Strong approximations of Brownian sheet by uniform transport processes

Xavier Bardina^{*}, Marco Ferrante[‡] and Carles Rovira[†]

*Departament de Matemàtiques, Facultat de Ciències, Edifici C, Universitat Autònoma de Barcelona, 08193 Bellaterra. Xavier.Bardina@uab.cat

[‡]Dipartimento di Matematica "Tullio Levi-Civita", Università degli Studi di Padova, Via Trieste 63, 35121 Padua, Italy. marco.ferrante@unipd.it

[†]Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08007 Barcelona. carles.rovira@ub.edu

Abstract

Many years ago, Griego, Heath and Ruiz-Moncayo proved that it is possible to define realizations of a sequence of uniform transform processes that converges almost surely to the standard Brownian motion, uniformly on the unit time interval. In this paper we extend their results to the multi parameter case. We begin constructing a family of processes, starting from a set of independent standard Poisson processes, that has realizations that converge almost surely to the Brownian sheet, uniformly on the unit square. At the end the extension to the *d*-parameter Wiener processes is presented.

1 Introduction

Let $W = \{W(s,t) : (s,t) \in [0,1]^2\}$ be a Brownian sheet, i.e. a zero mean real continuous Gaussian process with covariance function $E[W_{(s_1,t_1)}W_{(s_2,t_2)}] =$ $(s_1 \wedge s_2)(t_1 \wedge t_2)$ for any $(s_1,t_1), (s_2,t_2) \in [0,1]^2$. The purpose of this paper is to find strong approximations of the Brownian sheet by processes constructed from a family of independent standard Poisson processes. In particular, we seek an extension, first in the two-parameter case, of a result proved by Griego, Heath and Ruiz-Moncayo [11], where the authors present realizations of a sequence of the uniform transform processes that converges almost surely to the standard Brownian motion, uniformly on the unit time interval. These results are not just interesting from a purely mathematical point of view, but are of great interest in order to provide sound approximation strategies to solutions of stochastic differential equations (in the classical Brownian motion case) and of stochastic

^{*}X. Bardina is supported by the grantMTM2015-67802-P from SEIDI, Ministerio de Economia y Competividad.

[†]C. Rovira is supported by the grant MTM2015-65092-P from MINECO/FEDER, UE.

partial differential equations (in the present case of multi-parameter Wiener processes).

Griego, Heath and Ruiz-Moncayo [11] deal with a sequence of uniform transport processes

$$X_n(t) = \frac{1}{\sqrt{n}} (-1)^A \int_0^{tn} (-1)^{N(u)} du, \qquad (1)$$

where $N = \{N(t), t \ge 0\}$ is a standard Poisson process and A is a random variable with law Bernoulli $(\frac{1}{2})$ independent of N. To pass for the convergence in distribution, proved by Pinsky in [12] to the almost sure convergence, they make use of an embedding result due to Skorokhod (see [13], page 163).

In the literature, there are some extensions of this result of almost sure convergence. In [9] Gorostiza and Griego extended the result of [1]) to the case of a diffusions. Again Gorostiza and Griego [10] and Csörgő and Horváth [5] obtained the rate of convergence of the approximation sequence. More recently, Garzón, Gorostiza and León [6] defined a sequence of processes that converges strongly to fractional Brownian motion uniformly on bounded intervals, for any Hurst parameter $H \in (0, 1)$ and computed the rate of convergence. In [7] and [8] the same authors deal with subfractional Brownian motion and fractional stochastic differential equations. Finally in [1], Bardina, Binotto and Rovira proved the strong convergence to a complex Brownian motion. As far as we know, our work is the first extension to the multiparameter case.

To the best of our knowledge, in the case of the Brownian sheet, or more generally of the d-parameter Wiener processes, similar results have been proved just for the weak convergence. For instance, Bardina and Jolis [2] prove that the process

$$\frac{1}{n}\int_0^{tn}\int_0^{sn}\sqrt{xy}(-1)^{N(x,y)}dxdy,$$

where $\{N(x, y), x \ge 0, y \ge 0\}$ is a Poisson process in the plane, converges in law to a Brownian sheet when n goes to infinity and Bardina, Jolis and Rovira [3] extended this result to get the weak convergence to the d-parameter Wiener processes.

The present paper fills the gap in the case of a multi-parameter Brownian motion. Indeed, in the next section we construct a uniform like transport process which converges almost surely to the Brownian sheet and we present in the last section the easy extension to the general case of a *d*-parameter Wiener process.

