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Externalities and the (pre)nucleolus in cooperative games ☆

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ABSTRACT

In most economic applications of cooperative games, externalities prevail: the worth of a coalition depends on how the other players are organized. We show that there is a unique natural way of extending the prenucleolus to games with coalitional externalities. This is in contrast to the Shapley value and the core for which many different extensions have been proposed.

1. Introduction

There is an abundance of economic situations where the worth of a coalition depends on how the other players are organized. In such situations a game with externalities associates with each coalition and each possible partition of the other players a worth of that (embedded) coalition. The literature on coalitional games with externalities is still relatively limited compared to the solid foundations of the theory of coalitional games without externalities.

For classic coalitional games, the most applied set-valued solution concept is the core and the three most applied single-valued solution concepts are the Shapley value, the prenucleolus and the nucleolus. For both the core and the Shapley value many different extensions were proposed to games with externalities. For instance, for the core the recursive approach by Kóczy (2007) and the expectation formation approach by Bloch and van den Nouweland (2014) and for the Shapley value the average approach by Macho-Stadler et al. (2007), the marginality approach by de Clippel and Serrano (2008), the utilization of reduction and consistency by Dutta et al. (2010), and the Harsanyi (1959) dividends by Macho-Stadler et al. (2010) and Huettner and Casajus (2019). All these contributions provide *families* of extensions.[1] To date, an extension of the (pre)nucleolus is missing in the literature.[2]

We provide a natural extension of the prenucleolus from coalitional games without externalities to games with externalities: for each embedded coalition consisting of the coalition and partition of the other players, we measure the excess of this embedded coalition as the difference between the worth of the embedded coalition minus what the coalition gets in the allocation (which equals the sum of the allotments of the players in the coalition). For each allocation, then we rearrange the excesses of all embedded coalitions in non-increasing order. The prenucleolus is then simply the set of efficient allocations which lexicographically minimize the rearranged excesses of all embedded coalitions. We show that (i) the prenucleolus is unique and (ii) the prenucleolus of a game with externalities coincides with prenucleolus of the following associated game without externalities: for each coalition we take the maximal worth among all possible organizations of the other players. Indeed, Fact (ii) is our key contribution. In the spirit of de Clippel and Serrano (2008), we obtain a unique "externality free" extension of the prenucleolus.

We also present an axiomatic foundation of the new solution concept. Indeed, we can adapt the properties used by Sobolev (1975) in the well known characterization of the prenucleolus, namely anonymity, covariance and the reduced game property, to games with externalities quite naturally. The reduced game property shapes a consistency principle and is of paramount importance in our result. Such a principle

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[1] Recently, Alonso-Meijide et al. (2019) provide a characterization of a class of Shapley values covering all the above families of extensions of the Shapley value.

[2] Kóczy (2018) provides a survey of the literature on partition function form games.

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states that in the event that some agents leave the game with the proposed payoffs and the remaining agents renegotiate the sharing in a reduced game, the payoffs do not change. We consider two natural ways to extend the Davis and Maschler (1965) reduced game to our framework. The first and more naive is a classic coalitional game without externalities that enables us to characterize the prenucleolus. The second inherits the externalities of the underlying game and yields a weaker property.

An additional important feature of the prenucleolus for games with externalities is the fact that, when the optimistic core (Shenoy, 1979) is non-empty it is in the core of the game with respect to any expectation formation rule (Bloch and van den Nouweland, 2014). Indeed, many different core notions have been introduced for games with externalities and our solution concept is a point in all of them when the smallest core is non-empty. This is in contrast with the proposed extensions of the Shapley value outlined above, which can prescribe payoff vectors that some coalitions may block based on their expected outside options.

We also introduce a family of extensions of the nucleolus from coalitional games without externalities to games with externalities. Contrary to the Shapley value and the prenucleolus, the core and the nucleolus might be empty. More precisely, for coalitional games without externalities the core and the nucleolus are empty when the set of individually rational and efficient allocations is empty. For coalitional games with externalities, individual rationality depends on the possible partition of the other players. We introduce a general individual rationality constraint for such situations and define and characterize the nucleolus using our main result for the prenucleolus.

We proceed as follows. In Section 2 we extend the prenucleolus from classic games to games with externalities and present our main result, its equivalence with the prenucleolus of an associated game without externalities. We also provide some intermediate results like the characterization by means of balanced collections. In Section 3 we present an axiomatic characterization of the prenucleolus in the spirit of Sobolev (1975). Section 4 contains a family of extensions of the nucleolus. Section 5 discusses other interesting properties of the prenucleolus.

2. The prenucleolus

Let $\mathcal{N}$ stand for the nonempty set of potential players. Let $N \subset \mathcal{N}$ be a finite nonempty set of players. The set of partitions of N is denoted by $\mathcal{P}(N)$.[3] An *embedded coalition* of N is a pair (S, P) where $S \subseteq N$ and $P \in \mathcal{P}(N \setminus S)$. We denote by $\mathcal{EC}^N$ the set of all embedded coalitions of N. Note that

$$P \in \mathcal{P}(N) \Leftrightarrow (S, P \backslash \{S\}) \in \mathcal{EC}^N \text{ for all } S \in P.$$

A *coalitional game with externalities (or for short, game)* is a pair (N, v) consisting of a finite set of players $N \subset \mathcal{N}$ and a partition function $v : \mathcal{EC}^N \to \mathbb{R}$, satisfying $v(\emptyset, P) = 0$, for every $P \in \mathcal{P}(N)$. The set of all games is denoted by $\mathcal{G}$. Given $(N, v) \in \mathcal{G}$, we say that (N, v) is a *coalitional game without externalities* if for all $(S, P), (S, Q) \in \mathcal{EC}^N$, $v(S, P) = v(S, Q)$. In this case we may simply write $v(S)$. The set of all coalitional games without externalities is denoted by $\mathcal{CG}$.

