

# HANKEL OPERATORS ON EXPONENTIAL BERGMAN SPACES

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ABSTRACT. We completely describe the boundedness and compactness of Hankel operators with general symbols acting on Bergman spaces with exponential type weights.

## 1. INTRODUCTION

Hankel operators acting on Bergman spaces is an important area of research in the theory of operators acting on spaces of analytic functions. Most of the theory of Hankel operators on standard Bergman spaces is well understood, but not so much is known for large Bergman spaces. The function and operator theory acting on large Bergman spaces on the unit disc  $\mathbb{D}$  of the complex plane  $\mathbb{C}$  is just developing, and it is our purpose to study big Hankel operators acting on such spaces. For a strictly subharmonic function  $\varphi$  on  $\mathbb{D}$  and  $0 < p \leq \infty$ , let  $L_\varphi^p$  consist of those Lebesgue measurable functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that

$$\|f\|_{L_\varphi^p} = \left\{ \int_{\mathbb{D}} |f(z)e^{-\varphi(z)}|^p dA(z) \right\}^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,$$
$$\|f\|_{L_\varphi^\infty} = \sup_{z \in \mathbb{D}} |f(z)|e^{-\varphi(z)} < \infty, \quad p = \infty,$$

and consider the weighted Bergman space  $A_\varphi^p = L_\varphi^p \cap H(\mathbb{D})$ . Here  $H(\mathbb{D})$  denotes the set of all holomorphic functions in  $\mathbb{D}$  and  $dA$  is the Lebesgue area measure on  $\mathbb{C}$ . We also use  $L^p$  to stand for the usual Lebesgue space  $L^p(\mathbb{D}, dA)$ .

In this paper we are interested in  $A_\varphi^p$  with weight function  $\varphi \in \mathcal{W}_0$  which was first introduced in [10]. To describe  $\mathcal{W}_0$  precisely, let  $C_0$  be the family of all continuous functions  $\rho$  on  $\mathbb{D}$  satisfying  $\lim_{|z| \rightarrow 1} \rho(z) = 0$ . Set

$$\mathcal{L} = \left\{ \rho : \mathbb{D} \rightarrow \mathbb{R} : \rho \in C_0, \quad \|\rho\|_L = \sup_{z, w \in \mathbb{D}, z \neq w} \frac{|\rho(z) - \rho(w)|}{|z - w|} < \infty \right\},$$

and let  $\mathcal{L}_0$  consist of those  $\rho \in \mathcal{L}$  with the property that for each  $\varepsilon > 0$  there is a compact subset  $E \subset \mathbb{D}$  with

$$|\rho(z) - \rho(w)| \leq \varepsilon |z - w|$$

whenever  $z, w \in \mathbb{D} \setminus E$ . The class  $\mathcal{W}_0$  is defined as

$$\mathcal{W}_0 = \left\{ \varphi \in C^2(\mathbb{D}) : \Delta\varphi > 0, \text{ and } \exists \rho \in \mathcal{L}_0 \text{ such that } \frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \right\}.$$

Here and afterward, the expression  $A \simeq B$  means there exist two positive constants  $c_1$  and  $c_2$  independent of the functions being considered such that  $c_1 A \leq B \leq c_2 A$ .

It is easy to verify that  $A_\varphi^p$  is a Banach space when  $1 \leq p \leq \infty$ , and  $A_\varphi^2$  is a Hilbert space. These spaces are also called large Bergman spaces because it usually contains all the standard Bergman spaces. Examples of weighted Bergman spaces with  $\varphi \in \mathcal{W}_0$  includes

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exponential Bergman spaces, double exponential weighted Bergman spaces, and also some non-radial weighted Bergman spaces (see [10, 16]). With the Bergman reproducing kernel  $K(\cdot, \cdot)$  on  $A_\varphi^2$  one can define the Bergman projection  $P$  as

$$P(g)(z) = \int_{\mathbb{D}} g(\xi) K(z, \xi) e^{-2\varphi(\xi)} dA(\xi).$$

For  $1 \leq p \leq \infty$ ,  $P$  is bounded from  $L_\varphi^p$  to  $A_\varphi^p$ , and  $P|_{A_\varphi^p}$ , the restriction on  $A_\varphi^p$ , is just the identity operator  $\text{Id}$  (see [10] for details).

Given some symbol function  $f$ , one defines the so-called Hankel operator  $H_f$  as

$$(1.1) \quad H_f(g) = (\text{Id} - P)(fg).$$

From [10] we know that

$$\Gamma = \left\{ \sum_{j=1}^N a_j K(\cdot, z_j) : N \in \mathbb{N}, a_j \in \mathbb{C}, z_j \in \mathbb{D}, \text{ for } 1 \leq j \leq N, \right\}$$

is dense in  $A_\varphi^p$ . Therefore, to let  $H_f$  make sense on  $\Gamma$  we naturally consider those  $f$  in the symbol class  $\mathcal{S}$  defined as

$$\mathcal{S} = \{f \text{ measurable on } \mathbb{D} : fg \in L_\varphi^1 \text{ for } g \in \Gamma\}$$

(from Theorem 3.3 in [10],  $\|K(\cdot, z)\|_{L_\infty} < \infty$  so that  $P(fg)(z)$  is well defined for  $f \in \mathcal{S}$ ,  $g \in \Gamma$  and  $z \in \mathbb{D}$ ). The purpose of this work is, for  $1 \leq p, q < \infty$ , to characterize those  $f \in \mathcal{S}$  such that  $H_f$  is bounded (or compact) as an operator acting from  $A_\varphi^p$  to  $L_\varphi^q$ . The descriptions obtained are presented in section 4.

As in [10], we write  $\mathcal{BDK}$  to be the weight class introduced by Borichev, Dhuez and Kellay in [3]. We know  $\mathcal{BDK} \subset \mathcal{W}_0$  and  $\mathcal{W}_0 \setminus \mathcal{BDK} \neq \emptyset$ . The Bergman space  $A_\varphi^p$  with  $\varphi \in \mathcal{BDK}$  have been studied in [2, 3, 6, 7, 9, 16, 17].

Given Banach spaces  $X$  and  $Y$ , and some linear operator from  $X$  to  $Y$ , we use  $\|\cdot\|_X$  and  $\|T\|_{X \rightarrow Y}$  respectively to stand for the norm on  $X$ , and the operator norm of  $T$ . Throughout this paper, we use  $C$  to denote positive constants whose value may change from line to line, but do not depend on functions being considered.

## 2. SOME PRELIMINARY

We are going to present some basic conclusions that will be used in the following sections. Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ . We define a distance  $d_\rho(z, w)$  on  $\mathbb{D}$  as

$$d_\rho(z, w) = \inf_{\gamma} \int_0^1 |\gamma'(t)| \frac{dt}{\rho(\gamma(t))},$$

where the infimum is taken over all piecewise  $C^1$  curves  $\gamma : [0, 1] \rightarrow \mathbb{D}$  with  $\gamma(0) = z$  and  $\gamma(1) = w$ . It is mentioned in [5] that  $d_\rho(\cdot, \cdot)$  is equivalent to the Bergman distance  $\beta_\varphi(\cdot, \cdot)$  induced by the Bergman metric  $\frac{1}{2} \frac{\partial^2 \log K(z, z)}{\partial z \partial \bar{z}} dz \otimes d\bar{z}$ .

The estimates on the Bergman kernel play an important role in our analysis. The following lemma comes from [10].

**Lemma 2.1.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ . There are positive constants  $C_1, C_2, \sigma$  and  $d$  such that*

$$|K(z, w)| \leq C_1 \frac{e^{\varphi(z)+\varphi(w)}}{\rho(z)\rho(w)} e^{-\sigma d_\rho(z, w)} \quad \text{for } z, w \in \mathbb{D},$$

and

$$|K(z, w)| \geq C_2 \frac{e^{\varphi(z)} e^{\varphi(w)}}{\rho(z)\rho(w)} \quad \text{for } d_\rho(z, w) \leq d.$$

For  $K_z(\cdot) = K(\cdot, z) \in H(\mathbb{D})$  and  $0 < p \leq \infty$ , with Lemma 2.1 and an elementary calculation as that of Corollary 3.2 in [10] we obtain

$$(2.1) \quad \|K_z\|_{L_\varphi^p} \simeq e^{\varphi(z)} \rho(z)^{\frac{2}{p}-2}.$$

Write  $k_{z,p} = \frac{K_z}{\|K_z\|_{L_\varphi^p}}$  to denote the normalized reproducing kernels in  $A_\varphi^p$ .