2 Main result

Let us built up our approximation processes. Following some ideas of Bass and Pyke [4], we will start defining a suitable partition of the unit square For any n and fixed $\lambda > 0$, we can consider the partition of the unit square $[0, 1]^2$ in disjoint rectangles

$$\left([0,\frac{1}{n^{\lambda}}]\times[0,1]\right)\cup\Big(\bigcup_{k=2}^{\left\lfloor n^{\lambda}\right\rfloor}(\frac{k-1}{n^{\lambda}},\frac{k}{n^{\lambda}}]\times[0,1]\Big)\cup\Big((\frac{\left\lfloor n^{\lambda}\right\rfloor}{n^{\lambda}},1]\times[0,1]\Big)$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. If $W = \{W(s,t) : (s,t) \in [0,1]^2\}$ is a Brownian sheet on the unit square, let W^k denotes its restriction to each of the above defined rectangles $(\frac{k-1}{n^{\lambda}}, \frac{k}{n^{\lambda}}] \times [0,1]$. That is, we define

$$W^k(t) := W(\frac{k}{n^{\lambda}}, t) - W(\frac{k-1}{n^{\lambda}}, t),$$

for $k \in \{1, 2, \dots, \lfloor n^{\lambda} \rfloor\}$. Thus, for any $l \in \{1, 2, \dots, \lfloor n^{\lambda} \rfloor\}$ and $t \in [0, 1]$

$$W(\frac{l}{n^{\lambda}},t) = \sum_{k=1}^{l} W^{k}(t).$$

Moreover, putting $\tilde{W}^k(t) := n^{\frac{\lambda}{2}} W^k(t)$, we obtain a family

$$\{\tilde{W}^k; k \in \{1, 2, \dots \lfloor n^\lambda \rfloor\}\}$$

of independent standard Brownian motions defined in [0, 1].

From the paper of Griego, Heath and Ruiz-Moncayo [11] we know that there exist realizations of processes of type (1) that converge strongly and uniformly on bounded time intervals to Brownian motion. Using an approximation sequence for each one of the standard Brownian motions $\{\tilde{W}^k; k \in \{1, 2, ..., \lfloor n^\lambda \rfloor\}\}$, we will approximate the Brownian sheet by a process W_n such that for any $l \in \{1, 2, ..., \lfloor n^\lambda \rfloor\}$ and $t \in [0, 1]$

$$W_n(\frac{l}{n^{\lambda}}, t) = \sum_{k=1}^l \frac{1}{n^{\frac{1+\lambda}{2}}} (-1)^{A_k} \int_0^{nt} (-1)^{N_k(u)} du,$$

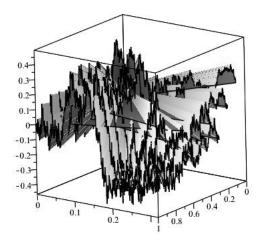
where $\{N_k, k \ge 1\}$ is a family of independent standard Poisson processes and $\{A_k, k \ge 1\}$ is a sequence of independent random variables with law Bernoulli $(\frac{1}{2})$, independent of the Poisson processes.

Furthermore, using linear interpolation, we can define $W_n(s,t)$ on the whole unit square as follows:

$$W_{n}(s,t) = \sum_{k=1}^{\lfloor sn^{\lambda} \rfloor} \frac{1}{n^{\frac{1+\lambda}{2}}} (-1)^{A_{k}} \int_{0}^{tn} (-1)^{N_{k}(u)} du$$

$$+ (sn^{\lambda} - \lfloor sn^{\lambda} \rfloor) \frac{1}{n^{\frac{1+\lambda}{2}}} (-1)^{A_{\lfloor sn^{\lambda} \rfloor + 1}} \int_{0}^{tn} (-1)^{N_{\lfloor sn^{\lambda} \rfloor + 1}(u)} du,$$
(2)

for any $(s,t) \in [0, \frac{\lfloor n^{\lambda} \rfloor}{n^{\lambda}}] \times [0,1]$ and $W_n(s,t) = W_n(\frac{\lfloor n^{\lambda} \rfloor}{n^{\lambda}},t)$ for any $(s,t) \in [\frac{\lfloor n^{\lambda} \rfloor}{n^{\lambda}},1] \times [0,1].$



Simulation of W_n with $\lambda := \frac{1}{5.0001}$ and $n := 10^{\frac{1}{\lambda}}$.

The main result of this paper states as follows:

Theorem 2.1. There exists realizations of the process $\{W_n(s,t), (s,t) \in [0,1]^2\}$ with $\lambda \in (0, \frac{1}{5})$ on the same probability space as a Brownian sheet $\{W(s, t), (s, t) \in W(s, t), (s, t) \in W(s, t)\}$ $[0,1]^2$ such that

$$\lim_{n \to \infty} \max_{0 \le s, t \le 1} |W_n(s, t) - W(s, t)| = 0 \quad a.s.$$

3 Proof of the main result

The key of the proof of the convergence result in [11] is a Skorokhod's result (see [13] page 163) of a reproduction of independent random variables by evaluating Brownian motion at random times. In our proof, we will use the following obvious extension of Skorokhod's theorem:

Theorem 3.1. Suppose that $\{\xi_1^k, \xi_2^k, \ldots, \xi_n^k; 1 \le k \le m\}$ are independent ran-dom variables such that $E(\xi_i^k) = 0$ and $Var(\xi_i^k) < \infty, 1 \le i \le n; 1 \le k \le m$ and that $\omega^1(t), \ldots, \omega^m(t)$ are a family of m independent Brownian processes. Then, there exist nonegative independent random variables $\tau_1^k, \tau_2^k, \ldots, \tau_n^k, 1 \le k \le m$, for which the variables

$$\omega^{k}(\tau_{1}^{k}), \omega^{k}(\tau_{1}^{k}+\tau_{2}^{k})-\omega^{k}(\tau_{1}^{k}), \dots, \omega^{k}(\tau_{1}^{k}+\tau_{2}^{k}+\dots+\tau_{n}^{k})-\omega(\tau_{1}^{k}+\tau_{2}^{k}+\dots+\tau_{n-1}^{k}),$$

have the same distribution as do $\xi_1^k, \xi_2^k, \ldots, \xi_n^k; 1 \le k \le m$. Also:

- (a) $E(\tau_i^k) = Var(\xi_i^k).$
- (b) There exist a constant $L_p > 0$ such that $E(\tau_i^k)^p \leq L_p E(\xi_i^k)^{2p}$. (c) For any h > 0, if $|\xi_i^k| \leq h$, then $|\omega^k(s) \omega^k(\sum_i^l \tau_i^k)| \leq h$ for $s \in [\sum_i^l \tau_i^k, \sum_i^{l+1} \tau_i^k]$.

Proof of Theorem 2.1. The proof of our main result follows the method

presented by Griego, Heath and Ruiz-Moncayo [11]. For each n and fixed $\lambda > 0$, consider $\{\eta_1^{(n)k}, \eta_2^{(n)k}, \ldots, \eta_{2n^2}^{(n)k}; 1 \le k \le \lfloor n^\lambda \rfloor\}$ families of independent and identically distributed random variables with exponential distribution of parameter 2n, and $\{K_1^k, K_2^k, \ldots, K_{2n^2}^k; 1 \le k \le \lfloor n^\lambda \rfloor\}$ sequences of independent and identically distributed random variables so that $P(K_i^k = 1) = P(K_i^k = -1) = \frac{1}{2}$. Then, $\{K_1^k \eta_1^{(n)k}, K_2^k \eta_2^{(n)k}, \ldots, K_{2n^2}^k \eta_{2n^2}^{(n)k}; 1 \le k \le [n^\lambda]\}$ are families of centered independent and identically distributed random variables $\{K_1^k, K_2^k, \ldots, K_{2n^2}^k, \ldots, K_{$

dom variables with common $Var(K_i^k \eta_i^{(n)k}) = \frac{1}{2n^2}$. By Theorem 3.1, the extension of the Skorokhod embedding theorem, for each $n \ge 0$ and $1 \le k \le \lfloor n^\lambda \rfloor$ there exist independent families of stopping times $\sigma_1^{(n)k}, \sigma_2^{(n)k}, \ldots, \sigma_{2n^2}^{(n)k}$ that are nonnegative, independent and identically distributed random variables such that the families

$$\tilde{W}^k(\sigma_1^{(n)k}), \tilde{W}^k(\sigma_1^{(n)k} + \sigma_2^{(n)k}), \dots, \tilde{W}^k(\sum_{i=1}^{2n^2} \sigma_i^{(n)k})$$

has the same law as

$$K_1^k \eta_1^{(n)k}, K_1^k \eta_1^{(n)k} + K_2^k \eta_2^{(n)k}, \dots, \sum_{i=1}^{2n^2} K_i^k \eta_i^{(n)k}.$$

Define now, for each $n \in \mathbb{N}$, $1 \le k \le \lfloor n^{\lambda} \rfloor$ and $i \in \{1, \ldots, 2n^2\}$

$$\gamma_i^{(n)k} = \frac{1}{n} |\tilde{W}^k(\sum_{j=0}^i \sigma_j^{(n)k}) - \tilde{W}^k(\sum_{j=0}^{i-1} \sigma_j^{(n)k})|.$$

Note that $\{\gamma_i^{(n)k} : 1 \leq i \leq 2n^2\}$ are families of independent and identically distributed random variables with exponential distribution of parameter $2n^2$.