Our purpose is to introduce a point-valued solution concept for coalitional games with externalities. Given a game $(N, v) \in \mathcal{G}$, an allocation for (N, v) is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$. We denote by $X(N, v)$ the set of all efficient allocations (or preimputations) for (N, v), i.e., $X(N, v) = \left\{x \in \mathbb{R}^N : x(N) = v(N, \emptyset)\right\}$.[4] Given a game $(N, v) \in \mathcal{G}$, an embedded coalition $(S, P) \in \mathcal{EC}^N$, and an efficient allocation $x \in X(N, v)$, the *excess* of (S, P) at x is defined by

$$e(S, P, x, v) = v(S, P) - x(S).$$

It measures the dissatisfaction of coalition S at x when the complementary coalition is organized according to P.

Given $m \in \mathbb{N}$, let $\mathbb{R}^m_{\geq}$ denote the set of all vectors $x \in \mathbb{R}^m$ such that $x_1 \geq x_2 \geq \cdots \geq x_m$, i.e. the coordinates of x are arranged in non-increasing order. Let $\precsim$ denote the lexicographical ordering on $\mathbb{R}^m_{\geq}$: for every $x, y \in \mathbb{R}^m_{\geq}$, $x \precsim y$ means that either $x = y$ or there is $1 \leq t \leq m$, such that $x_i = y_i$ for every $1 \leq i < t$ and $x_t < y_t$. We write $x \prec y$ if $x \precsim y$ and $y \not\precsim x$.

Define $c(n) = \left|\mathcal{EC}^N\right|$ where $n = |N|$. For a given $(N, v) \in \mathcal{G}$ and $x \in X(N, v)$, we are going to build a vector with all the excesses arranged in non-increasing order. Formally, the vector of ordered excesses is defined as follows[5]:

$$\theta(x, v) \in \mathbb{R}^{c(n)}_{\geq}, \text{ where } \left\{\theta_i(x, v) : 1 \leq i \leq c(n)\right\} = \left\{e(S, P, x, v) : (S; P) \in \mathcal{EC}^N\right\}$$
$$(\text{and } \theta_1(x, v) \geq \theta_2(x, v) \geq \cdots \geq \theta_{c(n)}(x, v)).$$

Definition 1. The *prenucleolus* of a game with externalities is the set of efficient allocations which lexicographically minimize the ordered vector of excesses:

$$\eta(N, v) = \{x \in X(N, v) : \theta(x, v) \precsim \theta(y, v) \text{ for all } y \in X(N, v)\}.$$

The first step is to show that the prenucleolus is a well defined solution. Let $\mathcal{H} \subseteq \mathcal{G}$. Formally, a *(single-valued) solution on $\mathcal{H}$* is a mapping f that assigns an allocation $f(N, v) \in \mathbb{R}^N$ to every game $(N, v) \in \mathcal{H}$.

The second step will relate the prenucleolus of games with externalities to the classic prenucleolus (Schmeidler, 1969) of a particular game without externalities. The latter is a well known solution on $\mathcal{CG}$ that we denote by η^* and can be defined for every $(N, v) \in \mathcal{CG}$ by $\eta^*(N, v) = \eta(N, v)$ (where we use the first step, or simply η^* is the restriction of η to $\mathcal{CG}$). This result is one of our key insights: the prenucleolus of a partition function form game is uniquely defined and given by the prenucleolus of its associated "externality-free" max-game where for any coalition S its worth is equal to the maximum of the worths $v(S, P)$ where P is any possible organization of the other players. Formally, for any $(N, v) \in \mathcal{G}$, let $(N, v^{\max}) \in \mathcal{CG}$ be defined for all $S \subseteq N$ by,

$$v^{\max}(S) = \max\{v(S, P) : P \in \mathcal{P}(N \setminus S)\}.$$

The following is our main result.

Theorem 1.

(i) The prenucleolus is a (single-valued) solution on $\mathcal{G}$.
(ii) For all $(N, v) \in \mathcal{G}$, we have $\eta(N, v) = \eta^*(N, v^{\max})$.

Proof. (i): It follows from Corollary 4.6 of Justman (1977), because $X(N, v)$ is a convex subset of $\mathbb{R}^N$ and for every $(S, P) \in \mathcal{EC}^N$, $e(S, P, x, v)$ is a linear function on $X(N, v)$.[6]

(ii): In order to show (ii), we recall Kohlberg's (1971) characterization of the prenucleolus of a characteristic function game.[7]

[3] By convenience, let $\emptyset$ be the only partition in $\mathcal{P}(\emptyset)$.

[4] For every $x \in \mathbb{R}^N$ and $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$.

