiality and probability of being corruptible

In this section we study the model taking into account three natural alternative considerations: i) the personal vote is anonymous; and ii) voting is performed sequentially; and iii) each member has a probability of being corruptible.

3.5.1. Anonymous voting

In this case, the interest group cannot directly compensate the voters who vote for B, because he does not know who they are. However, we assume that the final score is public. Hence, let's assume that the interest group promises that it will randomly compensate as many members as votes obtained the option B.

It is easy to see that the analysis of equilibria in pure strategies remains intact. Let's consider mixed strategies. We focus once again in symmetric mixed strategy Nash equilibria. Notice as well that the situation in which all agents vote for A is not an equilibrium, since any agent has an incentive to deviate, because A passes anyway and he might get some compensation from voting for B.

Lemma 3.7. *Let i denote how many of $N-1$ members vote for B. If $i > 0$, then the remaining member, in expected terms, gets the same from the interest group whether he votes for A or for B.*

Proof. It suffices to check that:

$$\frac{1}{i+1}\frac{\binom{N-1}{i}}{\binom{N}{i+1}}=\frac{1}{i}\frac{\binom{N-1}{i-1}}{\binom{N}{i}},$$

which is trivial when written in factorial form and simplifying. The LHS is the expected payment made by the interest group to the member when he votes for B and the RHS when he votes for A. ■

Voting for B instead of voting for A now has two effects: i) it is more likely to be chosen by the interest group; and ii) less money is distributed to the selected members. And the previous Lemma proves that if $i>0$, this two effects cancel each other.

However, if $i=0$, it is obvious that this is not the case, since if the remaining member votes for A, it is certain the he will not receive any payment. Hence, if p continues to denote the probability with which every agent votes for B, the expected gain from voting for B instead of for A in the eyes of a single member of the committee is: $\frac{(1-p)^{N-1}}{N}$. And the equation describing symmetric completely mixed strategy Nash equilibria is:

$$H\binom{N-1}{\frac{N-1}{2}}p^{\frac{N-1}{2}}(1-p)^{\frac{N-1}{2}}=\frac{(1-p)^{N-1}}{N}. \tag{3.14}$$

And now, due to the nature of the functions involved, it is easier than before to characterize symmetric mixed strategy Nash equilibria.

Proposition 3.12. *In the game with anonymous voting there are exactly two symmetric mixed strategy Nash equilibria; one with $p\in(0,1/2)$ and one with $p=1$.*

Proof. Moving all terms to one side we have that Equation 3.14 can be rewritten as

$$H(1-p)^{\frac{N-1}{2}}\left(\binom{N-1}{\frac{N-1}{2}}p^{\frac{N-1}{2}}-\frac{(1-p)^{\frac{N-1}{2}}}{N}\right)=0.$$

Hence, p defines a completely mixed strategy Nash equilibrium if, and only if,

$$\binom{N-1}{\frac{N-1}{2}}p^{\frac{N-1}{2}}=\frac{(1-p)^{\frac{N-1}{2}}}{N}, \tag{3.15}$$

or $p=1$. It is clear that the LHS of Equation 3.15 is increasing and the RHS decreasing inside $(0,1)$, so there can be only one solution in the interval. Furthermore, since at $p=0$ the RHS is greater than the LHS and at $p=1/2$ the relationship is clearly the opposite, there has to be a solution in the interval $(0,1/2)$. ■

We refer to the solution $p \in (0,1/2)$ as $p_1^a(H,N)$, where the super-index a makes reference to the anonymous case we are considering. Then it is easy to see that

$$p_1^a(H,N) < p_1(H,N),$$

where $p_1(H,N)$ is the solution in the same interval of the original model. It is just a consequence of the immediate fact that the RHS of Equation 3.14 is smaller than the RHS of Equation 3.7.

3.5.2. Sequential voting

We move on to briefly discuss how the game changes if the members of the committee take turns in order to vote and the votes are public. If the votes were not public sequentiality doesn't matter.

Now we deal with a dynamic game and, as it is usually done, we use as solution concept subgame perfect Nash equilibrium (SPNE). The characterization of equilibria is simple in this case.

Proposition 3.13. *If members vote sequentially, A passes for sure in any SPNE. In any SPNE, the first $(N-1)/2$ members vote for B and the rest vote for A.*

Proof. Just notice that the $(N+1)/2$ last members don't have a credible threat with which they can punish the other ones. Pivotality is no longer uncertain. Hence, equilibrium strategies are "I vote for B unless I am pivotal, then I vote for A". ■

At this point it is worth mentioning that we should take these results with a grain of salt. Both anonymous and sequential voting, given the results we have just derived, might look like better voting mechanisms and we might consider them as good policy advise in order to reduce outside pressure on voting institutions. However, they have their own issues. On one hand, anonymous voting erases partially accountability of the voting members, which in an institution whose members are elected, like in a parliament, can be a big issue. On the other hand, the results on sequential voting are very sensitive to uncertainty on whether everyone got the same offer or not, since buying the last voter can be extremely powerful; members of the committee are no longer symmetric given a certain order of voting.

3.5.3. Each member has a probability of being corruptible

Assume that each member of the committee is corruptible with probability η. Then, from the point of view of a corruptible member, the other ones now vote for B with probablity ηp. If $\eta = 1$ we clearly recover the baseline case.

Then the indifference condition describing completely mixed equilibria is:

$$H\frac{(N-1)!}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!}(\eta p)^{\frac{N-1}{2}}(1-\eta p)^{\frac{N-1}{2}} = \frac{1-(1-\eta p)^N}{\eta pN}. \tag{3.16}$$

From the study of the baseline case, we know that, if $N \geq 7$, the previous equation has exactly two solutions in the interval $[0, 1/\eta]$, and that these solutions are at opposite sides of $1/(2\eta)$.

Proposition 3.14. *If $N \geq 7$, then let $p_1(H,N)$ and $p_2(H,N)$ denote the completely mixed strategy Nash equilibria of the baseline case ($\eta = 1$). It holds that:*

1. *If $p_2(H,N) < \eta$, there are two symmetric completely mixed strategy Nash equilibria, $p_1^\eta(H,N) = \frac{p_1(H,N)}{\eta}$ and $p_2^\eta(H,N) = \frac{p_2(H,N)}{\eta}$, and three symmetric mixed strategy Nash equilibria (since, $p = 1$ constitutes an equilibrium).*
2. *If $p_2(H,N) = \eta$, there is one completely mixed strategy Nash equilibrium, $p_1^\eta(H,N) = \frac{p_1(H,N)}{\eta}$, and $p = 1$ is a symmetric equilibrium.*
3. *If $p_1(H,N) < \eta < p_2(H,N)$, there is only one symmetric mixed strategy Nash equilibrium, which is completely mixed, $p_1^\eta(H,N) = \frac{p_1(H,N)}{\eta}$.*
4. *If $p_1(H,N) = \eta$, there is only one symmetric Nash equilibrium, which corresponds to $p = 1$.*
5. *If $\eta < p_1(H,N)$, there is no symmetric completely mixed strategy equilibria, and there is only one symmetric equilibrium, which is given by $p = 1$.*

Proof. We know that Equation 3.16, which describes completely mixed equilibria, has exactly two solutions in the interval $[0, 1/\eta]$. By performing the change of variable $p' = \eta p$ we recover the equation of the baseline case, which we know that has exactly two solutions, $p_1(H,N)$ and $p_2(H,N)$. Then, in this problem, the only two solutions in the interval $[0, 1/\eta]$ are given by $p_1^\eta(H,N) = \frac{p_1(H,N)}{\eta}$ and $p_2^\eta(H,N) = \frac{p_2(H,N)}{\eta}$. All we have to do is check when these solution are in the interval $[0,1]$. Doing it is immediate and provides the classification stated by the proposition. ■

Notice in particular, that for some values of η we lose the equilibrium with $p = 1$. This happens when $p_1(H,N) < \eta < p_2(H,N)$, since in this situation, if the other $N-1$ members set $p = 1$, the best reply of the remaining member is to vote for A (hence, to set $p = 0$). This happens because the proportion of non-corruptible members is expected to be large enough as for a single member to be pivotal with high enough probability.

Proposition 3.15. *Given $1/2 < \eta$, there exists $N^*(H)$ such that, if $N \geq N^*(H)$, then there are two symmetric completely mixed strategy Nash equilibria.*

Proof. The result is an immediate consequence of the convergence of the solutions to $1/2$ in the baseline case ($\eta = 1$). ■

Similarly:

Proposition 3.16. *Given $\eta < 1/2$, there exists $N^*(H)$ such that, if $N \geq N^*(H)$, then there is no symmetric completely mixed strategy Nash equilibrium.*

3.6. Concluding remarks

We have studied a model of voting in committees that allows for the presence of an outside interest group in a position to exert influence. The committee has to choose whether to implement a reform or not, and all its members strictly prefer the reform to be implemented. We study how disruptive an interest group that prefers the status quo and that is willing to spend a certain budget buying votes can be. In a similar scenario, Dal Bò (2007) proves that the interest group can make its preferred option pass at virtually no cost using *pivotal bribes*. However, for such bribes to be credible, the interest group might need a big budget, even if in equilibrium it does not spend it all. Furthermore, the necessary budget grows unboundedly as the committee size grows large. Focusing on a different bribing scheme, *equal split bribes*, we show that if the interest group is willing to be uncertain about avoiding the reform to be implemented, it can be very disruptive employing few resources. In addition, the scheme that we study is simple in a couple of dimensions: *i)* it is anonymous; i.e. everyone gets the same offer, hence offers need not be stated individually; and *ii)* the bribe paid to each voter does not depend on which alternative passes.

We show that, in general, the game has three symmetric mixed strategy Nash equilibria. In one of these equilibria all members of the committee vote against the reform, which is an equilibrium sometimes deemed unrealistic in the literature, and that is often found in models like ours without expressive concerns. The other two equilibria are completely mixed and there is one in which reform is more likely to pass and one in which it is less likely to pass. We see numerically that for small and medium size committees, even in the equilibrium in which the reform is more likely to pass, there is a decent chance that it does not. We show that after a certain threshold, an increase in the size of the committee makes the probabilities that committee members use in the two completely mixed strategy Nash equilibria to *moderate*, i.e., to move towards $1/2$. However, the probability of the reform being implemented does not moderate at all; in fact, we see that in the limit, as the size of the committee grows large, the probability of the reform passing the voting goes to 1 or 0 depending on which of the two completely mixed equilibria is played. The uncertainty faced by the interest group can be viewed as the price to pay in order to try to buy an election with few resources.

Our results provide a possible explanation for the "Tullock paradox"(Tullock 1967; Tullock 1997), suggesting that members of a voting body might be tempted by lower than *necessary* offers because they might assess that they are not likely, individually, to be pivotal. They are also in line with the observation that there seems to be "few money in US politics", of which de Figueiredo and Richter (2014) and Ansolabehere et al. (2003) provide evidence.

At last, we see that under anonymity and sequentiality the potential influence of the interest group is reduced. However, both these alternative assumptions have other issues outside the model here considered. For example, in democratic institutions it might be desirable to make the members accountable for their actions, since lack of accountability has been empirically associated with corruption (Persson et al. 2003).

4. Information acquisition, voting and networks.

Abstract

A society of identical individuals must choose through elections one of two alternatives under uncertainty about the state of the world. Individuals can *(a)* choose the accuracy of their private signals about the state of the world at an increasing cost, and *(b)* send messages to other individuals to whom they are connected in some network. We show that the existence of a full (communication) network leads generically to two types of equilibria. First, there always exists an equilibrium in which only one citizen—a *dictator*—acquires information and everybody else votes equally based on such information, which is sent by the dictator to all other citizens via the network. Second, the only symmetric equilibrium that would exist without a network is also an equilibrium with a full network, but only if information acquisition costs are sufficiently high. This condition keeps at bay the extent of the positive externalities created by acquiring information that can be distributed at no cost.

Keywords: elections; information acquisition; networks; free-riding.

JEL Codes: D71, D72.

4.1. Introduction

Information plays an important role in elections. As pointed out already by Downs (1957), citizens will typically not find it worthwhile to incur a substantial effort to acquire (costly) information about the consequences of policy. The reason for this is that citizens expect the information they acquire to be consequential for the election outcome only with a small probability, which captures the events in which the citizens' vote will be pivotal. Yet, crucially, election outcomes can still maximize social welfare if society acquires collectively the right amount of information. Whether elections are an effective tool to implement socially optimal outcomes hinges on the technology citizens use to acquire information about the consequences of policy, as well as on the size of the population (see e.g. Martinelli, 2006).