We consider now the processes $\tilde{W}^{(n)k}(t), t \ge 0$, defined by piecewise linear interpolation in such a manner that

$$\tilde{W}^{(n)k}(\sum_{j=0}^{i}\gamma_{j}^{(n)k}) = \tilde{W}^{k}(\sum_{j=0}^{i}\sigma_{j}^{(n)k}),$$
(3)

 $\tilde{W}^{(n)k}(0) = 0$. Putting

$$W^{(n)k}(t) = \frac{1}{n^{\frac{\lambda}{2}}} \tilde{W}^{(n)k}(t),$$

we can define the process

$$W_n(\frac{l}{n^{\lambda}}, t) = \sum_{k=1}^l W^{(n)k}(t)$$

for any $0 \leq l \leq \lfloor n^{\lambda} \rfloor$, $t \in [0, 1]$. Finally, we can define $W_n(s, t)$ for any $(s, t) \in [0, 1]^2$ using again linear interpolation, that is, if $s \in (\frac{k-1}{n^{\lambda}}, \frac{k}{n^{\lambda}}]$, $W_n(s, t) = (\frac{k}{n^{\lambda}} - s)W_n(\frac{k-1}{n^{\lambda}}, t) + (s - (\frac{k-1}{n^{\lambda}}))W_n(\frac{k}{n^{\lambda}}, t)$ and $W_n(s, t) = W_n(\frac{\lfloor n^{\lambda} \rfloor}{n^{\lambda}}, t)$ if $s \geq \frac{\lfloor n^{\lambda} \rfloor}{n^{\lambda}}$. It is easy to check that W_n is a realization of the process (2).

Observe now that using Kolmogorov's inequality

$$P\left(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le l \le \lfloor n^\lambda \rfloor}} |\gamma_1^{(n)l} + \dots + \gamma_i^{(n)l} - \frac{i}{2n^2}| \ge \varepsilon\right)$$

$$\leq \sum_{1 \le l \le \lfloor n^\lambda \rfloor} P\left(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le i \le 2n^2}} |\gamma_1^{(n)l} + \dots + \gamma_i^{(n)l} - \frac{i}{2n^2}| \ge \varepsilon\right)$$

$$\leq \frac{n^\lambda}{\varepsilon^2} \sum_{i=1}^{2n^2} Var(\gamma_i^{(n)l}) = \frac{n^\lambda}{2\varepsilon^2 n^2} = \frac{1}{2\varepsilon^2 n^{2-\lambda}}.$$
(4)

By the same arguments and using that $Var(\sigma_i^{(n)l}) = \frac{3L_2}{2n^4}$, where L_2 is a constant obtained from Theorem 3.1, we have also that

$$P(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le l \le \lfloor n^\lambda \rfloor}} |\sigma_1^{(n)l} + \dots + \sigma_i^{(n)l} - \frac{i}{2n^2}| \ge \varepsilon) \le \frac{3L_2}{\varepsilon^2 n^{2-\lambda}}.$$
(5)

Since the Brownian sheet is α -Hölder continuous for any $\alpha < \frac{1}{2}$ and the Lipschitz continuity of $W^{(n)}$, we obtain that

$$\lim_{n \to \infty} \max_{\substack{0 \le s \le 1\\ 0 \le t \le 1}} |W^{(n)}(s,t) - W(s,t)| \\
= \lim_{n \to \infty} \max_{\substack{1 \le i \le 2n^2\\ 1 \le k \le [n^{\lambda}]}} |W^{(n)}(\frac{k}{n^{\lambda}}, \frac{i}{2n^2}) - W(\frac{k}{n^{\lambda}}, \frac{i}{2n^2})| \\
= \lim_{n \to \infty} \max_{\substack{1 \le i \le 2n^2\\ 1 \le k \le [n^{\lambda}]}} |\sum_{l=1}^k W^{(n)l}(\frac{i}{2n^2}) - \sum_{l=1}^k W^l(\frac{i}{2n^2})|.$$
(6)

Let us check that we can change $\sum_{l=1}^{k} W^{l}(\frac{i}{2n^{2}})$ by $\sum_{l=1}^{k} W^{l}(\sum_{j=0}^{i} \sigma_{j}^{(n)l})$ and $\sum_{l=1}^{k} W^{(n)l}(\frac{i}{2n^{2}})$ by $\sum_{l=1}^{k} W^{(n)l}(\sum_{j=0}^{i} \gamma_{j}^{(n)l})$. Set

$$A_{n,\delta} := \{ \max_{\substack{1 \le i \le 2n^2 \\ 1 \le l \le [n^{\lambda}]}} |\sigma_1^{(n)l} + \dots + \sigma_i^{(n)l} - \frac{i}{2n^2}| \ge \delta \};$$