For  $z \in \mathbb{D}$  and  $r > 0$ , set  $D(z, r) = \{w : |w - z| < r\}$  to be the Euclidean disc with center  $z$  and radius  $r$ . Write

$$B_\rho(z, r) = \{w \in \mathbb{D} : d_\rho(w, z) < r\} \text{ and } D^r(z) = D(z, r\rho(z)).$$

The following lemma is from [10].

**Lemma 2.2.** *Let  $\rho \in \mathcal{L}$  be positive. Then there exists  $\alpha > 0$  with the following properties:*

(i) *There exist constants  $C_1$  and  $C_2$  such that*

$$(2.2) \quad C_1 \rho(w) \leq \rho(z) \leq C_2 \rho(w)$$

*for  $z \in \mathbb{D}$  and  $w \in D^\alpha(z)$ .*

(ii) *There exists a constant  $B > 0$  such that*

$$(2.3) \quad D^r(z) \subseteq D^{Br}(w), \quad D^r(w) \subseteq D^{Br}(z)$$

*for  $w \in D^r(z)$  and  $0 < r \leq \alpha$ .*

(iii) *There exist positive constants  $c_1$  and  $c_2$  such that*

$$(2.4) \quad B_\rho(z, c_1 r) \subseteq D^r(z) \subseteq B_\rho(z, c_2 r)$$

*for  $z \in \mathbb{D}$  and  $0 < r \leq \alpha$ .*

Moreover, if  $\alpha$  is small enough, we can take  $C_1 = 1/2$ ;  $C_2 = 2$  in part (i) and  $B = 4$  in part (ii).

For our analysis we need a covering lemma which is almost identical to Lemma 3.1 of [8].

**Lemma 2.3.** *Let  $\rho \in \mathcal{L}$  be positive. There are positive constants  $\alpha$  and  $s$ , depending only on  $\|\rho\|_L$ , such that for  $0 < r \leq \alpha$  there exists a sequence  $\{z_j\}_{j=1}^\infty \subset \mathbb{D}$  satisfying*

- (i)  $\mathbb{D} = \cup_{j \geq 1} D^r(z_j)$ ;
- (ii)  $D^{sr}(z_j) \cap D^{sr}(z_m) = \emptyset$  for  $m \neq j$ ;
- (iii)  $\{D^{2\alpha}(z_j)\}_{j=1}^\infty$  is a covering of  $\mathbb{D}$  of finite multiplicity.

A sequence  $\{z_j\}_{j=1}^\infty$  satisfying (i)-(iii) of Lemma 2.3 will be called a  $(\rho, r)$ -lattice. Given some  $(\rho, r)$ -lattice  $\{z_j\}_{j=1}^\infty$ , by the statement (iii) of Lemma 2.3 we have some integer  $N$  so that

$$(2.5) \quad 1 \leq \sum_{j=1}^{\infty} \chi_{D^{Br}(z_j)}(z) \leq N \quad \text{for } z \in \mathbb{D}.$$

Here and afterward,  $\chi_E$  is the characteristic function of a subset  $E$  of  $\mathbb{D}$ . In what follows we always take  $\alpha > 0$  as that in Lemma 2.2 and Lemma 2.3. The next lemma has already been obtained for  $\varphi \in \mathcal{BDK}$  in Arroussi's dissertation [1].

**Lemma 2.4.** *Let  $\varphi \in \mathcal{W}_0$ ,  $0 < p \leq \infty$ , and let  $\{z_j\}_{j=1}^\infty$  be some  $(\rho, r)$ -lattice with  $0 < r \leq \alpha$ . Then for  $\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^p$ , we have  $\sum_{j=1}^\infty \lambda_j k_{z_j, p} \in A_\varphi^p$  with the norm estimate*

$$(2.6) \quad \left\| \sum_{j=1}^\infty \lambda_j k_{z_j, p} \right\|_{L_\varphi^p} \leq C \|\lambda\|_{\ell^p}.$$

*Proof.* We treat the case  $1 \leq p \leq \infty$  first. Let  $q$  be the conjugate exponent of  $p$ . For  $f \in H(\mathbb{D})$ , by Lemma 3.3 in [10] we have

$$(2.7) \quad |f(z)e^{-\varphi(z)}|^p \leq \frac{C}{\rho(z)^2} \int_{D^r(z)} |f(w)e^{-\varphi(w)}|^p dA(w), \quad z \in \mathbb{D}.$$

Hence,

$$\begin{aligned} \sum_{j=1}^\infty |K_z(z_j)e^{-\varphi(z_j)}|^q \rho(z_j)^2 &\leq C \sum_{j=1}^\infty \int_{D^r(z_j)} |K_z(\xi)e^{-\varphi(\xi)}|^q dA(\xi) \\ &\leq C \|K_z\|_{L_\varphi^q}^q. \end{aligned}$$

Then, for each  $N$ , Hölder's inequality implies

$$\begin{aligned} \sum_{j=1}^N |\lambda_j k_{z_j, p}(z)| &\leq \left( \sum_{j=1}^N |\lambda_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^N |k_{z_j, p}(z)|^q \right)^{\frac{1}{q}} \\ &\leq C \|\lambda\|_{\ell^p} \left( \sum_{j=1}^\infty |K_z(z_j)e^{-\varphi(z_j)}|^q \rho(z_j)^2 \right)^{\frac{1}{q}} \\ &\leq C \|\lambda\|_{\ell^p} \|K_z\|_{L_\varphi^q} < \infty. \end{aligned}$$

This implies that  $\sum_{j=1}^\infty \lambda_j k_{z_j, p}$  converges uniformly on compact subsets of  $\mathbb{D}$ . Furthermore, for any  $g \in A_\varphi^q$ ,

$$\begin{aligned} \sum_{j=1}^\infty \left| \langle \lambda_j k_{z_j, p}, g \rangle_{L_\varphi^2} \right| &= \sum_{j=1}^\infty \frac{|\lambda_j \overline{g(z_j)}|}{\|K_{z_j}\|_{L_\varphi^p}} \\ &\leq C \sum_{j=1}^\infty |\lambda_j| |g(z_j)e^{-\varphi(z_j)}| \rho(z_j)^{2-\frac{2}{p}} \\ &\leq C \|\lambda\|_{\ell^p} \left( \sum_{j=1}^\infty \int_{D^r(z_j)} |g(\xi)e^{-\varphi(\xi)}|^q dA(\xi) \right)^{\frac{1}{q}} \\ &\leq C \|\lambda\|_{\ell^p} \|g\|_{L_\varphi^q}. \end{aligned}$$

Therefore,

$$\left| \left\langle \sum_{j=1}^\infty \lambda_j k_{z_j, p}, g \right\rangle_{L_\varphi^2} \right| \leq \sum_{j=1}^\infty \left| \langle \lambda_j k_{z_j, p}, g \rangle_{L_\varphi^2} \right| \leq C \|\lambda\|_{\ell^p} \|g\|_{L_\varphi^q}.$$

Theorem 4.3 in [10] tells us that the dual of  $A_\varphi^p$  is  $A_\varphi^q$  for  $1 \leq p < \infty$  and the predual of  $A_\varphi^\infty$  is  $A_\varphi^1$ . From these we obtain (2.6) for  $1 \leq p \leq \infty$ .

For  $0 < p \leq 1$ , by  $(a+b)^p \leq a^p + b^p$  for  $a, b > 0$  we have

$$\left\| \sum_{j=1}^N \lambda_j k_{z_j, p} \right\|_{L_\varphi^p}^p \leq \sum_{j=1}^\infty |\lambda_j|^p \|k_{z_j, p}\|_{L_\varphi^p}^p = \|\lambda\|_{\ell^p}^p.$$

□

In our analysis, we are in need to use the notion of Carleson measures. Here is the definition.

**Definition 2.5.** *Suppose  $\mu$  is a positive Borel measure on  $\mathbb{D}$  and  $0 < p, q < \infty$ . If the embedding  $\text{Id} : A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi}d\mu)$  is continuous (or compact) then  $\mu$  is said to be a  $q$ -Carleson measure (or a vanishing  $q$ -Carleson measure) for  $A_\varphi^p$ .*

As on the classical Bergman spaces we are going to use  $\widehat{\mu}_r$  to characterize Carleson measures shown in the following proposition. For  $\varphi \in \mathcal{BDK}$ , the weight class introduced in [3], all conclusions in Lemma 2.6 except the estimate (2.9) were represented as Theorem 1 in [16] (although it is given there in a different form). Fortunately, the proof of that in [16] works well in the present setting with only one adjustment that the test function  $F_{a,n,p}(z)$  there should be replaced by

$$F_a(z) = k_{a,\infty}(z) \simeq \rho(a)^2 K_a(z) e^{-\varphi(a)},$$

because  $F_{a,n,p}(z)$  is available only when  $\varphi \in \mathcal{BDK}$  (particularly,  $\varphi$  must be radial), see [3] and [16].