The above insights have been derived in (theoretical) settings that rest on a crucial assumption: individuals acquire information that is private and can only be expressed through the citizens' vote in the election. However, the emergence of social networks in modern societies—and, more broadly, the advent of myriad of communication technologies—facilitates the dissemination of (previously acquired) information among the citizens at virtually no cost prior to elections. The possibility for (true) information to be spread among the citizens may affect the functioning of elections and have important consequences for collective action in democracy. A priori, such a possibility could generate two intertwined effects affecting citizens in a heterogeneous way, but it might as well lead to none of these effects. First, at the information acquisition stage, some, or even most, citizens may prefer *not* to learn themselves first-hand about the state of the world and instead free ride on others providing second-hand information to save the costs of information acquisition; we call these citizens *uninformed voters*. Second, at the voting stage, those citizen(s) who *did* acquire information about the state of the world may be pivotal *de facto* with a very large probability; we call these citizens *informed voters*. The reason for this is that uniformed citizens will replicate the vote of the informed citizens based on the information acquired by the latter, provided uninformed voters trust the messages sent by informed voters. An important question is whether such effects can be sustained in some or in all equilibria of an underlying game and, if so, what the consequences for welfare are.

In this paper, we take up the question how the *ex post* costless spread of information influences the incentives to *ex-ante* acquire such information, and thus how this, in turn, affects election outcomes. We start from Martinelli (2006) and consider a society made up of a finite (and, for simplicity, odd) number of *ex-ante* equal citizens who has to choose via voting with the majority rule one of two alternatives. All citizens agree that choosing one of the two alternatives is best for one of the two binary states of the world, so our setup is of common value. However, citizens may disagree in the likelihood they attach to each state of the world, since they receive different private independent signals about the true state of the world. The accuracy of any such state-conditional signal depends on the effort an individual incurs. Following Martinelli (2006), information acquisition costs are modeled as an increasing, convex function of accuracy.

To include the possibility of information spread, we then depart from Martinelli

(2006) by assuming that citizens can communicate whatever information they have acquired first-hand at no cost to any other citizen with whom they are linked in some (social) network, which we model as a non-directed graph. For most of our analysis, we focus on the case of a full network: i.e., each citizen can communicate for free with any other citizen in the society. Proceeding with the full network suffices to unravel novel mechanisms in information acquisition in elections that cannot be obtained if there is no network. The specifics of the network are common knowledge before information can be acquired, and for simplicity we assume the network to be given exogenously.

The timeline of the game we analyze is the following: In Stage 0, nature draws the state of the world. In Stage 1 (*information acquisition stage*), each voter chooses the quality of the information about the state of the world that s/he wants to acquire, at a cost that increases with accuracy. Then voters privately observe their signals of the chosen accuracy. In Stage 2 (*message stage*), voters send messages to the other voters with whom they are connected in the (communication) network. Each voter observes the messages sent to him/her, evaluates how truthful these messages are, and updates his/her beliefs about the state of the world and about "how much information" other voters might hold. In Stage 3 (*voting stage*), each citizen casts a vote (there is no abstention), the alternative with most votes is implemented, and payoffs are realized.

The above dynamic game is difficult to analyze. Yet, provided the network is full, the bulk of the analysis of the above dynamic game is tantamount to investigating a suitable static game—called *information game with communication network*—where (in the only step, viz. Stage 1) all citizens simultaneously choose the quality of the information about the state of the world and receive their private signals, and then *(a)* transmit their first-hand information truthfully to everybody else, and *(b)* vote sincerely, i.e., they vote for the alternative that is most likely given all the information (first-hand and second-hand) at their disposal. This simplification allows us to compare our results about the information game with communication network to the benchmark case established in Martinelli (2006), which corresponds to the variant of the above dynamic game where the network is empty (i.e., no message can be sent between voters).

To prove that a simplification of our dynamic setup that focuses on Stage 1 makes sense, we prove the following auxiliary results for the dynamic game. First, truthful communication in Stage 2 is weakly dominant (Lemma 4.2). That is, if citizens acquire any information at all first-hand, we can consider that they will simply share it with all the other citizens (provided the network is full, as otherwise there is no guarantee for truthful reporting to be sustained in equilibrium). This means that we do not need to explicitly model Stage 2.[1] Second, voting sincerely in Stage 3 weakly dominates all other voting strategies (Lemma 4.1).[2] This means that we can also abstract from Stage 3 for our analysis. To derive Lemmas 4.1 and 4.2, we make the (behavioral) assumption that citizens will believe any message sent to them. Proceeding with such an assumption therefore enables the simplification of our dynamic game and is in keeping with a naive interpretation of our setup in which where all agents have the same goal. A Nash equilibrium of the static game underlying Stages 0–1 (which we simply call an *equilibrium*) can therefore be extended to a perfect Bayesian equilibrium (PBE) of the whole dynamic game consisting of Stages 0–3 (Lemma 4.3) in which citzens sent truthful messages, believe all messages, and vote sincerely.

Our results with regard to information games with communication network identify two *possible* intertwined mechanisms for information acquisition and voting in the presence of a full (communication) network. We show that there always exists an equilibrium of our static game in which exactly one citizen—which we call a *dictator*—acquires a positive level of information about the state of the world and all citizens—including the informed voter (or dictator) who acquired first-hand information—vote according to the informed voter's signal (Theorem 4.2). Such an equilibrium—called a *dictator equilibrium*—yields the same outcome a one-person committee would attain, and thus its predictions are independent of the size of the population. The dictator equilibrium fails to exist if there is no (communication) network (Proposition 4.1), so it cannot be obtained in the benchmark case considered by Martinelli (2006).

With a full network, the citizen acting as dictator is content with his/her decision to acquire a higher information level than s/he would if there were no network since, given the strategies of all other citizens, s/he expects to be pivotal with probability one. At the same time, the citizens who do not acquire any first-hand information do not find it worth to do otherwise as long as the information acquisition cost function is convex (and increasing). This means that the accuracy of the information the citizenry attains collectively with a full network can in principle be lower or higher than the accuracy of the information the citizenry attains collectively when there is no communication network. We show that, depending on the information acquisition function and on the population size, the probability of implementing the right alternative will be smaller or larger in the dictator equilibrium than in the case with no network. This means that we cannot unambiguously say that (communication)

[1]Hence, our setup with a full network is akin to a setting where signals are publicly observed but whose costs are private.

[2]By means of an example we show that Lemma 4.1 fails to hold in general if the network is not full.

networks are either good (Proposition 4.3) or bad for welfare (Proposition 4.2).

Beyond the dictator equilibria, an information game with communication may have other equilibria in which more than one individual acquires a positive level of information. Under a full network, it can be a focal point to look for symmetric equilibria in which all agents acquire the same level of costly information. In comparison with dictator equilibria, if such an equilibrium exists there is no asymmetry in (first-hand) information at the population level, which could be beneficial in the long term for other issues such as income inequality. We show that the equilibrium information accuracy choices in Martinelli (2006) remain also equilibrium choices under a full network, provided that marginal information acquisition costs are sufficiently large (Theorem 4.3). Moreover, *(a)* this is the only symmetric equilibrium regardless of how large marginal information acquisition costs are (Proposition 4.4), and *(b)* symmetric equilibria do not generally exist if it is sufficiently cheap to acquire a further marginal bit of information (Proposition 4.5). With a full network, all individual signals (of the accuracy chosen without a network) become public for everyone, so it cannot be very cheap to acquire information for intermediate levels of information quality to be sustained in equilibrium. When such an equilibrium exists, the posterior about the state of the world is much more accurate with the possibility of communication than without such a possibility, conditional on the same (equilibrium) information accuracy levels. Yet, because the decision which alternative to implement is taken via the majority rule, the probability of implementing the correct alternative—i.e., the alternative that matches the state of the world—is the same in both situations (Proposition 4.6). This implies that, in equilibrium, the positive externalities created by acquiring information in the presence of a full network vanish, and thus no citizen underprovides information (compared to the case of no network).

Our results suggest that networks can have a dramatic impact on the structure of information acquisition in elections and/or in election outcomes, but also that neither of these phenomena need necessarily take place. This raises the question whether or not it is good for social welfare that such networks—which facilitate the spread of information—be regulated. Of course, in real life (social) networks can transmit false information, particularly in polarized societies, which can have further profound consequences for the quality of democracy. But our results warn us that even in the absence of conflict and polarization in the citizenry, social networks—or, more generally, the possibility for information to be spread among citizens at no cost—can have negative consequences for welfare.

The paper is organized as follows. In Section 4.2 we review the literature most closely connected to our paper. In Section 4.3 we introduce our model, set up the notation, and prove some results that justify our analysis of a simplified static game instead of the full dynamic game. In Section 4.4 we show the existence of dictator (asymmetric) equilibria in our static game. In Section 4.5 we show the existence of symmetric equilibria in our static game. In Section 4.6 we briefly discuss some extensions of our model and results. Section 4.7 concludes.

4.2. Related literature

Our paper enriches the understanding of how elections aggregate information and impact democratic performance. Condorcet (1785) conjectured that majority outcomes may be more reliable when more citizens exert their voting rights; in fact, the wisdom of majority may be infallible in arbitrarily large elections. Several strands of research have emerged since to investigate this conjecture, some of which our study aligns with, as elaborated in the following paragraphs.

The first formal explorations of Condorcet's thoughts (Black, 1958; Miller, 1986; Grofman and Feld, 1988; Young, 1988; Ladha, 1992, 1993; Berg, 1993; Paroush, 1998; McLean et al., 1994) generally consider settings in which information is spread exogenously among voters, and in which voters vote sincerely. A voter is said to vote sincerely when s/he votes for the alternative that is more likely from his/her point of view. Austen-Smith and Banks (1996) demonstrates that, even if all voters share the same goal, sincere voting might not be always rational; other papers further contributed to this understanding (Duggan and Martinelli, 2001; Mukhopadhaya, 2003). However, most of this literature focuses on environments in which voters do not share their private information. Information transmission across peers have become an important aspect in elections recently, so it is natural to investigate the role of strategic behavior and (strategic) communication under such circumstances. Li (2001) studies a model in which voters are assumed to perfectly see each others' efforts and signals, and investigates how committing to different aggregation rules might lead to different levels of effort. We prove that voting sincerely and reporting information truthfully can be rational if voters trust each other's messages.

Martinelli (2006), which is the closest paper to our work, questions the assumption that information is exogenously dispersed among voters. In both models, voters incur a cost to increase the precision of their private information about an alternative to be voted upon later. Unlike in Martinelli (2006), in our setup voters in our setup can send messages to each other about both the accuracy and the content of their signals. Gerardi and Yariv (2008) explore a model with information acquisition and communication, without votes but with one principal who aggregates the reported information. However, they focus on the mechanism design side of the problem, looking for ways to both extract as much information as possible and ensure truthful reporting. In addition, unlike in our setup, they assume that there are just two information levels (informed and non-informed); we have a continuum of possible information levels.

Bobkova and Klein (2020) investigate voting scenarios where privately informed voters can share their information as well, although not in a costly information acquisition environment, and under a unanimous voting rule. They show that information sharing can only be beneficial to an information centralizer principal when the accuracy of each voter's private information is sufficiently high. Our full network can be analogous to a non-strategic central planner eliciting private information from voters and recommending decisions. Our findings offer a different perspective: for each private information accuracy that emerges in a symmetric equilibrium without information sharing, we identify a class of cost functions that implement the same equilibrium assuming perfect observability.

Gersbach et al. (2020) study the role of information in committees. Like us, they consider information acquisition to be costly and examine its impact on the level of information to be acquired. However, Gersbach et al. (2020) do not consider the possibility of communication between committee members, which we do if we think of our full network as a full committee.

Apart from the analysis of information aggregation in elections, our paper contributes broadly to a literature of public goods provision. Indeed, information serves a public good in our model, since private signals and their inference are made available to all connected voters in equilibrium. Moreover, our assumption on costly acquisition deepens the incentives for free-riding on the provision of information by other players. Bramoullé and Kranton (2007) study the provision of public goods in networks, although in a complete information environment. They find the existence of equilibria with specialization, that is, equilibria in which only some individuals contribute and others free ride. Our results of existence of dictator equilibria is an analogue to their findings. Like us, Bramoullé and Kranton (2007) also find that specialization can be good or bad for society.

In our model, messages can be sent for free, so lying is costless. Yet, because there is no conflict of interest, we find that an equilibrium exists with full information revelation among players (a separating equilibrium in cheap talk games, as commonly denoted since Crawford and Sobel, 1982). Our assumption about messages being commonly believed resembles the assumption of naivety from receivers in the cheap-talk model in Chen (2011).

4.3. Model

An electorate of $2n+1$ citizens must choose one of two alternatives, A and B. We let $N := \{1,...,2n+1\}$ denote the set of all citizens (also called voters). All citizens agree that each of the alternatives is more suitable for one of two states of the world, so we consider a common value setup. The state of the world cannot be directly observed. *ex-ante*, citizens attach equal probabilities to any of the two possible states of the world, and this is common knowledge. A voter's utility $U(d,z)$ depends on the chosen alternative, $d \in \{A,B\}$, and the realized state of the world, $z \in \{z^A, z^B\}$. We normalize utilities so that for all citizens, $U(A,z^A) = U(B,z^B) = 1$ and $U(A,z^B) = U(B,z^A) = 0$. This means that citizens derive the same utility from implementing alternative A in state z^A and alternative B in state z^B. Citizens also derive the same utility level, yet a lower one, from implementing alternative A in state z^B or alternative B in state z^A.