we get

$$\begin{split} &P\Big(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le k \le \lfloor n^\lambda \rfloor}} |\sum_{l=1}^k W^l(\frac{i}{2n^2}) - \sum_{l=1}^k W^l(\sum_{j=0}^i \sigma_j^{(n)l})| > \varepsilon\Big) \\ &\le P(A_{n,\delta}) + P\Big(\{\max_{\substack{1 \le i \le 2n^2 \\ 1 \le k \le \lfloor n^\lambda \rfloor}} |\sum_{l=1}^k W^l(\frac{i}{2n^2}) - \sum_{l=1}^k W^l(\sum_{j=0}^i \sigma_j^{(n)l})| > \varepsilon\} \cap A_{n,\delta}^c\Big) \\ &\le \frac{3L_2}{\delta^2 n^{2-\lambda}} + P\Big(\max_{\substack{1 \le k \le \lfloor n^\lambda \rfloor}} \sum_{l=1}^k \sup_{|s-t| < \delta} |W^l(t) - W^l(s)| > \varepsilon\Big) \\ &\le \frac{3L_2}{\delta^2 n^{2-\lambda}} + n^\lambda P\Big(\sup_{|s-t| < \delta} |W^1(t) - W^1(s)| > \frac{\varepsilon}{n^\lambda}\Big) \\ &\le \frac{3L_2}{\delta^2 n^{2-\lambda}} + \frac{n^{\lambda(M+1)}}{\varepsilon^M} E\Big(\sup_{|s-t| < \delta} |W^1(t) - W^1(s)|^M\Big) \\ &\le \frac{3L_2}{\delta^2 n^{2-\lambda}} + C_M \frac{n^{\lambda(M+1)}}{\varepsilon^M} \Big(\delta \log \left(\frac{2}{\delta}\right)\Big)^{\frac{M}{2}}, \end{split}$$

where we have used (5), Markov's inequality and estimates for the modulus of continuity of the Brownian motion. Putting $\delta = n^{\frac{\lambda-1}{2}+\mu}$, for any $\lambda \in (0, \frac{1}{5})$, we can choose $\mu > 0$ such that $5\lambda - 1 + 2\mu < 0$. Then

$$\frac{\lambda - 1}{2} + \mu < 0$$

and there exists M large enough such that

$$\lambda(M+1) + \frac{M}{2} \left(\frac{\lambda - 1}{2} + \mu\right) = \frac{M}{4} \left(5\lambda - 1 + 2\mu\right) + \lambda < -1.$$

Thus, we get that

$$\sum_{n=1}^{\infty} P\Big(\max_{\substack{1 \leq i \leq 2n^2 \\ 1 \leq k \leq \lfloor n^\lambda \rfloor}} |\sum_{l=1}^k W^l(\frac{i}{2n^2}) - \sum_{l=1}^k W^l(\sum_{j=0}^i \sigma_j^{(n)l})| > \varepsilon \Big) < \infty.$$

Then by the Borel-Cantelli lemma we have

$$\lim_{n \to \infty} \max_{\substack{1 \le i \le 2n^2 \\ 1 \le k \le \lfloor n^\lambda \rfloor}} |\sum_{l=1}^k W^l(\frac{i}{2n^2}) - \sum_{l=1}^k W^l(\sum_{j=0}^i \sigma_j^{(n)l})| = 0, \quad a.s. \quad (7)$$

Using (4) and similar argument we get

$$\lim_{n \to \infty} \max_{\substack{1 \le i \le 2n^2 \\ 1 \le k \le \lfloor n^\lambda \rfloor}} |\sum_{l=1}^k W^{(n)l}(\frac{i}{2n^2}) - \sum_{l=1}^k W^{(n)l}(\sum_{j=0}^i \gamma_j^{(n)l})| = 0, \quad a.s. \quad (8)$$

Putting together (6), (7) and (8) and using that $\tilde{W}^k(t) := n^{\frac{\lambda}{2}} W^k(t)$ and (3), at the end we obtain

$$\begin{split} \lim_{n \to \infty} & \max_{\substack{0 \le s \le 1\\ 0 \le t \le 1}} |W^{(n)}(s,t) - W(s,t)| \\ &= \lim_{n \to \infty} & \max_{\substack{1 \le i \le 2n^2\\ 1 \le k \le \lfloor n^\lambda \rfloor}} |\sum_{l=1}^k W^{(n)l}(\sum_{j=0}^i \gamma_j^{(n)l}) - \sum_{l=1}^k W^l(\sum_{j=0}^i \sigma_j^{(n)l})| \\ &= \lim_{n \to \infty} & \max_{\substack{1 \le i \le 2n^2\\ 1 \le k \le \lfloor n^\lambda \rfloor}} |\frac{1}{n^{\frac{\lambda}{2}}} \sum_{l=1}^k \tilde{W}^l(\sum_{j=0}^i \sigma_j^{(n)l}) - \sum_{l=1}^k W^l(\sum_{j=0}^i \sigma_j^{(n)l})| \\ &= 0, \end{split}$$

a.s. and the proof is complete.