Given  $\mu$  as above and  $0 < r \leq \alpha$ , set

$$\widehat{\mu}_r(z) = \frac{\mu(D^r(z))}{|D^r(z)|},$$

where  $|D^r(z)|$  denotes the area measure of  $D^r(z)$ . Notice that  $|D^r(z)| \simeq \rho(z)^2$ .

**Proposition 2.6.** *Let  $\mu$  be a positive Borel measure on  $\mathbb{D}$ .*

(A) *For  $0 < p \leq q < \infty$ ,  $\mu$  is a  $q$ -Carleson measure for  $A_\varphi^p$  if and only if*

$$\sup_{z \in \mathbb{D}} \widehat{\mu}_r(z) \rho(z)^{2(1-\frac{q}{p})} < \infty$$

*for some (or any)  $r \in (0, \alpha]$ . And  $\mu$  is a vanishing  $q$ -Carleson measure for  $A_\varphi^p$  if and only if*

$$\lim_{|z| \rightarrow 1} \widehat{\mu}_r(z) \rho(z)^{2(1-\frac{q}{p})} = 0$$

*for some (or any)  $r \in (0, \alpha]$ .*

(B) *For  $0 < q < p < \infty$ ,  $\mu$  is a  $q$ -Carleson measure for  $A_\varphi^p$  if and only if  $\mu$  is a vanishing  $q$ -Carleson measure for  $A_\varphi^p$  if and only if*

$$\widehat{\mu}_r \in L^{\frac{p}{p-q}}$$

*for some (or any)  $r \in (0, \alpha]$ .*

When  $\mu$  is a  $q$ -Carleson measure for  $A_\varphi^p$ , there holds

$$(2.8) \quad \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi}d\mu)} \simeq \left\| (\widehat{\mu}_r)^{\frac{1}{q}} \rho(z)^{2(\frac{1}{q}-\frac{1}{p})} \right\|_{L^\infty} \quad \text{if } 0 < p \leq q < \infty,$$

and

$$(2.9) \quad \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi}d\mu)} \simeq \left\| (\widehat{\mu}_r)^{\frac{1}{q}} \right\|_{L^{\frac{pq}{p-q}}} \quad \text{if } 0 < q < p < \infty.$$

*Proof.* We only present the proof of the estimate (2.9). For this purpose we first prove

$$(2.10) \quad \left\| (\widehat{\mu}_r)^{\frac{1}{q}} \right\|_{L^{\frac{pq}{p-q}}} \leq C \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi}d\mu)}.$$

As in [16] we use an argument of Luecking (see [14]). Let  $\{z_j\}_{j=1}^\infty$  be some  $(\rho, r)$ -lattice, and take  $\{\phi_j\}_{j=1}^\infty$  to be a sequence of Rademacher functions on  $[0, 1]$ . For  $\lambda = \{\lambda_j\}_{j=1}^\infty \in \ell^p$  consider the function  $G_t$  defined as

$$G_t(z) = \sum_{j=1}^{\infty} \lambda_j \phi_j(t) k_{z_j, p}(z).$$

From Lemma 2.4 we know  $\|G_t\|_{A_\varphi^p} \leq C\|\lambda\|_{\ell^p}$ . If  $\mu$  is a  $q$ -Carleson measure for  $A_\varphi^p$ , then

$$\int_{\mathbb{D}} |G_t(z)|^q e^{-q\varphi(z)} d\mu(z) \leq \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)}^q \|\lambda\|_{\ell^p}^q.$$

Integrating with respect to  $t$  from 0 to 1, applying Fubini's theorem, and invoking Khintchine's inequality we obtain

$$\int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} |\lambda_j|^2 |k_{z_j, p}(z)|^2 \right)^{\frac{q}{2}} e^{-q\varphi(z)} d\mu(z) \leq C \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)}^q \|\lambda\|_{\ell^p}^q.$$

On the other hand, by Lemmas 2.1-2.3 and (2.7), one gets

$$\begin{aligned} \int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} |\lambda_j|^2 |k_{z_j, p}(z)|^2 \right)^{\frac{q}{2}} e^{-q\varphi(z)} d\mu(z) &\geq C \sum_{k=1}^{\infty} \int_{D^r(z_k)} \left( \sum_{j=1}^{\infty} |\lambda_j|^2 |k_{z_j, p}(z)|^2 \right)^{\frac{q}{2}} e^{-q\varphi(z)} d\mu(z) \\ &\geq C \sum_{k=1}^{\infty} \int_{D^r(z_k)} |\lambda_k|^q |k_{z_k, p}(z)|^q e^{-q\varphi(z)} d\mu(z) \\ &\geq C \sum_{k=1}^{\infty} |\lambda_k|^q \rho(z_k)^{2-\frac{2q}{p}} \widehat{\mu}_r(z_k). \end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} |\lambda_k|^q \left( \rho(z_k)^{2-\frac{2q}{p}} \widehat{\mu}_r(z_k) \right) \leq C \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)}^q \|\lambda\|_{\ell^p}^q.$$

By the duality between  $\ell^{p/q}$  and  $\ell^{p/(p-q)}$  we have

$$(2.11) \quad \left( \sum_{k=1}^{\infty} \rho(z_k)^2 \widehat{\mu}_r(z_k)^{\frac{p}{p-q}} \right)^{\frac{p-q}{p}} \leq C \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)}^q.$$

Meanwhile, it is easy to verify that, for  $z \in D^r(z_k)$ ,

$$\rho(z_k)^2 \widehat{\mu}_r(z)^{\frac{p}{p-q}} \leq C \sum_{j: D^r(z_j) \cap D^r(z_k) \neq \emptyset} \rho(z_j)^2 \widehat{\mu}_r(z_j)^{\frac{p}{p-q}}.$$

Therefore,

$$\|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} \leq \sum_{k=1}^{\infty} \int_{D^r(z_k)} \widehat{\mu}_r(z)^{\frac{p}{p-q}} dA(z) \leq C \sum_{j=1}^{\infty} \rho(z_j)^2 \widehat{\mu}_r(z_j)^{\frac{p}{p-q}}.$$

This and (2.11) imply (2.10).

To prove the other direction, for  $f \in H(\mathbb{D})$  applying (2.7) and Hölder's inequality to obtain

$$\begin{aligned}
 & \int_{\mathbb{D}} |f(z)|^q e^{-q\varphi(z)} d\mu(z) \\
 & \leq \sum_{j=1}^{\infty} \int_{D^r(z_j)} |f(z)|^q e^{-q\varphi(z)} d\mu(z) \\
 & \leq C \sum_{j=1}^{\infty} \widehat{\mu}_r(z_j) \rho(z_j)^2 \sup_{\xi \in D^r(z_j)} |f(\xi)|^q e^{-q\varphi(\xi)} \\
 & \leq C \left( \sum_{j=1}^{\infty} \widehat{\mu}_r(z_j)^{\frac{p}{p-q}} \rho(z_j)^2 \right)^{\frac{p-q}{p}} \left( \sum_{j=1}^{\infty} \rho(z_j)^2 \sup_{\xi \in D^r(z_j)} |f(\xi)|^p e^{-p\varphi(\xi)} \right)^{\frac{q}{p}} \\
 & \leq C \left( \sum_{j=1}^{\infty} \widehat{\mu}_r(z_j)^{\frac{p}{p-q}} \rho(z_j)^2 \right)^{\frac{p-q}{p}} \left( \sum_{j=1}^{\infty} \int_{D^{2r}(z_j)} |f(\zeta)|^p e^{-p\varphi(\zeta)} dA(\zeta) \right)^{\frac{q}{p}} \\
 & \leq C \|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}} \|f\|_{L^p}^q.
 \end{aligned}$$

This means

$$\|\text{Id}\|_{A_{\varphi}^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)} \leq C \|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}}^{\frac{1}{q}} = C \left\| (\widehat{\mu}_r)^{\frac{1}{q}} \right\|_{L^{\frac{pq}{p-q}}}.$$