[5] Here identical numbers appear multiple times, i.e. we could have $\{2, 2, 2, 1, 1, 0, \ldots\}$.

[6] For completeness, we include here Justman's result. Let X be a convex subset of a linear space and $U = \{u_i\}_{i=1}^m$ a set of convex real valued functions on X. For each $x \in X$, let $\theta(x)$ be the vector in $\mathbb{R}^m$ whose coordinates are $\{u_i(x)\}_{i=1}^m$ arranged in non-increasing order. Define $N(X, U) = \{x \in X : \theta(x) \precsim \theta(y), \forall y \in X\}$. Then $N(X, U)$ is convex and $u_i(x) = u_i(y)$ for every $x, y \in N(X, U)$ and $1 \leq i \leq m$.

[7] Note that Kohlberg's characterization deals with the nucleolus on the set of characteristic function games with non-empty imputation set. The result we present here is actually Theorem 5.2.6 of Peleg and Sudhölter (2007).

Definition 2. Let $(N, v) \in \mathcal{CG}$. For every $x \in \mathbb{R}^N$ with $x(N) = v(N)$ and $\alpha \in \mathbb{R}$, define

$$\mathcal{D}(\alpha, x, v) = \{S \subseteq N : v(S) - x(S) \geq \alpha\}.$$

A vector $x \in \mathbb{R}^N$ with $x(N) = v(N)$ is said to have *Property I* with respect to (N, v) if the following condition is satisfied for every $\alpha \in \mathbb{R}$ where $\mathcal{D}(\alpha, x, v) \neq \emptyset$: If $y \in \mathbb{R}^N$ is such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{D}(\alpha, x, v)$, then $y(S) = 0$ for every $S \in \mathcal{D}(\alpha, x, v)$.

Theorem 2. *(Kohlberg, 1971) Let $(N, v) \in \mathcal{CG}$ and $x \in \mathbb{R}^N$ with $x(N) = v(N)$. Then $x = \eta^*(N, v)$ if and only if x has Property I with respect to (N, v).*

Let $(N, v) \in \mathcal{G}$, for every $x \in X(N, v)$ and $\alpha \in \mathbb{R}$, define $\mathcal{A}(\alpha, x, v) = \mathcal{D}(\alpha, x, v^{\max})$. Take $x = \eta(N, v)$, let $\alpha \in \mathbb{R}$ be such that $\mathcal{A}(\alpha, x, v) \neq \emptyset$ and $y \in \mathbb{R}^N$ such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{A}(\alpha, x, v)$. We denote by $\mathcal{B}(\alpha, x, v)$ the set of embedded coalitions whose excesses at x are no less than α, i.e., $\mathcal{B}(\alpha, x, v) = \{(S, P) \in \mathcal{EC}^N : e(S, P, x, v) \geq \alpha\}$. Note that $\mathcal{B}(\alpha, x, v)$ contains the embedded coalitions whose excesses at x are the first coordinates of $\theta(x, v)$. Define $z_\epsilon = x + \epsilon y$, where $\epsilon > 0$. Note that $z_\epsilon \in X(N, v)$. We choose $\epsilon^* > 0$ such that for every $(S, P) \in \mathcal{B}(\alpha, x, v)$ and every $(T, Q) \notin \mathcal{B}(\alpha, x, v)$,

$$e(S, P, z_{\epsilon^*}, v) > e(T, Q, z_{\epsilon^*}, v). \quad (1)$$

In other words, we choose $\epsilon^* > 0$ in such a way that the excesses of the embedded coalitions in $\mathcal{B}(\alpha, x, v)$ are in the first positions of $\theta(z_{\epsilon^*}, v)$. Next, for every $(S, P) \in \mathcal{B}(\alpha, x, v)$,

$$e(S, P, z_{\epsilon^*}, v) \leq e(S, P, x, v), \quad (2)$$

because $y(S) \geq 0$ for every $S \in \mathcal{A}(\alpha, x, v)$ and $(S, P) \in \mathcal{B}(\alpha, x, v)$ implies $S \in \mathcal{A}(\alpha, x, v)$.

Finally, suppose that there is $S \in \mathcal{A}(\alpha, x, v)$ such that $y(S) > 0$. Then, by (1) and (2), $\theta(z_{\epsilon^*}, v) \prec \theta(x, v)$ which contradicts our assumption. We have shown that $y(S) = 0$ for every $S \in \mathcal{A}(\alpha, x, v)$, i.e., x has Property I with respect to $(N, v^{\max})$. Then, by Theorem 2, $x = \eta^*(N, v^{\max})$ as desired. □

Remark 1. One may be dissatisfied with the fact to minimize the vector of excesses of all embedded coalitions. In other words, one could be dissatisfied with the fact that the excess of the same coalition may vary with the coalitions formed in its complement. After all, these different "complaints" are not compatible. However, a similar criticism applies to the prenucleolus of coalitional games without externalities: why should one take into account the excess of all coalitions while clearly any given player can belong to only one of these? Therefore, the prenucleolus could be chosen by a third party who has no idea what coalition structure may arise and at the same time desires to keep dissatisfaction at its lowest in a worst-case scenario analysis. It is with that interpretation in mind that our definition is the natural extension of the prenucleolus from coalitional games without externalities to coalitional games with externalities.