4 Extension to the *d*-parameter Wiener process

Suppose $d \geq 2$ and consider $[0,1]^d \subset \mathbb{R}^d$. Define on $[0,1]^d$ the partial ordering

$$s^1 \le s^2 \Leftrightarrow s_i^1 \le s_i^2$$
 for any $1 \le i \le d$

and denote $I_0 = \{s \in [0,1]^d : s_1 \cdots s_d = 0\}$. Let (Ω, \mathcal{F}, Q) be a complete probability space and let $\{\mathcal{F}_s, s \in [0,1]^d\}$ be a family of sub- σ -fields of \mathcal{F} such that: $\mathcal{F}_{s^1} \subseteq \mathcal{F}_{s^2}$ for any $s^1 \leq s^2$. Fix $t \in [0,1]^d$ we also consider $\mathcal{F}_t^T :=$ $\bigvee_{i=1}^d \mathcal{F}_{(1,\cdots,1,t_i,1,\cdots,1)}$ (the σ -field generated by all the past of t). Given s < twe denote by $\Delta_s X(t)$ the increment of the process X over the rectangle (s,t] = $\prod_{i=1}^d (s_i, t_i] \subset \mathbb{R}^d$.

Then, a *d*-parameter continuous process $W = \{W(s); s \in [0, 1]^d \subset \mathbb{R}^d_+\}$ is called a *d*-parameter $\{\mathcal{F}_t\}$ -Wiener process if it is $\{\mathcal{F}_t\}$ -adapted, null on I_0 and for any s < t the increment $\Delta_s W(t)$ is independent of \mathcal{F}_s^T and is normally distributed with zero mean and variance $\prod_{i=1}^d (t_i - s_i)$.

In order to built up the approximation sequence we can deal now with the disjoint rectangles

$$\prod_{i=1}^{d-1} \left(\frac{k_i - 1}{n^{\lambda}}, \frac{k_i}{n^{\lambda}}\right] \times [0, 1]$$

where $1 \leq k_1, \ldots, k_{d-1} \leq \lfloor n^{\lambda} \rfloor$ and the convention that the intervals are leftclosed at the zero end points. Set $W^{k_1,\ldots,k_{d-1}}$ the restriction of the *d*-parameter Wiener process to each of these rectangles, that is, we define

$$W^{k_1,\ldots,k_{d-1}}(t) := \Delta_{(\frac{k_1-1}{n^{\lambda}},\ldots,\frac{k_{d-1}-1}{n^{\lambda}},t)} W(\frac{k_1}{n^{\lambda}},\ldots,\frac{k_{d-1}}{n^{\lambda}},t),$$

for $1 \leq k_1, \ldots, k_{d-1} \leq \lfloor n^{\lambda} \rfloor$. Thus, for any $1 \leq l_1, \ldots, l_{d-1} \leq \lfloor n^{\lambda} \rfloor$ and $t \in [0, 1]$

$$W(\frac{l_1}{n^{\lambda}}, \dots, \frac{l_{d-1}}{n^{\lambda}}, t) = \sum_{k_1=1}^{l_1} \dots \sum_{k_{d-1}=1}^{l_{d-1}} W^{k_1, \dots, k_{d-1}}(t)$$

and thus $\tilde{W}^{k_1,\ldots,k_{d-1}}(t) := n^{(d-1)\frac{\lambda}{2}}W^{k_1,\ldots,k_{d-1}}(t)$ is a family of independent standard Brownian motions defined in [0, 1]. As in the 2-dimensional case we will approximate the *d*-dimensional Wiener process by a process W_n such that for any $1 \leq l_1,\ldots,l_{d-1} \leq \lfloor n^{\lambda} \rfloor$ and $t \in [0, 1]$

$$W_n(\frac{l_1}{n^{\lambda}},\dots,\frac{l_{d-1}}{n^{\lambda}},t)$$

= $\sum_{k_1=1}^{l_1}\dots\sum_{k_{d-1}=1}^{l_{d-1}}\frac{1}{n^{\frac{1+(d-1)\lambda}{2}}}(-1)^{A_{k_1,\dots,k_{d-1}}}\int_0^{nt}(-1)^{N_{k_1,\dots,k_{d-1}}(u)}du,$

where $\{N_{k_1,\ldots,k_{d-1}}\}$ is a family of independent standard Poisson processes and $\{A_{k_1,\ldots,k_{d-1}}\}$ is a sequence of independent random variables with law Bernoulli $(\frac{1}{2})$ independent of the Poisson processes. We can define $W_n(s_1,\ldots,s_{d-1},t)$ for any $(s_1,\ldots,s_{d-1}) \in [0,1]^{d-1}$ using a d-1-dimensional linear interpolation.