From this and (2.10) we obtain (2.9).  $\square$

### 3. SOME $\bar{\partial}$ -ESTIMATES

By Lemma 2.1 and Lemma 2.2 (iii) we have some  $\alpha > 0$  such that  $K_z(\xi) = K(\xi, z)$  does not vanish for  $\xi \in D^\alpha(z)$ . Given any  $r \in (0, \alpha/3]$  and a  $(\rho, r)$ -lattice  $\{z_j\}_{j=1}^{\infty}$ , let  $\{\psi_j\}_{j=1}^{\infty}$  be some partition of unity subordinate to the covering  $\{D^r(z_j)\}_{j=1}^{\infty}$ . Precisely,

$$\psi_j \in C^\infty(\mathbb{D}), \text{ Supp } \psi_j \subset D^r(z_j) \text{ and } \psi_j \geq 0, \sum_{j=1}^{\infty} \psi_j = 1.$$

Set

$$G(z, \xi) = \frac{1}{(\xi - z)\rho(\xi)} \sum_{j=1}^{\infty} \frac{K_{z_j}(z)\psi_j(\xi)}{K_{z_j}(\xi)}.$$

Define an integral operator  $T$  as

$$T(f)(z) = \int_{\mathbb{D}} G(z, \xi) f(\xi) dA(\xi).$$

**Lemma 3.1.** *Let  $\varphi \in \mathcal{W}_0$  and  $1 \leq p \leq \infty$ . Then  $T$  is a bounded linear operator on  $L_{\varphi}^p$ .*

*Proof.* We will use interpolation to prove this lemma. By (2.7), and by Lemma 2.1, 2.2 we have

$$\begin{aligned}
 \sum_{j=1}^{\infty} \frac{|K_{z_j}(z)|\psi_j(\xi)}{|K_{z_j}(\xi)|} & \simeq \rho(\xi)^2 e^{-\varphi(z_j) - \varphi(\xi)} \sum_{j \in \{k: \xi \in D^r(z_k)\}} |K_z(z_j)|\psi_j(\xi) \\
 & \leq C e^{-\varphi(\xi)} \int_{D^{2r}(\xi)} |K_z(\zeta)| e^{-\varphi(\zeta)} dA(\zeta).
 \end{aligned}$$

Write

$$(3.1) \quad Q(z, \xi) = \frac{e^{-\varphi(\xi)}}{|\xi - z|\rho(\xi)} \int_{D^{2r}(\xi)} |K_{\zeta}(z)| e^{-\varphi(\zeta)} dA(\zeta).$$

We have

$$(3.2) \quad |G(z, \xi)| \leq C Q(z, \xi).$$

For  $f$  measurable on  $\mathbb{D}$ , set

$$T_1(f)(z) = \int_{D^r(z)} Q(z, \xi) f(\xi) dA(\xi)$$

and

$$T_2(f)(z) = \int_{\mathbb{D} \setminus D^r(z)} Q(z, \xi) f(\xi) dA(\xi).$$

To prove the conclusion of the lemma, from (3.2) we need only to prove that both  $T_1$  and  $T_2$  are bounded on  $L^p_\varphi$ . For  $T_1$ , by Lemma 2.2, we have

$$\begin{aligned} \|T_1(f)\|_{L^1_\varphi} &\leq \int_{\mathbb{D}} \left( \int_{\mathbb{D}} \chi_{D^r(z)}(\xi) Q(z, \xi) |f(\xi)| dA(\xi) \right) e^{-\varphi(z)} dA(z) \\ &= \int_{\mathbb{D}} |f(\xi)| \left( \int_{\mathbb{D}} \chi_{D^r(z)}(\xi) Q(z, \xi) e^{-\varphi(z)} dA(z) \right) dA(\xi) \\ &\leq \int_{\mathbb{D}} |f(\xi)| \left( \int_{D^{2r}(\xi)} Q(z, \xi) e^{-\varphi(z)} dA(z) \right) dA(\xi). \end{aligned}$$

Putting the expression of  $Q(z, \xi)$  inside and using (2.1), we obtain

$$\begin{aligned} \|T_1(f)\|_{L^1_\varphi} &\leq \int_{\mathbb{D}} |f(\xi)| e^{-\varphi(\xi)} \left( \int_{D^{2r}(\xi)} \frac{e^{-\varphi(z)}}{|\xi - z| \rho(\xi)} \int_{D^{2r}(\xi)} |K_\zeta(z)| e^{-\varphi(\zeta)} dA(\zeta) dA(z) \right) dA(\xi) \\ &\leq \int_{\mathbb{D}} |f(\xi)| e^{-\varphi(\xi)} \left( \int_{D^{2r}(\xi)} \frac{e^{-\varphi(z)}}{|\xi - z| \rho(\xi)} \|K_z\|_{L^1_\varphi} dA(z) \right) dA(\xi) \\ &\leq C \int_{\mathbb{D}} |f(\xi)| e^{-\varphi(\xi)} dA(\xi) \int_{D^{2r}(\xi)} \frac{1}{|\xi - z| \rho(\xi)} dA(z). \end{aligned}$$

Using polar coordinates, it is easy to see that

$$\int_{D^{2r}(\xi)} \frac{1}{|\xi - z|} dA(z) \leq C \rho(\xi),$$

so that, we finally obtain

$$\|T_1(f)\|_{L^1_\varphi} \leq C \int_{\mathbb{D}} |f(\xi)| e^{-\varphi(\xi)} dA(\xi) = C \|f\|_{L^1_\varphi},$$

proving that  $T_1$  is bounded on  $L^1_\varphi$ . Similarly,

$$\begin{aligned} \|T_1(f)\|_{L^\infty_\varphi} &= \sup_{z \in \mathbb{D}} e^{-\varphi(z)} \int_{D^r(z)} Q(z, \xi) |f(\xi)| dA(\xi) \\ &\leq C \|f\|_{L^\infty_\varphi} \sup_{z \in \mathbb{D}} \int_{D^r(z)} \frac{1}{|\xi - z| \rho(\xi)} \int_{D^{2r}(\xi)} |K_\zeta(z)| e^{-\varphi(z) - \varphi(\zeta)} dA(\zeta) dA(\xi) \\ &\leq C \|f\|_{L^\infty_\varphi} \sup_{z \in \mathbb{D}} \int_{D^r(z)} \frac{1}{|\xi - z| \rho(\xi)} dA(\xi) \\ &\leq C \|f\|_{L^\infty_\varphi}. \end{aligned}$$

Set  $M_{e^\varphi}$  to be the multiplier that  $M_{e^\varphi}(f) = f e^\varphi$ . It is easy to see  $M_{e^\varphi}$  is an isometry from  $L^p$  to  $L^p_\varphi$  with the inverse  $M_{e^{-\varphi}}$ . Therefore,  $M_{e^{-\varphi}} T_1 M_{e^\varphi}$  is bounded both on  $L^1$  and  $L^\infty$ . By interpolation,  $M_{e^{-\varphi}} T_1 M_{e^\varphi}$  is bounded on  $L^p$  which implies  $T_1$  is bounded on  $L^p_\varphi$ .



For  $T_2$ , applying Lemma 2.1, we have

$$\begin{aligned} |T_2 f(z)| &\leq \int_{\mathbb{D} \setminus D^r(z)} \frac{|f(\xi)| e^{-\varphi(\xi)}}{|\xi - z| \rho(\xi)} \left( \int_{D^{2r}(\xi)} |K(\zeta, z)| e^{-\varphi(\zeta)} dA(\zeta) \right) dA(\xi) \\ &\leq C \frac{e^{\varphi(z)}}{\rho(z)} \int_{\mathbb{D} \setminus D^r(z)} \frac{|f(\xi)| e^{-\varphi(\xi)}}{|\xi - z| \rho(\xi)} \left( \int_{D^{2r}(\xi)} \frac{e^{-\sigma d_\rho(\zeta, z)} dA(\zeta)}{\rho(\zeta)} \right) dA(\xi) \\ &\leq C \frac{e^{\varphi(z)}}{\rho(z)^2} \int_{\mathbb{D} \setminus D^r(z)} \frac{|f(\xi)| e^{-\varphi(\xi)}}{\rho(\xi)^2} \left( \int_{D^{2r}(\xi)} e^{-\sigma d_\rho(\zeta, z)} dA(\zeta) \right) dA(\xi). \end{aligned}$$