Prior to voting, citizens must decide on the quality of information about the state of the world they want to acquire. The quality (or accuracy) is modeled as some value $x \in \left[0, \frac{1}{2}\right]$ and a binary signal space $\{s^A, s^B\}$, with probability distributions such that $\mathbb{P}[s^A|z^A,x] = \mathbb{P}[s^B|z^B,x] = \frac{1}{2}+x$. Thus, the higher the choice of x, the higher the informativeness of any observed signal. However, acquiring information is costly: choosing x reduces the citizen's utility by some amount $C(x)$. We follow Martinelli (2006) and assume that $C(x)$ is strictly increasing, strictly convex and twice continuously differentiable for $x \in \left(0, \frac{1}{2}\right)$, and it satisfies $C(0) = 0$ and $C'(0) = 0$.

Once citizens have acquired *first-hand* information about the state of the world through their signals, they can send a message to the voters connected to them in some exogenously given network. In these messages, citizens can specify both *(a)* how much information about the state of the world they have acquired and *(b)* what signal they have received. Formally, we let $\Gamma : N \to \mathscr{P}(N)$ be the *(communication) graph* describing the network. That is, we assume that citizen $i \in N$ is connected to the set of voters $\Gamma(i)$, and we impose the condition that $j \in \Gamma(i)$ if and only if $i \in \Gamma(j)$, i.e., the graph is non-directed. Given the graph, voter i sends message $m_i^j = (x_i^j, s_i^j) \in [0, \frac{1}{2}] \times \{s^A, s^B\}$ to voter $j \in \Gamma(i)$. We let M_i denote the set of all the messages containing *second-hand* information that voter i has received from all citizens $j \in \Gamma(i)$.

After receiving the messages sent to them, voters update their beliefs about *(a)* the state of the world, and about *(b)* how much information the other voters have. In our setup, updating about the state of world alone is not sufficient. The reason for this is as follows: when casting a vote after acquiring first-hand information and receiving second-hand information from others, if a voter thinks that someone else has more information than him/her, s/he might find it beneficial to increase the probability of that agent being pivotal. This will be made clear in Example 4.1. To account for the fact that updating about the state of the world does not suffice for voting, we define a *node* of the game as some vector $\theta = ((z_i, P_i))_{i \in N}$, where $z_i \in \{z^A, z^B\}$ denotes the state of the world that voter i deems more likely, and P_i denotes the probability with which that agent is correct about the state of the world. We denote by Θ the set of all possible nodes of the game and by $\Delta(\Theta)$ the set of all probability distributions over Θ. After observing the messages, and given their prior beliefs, voters update the probability distribution from $\Delta(\Theta)$.

Finally, citizens cast their votes (no abstention occurs). The alternative that receives more votes is implemented, and payoffs are realized.

Summarizing, the timing of our political game, which consists of three main stages, is the following:

0. Nature draws the state of the world $z \in \{z^A, z^B\}$.

1. Information acquisition stage:
   a) Each voter $i \in N$ chooses quality of information $x_i \in \left[0, \frac{1}{2}\right]$.
   b) Each voter $i \in N$ observes signal $s_i \in \{s^A, s^B\}$ with precision $\frac{1}{2} + x_i$.

2. Message stage:
   a) Voters send messages to the voters connected with them according to Γ.
   b) Each voter observes the messages sent to him/her and updates his/her beliefs about which game node has been reached.

3. Voting stage:
   a) Each voter casts one vote, and the alternative with more votes is implemented.
   b) Voter i obtains payoff $U(d,z) - C(x_i)$ under state of the world z if $d \in \{A, B\}$ is implemented and s/he choose x_i in Stage 1.

In the above dynamic game, which we call **information game with messages** and denote by $\mathscr{G}$, a strategy for voter i consists of: *(i)* an information quality/accuracy $x_i \in [0, \frac{1}{2}]$; *(ii)* a message $m_i(x_i, s_i)$ for any chosen information level, $x_i \in [0, \frac{1}{2}]$, and for any signal received, $s_i \in \{s^A, s^B\}$; *(iii)* mappings α_A^i and α_B^i from the set of probability distributions over the set of nodes, $\Delta(\Theta)$, to a probability of choosing alternative A or B, respectively, i.e., for $d \in \{A, B\}$,

$$\alpha_d^i : \Delta(\Theta) \to [0, 1].$$

Given that all citizens vote (there is no abstention), we have $\alpha_A^i + \alpha_B^i = 1$ for all $i \in N$. Hence, one of the mappings is enough to describe the strategy of a voter.

From now on, unless stated otherwise we consider a full comunication network, i.e., we make the following assumption:

Assumption 4.1 (Full network)**.** $\Gamma(i) = N \setminus \{i\}$ *for every voter* $i \in N$.

Under Assumption 4.1, every citizen sends a message to any other citizen. Later we discuss other network structures and the relevance of our choice for the communication network.

Next, we adapt the definition of sincere voting of Austen-Smith and Banks (1996) to our setup, and we show that voting sincerely is a dominant strategy for all voters in Stage 3 under appropriate assumptions.

Definition 4.1. *A voting strategy* (α_A^i, α_B^i) *for voter i is sincere if it always selects the alternative which maximizes the expected individual utility given individual i's information.*

A sincere voting strategy is uniquely defined for any given information unless both alternatives yield the same expected utility given such an information.

In our game, citizens update their beliefs about the node of the game that has been reached, so the appropriate solution concept is Perfect Bayesian Equilibrium. Yet, in the next three lemmas we justify that we can simplify the analysis of game $\mathscr{G}$ to the analysis of some suitable static game capturing the main elements of game $\mathscr{G}$, and then use Nash equilibrium as our solution concept. To be able to carry out such a simplification, we momentarily make the following assumption:

Assumption 4.2 (Commonly believed messages)**.** *It is common knowledge that voters believe that the information sent to them is truthfully reported by the other citizens (in short, we say that messages are commonly believed).*

Assumption 4.2 can be seen as a behavioral tenet that is in keeping with our common value setup. As we shall see, it renders such a setup solvable while keeping the main features of the entire setup as we introduced it above. Assumption 4.2 allows voters not to truthfully communicate their information to others (on or off the equilibrium path), yet this fact will not be learned by the citizens receiving such messages. Given Assumption 4.2, the voters' updating process with regard to the probability distribution of the game nodes that have been reached is straightforward: each voter will allocate probability one to the node matching his/her first-hand information (which is private and has been acquired by himself/herself) and his/her second-hand information (which s/he has received from other citizens).

In the following, we prove a number of lemmas that will enable us to simplify the dynamic setup introduced above. The first lemma considers the incentives to vote sincerely, as set out in Definition 4.1.

Lemma 4.1 (Sincere voting)**.** *Under Assumptions 4.1 and 4.2, voting sincerely is weakly dominant for all voters in Stage 3 of game $\mathcal{G}$ regardless of history until the end of Stage 2.*

Proof. Consider some voter $i \in N$. No matter whether or not s/he sent a truthful message to any other citizen, under Assumption 4.2 his/her probability distribution over Θ puts probability one to the node corresponding to his/her private information (information acquisition level and signal) and the messages s/he received. Conditional on being in that node of the game, no other citizen can have a higher probability than him of being right. The reason for this is as follows: being in a full network in which all citizens receive messages from all other citizens, under Assumption 4.2 there is no citizen different from voter i who has a higher level of information from voter i's perspective. If voter i did truthfully reveal his/her private information in Stage 1, then s/he believes all other citizens to have his/her same level of information. If voter i did not truthfully reveal his/her private information in Stage 1, then s/he believes all other citizens to have a lower level of information. Finally, since citizen i's vote only matters when breaking a tie, voting for whichever alternative s/he thinks is more likely (given his/her information about the nodes of the game) is weakly-dominant. ∎

Next, we show that if messages are commonly believed and all voters vote sincerely in Stage 3, it must also be the case that in Stage 2 all citizens report truthfully both the signals they received and the information levels they acquired in Stage 1.

Lemma 4.2 (Truthful reporting)**.** *Let Assumptions 4.1 and 4.2 be satisfied. Then, for any history of game $\mathcal{G}$ until the end of Stage 1, no citizen has strict incentives to send a message different from (x_i, z_i) in Stage 2, where $x_i \in \left[0, \frac{1}{2}\right]$ is the level of information acquired in Stage 1 and $z_i \in \{z^A, z^B\}$ is the signal received in this same stage by citizen i.*

Proof. We start noting that by Lemma 4.1, under Assumptions 4.1 and 4.2 citizens anticipate that all of them will vote sincerely in Stage 3. Then consider some voter $i \in N$. Due to Assumption 4.2, voter i believes that all the messages s/he received were truthfully reported. We proceed by contradiction, so we suppose that citizen i sends some message m_i in Stage 2 to some citizen j that differs from (x_i, s_i). Then, Assumption 4.2 and the fact that the communication graph is complete imply that citizen j will have, from the perspective of voter i, an overall lower information

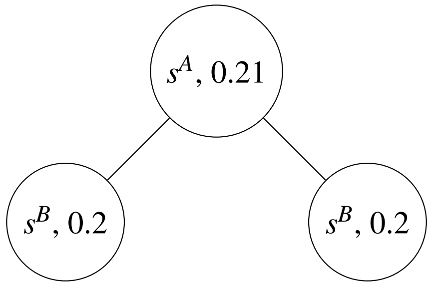

Figure 4.1.: Network of Example 4.1 with the information levels and signals.

level than i. Now recall that we are assuming that citizen j will vote sincerely in Stage 3, i.e., s/he will use whatever information s/he gathered in Stage 2. This implies that there is some (possibly zero) probability that citizen j will vote in Stage 3 for alternative $z \in \{z^A, z^B\}$ when, according to the information citizen i holds, i.e., $(x_i, s_i) \times M_i$, alternative $z' \in \{z^A, z^B\} \setminus \{z\}$ should be implemented. Clearly, citizen j's vote can either have no effect on the election outcomes or break a tie and yield the implementation of z. Hence, by not truthfully reporting (x_i, z_i) to all other citizens, voter i expects a utility that is lower than, or equal to, the one s/he expects if s/he sends message (x_i, z_i) to all citizens. Hence, given $(x_i, s_i) \times M_i$, citizen i has no strong incentive to send false information in Stage 3, provided of course that Assumption 4.2 holds. ■

Lemmas 4.1 and 4.2 imply that commonly believing all messages, i.e., Assumption 4.2, *can* be compatible with a PBE of the information game with messages consisting of Stages 1–3, viz. game $\mathscr{G}$. The reason for this is as follows: we have seen that, given any information levels acquired in Stage 1, commonly believing messages induces truthful reporting in Stage 2. We have also proved that these beliefs (weakly) induce sincere voting in Stage 3. It remains to be seen if, given truthful reporting, common believing and sincere voting, there exist equilibrium information levels in Stage 1.

It is important to note that Lemmas 4.1 and 4.2 may not carry over to all communication networks. For example, take a look at the following instance of our model:

Example 4.1. *There are three voters ($n = 1$), so $N = \{1, 2, 3\}$. Voter 1 is connected with voters 2 and 3 in graph Γ, but the latter two voters are not connected to each other in the graph. Suppose that voter 1 acquires information level $x_1 = 0.3$ and receives signal $s_1 = s^A$, while voters 2 and 3 acquire information level $x_2 = x_3 = 0.2$ and both receive $s_2 = s_3 = s^B$. This information is summarized in Figure 4.1.*

Suppose now that the three voters submit truthful messages and that Assumption 4.2 holds. With the information they hold, voters 2 *and* 3 *think that* $z = z^A$ *is the most likely state of the world. However, voter* 1*, who has more information than them, thinks that* $z = z^B$ *is actually more likely. Hence, if voter* 3 *votes sincerely, i.e., if s/he votes for A, it is strictly better for voter* 2 *to vote for B than to vote for A, since then s/he makes voter* 1 *pivotal. In other words, in this example, voting sincerely might not be dominant for all voters, and an equivalent to Lemma 4.1 cannot be extended in general that do not satisfy Assumption 4.1.*

Furthermore, if we assume that all voters vote sincerely and that Assumption 4.2 holds, then voter 1 *strictly prefers to submit a truthful message to one of the voters and the message* (x_2, s^B) *to the other one, instead of submitting a truthful message to both voters. Similarly as before, this ensures that voter* 1 *is pivotal, which is better for everyone since s/he has more information. Hence, Lemma 4.2 cannot be extended in general to non-full networks.*

Henceforth, we assume that voters submit truthful messages in Stage 2 and vote sincerely in Stage 3 no matter the history of the game. This enables us to focus on a static reduced version of the dynamic game $\mathscr{G}$ in which each voter can perfectly observe the signals and the information levels of the other voters and voting is sincere. The timing of this static game, denoted by $\mathscr{G}^1$ and called **information game with communication**, is:

0. Nature draws the state of the world $z \in \{z^A, z^B\}$.

1. Acquisition, observation, and voting stage:
   a) Each voter $i \in N$ chooses quality of information $x_i \in \left[0, \frac{1}{2}\right]$.
   b) Each voter observes signal $s_i \in \{s^A, s^B\}$ with precision $\frac{1}{2} + x_i$.
   c) Each voter observes the signals and information levels of the voters connected with him/her according to Γ.
   d) Each voter votes sincerely, given $(x_i, s_i)_{i \in N}$.
   e) Voter i obtains payoff $U(d,z) - C(x_i)$ under state of the world z, if $d \in \{A, B\}$ is implemented and x_i was chosen by such a voter.