Remark 2. Two interesting instances in which the max-game is specially simple are situations where externalities are all negative or positive. A game has negative externalities if for every $(S, P) \in \mathcal{EC}^N$ and every $T, Q \in P$, $v(S, P) \geq v(S, (P \setminus \{T, Q\}) \cup \{T \cup Q\})$. A game has positive externalities if for every $(S, P) \in \mathcal{EC}^N$ and every $T, Q \in P$, $v(S, P) \leq v(S, (P \setminus \{T, Q\}) \cup \{T \cup Q\})$. Then, for games with negative externalities $v^{\max}(S) = v(S, \{\{i\} : i \in N \setminus S\})$ and for games with positive externalities $v^{\max}(S) = v(S, \{N \setminus S\})$ (where $S \subseteq N$).

Remark 3. Implicitly, we assume that the grand coalition is the most efficient organization of players. Convex games as defined by Hafalir (2007) and superadditive games introduced in Alonso-Meijide et al. (2022) are two classes of games where this happens. In case the grand coalition is not the most efficient organization of the set of players, the set of preimputations $X(N, v)$ should be replaced by $\{x \in \mathbb{R}^N : x(S) = v(S, P \setminus \{S\}), \forall S \in P\}$, where $P \in \mathcal{P}(N)$ is such that for every $Q \in \mathcal{P}(N)$, $\sum_{S \in P} v(S, P \setminus \{S\}) \geq \sum_{S \in Q} v(S, Q \setminus \{S\})$.[8] This leads to the prenucleolus of the max-game with the coalition structure P as defined by Aumann and Dreze (1974).

Remark 4. Our approach can be used to generalize the prekernel, a superset of the prenucleolus, to games with externalities by just considering the excesses to all embedded coalitions. Then, the maximal surplus of an agent over another leads trivially to the max-game.

3. Foundation

The purpose of this section is to present an axiomatic foundation of the solution introduced above. The first property we would like to impose on a solution is the classic anonymity.

Anonymity: A solution f is *anonymous* if for every $(N, v) \in \mathcal{G}$ and every injection $\pi : N \to \mathcal{N}$,

$$f(\pi(N), \pi v) = \pi(f(N, v)),$$

where $(\pi(N), \pi v) \in \mathcal{G}$ is defined for every $(S, P) \in \mathcal{EC}^N$, by $v(S, P) = \pi v(\pi(S), \pi(P))$ with $\pi(P) = \{\pi(T) : T \in P\}$.

In words, anonymity states that relabeling of players should not affect the solution.

The next property is a natural generalization of a classic property.

Covariance: A solution f is *covariant* if for every $(N, v) \in \mathcal{G}$, $\alpha > 0$, and $\beta \in \mathbb{R}^N$,

$$f(N, \alpha v \oplus \beta) = \alpha f(N, v) + \beta,$$

where $(N, \alpha v \oplus \beta) \in \mathcal{G}$ is defined for every $(S, P) \in \mathcal{EC}^N$, by $(\alpha v \oplus \beta)(S, P) = \alpha v(S, P) + \beta(S)$.

Note that covariance entails linearity of an arbitrary game with an inessential game.[9]

Next, we present the most important property of the characterization result which states that a solution should not be affected if a coalition renegotiates the sharing in a particular subgame. Given $(N, v) \in \mathcal{G}$, $\emptyset \neq S \subseteq N$, and $x \in \mathbb{R}^N$. The *reduced game* with respect to S and x is denoted by $(S, v_{S,x}) \in \mathcal{CG}$ and is defined for every $T \subseteq S$ by

$$v_{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(N, \emptyset) - x(N \setminus S) & \text{if } T = S, \\ \max\{v(R, Q) - x(R \setminus T) : (R, Q) \in \mathcal{EC}^N \text{ and } R \cap S = T\} & \text{otherwise.} \end{cases}$$

The reduced game $v_{S,x}$ is a coalitional game without externalities and in Section 5 we discuss a reduced game inheriting externalities. The idea behind the above reduced game is that if agents in $N \setminus S$ leave the game with the payoff proposed by x, the remaining agents interact in a new coalitional game without externalities. In the latter game, the worth of the grand coalition, S, is determined by the remainder $v(N, \emptyset) - x(N \setminus S)$ and every other coalition $T \neq \emptyset$ assesses its worth by taking the maximum over all possible embedded coalitions obtained when some agents in $N \setminus S$ may join coalition T. Note that this coincides with the Davis and Maschler (1965) reduced game with the exception that instead the worth of a bare coalition, say R, we consider the worth of every embedded coalition of the type (R, Q).

Reduced Game Property: A solution f satisfies the *reduced game property* if for all $(N, v) \in \mathcal{G}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$ (where

[8] If P is not unique, we could use a tie-breaking rule.

[9] An inessential game is built from any vector $\beta \in \mathbb{R}^N$, by assigning to each coalition $S \subseteq N$ the worth $\beta(S)$.

$x = f(N, v)$), we have

$$x_i = f_i\left(S, v_{S,x}\right).$$

If a solution meets the reduced game property, then the payoffs remain unaffected when agents in a coalition renegotiate in the reduced game.