On the other hand,  $d_\rho(\cdot, \cdot)$  is a distance on  $\mathbb{D}$ . From Lemma 2.2, part (iii), there is some constant  $C$  such that, for  $\zeta \in D^{2r}(\xi)$ ,

$$d_\rho(\xi, z) \leq d_\rho(\xi, \zeta) + d_\rho(\zeta, z) \leq C + d_\rho(\zeta, z).$$

Thus, for  $\zeta \in D^{2r}(\xi)$ , we have  $e^{-\sigma d_\rho(\zeta, z)} \leq C e^{-\sigma d_\rho(\xi, z)}$ . It follows that

$$|T_2 f(z)| \leq C \frac{e^{\varphi(z)}}{\rho(z)^2} \int_{\mathbb{D} \setminus D^r(z)} |f(\xi)| e^{-\varphi(\xi)} e^{-\sigma d_\rho(\xi, z)} dA(\xi).$$

With this estimate and [10, Corollary 3.1] we obtain

$$\begin{aligned} \|T_2 f\|_{L_\varphi^1} &\leq C \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |f(\xi)| e^{-\varphi(\xi)} e^{-\sigma d_\rho(\xi, z)} dA(\xi) \right) \frac{dA(z)}{\rho(z)^2} \\ &= C \int_{\mathbb{D}} |f(\xi)| e^{-\varphi(\xi)} \left( \int_{\mathbb{D}} \frac{e^{-\sigma d_\rho(\xi, z)}}{\rho(z)^2} dA(z) \right) dA(\xi) \\ &\leq C \|f\|_{L_\varphi^1}. \end{aligned}$$

Similarly, for  $p = \infty$  we have

$$\begin{aligned} \|T_2 f\|_{L_\varphi^\infty} &\leq \sup_{z \in \mathbb{D}} \frac{1}{\rho(z)^2} \int_{\mathbb{D} \setminus D^r(z)} |f(\xi)| e^{-\varphi(\xi)} e^{-\sigma d_\rho(\xi, z)} dA(\xi) \\ &\leq C \|f\|_{L_\varphi^\infty} \sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{e^{-\sigma d_\rho(\xi, z)}}{\rho(z)^2} dA(\xi) \\ &\leq C \|f\|_{L_\varphi^\infty}. \end{aligned}$$

With the same approach for  $T_1$ , by interpolation we know that  $T_2$  is bounded on  $L_\varphi^p$  as well. □

Set  $C_c^\infty$  to be the family of all  $C^\infty$  functions with compact support in  $\mathbb{D}$ . Given  $f$  Lebesgue measurable on  $\mathbb{D}$ , for  $z = x + iy$  one can define the weak derivative  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ , see [4]. Set  $\frac{\partial f}{\partial z} = \frac{1}{2} \left\{ \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right\}$  and  $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left\{ \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right\}$ . Since we deal with functions of one complex variable, we can use  $\bar{\partial} f$  to stand for  $\frac{\partial f}{\partial \bar{z}}$  for short.

**Theorem 3.2.** *Let  $\varphi \in \mathcal{W}_0$ . Given  $f$  be measurable on  $\mathbb{D}$  such that  $\rho f \in L_\varphi^1$ , set*

$$(3.3) \quad u(z) = \sum_{j=1}^{\infty} K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z) K_{z_j}(\xi)} f(\xi) dA(\xi).$$

*Then  $u$  solves the equation  $\bar{\partial} u = f$  weakly in  $\mathbb{D}$ . Furthermore, for  $1 \leq p < \infty$  there is some constant  $C > 0$  such that*

$$(3.4) \quad \|u\|_{L_\varphi^p} \leq C \|\rho f\|_{L_\varphi^p}.$$

*Proof.* For a function  $f$  with  $\rho f \in L^p_\varphi$ , one has  $u(z) = T(f\rho)(z)$ . Then Lemma 3.1 implies  $\|u\|_{L^p_\varphi} \leq C\|f\rho\|_{L^p_\varphi}$  which gives (3.4).

For  $f \in C^1(\mathbb{D})$ , Cauchy-Pompeiu formula tells us that (see Theorem 2.1.2 from [4])

$$(3.5) \quad \frac{\partial}{\partial \bar{z}} \int_{\mathbb{D}} \frac{f(\xi)}{\xi - z} dA(\xi) = f(z) \quad \text{for } z \in \mathbb{D}.$$

Then for  $\phi \in C_c^\infty(\mathbb{D})$  and  $f \in L^1_{loc}$ , (3.5) and the fact that  $K_{z_j} \in H(\mathbb{D})$  imply

$$\left\langle K_{z_j}(\cdot) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - \cdot)K_{z_j}(\xi)} f(\xi) dA(\xi), \frac{\partial \phi}{\partial z} \right\rangle_{L^2} = -\langle f\psi_j, \phi \rangle_{L^2}.$$

Set

$$U(z) = \sum_{j=1}^{\infty} |K_{z_j}(z)| \int_{\mathbb{D}} \frac{\psi_j(\xi)}{|(\xi - z)K_{z_j}(\xi)|} |f(\xi)| dA(\xi).$$

We have

$$|u(z)| \leq U(z).$$

By the fact that  $\text{Supp } \psi_j \subset D^r(z_j)$ , applying Lemma 2.1 and Corollary 3.1 from [10] to get

$$\begin{aligned} U(z) &\leq C \sum_{j=1}^{\infty} \frac{e^{\varphi(z_j)+\varphi(z)}}{\rho(z_j)\rho(z)} \int_{D^r(z_j)} \frac{\psi_j(\xi)}{|\xi - z|} |f(\xi)| \frac{\rho(z_j)\rho(\xi)}{e^{\varphi(z_j)+\varphi(\xi)}} dA(\xi) \\ &\leq C \sum_{j=1}^{\infty} \frac{e^{\varphi(z)}}{\rho(z)} \int_{D^r(z_j)} \frac{\psi_j(\xi)}{|\xi - z|} |\rho(\xi)f(\xi)e^{-\varphi(\xi)}| dA(\xi) \\ &= C \sum_{j=1}^{\infty} \frac{e^{\varphi(z)}}{\rho(z)} \int_{\mathbb{D}} \frac{\psi_j(\xi)}{|\xi - z|} |\rho(\xi)f(\xi)e^{-\varphi(\xi)}| dA(\xi). \end{aligned}$$

Write  $\Omega = \text{Supp } \phi$  which is compact. Then,

$$\begin{aligned} &\int_{\mathbb{D}} U(z) \left| \frac{\partial \phi}{\partial z}(z) \right| dA(z) \\ &\leq C \int_{\Omega} \left| \frac{\partial \phi}{\partial z}(z) \right| dA(z) \sum_{j=1}^{\infty} \frac{e^{\varphi(z)}}{\rho(z)} \int_{\mathbb{D}} \frac{\psi_j(\xi)}{|\xi - z|} |\rho(\xi)f(\xi)e^{-\varphi(\xi)}| dA(\xi) \\ &\leq C \left\| \frac{e^{\varphi(z)}\bar{\partial}\phi}{\rho(z)} \right\|_{L^\infty(\Omega)} \int_{\mathbb{D}} |\rho(\xi)f(\xi)e^{-\varphi(\xi)}| dA(\xi) \sum_{j=1}^{\infty} \psi_j(\xi) \int_{\Omega} \frac{1}{|\xi - z|} dA(z) \\ &\leq C \left\| \frac{e^{\varphi(z)}\bar{\partial}\phi}{\rho(z)} \right\|_{L^\infty(\Omega)} \int_{\mathbb{D}} |\rho(\xi)f(\xi)e^{-\varphi(\xi)}| \sum_{j=1}^{\infty} \psi_j(\xi) dA(\xi) \\ &\leq C \left\| \frac{e^{\varphi(z)}\bar{\partial}\phi}{\rho(z)} \right\|_{L^\infty(\Omega)} \int_{\mathbb{D}} |\rho(\xi)f(\xi)e^{-\varphi(\xi)}| dA(\xi) \\ &< \infty. \end{aligned}$$

Hence, we can apply Fubini's theorem to obtain

$$\begin{aligned} \int_{\mathbb{D}} u(z) \frac{\partial \bar{\phi}}{\partial z}(z) dA(z) &= \int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} f(\xi) dA(\xi) \right) \frac{\partial \bar{\phi}}{\partial z}(z) dA(z) \\ &= \sum_{j=1}^{\infty} \int_{\mathbb{D}} \left( K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} f(\xi) dA(\xi) \right) \frac{\partial \bar{\phi}}{\partial z}(z) dA(z). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\langle u, \frac{\partial \phi}{\partial z} \right\rangle_{L^2} &= \sum_{j=1}^{\infty} \left\langle K_{z_j}(\cdot) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - \cdot)K_{z_j}(\xi)} f(\xi) dA(\xi), \frac{\partial \phi}{\partial z} \right\rangle_{L^2} \\ &= - \sum_{j=1}^{\infty} \langle f \psi_j, \phi \rangle_{L^2} \\ &= - \langle f, \phi \rangle_{L^2}. \end{aligned}$$