The main theorem reads as follows:

Theorem 4.1. There exists realizations of the process $\{W_n(s_1, \ldots, s_d) : (s_1, \ldots, s_d) \in [0, 1]^d\}$ with $\lambda \in (0, \frac{1}{5(d-1)})$ on the same probability space as a d-dimensional Wiener process $\{W(s_1, \ldots, s_d) : (s_1, \ldots, s_d) \in [0, 1]^d\}$ such that

$$\lim_{n \to \infty} \max_{0 \le s_1, \dots, s_d \le 1} |W_n(s_1, \dots, s_d) - W(s_1, \dots, s_d)| = 0 \quad a.s$$

Sketch of the Proof: the proof follows the same steps that for the Brownian sheet. For each $n \geq 0$ and $1 \leq k_1, \ldots, k_{d-1} \leq \lfloor n^{\lambda} \rfloor$ there exist independent families of stopping times $\sigma_1^{(n)k_1,\ldots,k_{d-1}}, \ldots, \sigma_{2n^2}^{(n)k_1,\ldots,k_{d-1}}$ that are nonnegative, independent and identically distributed random variables that we can use to define, for each n and $1 \leq k_1, \ldots, k_{d-1} \leq \lfloor n^{\lambda} \rfloor$,

$$\gamma_i^{(n)k_1,\dots,k_{d-1}} = \frac{1}{n} |\tilde{W}^{k_1,\dots,k_{d-1}}(\sum_{j=0}^i \sigma_j^{(n)k_1,\dots,k_{d-1}}) - \tilde{W}^{k_1,\dots,k_{d-1}}(\sum_{j=0}^{i-1} \sigma_j^{(n)k_1,\dots,k_{d-1}})|,$$

for any $i \in \{1, ..., 2n^2\}$.

We consider now the processes $\tilde{W}^{(n)k_1,\ldots,k_{d-1}}(t), t \ge 0$, defined by piecewise linear interpolation in such a manner that

$$\tilde{W}^{(n)k_1,\dots,k_{d-1}}(\sum_{j=0}^i \gamma_j^{(n)k_1,\dots,k_{d-1}}) = \tilde{W}^{k_1,\dots,k_{d-1}}(\sum_{j=0}^i \sigma_j^{(n)k_1,\dots,k_{d-1}})$$

 $\tilde{W}^{(n)k_1,\ldots,k_{d-1}}(0) = 0$. Then we can define the process

$$W^{(n)}(\frac{l_1}{n^{\lambda}},\dots,\frac{l_{d-1}}{n^{\lambda}},t) = \sum_{k_1=1}^{l_1}\dots\sum_{k_{d-1}=1}^{l_{d-1}} W^{k_1,\dots,k_{d-1}}(t)$$

for any $l_1, \ldots, l_{d-1} \in [0, \lfloor n^{\lambda} \rfloor]^{d-1}, t \in [0, 1]$. Finally, we can define $W^{(n)}(s, t)$ for any $(s, t) \in [0, 1]^2$ using d - 1-dimensional linear interpolation.

Following the Brownian sheet case we get the following estimates:

$$P(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le l_1, \dots, l_{d-1} \le \lfloor n^\lambda \rfloor}} |\gamma_1^{(n)l_1, \dots, l_{d-1}} + \dots + \gamma_i^{(n)l_1, \dots, l_{d-1}} - \frac{i}{2n^2}| \ge \varepsilon)$$
$$\le \frac{1}{2\varepsilon^2 n^{2-\lambda(d-1)}}$$

and

$$P(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le l_1, \dots, l_{d-1} \le \lfloor n^\lambda \rfloor}} |\sigma_1^{(n)l_1, \dots, l_{d-1}} + \dots + \sigma_i^{(n)l_1, \dots, l_{d-1}} - \frac{i}{2n^2}| \ge \varepsilon)$$
$$\le \frac{3L_2}{\varepsilon^2 n^{2-\lambda(d-1)}}$$

and

$$\begin{split} & P\Big(\max_{\substack{1 \le i \le 2n^2 \\ 1 \le k_1, \dots, k_{d-1} \le \lfloor n^\lambda \rfloor}} |\sum_{l_1=1}^{k_1} \dots \sum_{l_{d-1}=1}^{k_{d-1}} W^{l_1, \dots, l_{d-1}}(\frac{i}{2n^2}) \\ & -\sum_{l_1=1}^{k_1} \dots \sum_{l_{d-1}=1}^{k_{d-1}} W^{l_1, \dots, l_{d-1}}(\sum_{j=0}^i \sigma_j^{(n)l})| > \varepsilon \Big) \\ \le & \frac{3L_2}{\delta^2 n^{2-\lambda(d-1)}} \\ & + P\Big(\max_{\substack{1 \le k_1, \dots, k_{d-1} \le \lfloor n^\lambda \rfloor}} \sum_{l_{1=1}=1}^{k_1} \dots \sum_{l_{d-1}=1}^{k_{d-1}} \\ & \sup_{|s-t| < \delta} |W^{l_1, \dots, l_{d-1}}(t) - W^{l_1, \dots, l_{d-1}}(s)| > \varepsilon \Big) \\ \le & \frac{3L_2}{\delta^2 n^{2-\lambda}} + n^{\lambda(d-1)} P\Big(\sup_{|s-t| < \delta} |W^l(t) - W^l(s)| > \frac{\varepsilon}{n^{\lambda(d-1)}}\Big) \\ \le & \frac{3L_2}{\delta^2 n^{2-\lambda(d-1)}} + C_M \frac{n^{\lambda(d-1)(M+1)}}{\varepsilon^M} \Big(\delta \log\left(\frac{2}{\delta}\right)\Big)^{\frac{M}{2}}. \end{split}$$