In game $\mathscr{G}^1$, a strategy for voter i is a choice of information quality x_i, with $x_i \in \left[0, \frac{1}{2}\right]$. To find Nash equilibria of game $\mathscr{G}^1$, we need a tie-breaking rule for the case when the voters' posterior is the same for both states of the world. To introduce our tie-breaking rule, some further notation comes in handy. Let $\boldsymbol{x} := (x_i)_{i \in N}$ represent a vector of information qualities and $\boldsymbol{s} := (s_i)_{i \in N}$ represent a vector of signals, one for each citizen. Let also $S_A(\boldsymbol{s})$ and $S_B(\boldsymbol{s})$ represent the set of components of $\boldsymbol{s}$ that are equal to s^A and s^B, respectively. The state-conditioned probabilities of $\boldsymbol{s}$ under $\boldsymbol{x}$ are as follows:[3]

$$\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A] = \prod_{j\in S_A(\boldsymbol{s})}\left(\frac{1}{2}+x_j\right)\prod_{k\in S_B(\boldsymbol{s})}\left(\frac{1}{2}-x_k\right), \tag{4.1}$$

$$\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^B] = \prod_{j\in S_A(\boldsymbol{s})}\left(\frac{1}{2}-x_j\right)\prod_{k\in S_B(\boldsymbol{s})}\left(\frac{1}{2}+x_k\right). \tag{4.2}$$

For the remainder of our analysis of game $\mathscr{G}^1$ we shall impose the following tie-breaking rule:

Assumption 4.3 (Tie-breaking rule)**.** *For any citizen $i \in N$, if his/her posterior is completely uninformative, i.e., if $\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A] = \mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^B]$, then such a citizen votes for the alternative that is most suitable for the state of the world that matches the signal s/he received.*[4]

Assumption 4.3 implies that, in game $\mathscr{G}^1$, it is common knowledge among all citizens that any voter i will vote for alternative A if $\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A] > \mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A]$, will vote for alternative B if $\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^B] > \mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A]$, and will votes for alternative s_i if $\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A] = \mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^B]$. Note that since we are considering a full network, viz. Assumption 4.1, all citizens will vote for the same alternative unless $\mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^A] = \mathbb{P}[\boldsymbol{s}|\boldsymbol{x},z^B]$.

Using Lemmas 4.1 and 4.2 and the discussion after such results enables us to obtain the following result:

Lemma 4.3 (Connection between games)**.** *Let $(x_i^*)_{i\in N}$ be a Nash equilibrium of the information game, viz. $\mathscr{G}^1$. Then, a PBE of the information game with messages, viz. $\mathscr{G}$, exists in which citizens acquire information levels $(x_i^*)_{i\in N}$, send truthful messages, and vote sincerely, and then messages are commonly believed.*

Our next task is therefore to explore the set of Nash equilibria of game $\mathscr{G}^1$. In any Nash equilibria of this (static) game, which we denote typically as $(x_i^*)_{i\in N}$, each voter i chooses $x_i^* \in \left[0,\frac{1}{2}\right]$ to maximize his/her *ex-ante* payoff, given that other voters choose x_{-i}^*. This payoff can be written as a function of $(x_i)_{i\in N}$ as

$$G(x_i,x_{-i}) := \frac{1}{2}\left\{P_\alpha(A|x_i,x_{-i},z^A)+P_\alpha(B|x_i,x_{-i},z^B)\right\} - C(x_i),$$

where $P_\alpha(A|\boldsymbol{x},z^A)$ and $P_\alpha(B|\boldsymbol{x},z^B)$ represent the probabilities of alternative A and B winning the election under state z^A and z^B respectively, given $\boldsymbol{x}$.

[3] In Equations (4.1) and (4.2), we adopt the convention that the empty product equals one.

[4] Other tie-breaking rules such as randomizing between states or choosing arbitrarily one of the states when a posterior tie occurs would not lead to different results. Hence, in the statements of the results we do not make it explicit that we are considering Assumption 4.3.

The full network—viz. Assumption 4.1—represents one polar case of our general setup. Another polar case is the empty network.

Assumption 4.4 (Empty network)**.** *$\Gamma(i) = \emptyset$ for every voter $i \in N$.*

Under Assumption 4.4, no citizen can send any message to other citizens. In such a case, game $\mathscr{G}$ is akin to the dynamic game obtained from $\mathscr{G}$ featuring only Stages 0, 1, and 3. If we additionally assume sincere voting, the analysis of the latter dynamic game reduces to the analysis of a static game consisting of Stages 0 and 1. We call such a static game **information game without communication** and denote it by $\mathscr{G}^2$. It has been analyzed in Martinelli (2006), which is our starting point.[5]

Theorem 4.1 (Martinelli (2006); Gersbach et al. (2020))**.** *Game $\mathscr{G}^2$ has a unique symmetric equilibrium, denoted by $(x_i^*)_{i\in N}$, where $x_i^* = x^*(n) = x^*$ solves the following equation*

$$\binom{2n}{n}\left(\frac{1}{4}-(x^*)^2\right)^2 = C'(x^*). \tag{4.3}$$

Moreover, $x^*(n)$ is strictly decreasing in n and

$$\lim_{n\to\infty} P[\textit{right alternative is chosen}] = \phi(2\sqrt{2k}), \tag{4.4}$$

where

$$\sqrt{2}\phi(2\sqrt{2k}) = kC''(0). \tag{4.5}$$

We shall use the results in Theorem 4.1 as benchmarks for our subsequent endeavours.[6]

4.4. Dictator equilibria

In this section we show that game $\mathscr{G}^1$ always has an equilibrium in which exactly one voter acquires information about the state of the world and all other voters vote according to this voter's signal. It is in this sense that we refer to equilibria with such a property as *dictator equilibria*.

[5]In the polar case of a full network, viz. Assumption 4.1, truth-telling in the message stage is compatible with equilibrium behavior (Lemma 4.2). This enables us to think of our framework under Assumption 4.1 as one in which signals and information acquisition levels are public, as opposed to our framework under Assumption 4.4 in which both signals and information acquisition levels are private.

[6]For the analysis of game $\mathscr{G}^2$ we also consider Assumption 4.3.

Theorem 4.2. *In game $\mathscr{G}^1$, a dictator equilibrium $(x_i^*, 0, \ldots, 0)$ always exist for any citizen $i \in N$. In this equilibrium, citizen i acquires perfect information if and only if $C'(1/2) \geq 1$.*

Proof. We want to show that $(x_i^*, 0, \ldots, 0)$ is an equilibrium of game $\mathscr{G}^1$ for some value of x_i^* and any citizen $i \in N$. That is, any $j \in N \backslash \{i\}$ is a voter for which $x_j^* = 0$.

We start by analyzing voter i's best response when no other citizen acquires information, i.e., when $x_{-i}^* = (\overbrace{0, \ldots, 0}^{2n})$. If voter i chooses $x_i > 0$, then whichever signal s/he receives is the alternative that will be chosen. The reason for this is two-fold: on the one hand, all signals and information acquisition levels are public; on the other, all citizens vote sincerely. Accordingly, if $x_i > 0$, voter i's *ex-ante* expected payoff given x_{-i}^* is

$$G(x_i, x_{-i}^*) = \left(\frac{1}{2} + x_i\right) - C(x_i).$$

It is easy to verify that $G(0, x_{-i}^*) = 1/2$, since in such a case no citizen acquires any information. Since $C'(0) = 0$, we obtain $G'(0, x_{-i}^*) = 1 > 0$, so $x_i = 0$ cannot be a best reply to x_{-i}^*. Hence, voter i's best response to x_{-i}^* is either interior ($x_i^* < 1/2$), if $C'(1/2) > 1$, or is $x_i^* = 1/2$, if $C'(1/2) \leq 1$. In the former case, the interior solution x_i^* corresponds to the information acquisition level that solves the following equation:

$$1 = C'(x_i^*). \tag{4.6}$$

Next, we take as given the optimal choice of $x_i^* > 0$ (depending on the value of $C'(1/2)$), and verify that no voter $j \in N \setminus \{i\}$ wishes to deviate from $x_j = 0$ to $x_j > 0$, taking also as given that $x_k = 0$ for all $k \in N \setminus \{i, j\}$. First of all, acquiring $x_j \in (0, x_i^*)$ is strictly dominated by $x_j = 0$. This follows from the fact that *(a)* $C(x)$ is strictly increasing for $x \in (0, 1/2)$, and that *(b)* if voter j acquires an information level lower than x_i^*, the alternative chosen by all citizens (including voter j) will continue to be the one that matches citizen i's signal. We split the remainder of the proof in two cases, depending on the value of the derivative of the cost function at $1/2$.

Case I: $C'(1/2) \leq 1$. In this case, voter i acquires full information, i.e., $x_i^* = 1/2$. Since the correct alternative is therefore chosen with probability one and C is strictly increasing, it is clear that voter j strictly prefers to not acquire any information. Therefore, $x_i^* = 1/2$ and $x_j^* = 0$ for all $j \in N \setminus \{i\}$ is a Nash equilibrium of game $\mathscr{G}^1$.

Case II: $C'(1/2) > 1$. In this case, voter i acquires an interior level of information, i.e., $0 < x_i^* < \frac{1}{2}$, which solves Equation (4.6). If voter j deviates from choosing $x_j = 0$ to $x_j \in (x_i^*, 1/2]$, then whichever alternative matching voter j's signal will be chosen by all citizens. The reason for this is the same as in Case I. Accordingly, voter j's *ex-ante* expected payoff for $x_j \in (x_i^*, 1/2]$ is

$$G(x_j, x_{-j}^*) = \left(\frac{1}{2} + x_j\right) - C(x_j).$$

It then suffices to note that for $x_j \in (x_i^*, 1/2]$,

$$\frac{\partial G(x_j, x_{-j}^*)}{\partial x_j} = 1 - C'(x_j) < 1 - C'(x_i^*) = 0,$$

where the inequality follows from the fact that C is strictly convex and the least equality is equivalent to Equation (4.6). Hence, for voter j no deviation from choosing $x_j = 0$ to $x_j \in (x_i^*, 1/2]$ is profitable.

Therefore, it remains to verify the case where citizen j deviates from $x_j = 0$ to $x_j = x_i^*$. There are two cases. First, if $s_i = s_j$, then all citizens vote according to citizen i's and citizen j's signals. Therefore, the alternative corresponding to such signals is chosen with probability one. Second, if $s_i \neq s_j$, citizens vote according to their own signal due to Assumption 4.3. In particular, citizen i votes for alternative s_i and citizen j votes for alternative s_j. This means that both votes cancel each other out, so the election outcome depends on the remaining $2n-1$ citizens. Hence, the probability of alternative A (B) winning under state z^A (z^B) is the probability of at least n over $2n-1$ voters obtaining signals s_A (s^B). This leads to:

$$\begin{aligned}
G(x_j; x_{-j}^*) &= \left(\frac{1}{2} + x_i^*\right)^2 + 2\left(\frac{1}{2} + x_i^*\right)\left(\frac{1}{2} - x_i^*\right)\sum_{k=n}^{2n-1}\binom{2n-1}{k}\left(\frac{1}{2}\right)^{2n-1} - C(x_i^*),\\
&= \left(\frac{1}{2} + x_i^*\right)^2 + 2\left(\frac{1}{2} + x_i^*\right)\left(\frac{1}{2} - x_i^*\right)\left(\frac{1}{2}\right)^{2n-1}\sum_{k=n}^{2n-1}\binom{2n-1}{k} - C(x_i^*),\\
&= \left(\frac{1}{2} + x_i^*\right)^2 + 2\left(\frac{1}{2} + x_i^*\right)\left(\frac{1}{2} - x_j^*\right)\left(\frac{1}{2}\right)^{2n-1} 2^{2(n-1)} - C(x_i^*),\\
&= \left(\frac{1}{2} + x_i^*\right)^2 + \left(\frac{1}{2} + x_i^*\right)\left(\frac{1}{2} - x_i^*\right) - C(x_i^*),\\
&= \left(\frac{1}{2} + x_i^*\right) - C(x_i^*).
\end{aligned}$$

The expected utility given by the above expression increases if citizen j chooses any $x_j' < x_i^*$ instead of $x_j = x_i^*$. The reason for this is that the probability of choosing the right alternative will still be $\left(\frac{1}{2} + x_i^*\right)$, but the private cost of learning incurred by voter j will be smaller since C is strictly increasing for $x \in (0, 1/2)$. To sum up, it is a best response for any citizen $j \in N \setminus \{i\}$ to choose $x_j^* = 0$, given x_i^* satisfying Equation (4.6) and $x_k^* = 0$ for all $k \in N \setminus \{i, j\}$, which completes the proof of the theorem. ∎

Theorem 4.2 reveals that a communication network—which we model by Γ—can lead to the endogenous sorting of the population in terms of information: Only a small number of citizens—in the theorem, only one citizen—will decide to actively seek first-hand information and be *informed* about the consequences of policy, and thus they become the sole experts in the society. The rest of the society is *uninformed* and relies on the second-hand information provided by the expert(s). While in the polar case of a full network all citizens have the same information *ex post*, uninformed voters only have such an information because it is transmitted to them by the informed voter(s), which justifies the use of the term "uninformed".