With this we know  $\frac{\partial u}{\partial \bar{z}} = f$  weakly. □

#### 4. HANKEL OPERATORS FROM $A_{\varphi}^p$ TO $L_{\varphi}^q$

Recall that

$$\Gamma = \left\{ \sum_{j=1}^N a_j K_{z_j} : N \in \mathbb{N}, a_j \in \mathbb{C}, z_j \in \mathbb{D}, \text{ for } 1 \leq j \leq N, \right\}$$

and

$$\mathcal{S} = \{f \text{ measurable on } \mathbb{D} : fg \in L_{\varphi}^1 \text{ for } g \in \Gamma\}.$$

Corollary 4.2 from [10] tells us that  $\Gamma$  is dense in  $A_{\varphi}^p$  for all  $0 < p < \infty$ . Hence, for  $f \in \mathcal{S}$  the Hankel operator  $H_f$  is densely defined on  $A_{\varphi}^p$ . Therefore, a function  $f \in \mathcal{S}$  can be used as the symbol to define a Hankel operator on  $A_{\varphi}^p$ .

The following lemma sets up a bridge between Hankel operators and the solution to  $\bar{\partial}$ -equation in Theorem 3.2.

**Lemma 4.1.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and suppose that  $f \in \mathcal{S}$  with  $\rho \bar{\partial} f \in \mathcal{S}$ , where the derivative is in the weak sense. Then for  $g \in \Gamma$  there holds*

$$(4.1) \quad H_f(g) = u - P(u),$$

where

$$(4.2) \quad u(z) = \sum_{j=1}^{\infty} K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} g(\xi) \bar{\partial} f(\xi) dA(\xi).$$

*Proof.* Since  $\rho \bar{\partial} f \in \mathcal{S}$ , for  $g \in \Gamma$  we have  $g\rho \bar{\partial} f \in L_{\varphi}^1$ . For  $u$  defined as in (4.2), Theorem 3.2 implies  $u \in L_{\varphi}^p$  with

$$(4.3) \quad \|u\|_{L_{\varphi}^p} \leq C \|g(\rho \bar{\partial} f)\|_{L_{\varphi}^p}.$$

Meanwhile,  $fg \in L_{\varphi}^1$  for  $g \in \Gamma$ . Then,  $fg - u \in L_{\varphi}^1$ , and Theorem 3.2 tells us that  $\bar{\partial}(fg - u) = g\bar{\partial}f - \bar{\partial}u = 0$ , showing that  $fg - u \in A_{\varphi}^1$ . Since  $P|_{A_{\varphi}^1} = \text{Id}$ , we have

$$P(fg - u) = fg - u.$$

Therefore,

$$H_f(g) - (u - P(u)) = fg - P(fg) - (u - P(u)) = (fg - u) - P(fg - u) = 0,$$

from which (4.1) follows. □

To characterize the boundedness (or compactness) of Hankel operators  $H_f$ , we need an auxiliary function  $G_{q,r}(f)$  which is an analogue of the one first introduced in [13], when

Luecking studied Hankel operators on the standard Bergman space  $A^p$ . Let  $q \geq 1$  and  $0 < r \leq \alpha$ . For  $f \in L^q_{loc}$  we define  $G_{q,r}(f)$  to be

$$G_{q,r}(f)(z) = \inf \left\{ \left( \frac{1}{|D^r(z)|} \int_{D^r(z)} |f - h|^q dA \right)^{\frac{1}{q}} : h \in H(D^r(z)) \right\}, \quad z \in \mathbb{D}.$$

For  $f \in L^1_{loc}(\mathbb{D})$ ,  $1 \leq q < \infty$  and  $0 < r \leq \alpha$ , write

$$M_{q,r}(f)(z) = \left\{ \frac{1}{|D^r(z)|} \int_{D^r(z)} |f|^q dA \right\}^{\frac{1}{q}}$$

to be the  $q$ -th mean of  $|f|$  over  $D^r(z)$ .

Our analysis on the Hankel operator going from  $A^p_\varphi$  to  $L^q_\varphi$  will be carried out in two cases:  $1 \leq p \leq q < \infty$  and  $1 \leq q < p < \infty$ .

**Theorem 4.2.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and let  $1 \leq p \leq q < \infty$ . Set  $s = \frac{1}{q} - \frac{1}{p}$ . Then for  $f \in \mathcal{S}$ , the following statements are equivalent:*

- (A)  $H_f : A^p_\varphi \rightarrow L^q_\varphi$  is bounded;
- (B) For some (or any)  $0 < r \leq \alpha$ ,  $\rho^{2s} G_{q,r}(f) \in L^\infty$ ;
- (C)  $f$  admits a decomposition  $f = f_1 + f_2$ , where  $f_1 \in C^1(\mathbb{D})$  satisfying

$$(4.4) \quad \rho^{2s+1} |\bar{\partial} f_1| \in L^\infty,$$

and  $f_2$  has the property that, for some (or any)  $0 < r \leq \alpha$ ,

$$(4.5) \quad \rho^{2s} M_{q,r}(f_2) \in L^\infty.$$

Furthermore, for  $0 < r \leq \alpha$ ,

$$(4.6) \quad \|H_f\|_{A^p_\varphi \rightarrow L^q_\varphi} \simeq \|\rho^{2s} G_{q,r}(f)\|_{L^\infty}.$$

*Proof.* (A)  $\Rightarrow$  (B). For  $\alpha$  as in Lemma 2.2, Lemma 2.1 tells us that there is some constant  $C > 0$  such that

$$\inf_{\xi \in D^\alpha(z)} |k_{z,p}(\xi)| \geq C \rho(z)^{-\frac{2}{p}} e^{\varphi(\xi)} > 0 \quad \text{for } z \in \mathbb{D}.$$

Then,  $\frac{1}{k_{z,p}} P(fk_{z,p}) \in H(D^r(z))$ , and

$$(4.7) \quad \begin{aligned} \|H_f(k_{z,p})\|_{L^q_\varphi}^q &= \int_{\mathbb{D}} |fk_{z,p}(\xi) - P(fk_{z,p})(\xi)|^q e^{-q\varphi(\xi)} dA(\xi) \\ &\geq \int_{D^r(z)} |k_{z,p}(\xi)|^q \left| f(\xi) - \frac{1}{k_{z,p}(\xi)} P(fk_{z,p})(\xi) \right|^q e^{-q\varphi(\xi)} dA(\xi) \\ &\geq C \rho(z)^{-\frac{2q}{p}} \int_{D^r(z)} \left| f(\xi) - \frac{1}{k_{z,p}(\xi)} P(fk_{z,p})(\xi) \right|^q dA(\xi) \\ &\geq C \{ \rho(z)^{2s} G_{q,r}(f)(z) \}^q. \end{aligned}$$

On the other hand,  $\|H_f(k_{z,p})\|_{L^q_\varphi}^q \leq \|H_f\|_{A^p_\varphi \rightarrow L^q_\varphi}^q \|k_{z,p}\|_{L^p_\varphi}^q = \|H_f\|_{A^p_\varphi \rightarrow L^q_\varphi}^q$ . Therefore, we have

$$(4.8) \quad \rho(z)^{2s} G_{q,r}(f)(z) \leq C \|H_f\|_{A^p_\varphi \rightarrow L^q_\varphi} \quad \text{for all } z \in \mathbb{D}.$$

From this, the statement (B) follows.