Theorem 3. *Let $\mathcal{N}$ be infinite. The prenucleolus η is the only solution on $\mathcal{G}$ satisfying anonymity, covariance, and the reduced game property.*

Before we continue, it is helpful to recall the characterization of η^* by Sobolev (1975). In order to present it, we can consider variants of the three properties we have introduced above that only apply to games without externalities. That is, let *anonymity**, *covariance**, and the *reduced game property** be the restrictions of anonymity, covariance, and the reduced game property to solutions on $\mathcal{CG}$, respectively.[10]

Theorem 4. *(Sobolev, 1975) Let $\mathcal{N}$ be infinite. The prenucleolus η^* is the only solution on $\mathcal{CG}$ satisfying anonymity*, covariance*, and the reduced game property*.*

We are now in the position to show our characterization result.

Proof of Theorem 3. *Existence:* Using *(ii)* in Theorems 1 and 4 we get the existence from the following observations. Let $(N, v) \in \mathcal{G}$. First, for every injection $\pi : N \to \mathcal{N}$, $\pi(v^{\max}) = (\pi v)^{\max}$. Second, for every $\alpha > 0$ and $\beta \in \mathbb{R}^N$, $(\alpha v \oplus \beta)^{\max} = \alpha v^{\max} + \beta$. Third, if $x = \eta(N, v)$ and $\emptyset \neq S \subseteq N$, then $\left(v_{S,x}\right)^{\max} = (v^{\max})_{S,x}$.

Uniqueness: Let $(N, v) \in \mathcal{G}$ and f be a solution on $\mathcal{G}$ satisfying the three properties. If $x = f(N, v)$, then from the definition of the reduced game it follows that for every $(S, P) \in \mathcal{EC}^N$, $v_{N,x}(S, P) = v^{\max}(S)$. Then, by the reduced game property

$$f(N, v) = f(N, v_{N,x}) = f(N, v^{\max}).$$

Finally, using Theorem 4 and the fact that $(N, v^{\max}) \in \mathcal{CG}$,

$$f(N, v^{\max}) = \eta^*(N, v^{\max}) = \eta(N, v),$$

where the last equality follows from *(ii)* in Theorem 1. □

4. The nucleolus

Given $(N, v) \in \mathcal{CG}$, an allocation $x \in \mathbb{R}^N$ is individually rational if for all $i \in N$, $x_i \geq v(\{i\})$. An individually rational and efficient allocation of a game without externalities is called an imputation. It is well known that the prenucleolus of coalitional games without externalities may not prescribe an individually rational allocation in games with non-empty imputation set. The nucleolus can be seen as the natural solution to this issue. It is the allocation which lexigographically minimizes the ordered vector of excesses on the set of imputations. However, this leads to a solution which is well defined only for games with non-empty imputation set.

When considering coalitional games with externalities, the worth of $\{i\}$ might depend on which coalitions the other agents will form. Below we consider a general formulation. An *individual rationality constraint* is a solution c on $\mathcal{G}$ such that for all $(N, v) \in \mathcal{G}$,

$$c_i(N, v) \in \left[\min_{P \in \mathcal{P}(N\setminus\{i\})} v(\{i\}, P), \max_{P \in \mathcal{P}(N\setminus\{i\})} v(\{i\}, P)\right] \text{ for all } i \in N.$$

This just requires that the individual rationality constraint shall be for every player between its pessimistic and optimistic worth. Now given an individual rationality constraint c, we define the set of c-imputations by $X_c(N, v) = \{x \in X(N, v) : x_i \geq c_i(N, v) \text{ for all } i \in N\}$.

Definition 3. Given an individual rationality constraint c and $(N, v) \in \mathcal{G}$ such that $X_c(N, v) \neq \emptyset$, the c-*nucleolus* of a game with externalities is defined by

$$\eta^c(N, v) = \left\{x \in X_c(N, v) : \theta(x, v) \precsim \theta(y, v) \text{ for all } y \in X_c(N, v)\right\}.$$

The c-nucleolus coincides with the nucleolus on the set of coalitional games without externalities with a non-empty set of imputations. Whenever the c-nucleolus is defined it is single-valued and for any game, it lexicographically minimizes the ordered vector of excesses on the set of c-imputations for the associated max-game.

Proposition 1. *Let c be an individual rationality constraint. Then for all $(N, v) \in \mathcal{G}$ such that $X_c(N, v) \neq \emptyset$ we have*

(i) *$\eta^c(N, v)$ is single-valued; and*
(ii) *$\eta^c(N, v) = \left\{x \in X_c(N, v) : \theta(x, v^{\max}) \precsim \theta(y, v^{\max}) \text{ for all } y \in X_c(N, v)\right\}$.*

Proof. The proof follows the same lines as that of Theorem 1. For (i), Justman's Corollary 4.6 applies. To show (ii), one must replace $X(N, v)$ by $X_c(N, v)$ and $\eta(N, v)$ by $\eta^c(N, v)$ and use the characterization of the nucleolus by Kohlberg (1971)[11] instead of its adaptation for the prenucleolus that we presented therein. □

Remark 5. For $\mathcal{CG}$ with non-empty set of imputations, Snijders (1995) provides an axiomatization of the nucleolus via a reduced game property saving the chosen imputation (restricted to a coalition) as imputation of the reduced game; see also Potters (1991) for related issues.