A possible (negative) consequence about the separation between experts and non-experts is the increase in information inequality in the citizenry. This can occur in our common value setup, and thus in the absence of political polarization. If, for reasons outside our model, exerting some effort to acquire first-hand information could help individuals develop (payoff-relevant) abilities in the mid- or long-term, the skill gap in society could become larger, thereby increasing income inequality. The sorting of the population in terms of information can also have dramatic effects on how much information is acquired at the aggregate level and later expressed through election outcomes.

To *(a)* see why the effects just discussed *can* be attributed to the communication network, and to *(b)* estimate the magnitude of such effects created by communication networks on the aggregate information levels, consider as a benchmark the case of an empty network—viz. Assumption 4.4—which as we saw leads to game $\mathscr{G}^2$. It then turns out that the equilibrium of Theorem 4.2 vanishes under an empty network.

Proposition 4.1. *In game $\mathscr{G}^2$, there is no equilibrium $(x_i, 0, \ldots, 0)$ for any citizen $i \in N$, provided $n \geq 1$.*

Proof. Suppose that voter i chooses $x_i \in (0, 1/2]$ and the remainder voters acquire no information at all, i.e., voters $j \in N \setminus \{i\}$ choose $x_j = 0$. We inquire if for some voter $j \in N \setminus \{i\}$, $x_j = 0$ can be the best response to $(x_k)_{k \in N \setminus \{j\}}$. Citizen k's vote only matters when there is a tie among the other voters,for $x_j = 0$ to be a best response it is necessary that

$$\begin{aligned} C'(0) &\geq \left(\frac{1}{2} + x_i\right) \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} + \left(\frac{1}{2} - x_i\right) \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} \\ &= \binom{2n-1}{n} \left(\frac{1}{2}\right)^{2n} > 0. \end{aligned} \tag{4.7}$$

To derive the right-hand side of the above inequality we have used Assumption 4.3, but other tie-breaking rules would lead to the same result. However, the assumption that $C'(0) = 0$ leads to a contradiction with Inequality (4.7), which finishes the proof of the proposition. ■

Proposition 4.1 shows that the dictator equilibria are not an artifact of the assumption that, in contrast to Martinelli (2006), we look for asymmetric equilibria of game $\mathscr{G}^1$. Such equilibria exist because the (full) network allows $2n$ citizens to rely on the information acquired by the dictator at no cost, which is the main first effect the (full) network can have on information acquisition and voting. The second main effect is that the dictator has all the voting power—i.e., s/he is pivotal in the election with probability one. Therefore, the dictator reckons it is worth incurring a higher information acquisition level compared to the case where all other citizens acquired some information (see Theorem 4.1). These two effects lead to the increase of information inequality, as discussed above.

A further important consequence of Theorem 4.2 is that the level of information acquisition the dictator acquires in a dictator equilibrium is independent of the size of the population, $2n+1$. Moreover, such an information level depends on the behavior of function $C(x)$ away from $x = 0$. By contrast, we know from Theorem 4.1 that in large populations—i.e., if we let n go to infinity—only the behavior of $C(x)$ around $x = 0$ matters. An ensuing question is therefore how in terms of welfare the equilibria of Theorem 4.2 compare to the equilibrium of Theorem 4.1. We (partially) undertake such a comparison in the following two propositions for sufficiently large populations.

Proposition 4.2. *Suppose $C''(0) = 0$ and $C'(1/2) > 1$. Then there exists $n^* \in \mathbb{N}$ such that, if $n \geq n^*$, the probability of choosing the right alternative is higher in the symmetric equilibrium of the information game without communication, $\mathscr{G}^2$, than in the dictator equilibrium of the information game with communication, $\mathscr{G}^1$.*

Proof. On the one hand, if $C'(1/2) > 1$ it follows from Theorem 4.2 that, in game $\mathscr{G}^1$, the dictator does not acquire perfect information and such information level is independent of n. Hence, the probability that the right alternative is implemented is bounded away uniformly from one, say it is below $1-\varepsilon$ with $\varepsilon > 0$, for all $n \in \mathbb{N}$.

On the other hand, if $C''(0) = 0$, it follows from Theorem 4.1—see Equations (4.4) and (4.5)—that the probability that the right alternative is implemented converges to one as n goes to infinity. Therefore, given ε, we can choose $n^* := n^*(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n^*$, the probability of choosing the right alternative in the unique symmetric equilibrium of $\mathscr{G}^2$ is higher than $1-\varepsilon$. ■

Proposition 4.2 shows that communication networks can have negative effects on welfare, when this is measured by the probability of reaching the right alternative. This will be the case if it is cheap to acquire just a bit of information ($C''(0) = 0$)

and expensive to acquire perfect information ($C'(1/2) > 1$). Moreover, Theorem 3 in Martinelli (2006) shows that if $C''(0) = 0$ then, in the limit as n goes to infinity, aggregate information costs convergence to zero. Using arguments analogous to those used for the proof of Proposition 4.2 one can then see that the conclusions of such a result remain if welfare accounts for both the probability of choosing the right alternative and the aggregate costs of information acquisition.

Proposition 4.3. *Suppose $C''(0) > 0$ and $C'(1/2) < 1$. Then there exists $n^* \in \mathbb{N}$ such that, if $n \geq n^*$, the probability of choosing the right alternative is lower in the symmetric equilibrium of the information game without communication, $\mathscr{G}^2$, than in the dictator equilibrium of the information game with communication, $\mathscr{G}^1$.*

Proof. On the one hand, if $C''(0) > 0$, it follows from Theorem 4.1 that the probability that the right alternative is implemented in the limit as n goes to infinity is bounded away from one, say it is below $1-\varepsilon$ with $\varepsilon > 0$—see Equation (4.4).

On the other hand, if $C'(1/2) = 1$ it follows from Theorem 4.2 that, in game $\mathscr{G}^1$, the dictator will acquire perfect information no matter the population size. Hence, the probability that the right alternative is implemented is one and independent of n. This implies that given ε, we can choose $n^* := n^*(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n^*$, the probability of choosing the right alternative in the dictator equilibrium of $\mathscr{G}^1$ is higher than $1-\varepsilon$. ■

Proposition 4.3 is concerned with the case when it is expensive to acquire just a bit of information ($C''(0) > 0$) and cheap to acquire perfect information ($C'(1/2) > 1$), and it is in sharp contrast with Proposition 4.2. It shows that communication networks can also have positive effects on welfare measured by the probability of reaching the right alternative. This means that, in general, we cannot predict whether communication networks will foster the common good or not.

4.5. Symmetric equilibria

In this section, we study the existence of equilibria of game $\mathscr{G}^2$ where all voters acquire the same (positive) level of information, which stands in contrast to the dictator equilibria analyzed in the previous section where only one citizen acquires a positive level of information. As we shall see, the former equilibria, if they exist, are intimately related to the symmetric equilibrium of game $\mathscr{G}^2$, yet for reasons different than in Martinelli (2006). Specifically, we next show that if information is sufficiently expensive to acquire for the citizens, the symmetric equilibrium of game $\mathscr{G}^2$ identified by Theorem 4.1 is also an equilibrium of game $\mathscr{G}^1$. One interpretation of this result is that communication networks allowing the costless spread of information *can* be neutral with respect to individual efforts to become informed and with respect to election outcomes. To derive the main result of this section—Theorem 4.3 below—we impose an additional assumption on the information acquisition function beyond the assumptions made in the previous section.

Assumption 4.5 (Information acquisition function)**.** *Information acquisition function C satisfies that $C \in \mathscr{C}^3([0,1/2])$ and $C'(x), C''(x), C'''(x) \geq 0$ for all $x \in [0,1/2]$.*

To show our main results, we rely on a number of technical lemmas, which we prove next. The first lemma, viz Lemma 4.4 below, gives sufficient conditions for the solution of one maximization problem in some interval to be higher than, or equal to, the solution of another maximization problem in a larger interval. In the proof of Theorem 4.3, we partition the set of information acquisition costs $x \in [0,1/2]$ in different intervals and for each such region we ensure by means of Lemma 4.4 that no citizen has an incentive to deviate from the equilibrium information acquisition cost.

Lemma 4.4. *Let C be an information acquisition function satisfying Assumption 4.5 and let $b_1, b_2 > 0$ be some constants satisfying $b_2 > b_1$, $c_1 > c_2$, and*

$$\hat{x} = \frac{c_1 - c_2}{b_2 - b_1} \in (0,1/2). \tag{4.8}$$

Define functions $y_1(x) := c_1 + b_1 x - C(x)$ and $y_2(x) := c_2 + b_2 x - C(x)$, which cross at $\hat{x}$. Then assume that

$$y_1'(\hat{x}) < 0 \tag{4.9}$$

and

$$y_2'(\hat{x}) < -y_1'(\hat{x}). \tag{4.10}$$

If there exists $\bar{x} \in [\hat{x}, 1/2]$ such that

$$y_2'(\bar{x}) < 0, \tag{4.11}$$

it must be the case that

$$\max_{[0,\hat{x}]} y_1(x) \geq \max_{[0,\bar{x}]} y_2(x). \tag{4.12}$$

Proof. We start by noting that for $k \in \{1,2\}$, the assumptions on $C(\cdot)$, including Assumption 4.5, guarantee that

$$y_k'(0) = b_k - C'(0) = b_k > 0$$

and

$$y_k''(x) = -C''(x) \leq 0. \tag{4.13}$$

Moreover, Conditions (4.9), (4.11), and (4.13) imply

$$y_1'(x) < 0 \text{ for all } x \geq \hat{x}$$

and

$$y_2'(x) < 0 \text{ for all } x \geq \bar{x}.$$

Therefore, for $k \in \{1,2\}$,

$$x_k := \arg \max_{x \in [0,1/2]} y_k(x),$$

is well defined and satisfies

$$b_k = C'(x_k). \tag{4.14}$$

Moreover, Conditions (4.9) and (4.11) imply respectively that

$$x_1 \in (0, \hat{x})$$

and

$$x_2 \in (0, \bar{x}), \tag{4.15}$$

while the assumption that $C'(\cdot)$ is increasing for $(0,1/2)$ together with $b_2 > b_1$ and Equation (4.14) imply that

$$x_1 < x_2. \tag{4.16}$$

Finally, note that for all $x \in [0,1/2]$,

$$y_2'(x) - y_1'(x) = b_2 - b_1 > 0. \tag{4.17}$$

Next, we distinguish two cases.

Case I: $y_2'(\hat{x}) \leq 0$.

In this case,

$$\max_{[0,\bar{x}]} y_2(x) = \max_{[0,\hat{x}]} y_2(x) < \max_{[0,\hat{x}]} y_1(x),$$

where the equality follows from $y_2'(\hat{x}) \leq 0$ and Condition (4.13), which ensure that $x_2 \leq \hat{x}$. The inequality is explained as follows. Since $y_2(x)$ is maximized at $x = x_2$, it suffices for the inequality to hold that

$$y_1(x_2) \geq y_2(x_2). \tag{4.18}$$

Finally,

$$y_1(x_2) = y_1(\hat{x}) + \int_{\hat{x}}^{x_2} y_1'(x)dx = y_2(\hat{x}) - \int_{x_2}^{\hat{x}} y_1'(x)dx$$
$$> y_2(\hat{x}) - \int_{x_2}^{\hat{x}} y_2'(x)dx = y_2(x_2),$$

where the first and third equality follow from the fundamental theorem of calculus, the second equality holds since $y_1(x)$ and $y_2(x)$ cross at $\hat{x}$, and the inequality is due to Condition (4.17).

Case II: $y_2'(\hat{x}) > 0$.