(B)  $\Rightarrow$  (C). Suppose  $\|\rho^{2s} G_{q,r}(f)\|_{L^\infty} < \infty$  for some  $r \in (0, \alpha]$ . Fix a  $(\rho, \frac{r}{2})$ -lattice  $\{z_j\}_{j=1}^\infty$ , and take  $\{\psi_j\}_{j=1}^\infty$  to be a partition of the unity subordinate to  $\{D^{\frac{r}{2}}(z_j)\}_{j=1}^\infty$ ,

satisfying  $\rho(z_j)|\bar{\partial}\psi_j| \leq C$  for  $j = 1, 2, \dots$ . With a normal family argument we may find some function  $h_j \in H(D^r(z_j))$  such that

$$(4.9) \quad \frac{1}{|D^r(z_j)|} \int_{D^r(z_j)} |f - h_j|^q dA = G_{q,r}^q(f)(z_j), \quad j = 1, 2, \dots$$

Set

$$f_1(z) = \sum_{j=1}^{\infty} h_j(z)\psi_j(z) \in C^\infty(\mathbb{D})$$

and  $f_2 = f - f_1$ . Define  $J_z = \{j : z \in D^r(z_j)\}$ . Then,  $\rho(z_j) \simeq \rho(z)$  for  $j \in J_z$ , and

$$(4.10) \quad |J_z| := \sum_{j=1}^{\infty} \chi_{D^r(z_j)}(z) \leq C.$$

As that on pages 254-255 in [14], for  $z \in \mathbb{D}$  there holds

$$(4.11) \quad \rho(z) |\bar{\partial}f_1(z)| \leq C \sum_{j \in J_z} G_{q,r}(f)(z_j).$$

This implies

$$(4.12) \quad \rho(z)^{2s+1} |\bar{\partial}f_1(z)| \leq C \|\rho^{2s} G_{q,r}(f)\|_{L^\infty} \quad \text{for } z \in \mathbb{D}.$$

On the other hand,  $f_2(z) = \sum_{j=1}^{\infty} (f(z) - h_j(z))\psi_j(z)$ , and by (2.5) only at most  $N$  terms are not zero in this summation. Hölder's inequality implies

$$|f_2(z)|^q \leq C \sum_{j=1}^{\infty} |f(z) - h_j(z)|^q \psi_j(z).$$

Then, by (4.9),

$$(4.13) \quad \begin{aligned} M_{q,r}(f_2)(z) &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{|D^r(z)|} \int_{D^r(z)} |f - h_j|^q \psi_j dA \right)^{\frac{1}{q}} \\ &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{|D^r(z)|} \int_{D^r(z) \cap D^{r/2}(z_j)} |f - h_j|^q dA \right)^{\frac{1}{q}} \\ &\leq C \sum_{j \in J_z} G_{q,r}(f)(z_j). \end{aligned}$$

Hence,

$$(4.14) \quad \rho(z)^{2s} M_{q,r}(f_2)(z) \leq C \|\rho^{2s} G_{q,r}(f)\|_{L^\infty} \quad \text{for } z \in \mathbb{D}.$$

Something more, the condition (4.5) is independent of  $r \in (0, \alpha]$ . We reach the condition (C) from (4.12) and (4.14).

(C)  $\Rightarrow$  (A). If we set  $d\mu = |f_2|^q dA$ , then

$$(4.15) \quad \widehat{\mu}_r(z)^{\frac{1}{q}} = M_{q,r}(f_2)(z).$$

The assumption (4.5) and Proposition 2.6, imply that  $\mu$  is a  $q$ -Carleson measure for  $A_\varphi^p$  with  $\|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)} \simeq \|\rho^{2s} M_{q,r}(f_2)\|_{L^\infty}$ . By the boundedness of the Bergman

projection on  $L_\varphi^q$ ,

$$\begin{aligned}
(4.16) \quad \|H_{f_2}g\|_{L_\varphi^q} &\leq C \|f_2g\|_{L_\varphi^q} \\
&= C \left( \int_{\mathbb{D}} |g|^q e^{-q\varphi} d\mu \right)^{\frac{1}{q}} \\
&\leq C \|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)} \|g\|_{L_\varphi^p} \\
&\leq C \|\rho^{2s} M_{q,r}(f_2)\|_{L^\infty} \|g\|_{L_\varphi^p}.
\end{aligned}$$

Next, we suppose that  $f_1$  satisfies (4.4). With the fact that  $s \leq 0$  and  $\rho^{2s+1}\bar{\partial}f \in L^\infty$ , we know  $\rho|\bar{\partial}f_1| \in L^\infty$ . Now, for  $g \in \Gamma$ , take  $u$  as in (4.1) so that

$$u(z) = \sum_{j=1}^{\infty} K_{z_j}(z) \int_{\mathbb{D}} \frac{\psi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} g(\xi) \bar{\partial}f_1(\xi) dA(\xi).$$

Theorem 3.2 and Lemma 4.1 tell us

$$H_{f_1}(g) = u - P(u) \quad \text{and} \quad \|u\|_{L_\varphi^q} \leq C \|g(\rho\bar{\partial}f_1)\|_{L_\varphi^q}.$$

From the boundedness of  $P$  on  $L_\varphi^q$  we obtain

$$(4.17) \quad \|H_{f_1}g\|_{L_\varphi^q} \leq (1 + \|P\|_{L_\varphi^q \rightarrow L_\varphi^q}) \|u\|_{L_\varphi^q} \leq C \|g(\rho\bar{\partial}f_1)\|_{L_\varphi^q}.$$

Meanwhile, if we consider the measure  $d\nu = [\rho|\bar{\partial}f_1|]^q dA$ , it is easy to see that

$$(4.18) \quad \widehat{\nu}_r(z)^{\frac{1}{q}} \leq C \sup_{\xi \in D^r(z)} \rho(\xi) |\bar{\partial}f_1(\xi)|.$$

Hence,  $\rho(z)^{2s} \widehat{\nu}_r(z)^{\frac{1}{q}} \leq C \|\rho^{2s+1} |\bar{\partial}f_1|\|_{L^\infty}$ . It follows from (4.4) and Proposition 2.6 that  $\nu$  is a  $q$ -Carleson measure for  $A_\varphi^p$  with  $\|\text{Id}\|_{A_\varphi^p \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\nu)} \simeq \|\rho^{2s+1} |\bar{\partial}f_1|\|_{L^\infty}$ . Then

$$\|g(\rho\bar{\partial}f_1)\|_{L_\varphi^q} \leq C \|\rho^{2s+1} |\bar{\partial}f_1|\|_{L^\infty} \cdot \|g\|_{L_\varphi^p}.$$

Hence,

$$\|H_{f_1}g\|_{L_\varphi^q} \leq C \|\rho^{2s+1} |\bar{\partial}f_1|\|_{L^\infty} \cdot \|g\|_{L_\varphi^p}.$$

With this and (4.16), we obtain

$$(4.19) \quad \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q} \leq C \left\{ \|\rho^{2s+1} |\bar{\partial}f_1|\|_{L^\infty} + \|\rho^{2s} M_{q,r}(f_2)\|_{L^\infty} \right\}.$$

This gives the implication (C)  $\Rightarrow$  (A) finishing the proof of the equivalence among (A), (B) and (C). The norm estimates (4.6) come from (4.8), (4.12), (4.14) and (4.19).  $\square$

The next result describes the compactness of  $H_f$  when  $p \leq q$ . For  $q \geq 1$ , we understand that  $H_f : A_\varphi^p \rightarrow L_\varphi^q$  is compact if and only if whenever  $\{g_m\}_{m=1}^\infty$  is a bounded sequence in  $A_\varphi^p$  converging to zero on compact subsets of  $\mathbb{D}$ , it follows that  $\|H_f g_m\|_{L_\varphi^q}$  tends to zero.

**Theorem 4.3.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and let  $1 \leq p \leq q < \infty$ . Set  $s = \frac{1}{q} - \frac{1}{p}$ . Then for  $f \in \mathcal{S}$ , the following statements are equivalent:*

- (A)  $H_f : A_\varphi^p \rightarrow L_\varphi^q$  is compact;
- (B) For some (or any)  $0 < r \leq \alpha$ ,  $\lim_{|z| \rightarrow 1} \rho^{2s} G_{q,r}(f)(z) = 0$ .
- (C)  $f$  admits a decomposition  $f = f_1 + f_2$ , where  $f_1 \in C^1(\mathbb{D})$  satisfying

$$(4.20) \quad \lim_{|z| \rightarrow 1} \rho(z)^{2s+1} |\bar{\partial}f_1(z)| = 0,$$

and

$$(4.21) \quad \lim_{|z| \rightarrow 1} \rho(z)^{2s} M_{q,r}(f_2)(z) = 0$$

for some (or any)  $0 < r \leq \alpha$ .

*Proof.* Let  $H_f$  be compact from  $A_\varphi^p$  to  $L_\varphi^q$ . It is easy to see that  $\{k_{z,p} : z \in \mathbb{D}\}$  tends to 0 weakly in  $A_\varphi^p$  as  $|z| \rightarrow 1$ . Then, for  $0 < r \leq \alpha$  fixed, from (4.7) we have

$$\rho(z)^{2s} G_{q,r}(f)(z) \leq C \|H_f(k_{z,p})\|_{L_\varphi^q} \rightarrow 0$$

as  $|z| \rightarrow 1$ . So, (A) implies (B).