Remark 6. Obviously, for two individual rationality constraints c and c' such that $c \geq c'$, we always have $X_c(N, v) \subseteq X_{c'}(N, v)$ and both the c-nucleolus is well-defined for every game where the c'-nucleolus is defined and $\eta^{c'}(N, v) \precsim \eta^c(N, v)$ whenever the c-nucleolus is defined. Thus, the pessimistic nucleolus whereby $\underline{c}_i(N, v) = \min_{P \in \mathcal{P}(N\setminus\{i\})} v(\{i\}, P)$ is the c-nucleolus both (i) defined on the largest set of games and (ii) lexicographically dominating all other c-nucleoli.

5. Discussion

In this section we discuss another interesting property of the prenucleolus and its relation to different notions of the core introduced in the literature.

It could be reasonable to define a reduced game which inherits externalities from the original game. Formally, given $(N, v) \in \mathcal{G}$, $\emptyset \neq S \subseteq N$, and $x \in \mathbb{R}^N$. The *reduced game with externalities* with respect to S and x is denoted by $\left(S, v^{S,x}\right) \in \mathcal{G}$ and is defined for every $(T, P) \in \mathcal{EC}^S$ by[12]

$$v^{S,x}(T, P) = \begin{cases} 0 & \text{if } T = \emptyset, \\ v(N, \emptyset) - x(N \setminus S) & \text{if } T = S, \\ \max\left\{v(R, Q) - x(R \setminus T) : (R, Q) \in \mathcal{EC}^N, R \cap S = T, Q \cap S = P\right\} & \text{otherwise.} \end{cases}$$

The idea behind the above reduced game is that when $N \setminus S$ leave the game with the payoff proposed by x, the remaining agents interact in a new coalitional game with externalities. In the latter game, the

[10] Orshan (1993) shows that in Theorem 3 anonymity may be replaced by equal treatment. For coalitional games with externalities there exist different notions of equal treatment and we have decided to keep anonymity as in Sobolev (1975).

[11] For completeness, we include here Kohlberg's result. Let $(N, v) \in \mathcal{CG}$, such that $v(N) \geq \sum_{i\in N} v(\{i\})$. A vector $x \in \mathbb{R}^N$ with $x(N) = v(N)$ and $x_i \geq v(\{i\})$ for every $i \in N$ is the nucleolus of (N, v) if and only if the following condition is satisfied for every $\alpha \in \mathbb{R}$ where $\mathcal{D}(\alpha, x, v) \neq \emptyset$: If $y \in \mathbb{R}^N$ is such that $y(N) = 0$ and $y(S) \geq 0$ for every $S \in \mathcal{D}(\alpha, x, v) \cup \{\{i\} : x_i = v(\{i\})\}$, then $y(S) = 0$ for every $S \in \mathcal{D}(\alpha, x, v)$.

[12] Given $(R, Q) \in \mathcal{EC}^N$, let $Q \cap S = \{U \cap S : U \in Q\}$.

worth of the grand coaltion, $(S, \emptyset)$, is determined by the remainder $v(N, \emptyset) - x(N\setminus S)$. Otherwise, in the event that coalition structure $P \cup \{T\}$ emerges, coalition T assesses its worth by taking the maximum over all possible ways in which some agents in $N\setminus S$ may join T and some others may form new coalitions or join any of the coalitions in P, assuming that agents that join coalition T are paid according to x. The above reduced game yields another version of the well known reduced game property.

Weak Reduced Game Property: A solution f satisfies the *weak reduced game property* if for all $(N, v) \in \mathcal{G}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$ (where $x = f(N, v)$), we have

$$x_i = f_i\left(S, v^{S,x}\right).$$

It is easy to see how the above property also generalizes the reduced game property, introduced for point-valued solutions by Sobolev (1975). Indeed, the two versions of the reduced game property proposed here coincide for coalitional games without externalities. The difference between the two properties is the fact that the former is not affected by the externalities of the original game because it takes the maximum over all possible partitions while the latter takes the maximum only among those partitions that are consistent with the coalitional organization of the players in the reduced game. We next present our findings related to this property. First, we see that η has this property.

Proposition 2. *The prenucleolus satisfies the weak reduced game property.*

Proof. Let $(N, v) \in \mathcal{G}$, $x = \eta(N, v)$, and $\emptyset \neq S \subseteq N$. Using the definition of both reduced games, we can write for every $T \subseteq S$,

$$\left(v^{S,x}\right)^{\max}(T) = v_{S,x}(T). \tag{3}$$

Then, by *(ii)* in Theorem 1, for every $i \in S$

$$\eta_i\left(S, v^{S,x}\right) = \eta_i\left(S, \left(v^{S,x}\right)^{\max}\right) = \eta_i\left(S, v_{S,x}\right) = x_i,$$

where the second equality holds by (3) and the third is because the prenucleolus satisfies the reduced game property (Theorem 3). □

Second, as it is the case for games without externalities, covariance and the weak reduced game property imply efficiency.

Proposition 3. *Let f be a solution satisfying covariance and the weak reduced game property. Then, for every $(N, v) \in \mathcal{G}$, $f(N, v) \in X(N, v)$.*

We omit the proof as it is a straightforward adaptation of the original one by Sobolev (1975).