In this case, $y_2'(\hat{x}) > 0$ and Inequality (4.11) imply that

$$x_2 \in (\hat{x}, \bar{x}). \tag{4.19}$$

We claim (and show next) that

$$|x_1 - \hat{x}| \geq |x_2 - \hat{x}|. \tag{4.20}$$

Indeed, we obtain

$$C'(\hat{x}) > \frac{b_1 + b_2}{2} = \frac{C'(x_1) + C'(x_2)}{2} \geq C'\left(\frac{x_1 + x_2}{2}\right), \tag{4.21}$$

where the first inequality can be derived using the definitions of $y_1(\cdot)$ and $y_2(\cdot)$ in Condition (4.10), the equality follows from Equation (4.14), and the last inequality holds since $C'(\cdot)$ is convex in $(0,1/2)$—see Assumption 4.5. Since $C'(\cdot)$ is also strictly increasing in $(0,1/2)$, it follows from (4.21) that

$$\hat{x} > \frac{x_1 + x_2}{2}.$$

The above inequality together with Inequality (4.16) shows that the claim made in (4.20) does indeed hold.

Let $x \in [\hat{x}, x_2]$. Then $y_2'(x) \geq 0$, and from (4.20), we obtain that

$$x_1 < \hat{x} - (x - \hat{x}) \tag{4.22}$$

and, hence,

$$y_1'(\hat{x} - (x - \hat{x})) < 0. \tag{4.23}$$

Moreover,

$$
\begin{aligned}
-y_1'(\hat{x}-(x-\hat{x})) = -y_1'(\hat{x}) - \int_{\hat{x}-(x-\hat{x})}^{\hat{x}} C''(t)dt > y_2'(\hat{x}) - \int_{\hat{x}}^{x} C''(t)dt \\
\geq y_2'(\hat{x}) - \int_{\hat{x}}^{x} C''(t)dt = y_2'(x),
\end{aligned}
\tag{4.24}
$$

where the two equalities follow from the fundamental theorem of calculus, the first inequality is implied by Condition (4.10), and the second inequality is due to the following three facts: *(a)* $x \geq \hat{x}$; *(b)* $\hat{x} - (\hat{x} - (x - \hat{x})) = x - \hat{x}$; and *(c)* $C''(x)$ is non-decreasing for $x \in (0, 1/2)$.

Finally, we claim (and show next) that

$$
y_1(\hat{x} - (x_2 - \hat{x})) \geq y_2(x_2), \tag{4.25}
$$

which implies Condition (4.12) and finishes the proof. Therefore, it remains to show Inequality (4.25). To do so, we start noting that if we use the fundamental theorem of calculus we can write

$$
y_1(\hat{x} - (x_2 - \hat{x})) = y_1(\hat{x}) + \int_{\hat{x}}^{\hat{x}-(x_2-\hat{x})} y'(x)dx = y_1(\hat{x}) + \int_{\hat{x}-(x_2-\hat{x})}^{\hat{x}} -y_1'(x)dx \tag{4.26}
$$

and

$$
y_2(x_2) = y_2(\hat{x}) + \int_{\hat{x}}^{x_2} y_2'(x)dx. \tag{4.27}
$$

Using (4.24) for all $x \in [\hat{x}, x_2]$ and noting that $\hat{x} - (\hat{x} - (x_2 - \hat{x})) = x_2 - \hat{x}$,

$$
\int_{\hat{x}-(x_2-\hat{x})}^{\hat{x}} -y_1'(x)dx > \int_{\hat{x}}^{x_2} y_2'(x)dx.
$$

This last inequality, together with Equation (4.8) and Equations (4.26)–(4.27), implies that the claim made in Condition (4.25) is correct. ■

The second lemma, viz. Lemma 4.5, shows a property of the information acquisition function under Assumption 4.5.

Lemma 4.5. *Let C be an information acquisition function satisfying Assumption 4.5 and let $x^* \in (0, 1/2]$ be given. For all $x \geq x^*$,*

$$
\frac{C'(x)}{C'(x^*)} \geq \frac{x}{x^*}. \tag{4.28}
$$

Proof. Inequality (4.28) is equivalent to $x^* C'(x) \geq x C'(x^*)$. For all $x \in (0, 1/2]$, define

$$h(x) := x^* C'(x) - x C'(x^*),$$

which implies

$$h'(x) = x^* C''(x) - C'(x^*)$$

and, by Assumption 4.5,

$$h''(x) = x^* C'''(x) \geq 0. \tag{4.29}$$

By definition of function h and since, by assumption, $C'(0) = 0$,

$$h(0) = h(x^*) = 0. \tag{4.30}$$

Moreover, we claim (and prove below) that

$$h'(x^*) = x^* C''(x^*) - C'(x^*) \geq 0, \tag{4.31}$$

which together with Equation (4.29) implies that

$$h'(x) \geq 0 \text{ for all } x \geq x^*. \tag{4.32}$$

Then, for all $x \geq x^*$,

$$h(x) = h(x^*) + \int_x^{x^*} h'(x)dx = \int_x^{x^*} h'(x)dx \geq 0,$$

where the first equality follows the fundamental theorem of calculus, the second equality follows from Equation (4.30), and the inequality holds due to Equation (4.32).

Therefore, it remains to show Equation (4.31), which can be explained as follows:

$$C'(x^*) = C'(0) + \int_0^{x^*} C''(x)dx = \int_0^{x^*} C''(x)dx \leq \int_0^{x^*} C''(x^*)dx = x^* C''(x^*),$$

where the first equality follows from the fundamental theorem of calculus, the second equality follows from the assumption that $C'(0) = 0$, and the inequality holds since $C'''(x) \geq 0$ for all $x \in [0, 1/2]$ due to Assumption 4.5.

■

Before we prove the next lemma, we introduce some further notation. Consider $s^1, \ldots, s^k = s \in \{s^A, s^B\}$ to be $k > 0$ equal signals of accuracy x ($x \in [0, 1/2]$). Then we define Δ_k^x as the accuracy of one signal $s' \in \{s^A, s^B\} \setminus \{s\}$ guaranteeing that the posterior after observing signals $s_1, \ldots, s_k, s'$ is equal to the prior, i.e., each alternative is believed to occur with probability $1/2$. Formally,

$$\Delta_k^x := \frac{(0.5+x)^k}{(0.5+x)^k+(0.5-x)^k} - 0.5. \tag{4.33}$$

For completeness, we also define $\Delta_0^x := 0$. It is clear that Δ_k^x is increasing in both k and x. The reasons for this is that one needs signal s' to be more accurate to "compensate" either more signals of the same accuracy or the same number of signals of more intense accuracy.

The next lemma shows a property of the thresholds defined in (4.33).

Lemma 4.6. *Let $k \geq 1$. Then,*

$$\lim_{x \to 0} \frac{\Delta_k^x}{x} = k.$$

Proof. We have a $0/0$ indeterminacy. However, if we apply L'Hôpital's rule, we obtain for $k \geq 1$,

$$\lim_{x \to 0} \frac{\Delta_k^x}{x} = \lim_{x \to 0} \frac{k\left(0.25 - 1.x^2\right)^{k-1}}{\left((0.5-x)^k+(x+0.5)^k\right)^2} = k.$$

$\blacksquare$

Before we show the existence of symmetric equilibria, we prove that there is an upper bound to the number of such equilibria. Specifically, the next proposition shows that game $\mathscr{G}^1$ has at most one (symmetric) equilibrium in which all citizens acquire the same quality of information.

Proposition 4.4. *Given an information game with communication, $\mathscr{G}^1$, if $(x_i = x)_{i \in N}$ is an equilibrium, then $x_i = x^*$.*

Proof. Let $(x, \ldots, x) \in [0, 1/2]^{2n+1}$ be a strategy profile of game $\mathscr{G}^1$. Without loss of generality, we focus on voter 1's best response to the remaining $2n$ voters choosing information level $x \geq 0$. We know from Theorem 4.2 that $x_i = 0$ for all $i \in N$ never constitutes an equilibrium, so we can assume $x > 0$. Moreover, $x = 1/2$ cannot constitute an equilibrium either, since any voter would strictly prefer to acquire zero information and free ride on the information acquired by the others. Hence, we can assume $x \in (0, 1/2)$.

It is clear that a signal of accuracy $x > 0$ can never compensate two opposite signals of the same accuracy $x > 0$. Therefore, if voter 1 acquires information level x, his/her signal will only be followed by all the citizens (including voter 1) whenever among the remaining citizens there are as many s^A signals as there are s^B

signals. But this means that the utility of voter 1 when $x_1 = x \in [0, \Delta_2^x]$ and all other citizens choose information acquisition level x is

$$G(x_1, x, \ldots, x) = b(x) + \left(\frac{1}{2} + x_1\right)\binom{2n}{n}\left(\frac{1}{2} + x\right)^n\left(\frac{1}{2} - x\right)^n - C(x_1),$$

where $b(x)$ is some expression independent of x_1. By differentiating the above expression and equating it to zero, we obtain Equation (4.3). This means that $x = x^*$, which completes the proof of the proposition.

■

Before we proceed to the general case with arbitrary information acquisition function, $C(\cdot)$, and population size, $2n+1$, we pause to take a closer look at an example of our model where $n = 1$ and $C(x) = ax^2$. (Note that $C(\cdot)$ satisfies Assumption 4.5.) In this example, we can solve the equations of interest algebraically.

Proposition 4.5. *Let $C(x) = ax^2$ and $2n+1 = 3$ (equivalently, $n = 1$). Then game $\mathscr{G}^1$ has a symmetric equilibrium if and only if $a \geq 1$. If such an equilibrium exists, each individual acquires information level x^*, as in the unique symmetric equilibrium of the information game without communication, $\mathscr{G}^2$.*

Proof. From Proposition 4.4, we know that if a symmetric equilibrium exists, then $x_i = x^*$ for all $i \in \{1,2,3\}$. If $n = 1$ and $C(x) = ax^2$, we can solve Equation (4.3) explicitly and obtain

$$x^* = \frac{-a + \sqrt{a^2+1}}{2}. \tag{4.34}$$

Without loss of generality, we focus on voter 1's best response to voters 2 and 3 acquiring information level x^*. From the proof of Proposition 4.4, we know that $x_1 = x^*$ is voter 1's optimal choice in the interval $[0, \Delta_2^{x^*}]$. Accordingly, we next analyze voter 1's optimal choice in the interval $(\Delta_2^{x^*}, 1/2]$, and then we compare the optimal choices in both intervals. If $x_1 \in (\Delta_2^{x^*}, 1/2]$, all voters vote according to voter 1's signal independent of the other signals. This means that for $x_1 \in (\Delta_2^{x^*}, 1/2]$, voter 1's (expected) utility (given $x_2 = x_3 = x^*$) is

$$G(x_1, x^*, x^*) = \left(\frac{1}{2} + x_1\right) - C(x_1). \tag{4.35}$$

This is the same utility of the dictator voter in Theorem 4.2. Given that $1/(2a) > \Delta_2^{x^*}$, we know that the optimal choice of x_1 in $(\Delta_2^{x^*}, 1/2]$ that maximizes the expression in (4.35) is attained at $x_1 = 1/(2a)$ if $a \geq 1$ and at $x_1 = 1/2$ if $a < 1$. For the remainder of the proof we thus distinguish two cases.

Case 1: $a \geq 1$.

We prove that there is a symmetric equilibrium. To do so, it suffices to compare voter 1's expected utility if $x_1 = x^*$ to his/her utility if $x_1 = 1/2$. Given Expression (4.34), we have that

$$G(x^*, x^*, x^*) = 0.5a^3 - 0.5a^2\sqrt{a^2+1} + 0.5\sqrt{a^2+1} - 0.25a + 0.5 \tag{4.36}$$

and

$$G(1/(2a), x^*, x^*) = 0.5 + \frac{1}{4a}. \tag{4.37}$$

We claim (and show next) that if $a \geq 1$,

$$G(x^*, x^*, x^*) \geq G(1/(2a), x^*, x^*). \tag{4.38}$$

Using Equations (4.36) and (4.37), Condition (4.38) reads as

$$0.5a^3 - 0.5a^2\sqrt{a^2+1} + 0.5\sqrt{a^2+1} - 0.25a + 0.5 \geq 0.5 + \frac{1}{4a},$$

which can be rearranged as

$$f(a) := 2a^4 - 2a^3\sqrt{a^2+1} + 2a\sqrt{a^2+1} - a^2 \geq 1. \tag{4.39}$$

It is immediate to verify that $f(1) = 1$. Hence, Condition (4.38) will hold if we show that if $a \geq 1$,

$$f'(a) \geq 0. \tag{4.40}$$

To show this, we note that $f'(a) = 0$ is equivalent to

$$8a^3\sqrt{a^2+1} - 2a\sqrt{a^2+1} = 8a^4 + 2a^2 - 2. \tag{4.41}$$

Next, we apply some non-injective transformations to Equation (4.41), and we obtain

$$\begin{aligned}
\left(8a^3\sqrt{a^2+1} - 2a\sqrt{a^2+1}\right)^2 &= \left(8a^4 + 2a^2 - 2\right)^2 \\
4a^2(a^2+1) - 32a^4(a^2+1) + 64a^6(a^2+1) &= 64a^8 + 32a^6 - 28a^4 - 8a^2 + 4 \\
64a^8 + 32a^6 - 28a^4 + 4a^2 &= 64a^8 + 32a^6 - 28a^4 - 8a^2 + 4 \\
4a^2 &= -8a^2 + 4 \\
a &= \pm\frac{\sqrt{3}}{3}.
\end{aligned}$$

It is straightforward to verify that only $a = -\frac{\sqrt{3}}{3}$ is a solution of $f'(a) = 0$. Finally,

$f'(0) > 0$ implies Inequality (4.40). This concludes the proof of Case 1 and shows that if $a \geq 1$, game $\mathscr{G}^1$ has a symmetric equilibrium (in which the three voters acquire information level x^*).