We finish the proof, as in the Brownian sheet case, putting $\delta = n^{\frac{\lambda(d-1)-1}{2}+\mu}$, for any $\lambda \in (0, \frac{1}{5(d-1)})$, we can choose $\mu > 0$ such that $5\lambda(d-1) - 1 + 2\mu < 0$. Then

$$\frac{\lambda(d-1)-1}{2} + \mu < 0$$

and there exists M large enough such that

$$\lambda(M+1) + \frac{M}{2} \left(\frac{\lambda(d-1) - 1}{2} + \mu\right) = \frac{M}{4} \left(5\lambda(d-1) - 1 + 2\mu\right) + \lambda(d-1) < -1.$$

5 Conclusions and future work

In the present paper we have constructed a uniform transport like process that converges strongly to the multi-parameter Wiener process. Our result makes use of the 1-parameter result proved by Griego, Heath and Ruiz-Moncayo [11] and some ideas from of Bass and Pyke [4].

The interest of this class of results is not only purely mathematical, but also for their possible application in order to approximate the solutions to stochastic partial differential equations, which arise in many fields as physics, biology or finance. We plan to continue our investigation with the aim to extend the present result to these class of processes and to obtain results on the rate of convergence, that in the present case looks quite challenging, needed for the applications.

References

- Bardina, X.; Binotto, G.; Rovira, C. The complex Brownian motion as a strong limit of processes constructed from a Poisson process. J. Math. Anal. Appl. Vol.444 (2016), no. 1, 700–720.
- [2] Bardina, X.; Jolis, M. Weak approximation of the Brownian sheet from a Poisson process in the plane. *Bernoulli* Vol.6 (2000), no. 4, 653-665.
- [3] Bardina, X.; Jolis, M.; Rovira, C. Weak approximation of the Wiener process from a Poisson process: the multidimensional parameter set case. *Statist. Probab. Lett.* Vol.50 (2000), no. 3, 245-255.
- [4] Bass, R. F.; Pyke, R, Functional law of the iterated logarithm and uniform central limit theorem for partial-sum processes indexed by sets. Ann. Probab. Vol.12 (1984), no. 1, 13–34.
- [5] Csörgo, M.; Horváth, L. Rate of convergence of transport processes with an application to stochastic differential equations. *Probab. Theory Related Fields* Vol.78 (1988), no. 3, 379-387.
- [6] Garzón, J.; Gorostiza, L. G.; León, J. A. A strong uniform approximation of fractional Brownian motion by means of transport processes. *Stochastic Process. Appl.* Vol.119 (2009), no. 10, 3435-3452.

- [7] Garzón, J.; Gorostiza, L. G.; León, J. A strong approximation of subfractional Brownian motion by means of transport processes. In: Malliavin calculus and stochastic analysis, 335–360, Springer Proc. Math. Stat., 34, Springer, New York, (2013).
- [8] Garzón, J.; Gorostiza, L. G.; León, J. Approximations of Fractional Stochastic Differential Equations by means of transport processes. *Commun. Stoch. Anal.* Vol.5. No.3 (2011), 433-456.
- [9] Gorostiza, L.G. and Griego, R.J. Strong approximation of diffusion processes by transport processes. *Journal of Mathematics of Kyoto University* Vol.19 (1979), No. 1, 91-103.
- [10] Gorostiza, L.G. and Griego, R.J. Rate of convergence of uniform transport processes to Brownian motion and application to stochastic integrals. *Stochastics* Vol. 3 (1980), 291-303.
- [11] Griego, R.J., Heath, D. and Ruiz-Moncayo, A. Almost sure convergence of uniform trasport processes to Brownian motion. Ann. Math. Stat. Vol. 42 (1971), No. 3, 1129-1131.
- [12] Pinsky, M. Differential equations with a small parameter and the central limit theorem for functions defined on a finite Markov chain. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete Vol. 9 (1968), 101–111.
- [13] Skorokhod, A.V. Study in the Theory of Random Processes. Addison-Wesley, Reading, (1965).