Third, when $\mathcal{N}$ contains at most three potential players, anonymity, covariance, and the weak reduced game property characterize the prenucleolus.

Proposition 4. *Let $|\mathcal{N}| \leq 3$. The prenucleolus, η, is the only solution on $\mathcal{G}$ satisfying anonymity, covariance, and the weak reduced game property*

Proof. The existence has already been proved. For the uniqueness, note that if $|N| \leq 2$ then, $(N, v) \in \mathcal{CG}$. Since for games without externalities, the reduced game with externalities (as well as the reduced game) coincides with the Davis and Maschler reduced game we have the uniqueness by Theorem 4. Then, let $N = \{1, 2, 3\}$ and f be a solution on $\mathcal{G}$ satisfying the three properties with $f(N, v) \neq \eta(N, v)$ for some $(N, v) \in \mathcal{G}$.

Taking $\beta = -\eta(N, v)$, by covariance of f and η we have $f(N, v \oplus \beta) = f(N, v) + \beta$ and $\eta(N, v \oplus \beta) = 0$. Since $f(N, v) \neq \eta(N, v)$, we have $f(N, v) + \beta \neq 0$. Thus, without loss of generality, we can assume that $\eta(N, v) = 0$ and $f(N, v) \neq 0$. Let $x = f(N, v)$. Then by efficiency, $0 = v(N, \emptyset) = x(N)$. Note that $0 = \eta^*(N, v^{\max})$.

We can also assume that $x_i \neq 0$ for all $i \in N$. Otherwise, if $x_i = 0$ for some $i \in N$, by definition $\left(N\setminus\{i\}, v^{N\setminus\{i\},x}\right) = \left(N\setminus\{i\}, v^{N\setminus\{i\},0}\right)$. But these are two person games and we know that $f\left(N\setminus\{i\}, v^{N\setminus\{i\},x}\right) = \eta\left(N\setminus\{i\}, v^{N\setminus\{i\},x}\right)$. Then, since f and η satisfy the weak reduced game property, we get $x = 0$.

Without loss of generality, let $x_1 \geq x_2 \geq x_3$. Define

$$\mathcal{B} = \{S : x(S) > 0\}.$$

Since $\theta(x, v)$ is non-increasingly ordered

$$\theta_1(x, v) = v^{\max}\left(S_1^x\right) - x\left(S_1^x\right),$$

for some S_1^x. If $S_1^x \in \mathcal{B}$, then

$$\theta_1(x, v) = v^{\max}\left(S_1^x\right) - x\left(S_1^x\right) < v^{\max}\left(S_1^x\right) \leq \theta_1(0, v),$$

which is a contradiction to $0 = \eta^*(N, v^{\max})$. Hence, $S_1^x \notin \mathcal{B}$. Moreover, we can assume that $x\left(S_1^x\right) \neq 0$, otherwise we continue the reasoning with the coalition with next highest excess at x. That is, $x\left(S_1^x\right) < 0$. Two cases may arise.

Case 1: $x_1 \geq x_2 > 0 > x_3$.

Then, $\mathcal{B} = \{\{1\}, \{2\}, \{1, 2\}\}$ and $3 \in S_1^x$. Take $i \notin S_1^x$ and $w = v^{\{i,3\},x}$. By the weak reduced game property

$$x_{\{i,3\}} = f(\{i, 3\}, w) = \eta(\{i, 3\}, w),$$

where the second equality holds because two-person games are without externalities. Now, since the prenucleolus satisfies the standard property for two-person games we have

$$\theta_1(x, v) = w(3) - x_3 = w(i) - x_i = \max\left\{v^{\max}(S) - x(S) : i \in S \text{ and } 3 \notin S\right\},$$

where the first and third equalities hold by definition of the reduced game with externalities. Note that the maximum on the right hand side will be attained by some coalition in $\mathcal{B}$, say T. Therefore

$$\theta_1(x, v) < v^{\max}(T),$$

which contradicts the fact that $\theta(0, v) \precsim \theta(x, v)$ (definition of the prenucleolus).

Case 2: $x_1 > 0 > x_2 \geq x_3$.

Then, $\mathcal{B} = \{\{1\}, \{1, 2\}, \{1, 3\}\}$ and $1 \notin S_1^x$. Take $i \in S_1^x$ and $w = v^{\{1,i\},x}$. As before,

$$x_{\{1,i\}} = f(\{1, i\}, w) = \eta(\{1, i\}, w).$$

Using again the standard property for two-person games of the prenucleolus,

$$\theta_1(x, v) = w(i) - x_i = w(1) - x_1 = \max\left\{v^{\max}(S) - x(S) : 1 \in S \text{ and } i \notin S\right\}.$$