Case 2: $a < 1$.

We prove that there is not a symmetric equilibrium. To do so, we compare voter 1's expected utility if $x_1 = x^*$ to his/her utility if $x_1 = 1/2$. Similarly as before,

$$G(x^*, x^*, x^*) = 0.5a^3 - 0.5a^2\sqrt{a^2+1} + 0.5\sqrt{a^2+1} - 0.25a + 0.5, \qquad (4.42)$$

and

$$G(1/2, x^*, x^*) = 1 + \frac{a}{4}. \qquad (4.43)$$

We claim (and show next) that if $a \in (0,1)$,

$$G(1/2, x^*, x^*) > G(x^*, x^*, x^*). \qquad (4.44)$$

Using Equations (4.42) and (4.43), Condition (4.44) reads as

$$0.5a^3 - 0.5a^2\sqrt{a^2+1} + 0.5\sqrt{a^2+1} - 0.25a + 0.5 < 1 - \frac{a}{4}$$

which can be rearranged as

$$g(a) := 0.5a^3 - 0.5a^2\sqrt{a^2+1} + 0.5\sqrt{a^2+1} - 0.25a + 0.5 + a/4 < 1. \qquad (4.45)$$

It is immediate to check that $g(0) = g(1) = 1$. Hence, proving that that $g'(a) = 0$ only has one solution in $[0,1]$ and that it corresponds to a minimum of $g(a)$ suffices to show Condition (4.44). Note that

$$g'(a) = \frac{a\left(-1.5a^2 + 1.5a\sqrt{a^2+1} - 0.5\right)}{\sqrt{a^2+1}}.$$

Therefore, $g'(a) = 0$ if and only if,

$$-1.5a^2 + 1.5a\sqrt{a^2+1} - 0.5 = 0. \qquad (4.46)$$

Equation (4.46) can be solved using arguments analogous as those we use to solve Equation (4.41). If we do so, we obtain that $a = \sqrt{3}/3$ is the only solution to Equation (4.46) in the interval $[0,1]$. It is then straightforward to check that $g''(\sqrt{3}/3) > 0$. Accordingly, we have proved Inequality (4.45), which means that Condition (4.44) holds. This concludes the proof of Case 2 and shows that if $a < 1$, voter 1 strictly prefers to set $x_1 = 1/2$ rather than $x_1 = x^*$, and as a consequence game $\mathscr{G}^1$

has no symmetric equilibrium. ■

Proposition 4.5 together with Proposition 4.4 shows that, in general, game $\mathscr{G}^1$ may not have a symmetric equilibrium if it is sufficiently cheap to acquire information about the true state of the world. In such a case, a full network will necessarily generate inequality regarding first-hand information levels in the population. This inequality can be very extreme, since we know from Theorem 4.2 that a dictator equilibrium always exists for game $\mathscr{G}^1$.

Next, we move on to the general case regarding the information acquisition function and population size and show the main result of this section, namely, that game $\mathscr{G}^1$ has a symmetric equilibrium if the cost of marginally learning another bit of information about the true state of the world is sufficiently large. It means that a full network need not necessarily generate inequality regarding first-hand information.

Theorem 4.3. *Consider an information game with communication, $\mathscr{G}^1$, where the information acquisition cost $aC(x)$ satisfies Assumption 4.5. There exists $a^*(n)$ such that if $a \geq a^*(n)$, then $(x^*,\dots,x^*)$ is an equilibrium of $\mathscr{G}^2$.*

Proof. We assume that $2n$ voters choose to acquire information level x^* and analyze the best response of the remaining agent, who we consider without loss of generality to be voter 1. Voter 1 chooses x_1 to maximize

$$G(x_1,x^*,\dots,x^*) = \frac{1}{2}\left\{P_\alpha(A|x_1,x^*,\dots,x^*,z^A) + P_\alpha(B|x_1,x^*,\dots,x^*,z^B)\right\} - C(x_1).$$

We divide the proof in five steps.

Step 1. We derive an explicit expression for $G(x_1,x^*,\dots,x^*)$ for all $x_1 \in [0,1/2]$ and show that it is continuous in x_1 in the entire interval.

Function $G(x_1,x^*,\dots,x^*)$ is defined piecewise. Indeed, for $k \in \{0,\dots,n\}$, the restriction of $G(x_1,x^*,\dots,x^*)$ to $(\Delta^{x^*}_{2k},\Delta^{x^*}_{2k+2}]$ coincides in this interval with the following function:

$$\begin{aligned}G^k(x_1,x^*,\dots,x^*) = -C(x_1) + \sum_{i=k+1}^{n}\binom{2n}{n+i}\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}\\ +\left(\frac{1}{2}+x_1\right)\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^{n}\left(\frac{1}{2}-x^*\right)^{n} + \left(\frac{1}{2}+x_1\right)\cdot\\ \cdot\left(\sum_{i=1}^{k}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i} + \left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right)\right).\end{aligned} \tag{4.47}$$

Function $G^k(x_1,x^*,\dots,x^*)$ is voter 1's expected utility when the remaining voters choose to acquire information level x^* and voter 1' chooses an information that is accurate enough to offset $2k$ opposite signals of accuracy x^*.

To show the continuity of function $G(x_1,x^*,\dots,x^*)$ for all $x^1 \in [0,1/2]$, it suffices to show that for all $k \in \{0,\dots,n-1\}$,

$$G^k(\Delta^{x^*}_{2k+2},x^*,\dots,x^*) = G^{k+1}(\Delta^{x^*}_{2k+2},x^*,\dots,x^*). \tag{4.48}$$

To show Equation (4.48), note that:

$$G^{k+1}(x_1,x^*,\dots,x^*) - G^k(x_1,x^*,\dots,x^*) = -\binom{2n}{n+k+1}\left(\frac{1}{2}+x^*\right)^{n+k+1}\left(\frac{1}{2}-x^*\right)^{n-k-1}$$
$$+\left(\frac{1}{2}+x_1\right)\binom{2n}{n+k+1}\left(\frac{1}{2}+x^*\right)^{n+k+1}\left(\frac{1}{2}-x^*\right)^{n-k-1}$$
$$+\left(\frac{1}{2}+x_1\right)\binom{2n}{n+k+1}\left(\frac{1}{2}+x^*\right)^{n-k-1}\left(\frac{1}{2}-x^*\right)^{n+k+1}$$

Finally, $G^{k+1}(x_1,x^*,\dots,x^*) - G^k(x_1,x^*,\dots,x^*) = 0$ is a linear equation in x_1. Solving for x_1 leads to $x_1 = \Delta^{x^*}_{2k+2}$. This completes the proof of Step 1.

Step 2. We show that for $k \in \{1,\dots,n\}$, there exists $a_1^*(n,k)$ such that if $a \geq a_1^*(n,k)$,

$$\left.\frac{\partial G^{k-1}(x_1,x^*,\dots,x^*)}{\partial x_1}\right|_{x_1=\Delta^{x^*}_{2k}} < 0.$$

Define function

$$f^n_k(x^*) := (1-2k)\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^n\left(\frac{1}{2}-x^*\right)^n$$
$$+\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}+\left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right).$$

Then

$$\begin{aligned} f^n_k(0) &= (1-2k)\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}+\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(2\left(\frac{1}{2}\right)^{2n}\right) \\ &\leq (1-2k)\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}+2(k-1)\binom{2n}{n+1}\left(\frac{1}{2}\right)^{2n} \\ &= \left(\frac{1}{2}\right)^{2n}\left(-(2k-1)\binom{2n}{n}+2(k-1)\binom{2n}{n+1}\right) < 0. \end{aligned} \tag{4.49}$$

Next, we compute

$$
\frac{\partial G^{k-1}(x_1,x^*,\dots,x^*)}{\partial x_1}\bigg|_{x_1=\Delta_{2k}^{x^*}} = -aC'(\Delta_{2k}^{x^*}) + \binom{2n}{n}\left(\frac{1}{2}+x^*\right)^n\left(\frac{1}{2}-x^*\right)^n
$$
$$
+\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}+\left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right). \tag{4.50}
$$

From Equation (4.3),

$$
a = \frac{\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^n\left(\frac{1}{2}-x^*\right)^n}{C'(x^*)}. \tag{4.51}
$$

If we substitute (4.51) into (4.50), we obtain

$$
\frac{\partial G^{k-1}(x_1,x^*,\dots,x^*)}{\partial x_1}\bigg|_{x_1=\Delta_{2k}^{x^*}} = \left(1-\frac{C'(\Delta_{2k}^{x^*})}{C'(x^*)}\right)\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^n\left(\frac{1}{2}-x^*\right)^n
$$
$$
+\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}+\left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right).
$$

By Lemma 4.5, we know that

$$
\frac{C'(\Delta_{k+1}^{x^*})}{C'(x^*)} \geq \frac{\Delta_{k+1}^{x^*}}{x^*},
$$

which allows us to obtain

$$
\frac{\partial G^{k-1}(x_1,x^*,\dots,x^*)}{\partial x_1}\bigg|_{x_1=\Delta_{2k}^{x^*}} \leq \left(1-\frac{\Delta_{2k}^{x^*}}{x^*}\right)\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^n\left(\frac{1}{2}-x^*\right)^n
$$
$$
+\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}+\left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right) := g_k^n(x^*).
$$

Since $g_k^n(x^*)$ is continuous in x^*, Lemma 4.6 and (4.49) guarantee that

$$
\lim_{x^*\to 0} g_k^n(x^*) = f_k^n(0) < 0.
$$

Hence, there exists $\overline{x_1}(k,n) > 0$ such that,

$$
x^* \leq \overline{x_1}(k,n) \Rightarrow g_k^n(x^*) < 0. \tag{4.52}
$$

Finally, from Theorem 4.1 we know the following two properties: *(a)* x^* is decreasing in a, and *(b)* $\lim_{a\to\infty} x^* = 0$. Using *(a)* and *(b)* together with (4.52) implies that there is $a_1^*(n,k)$

$$a \geq a_1^*(n,k) \Rightarrow x^* \leq \overline{x_1}(k,n).$$

This completes the proof of Step 2.

Step 3. We show that given $k \in \{1,\ldots,n\}$, there exists $a_2^*(n,k)$ such that if $a \geq a_1^*(n,k)$, then

$$-\left.\frac{\partial G^{k-1}(x_1,x^*,\ldots,x^*)}{\partial x_1}\right|_{x_1=\Delta_{2k}^{x^*}} > \left.\frac{\partial G^{k}(x_1,x^*,\ldots,x^*)}{\partial x_1}\right|_{x_1=\Delta_{2k}^{x^*}}.$$

Define the following function:

$$\begin{aligned} h_k^n(x) := {} & (4k-2)\binom{2n}{n}\left(\frac{1}{2}+x\right)^n\left(\frac{1}{2}-x\right)^n \\ & -\binom{2n}{n+k}\left(\left(\frac{1}{2}+x\right)^{n+k}\left(\frac{1}{2}-x\right)^{n-k}+\left(\frac{1}{2}+x\right)^{n-k}\left(\frac{1}{2}-x\right)^{n+k}\right) \\ & -2\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x\right)^{n+i}\left(\frac{1}{2}-x\right)^{n-i}+\left(\frac{1}{2}+x\right)^{n-i}\left(\frac{1}{2}-x\right)^{n+i}\right). \end{aligned}$$

By some standard algebraic manipulations we obtain

$$\begin{aligned} h_k^n(0) &= (4k-2)\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} - 2\binom{2n}{n+k}\left(\frac{1}{2}\right)^{2n} - 4\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\frac{1}{2}\right)^{2n} \\ &= \left(\frac{1}{2}\right)^{2n}\left((4k-2)\binom{2n}{n} - 2\binom{2n}{n+k} - 4\sum_{i=1}^{k-1}\binom{2n}{n+i}\right) \\ &\geq \left(\frac{1}{2}\right)^{2n}\left((4k-2)\binom{2n}{n} - 2\binom{2n}{n+k} - 4(k-1)\binom{2n}{n+1}\right) \\ &\geq \left(\frac{1}{2}\right)^{2n}\left((4k-4)\binom{2n}{n} - (4k-4)\binom{2n}{n+1}\right) > 0. \end{aligned}$$

Next, note that

$$\begin{aligned} & -\left.\frac{\partial G^{k-1}(x_1,x^*,\ldots,x^*)}{\partial x_1}\right|_{x_1=\Delta_{2k}^{x^*}} + \left.\frac{\partial G^{k}(x_1,x^*,\ldots,x^*)}{\partial x_1}\right|_{x_1=\Delta_{2k}^{x^*}} \\ & = \left(2\frac{C'(\Delta_{2k}^{x^*})}{C'(x^*)}-2\right)\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^n\left(\frac{1}{2}-x^*\right)^n \\ & -\binom{2n}{n+k}\left(\left(\frac{1}{2}+x^*\right)^{n+k}\left(\frac{1}{2}-x^*\right)^{n-k}+\left(\frac{1}{2}+x^*\right)^{n-k}\left(\frac{1}{2}-x^*\right)^{n+k}\right) \end{aligned}$$

$$
\begin{aligned}
&-2\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}+\left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right)\\
&\geq -2\sum_{i=1}^{k-1}\binom{2n}{n+i}\left(\left(\frac{1}{2}+x^*\right)^{n+i}\left(\frac{1}{2}-x^*\right)^{n-i}+\left(\frac{1}{2}+x^*\right)^{n-i}\left(\frac{1}{2}-x^*\right)^{n+i}\right)\\
&-\binom{2n}{n+k}\left(\left(\frac{1}{2}+x^*\right)^{n+k}\left(\frac{1}{2}-x^*\right)^{n-k}+\left(\frac{1}{2}+x^*\right)^{n-k}\left(\frac{1}{2}-x^*\right)^{n+k}\right)\\
&+\left(2\frac{\Delta_{2k}^{x^*}}{x^*}-2\right)\binom{2n}{n}\left(\frac{1}{2}+x^*\right)^{n}\left(\frac{1}{2}-x^*\right)^{n}:=l_k^n(x^*),
\end{aligned}
$$

where to derive the equality we used (4.51) to substitute for a and the inequality follows from Lemma 4.5. Since $l_k^n(x^*)$ is continuous in x^*, Lemma 4.6 guarantees that

$$
\lim_{x^*\to 0} l_k^n(x^*) = h_k^n(0). \tag{4.53}
$$

Hence, there exists $\overline{x_2}(k,n) > 0$ such that,

$$
x^* \leq \overline{x_2}(k,n) \implies l_k^n(x^*) > 0. \tag{4.54}
$$

Finally, from Theorem 4.1 we know the following two properties: *(i)* x^* is decreasing in a, and *(ii)* $\lim_{a\to\infty} x^* = 0$. Using *(i)*–*(ii)* together with (4.54) implies that there is $a_2^*(n,k)$ such that

$$
a \geq a_2^*(n,k) \Rightarrow x^* \leq \overline{x_2}(k,n).
$$

This completes the proof of Step 3.

Step 4. We prove that given n, there exists $a_3^*(n)$ such that if $a \geq a_3^*(n)$, then

$$
\left.\frac{\partial G^n(x_1,x^*,\ldots,x^*)}{\partial x_1}\right|_{x_1=\Delta_{2k}^{x^*}} < 0.
$$

We know that $G^n(1/2,x^*,\ldots,x^*) = \left(\frac{1}{2}+x_1\right) - aC(x_1)$, so

$$
G^{n'}(1/2,x^*,\ldots,x^*) = 1 - aC'(1/2), \tag{4.55}
$$

and choosing $a_3^*(n) > 1/C'(1/2)$ suffices.

Step 5. Use Lemma 4.4 to show the theorem.

Given n, consider $a^*(n) = \max\{a_1(n,k), a_2(n,k), a_3(n)\}_{k\in\{1,\ldots,n\}}$. Then, the following three properties hold:

1. For all $k \in \{1,\ldots,n\}$, $G^{k-1'}(\Delta_{2k}^{x^*},x^*,\ldots,x^*) < 0$.

2. For all $k \in \{1,\dots,n\}$, $-G^{k-1'}(\Delta_{2k}^{x^*},x^*,\dots,x^*) > G^{k'}(\Delta_{2k}^{x^*},x^*,\dots,x^*)$.

3. $G^{n'}(1/2,x^*,\dots,x^*) < 0$.

From (4.47), it is clear that we can write

$$G^k(x_1,x^*,\dots,x^*) = c_k(x^*) + b_k(x^*)x_1 - C(x_1),$$

with $c_k(x^*) < c_{k-1}(x^*)$ and $b_k(x^*) > b_{k-1}(x^*)$ for $k \in \{1,\dots,n\}$. This, along with the three properties enumerated above, allows us to apply Lemma 4.4 to functions $G^k(x_1,x^*,\dots,x^*)$ and $G^{k+1}(x_1,x^*,\dots,x^*)$. For all $k \in \{0,\dots,n-1\}$, we therefore obtain

$$\max_{[0,\Delta_{2k}^{x^*}]} G^{k-1}(x_1,x^*,\dots,x^*) \geq \max_{[0,\Delta_{2k+2}^{x^*}]} G^k(x_1,x^*,\dots,x^*),$$

which implies that for all $k \in \{1,\dots,n\}$,

$$\max_{[0,\Delta_2^{x^*}]} G^0(x_1,x^*,\dots,x^*) \geq \max_{[0,\Delta_{2k+2}^{x^*}]} G^k(x_1,x^*,\dots,x^*) \tag{4.56}$$

Accordingly, it only remains to be shown that $x^* = \max_{[0,\Delta_2^{x^*}]} G^0(x_1,x^*,\dots,x^*)$, but we know this from the proof of Proposition 4.4. This completes the proof of the theorem. ■

Theorem 4.3 shows that game $\mathscr{G}^1$ has one symmetric equilibrium if there is a full communication network and information acquisition is sufficiently costly, and that in such an equilibrium all citizens acquire the information level they would acquire if no network existed. In a full network, information acquired by one individual becomes public and can therefore be used by everybody else. This means that under the symmetric equilibrium of Theorem 4.3, the accuracy of the posterior each citizen holds about the state of the world is higher than in the case of an empty network in which the information acquired by citizens is private. Yet, as we show in Proposition 4.6 below, the probability of choosing via voting with the majority rule the alternative that matches the state of the world is the same with a full network and with an empty network. It is clear that all else being equal individually citizens benefit from first-hand information being public (in a full network) compared to the case when it is private (in an empty network), so acquiring information creates positive externalities. But *in equilibrium* these externalities vanish since the effect of the information each citizen acquires being public is nullified. For this to happen, however, acquiring information has to be sufficiently expensive so that *off equilibrium* the extent of the positive externality is not very large.

Proposition 4.6. *Let $\mathscr{G}^1$ be an information game with communication and suppose that the unique symmetric equilibrium exists. Then, the probability of the right alternative being chosen via voting with the majority rule is the same as in the symmetric equilibrium of the information game without communication, $\mathscr{G}^2$.*[7]

Proof. In a symmetric equilibrium of the information game with communication, $\mathscr{G}^1$, given that all signals have the same quality, every voter votes for A if and only if more A signals were received collectively than B signals. This means that the election outcome would be the same if everyone just voted for the alternative coinciding with their own signal, which is precisely what happens when no communication is possible between the citizens. This finished the proof, since we know from Theorems 4.1 and 4.3 that the individual information level each citizen acquires is the same with an without the network. ∎

4.6. Extensions

There are some conceivable directions in which our analysis can be extended, all of which are left for further research. First, in Sections 4.4 and 4.5 we do not offer a full characterization of the set of equilibria of game $\mathscr{G}^1$. In particular, it remains an open question to see *(a)* whether game $\mathscr{G}^1$ can have equilibria in which a number of citizens higher than one and lower than $2n+1$ acquires the same level of information, with the remaining citizens free-riding on the information provided by the experts, and *(b)* whether larger information acquisition functions guarantee the existence of equilibria with less (first-hand) information inequality.

Second, our analysis in Sections 4.3–4.5 has mainly looked into the case of a full network. It is clear that this represents one polar case, another polar case being the empty network. But in reality other (types of) networks may also exist. As a first step, it therefore remains to be seen if the main insights of our paper would remain if we considered certain classes of networks such as star-like constellations or clusters of (fully connected) citizens in the form of bubbles.[8]

Third and last, our setup considers citizens with the same preferences, which suffices to generate novel insights about the role of communication networks. It is possible that in the case where preferences show some degree of conflict, some of our insights will remain while some will not. Whether this is the case or not remains to be elucidated.

[7] Proposition 4.6 holds as long as all individuals acquire the same level of information, even if such a level is not optimally chosen by the individuals.

[8] From a technical viewpoint, it would be interesting to show the existence of PBE for game $\mathscr{G}$ independent of the network, if such a result can at all hold.

4.7. Concluding remarks

We introduced and analyzed a model that allows investigating the effect of communication networks on information acquisition in elections. Our setup is non-ideological, in the sense that all agents agree on which alternative is right for each of the two possible states of the world. Agents can purchase costly information to get a signal about the state, and then can transmit this information to the other voters. We have seen that, given that agents *believe all the messages they receive*, then they do not have any incentive to send false messages. This allows us to focus on the static game of information acquisition.

We have studied two types of equilibria: i) *dictator equilibria* and ii) *symmetric equilibra*. In dictator equilibria only one agent acquires information, and such equilibria do not exist if there is no communication between the agents. We prove that in our setup such equilibria always exist and that, compared to the symmetric equilibrium of the case without communication, welfare might be enhanced or reduced. In symmetric equilibria; all agents acquire the same level of information. We prove that such equilibria might not exist and that, if they do, they must coincide with the symmetric equilibrium of the case without communication. Furthermore, we show that if *information is expensive enough*, then such an equilibrium exists. As a consequence, we demonstrate that, if the symmetric equilibrium exists, then the welfare is exactly the same as in the case without communication. Hence, in a situation in which information is expensive, and voters are expected to behave similarly, adding a communication network does not enhance welfare. In such a situation, if the communication network is expensive to *build*, then it is optimal not to do it.

In this work we have focused in scenarios in which voters "trust" each other (commonly believed messages), and we have shown that this attitude is rational. However, it would also be interesting to understand how other belief structures might shape equilibria. For example, if voters in general do not trust messages that state a "large" information level, the symmetric equilibrium might exist more often than in our model. Questions of this sort might be interesting to study further.

5. Concluding remarks

In this dissertation we have studied how different rules, or institutional design choices, might shape the incentives of the agents to whom they apply. Throughout the thesis, we have looked closely at a model of a job market (Chapter 2) and at two different models of voting (Chapters 3 and 4). We have seen that the effect of rules on incentives and, eventually, on equilibrium outcomes, is not always intuitive and might not go in the direction intended by the rules themselves. The three works that form the core of this thesis highlight the importance of understanding properly how agents might act and react to rules, and which consequences this might bring. In the following lines I summarize the main contributions of these works and some directions for further research.

In Chapter 2, we have provided axiomatizations for the firms-optimal, workers-optimal and the fair-division stable rules in the context of the multiple-partners assignment game. Axiomatizations of these rules in this generalized framework did not exist. Furthermore, the axioms we use are intuitive and make explicit reference to how *extra gains* in the valuations of firms and workers are shared. This allows us to address some novel comments on the manipulability of the firms-optimal and workers-optimal stable rules. In particular, we show that, even if firms that can hire more than one worker can manipulate the firms-optimal stable rules, they cannot do so by modifying their productivities with the workers in a constant way.

In Chapter 3, we have proved that an interest group with a limited budget *can* block a reform even when all the Parliament members are aligned and want the reform to pass. This is the case because voters take into account how likely their vote is to be pivotal. We have characterized the symmetric mixed-strategy Nash equilibria of the game, proving that there is only one such equilibrium such that the reform is more likely to pass, but that even in that equilibrium it might not pass. We have shown that when the electorate grows asymptotically large, the reform passes with probability either zero or one in any symmetric equilibrium; hence, in that case, the reform passes depending on which equilibrium is played. We have seen that anonymous voting makes capture more difficult but that the interest group still can induce a positive probability of the reform not passing. It might be interesting for future research to study how capture might occur with other voting procedures. For example, as seen in Section 3.2.3, in our model sequential voting ensures that the reform passes. However, with uncertainty on others' preferences/bribes, this might not be the case, since relying on the last voters becomes risky.

In Chapter 4, we have analyzed the introduction of a communication network into a model of voting with information acquisition, in which voters try to choose the right alternative without ideological concerns. We have seen that allowing voters to communicate between each other does not necessarily enhance the probability of choosing correctly. We have shown that, with communication, a new class of equilibria, which we call *dictator equilibria*, emerges and that, if information is costly enough, the symmetric equilibrium of the no communication case carries over. Hence, if information is costly enough, allowing for communication might not have any effect towards equilibrium play. As pointed out in Section 4.6, a natural next step is to study equilibria *in between* the symmetric and the dictator equilibria. Furthermore, it seems of interest to understand whether, and when, those equilibria might co-exist. We also want to look at the setting in which there are different *information groups*, that is: there are multiple groups of people that are only connected to the other people in the same group as them.

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