Once again, the maximum on the right hand side will be attained by some coalition in $\mathcal{B}$ which means that

$$\theta_1(x, v) < v^{\max}(T),$$

for some T. A contradiction to $\theta(0, v) \precsim \theta(x, v)$. □

Remark 7. When $\mathcal{N}$ is finite, Sudhölter (1993) has shown for coalitional games without externalities that Theorem 3 holds if and only if $|\mathcal{N}| \leq 3$. Thus, Proposition 4 is a generalization of this result to coalitional games with externalities. For $3 < |\mathcal{N}| < +\infty$, one can use the construction of Peleg and Sudhölter (2007, Remark 6.3.3, Exercises 6.3.2 and 6.3.3) to show that Proposition 4 does not hold: for instance, for $\mathcal{N} = \{1, \ldots, 6\}$, let $(\mathcal{N}, w)$ be the weighted majority (coalitional) game (without externalities) where $q = (3, 3, 1, 1, 1, 1)$ and for all $S \subseteq \mathcal{N}$, $w(S) = 1$ if $q(S) \geq 5$ and otherwise $w(S) = 0$. Then $\eta(\mathcal{N}, w) = \left(\frac{3}{10}, \frac{3}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right)$. Let $x = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$. Then $x \neq \eta(\mathcal{N}, w)$ and for all $S \subsetneq \mathcal{N}$, $\eta\left(S, w^{S,x}\right) = (x_i)_{i\in S}$. Now define the solution f on $\mathcal{CG}$ as follows: (i) $f(\mathcal{N}, w) = x$ and (ii) for all $(N, v) \in \mathcal{G}$, if there exist an injection $\pi : \mathcal{N} \to \mathcal{N}$, $\alpha > 0$ and $\beta \in \mathbb{R}^{\mathcal{N}}$ such that

$v^{\max} = \pi(\alpha w + \beta)$, then $f(N, v) = \pi(\alpha x + \beta)$, and otherwise $f(N, v) = \eta(N, v)$. Then f satisfies all the properties in Proposition 4. It is an open question whether Proposition 4 is true when the set of potential players is infinite.

Another well-known property of the prenucleolus of coalitional games without externalities is that it always lies in the core whenever the latter is non-empty. It is interesting to analyze the behavior of the prenucleolus as introduced here with respect to different notions of the core proposed in the literature. A way to pin down a particular core in the presence of externalities is to anticipate the coalitional reaction of the deviating players. This is precisely the approach of Bloch and van den Nouweland (2014) where a large class of core notions are studied in a common framework. Formally, an *expectation formation rule* is a mapping, f, that associates to every $S \subseteq N$ a partition of $N \setminus S$, i.e., for every $S \subseteq N$, $g(S, v) \in \mathcal{P}(N \setminus S)$.[13] Then, the *core* of $(N, v) \in \mathcal{G}$ with respect to the expectation formation rule g is defined by

$$C_g(N, v) = \{x \in X(N, v) \,:\, x(S) \geq v\,(S, g(S, v)) \quad \forall S \subseteq N\}\,.$$

The optimistic rule, g_o, originally proposed by Shenoy (1979) selects for every coalition, the most favorable partition, i.e., for every $(N, v) \in \mathcal{G}$ and $S \subseteq N$, $g_o(S, v) \in \arg\max_{P \in \mathcal{P}(N \setminus S)} v(S, P)$. The core with respect to the optimistic rule is called the *optimistic core*.

Proposition 5. *If the optimistic core is non-empty, then the prenucleolus belongs to the core of the game with respect to any expectation formation rule.*

Proof. Let $(N, v) \in \mathcal{G}$. Note that, for every $S \subseteq N$, $v(S, g_o(S, v)) = v^{\max}(S)$. That is, the optimistic core is the core of the coalitional game without externalities $(N, v^{\max})$. Then, by *(ii)* in Theorem 1 and the well-known fact that the prenuclolus of a coalitional game without externalities lies in the core whenever non-empty, we have that $\eta(N, v) \in C_{g_o}(N, v)$. Finally, since the optimistic core is contained in every other core (Bloch and van den Nouweland, 2014) we get the desired result. □

A natural follow up question is whether the prenucleolus is in the core of any expectation formation rule whenever non-empty. We show by a counter-example that the answer is negative.

Example 1. Let $N = \{1, 2, 3\}$ and $(N, v) \in \mathcal{G}$ be defined by[14]

$v(1; 2, 3) = 0$	$v(1; 23) = 1$	$v(12; 3) = 2$	
$v(2; 1, 3) = 0$	$v(2; 13) = 1$	$v(13; 2) = 1$	$v(N; \emptyset) = 2$
$v(3; 1, 2) = 2$	$v(3; 12) = 0$	$v(23; 1) = 1$	

and let also the expectation formation rule be such that, $g(1, v) = 23$, $g(2, v) = 13$, and $g(3, v) = 12$. That is, according to g each coalition expect the rest of agents to form a one coalition partition. Then it is easy to see that

$$C_g(N, v) = \{(1, 1, 0)\}.$$

However, using *(ii)* in Theorem 1 we can easily compute the prenucleolus

$$\eta(N, v) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right).$$

Still, one could wonder whether there is a necessary and sufficient condition on the expectation formation rule that guarantees the prenucleolus to be a core allocation (with respect to the expectation formation rule) whenever non-empty. This is another open question for future research.

[13] Implicitly, we are assuming that the grand coalition is the most efficient arrangement of a set of players.

[14] For the sake of clarity we omit brackets and only use commas between coalitions.

CRediT authorship contribution statement

Mikel Álvarez-Mozos: Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing. **Lars Ehlers:** Conceptualization, Formal analysis, Writing – original draft, Writing – review & editing.

Data availability

No data was used for the research described in the article.

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