Suppose now that (B) is holds for some  $r \in (0, \alpha]$ . From (4.11) and (4.13) we know

$$\rho(z)^{2s+1} |\bar{\partial} f_1(z)| \leq C \sum_{j \in J_z} \rho(z_j)^{2s} G_{q,r}(f)(z_j)$$

and

$$\rho(z)^{2s} M_{q,r}(f_2)(z) \leq C \sum_{j \in J_z} \rho(z_j)^{2s} G_{q,r}(f)(z_j).$$

From these estimates, the statement (C) follows easily.

Finally, we prove the implication (C)  $\Rightarrow$  (A). As in the proof of Theorem 4.2, we know that both  $d\mu = |f_2|^q dA$  and  $d\nu = [\rho |\bar{\partial} f_1|]^q dA$  are vanishing  $q$ -Carleson measures for  $A_\varphi^p$ . **With (2.7) we know that the unit ball of  $A_\varphi^p$  is a normal family.** Then, for any bounded sequence  $\{g_m\}$  in  $A_\varphi^p$  converging to zero uniformly on compact subsets of  $\mathbb{D}$ , we have

$$\|H_{f_2}(g_m)\|_{L_\varphi^q} \leq C \left( \int_{\mathbb{D}} |f_2|^q |g_m|^q e^{-q\varphi} dA \right)^{\frac{1}{q}} \rightarrow 0,$$

and, by (4.17),

$$\|H_{f_1}(g_m)\|_{L_\varphi^q} \leq C \|(\rho |\bar{\partial} f_1|) g_m\|_{L_\varphi^q} \rightarrow 0.$$

Then,  $\lim_{m \rightarrow \infty} \|H_f(g_m)\|_{L_\varphi^q} = 0$ , and this tells us that  $H_f$  is compact from  $A_\varphi^p$  to  $L_\varphi^q$ .  $\square$

Next, we proceed to characterize the boundedness and compactness in the case that  $1 \leq q < p < \infty$ .

**Theorem 4.4.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and let  $1 \leq q < p < \infty$ . Set  $s = \frac{1}{q} - \frac{1}{p}$ . Then for  $f \in \mathcal{S}$ , the following statements are equivalent.*

- (A)  $H_f : A_\varphi^p \rightarrow L_\varphi^q$  is bounded.
- (B)  $H_f : A_\varphi^p \rightarrow L_\varphi^q$  is compact.
- (C) For some (or any)  $0 < r \leq \frac{\alpha}{2}$ ,  $G_{q,r}(f)(z) \in L^{\frac{1}{s}}$ .
- (D)  $f$  admits a decomposition  $f = f_1 + f_2$ , where

$$(4.22) \quad f_1 \in C^1(\mathbb{D}), \quad \rho |\bar{\partial} f_1| \in L^{\frac{1}{s}}, \quad \text{and } M_{q,r}(f_2) \in L^{\frac{1}{s}}$$

for some (or any)  $0 < r \leq \alpha$ .

Furthermore, for  $0 < r \leq \frac{\alpha}{2}$  fixed,

$$(4.23) \quad \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q} \simeq \|G_{q,r}(f)\|_{L^{\frac{1}{s}}}.$$

*Proof.* (B) $\Rightarrow$ (A) is trivial. We need only to prove the implications (A) $\Rightarrow$ (C), (C) $\Rightarrow$ (D) and (D) $\Rightarrow$ (B).

(A) $\Rightarrow$ (C). For  $r \in (0, \alpha]$  fixed, take  $\{z_j\}_{j=1}^\infty$  to be some  $(r/4, \rho)$ -lattice. By Lemma 2.4, for  $\lambda = \{\lambda_j\} \in \ell^p$ , we have  $\left\| \sum_{j=1}^\infty \lambda_j k_{z_j,p} \right\|_{L_\varphi^p} \leq C \|\lambda\|_{\ell^p}$ . As in [14] again, take  $\{\phi_j\}_{j=1}^\infty$  to be a sequence of Rademacher functions in  $[0, 1]$ . From the boundedness of  $H_f$ , we have

$$\begin{aligned} \left\| H_f \left( \sum_{j=1}^\infty \lambda_j \phi_j(t) k_{z_j,p} \right) \right\|_{L_\varphi^q} &\leq \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q} \cdot \left\| \sum_{j=1}^\infty \lambda_j \phi_j(t) k_{z_j,p} \right\|_{L_\varphi^p} \\ &\leq C \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q} \|\lambda\|_{\ell^p}. \end{aligned}$$

Meanwhile, by Khintchine's inequality,

$$\begin{aligned} & \int_0^1 \left\| H_f \left( \sum_{j=1}^{\infty} \lambda_j \phi_j(t) k_{z_j, p} \right) \right\|_{L_\varphi^q}^q dt \\ &= \int_{\mathbb{D}} e^{-q\varphi(z)} dA(z) \int_0^1 \left| \sum_{j=1}^{\infty} \lambda_j \phi_j(t) H_f(k_{z_j, p})(z) \right|^q dt \\ &\simeq \int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} |\lambda_j|^2 |H_f(k_{z_j, p})(z)|^2 \right)^{\frac{q}{2}} e^{-q\varphi(z)} dA(z). \end{aligned}$$

This, together with the previous estimate, gives

$$\int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} |\lambda_j|^2 |H_f(k_{z_j, p})(z)|^2 \right)^{\frac{q}{2}} e^{-q\varphi(z)} dA(z) \leq C \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q}^q \|\lambda\|_{\ell^p}^q.$$

On the other hand,

$$\begin{aligned} & \int_{\mathbb{D}} \left( \sum_{j=1}^{\infty} |\lambda_j|^2 |H_f(k_{z_j, p})(z)|^2 \right)^{\frac{q}{2}} e^{-q\varphi(z)} dA(z) \\ &\geq C \sum_{k=1}^{\infty} \int_{D^r(z_k)} (|\lambda_k| |H_f(k_{z_k, p})(z)|)^q e^{-q\varphi(z)} dA(z) \\ &= C \sum_{k=1}^{\infty} |\lambda_k|^q \int_{D^r(z_k)} |f(z)k_{z_k, p}(z) - P(fk_{z_k, p})(z)|^q e^{-q\varphi(z)} dA(z). \end{aligned}$$

As in (4.7),

$$\int_{D^r(z_k)} |f(z)k_{z_k, p}(z) - P(fk_{z_k, p})(z)|^q e^{-q\varphi(z)} dA(z) \geq C \{\rho(z_k)^{2s} G_{q,r}(f)(z_k)\}^q.$$

Therefore, joining the previous estimates, we obtain

$$\sum_{k=1}^{\infty} |\lambda_k|^q \{\rho(z_k)^{2s} G_{q,r}(f)(z_k)\}^q \leq C \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q}^q \|\{\lambda_j\}\|_{\ell^{p/q}}^q.$$

By the duality between  $\ell^{p/q}$  and  $\ell^{p/(p-q)}$ , we have

$$\sum_{k=1}^{\infty} [G_{q,r}(f)(z_k)]^{\frac{pq}{p-q}} \rho(z_k)^2 = \sum_{k=1}^{\infty} [\rho(z_k)^{2s} G_{q,r}(f)(z_k)]^{\frac{pq}{p-q}} \leq C \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q}^{\frac{pq}{p-q}}.$$

This can be viewed as the discrete version of the statement (C). Since

$$(4.24) \quad G_{q, \frac{r}{2}}(f)(w) \leq C G_{q,r}(f)(z) \quad \text{for } w \in D^{\frac{r}{2}}(z),$$

we have

$$\begin{aligned} & \int_{\mathbb{D}} G_{q, \frac{r}{2}}(f)^{\frac{pq}{p-q}} dA \leq \sum_{k=1}^{\infty} \int_{D^{\frac{r}{2}}(z_k)} G_{q, \frac{r}{2}}(f)^{\frac{pq}{p-q}}(u) dA(u) \\ (4.25) \quad & \leq C \sum_{k=1}^{\infty} |D^{\frac{r}{2}}(z_k)| G_{q,r}(f)^{\frac{pq}{p-q}}(z_k) \\ & \leq C \|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q}^{\frac{pq}{p-q}}. \end{aligned}$$

This gives the statement (C).



(C) $\Rightarrow$ (D). As in the proof of Theorem 4.2, set  $f_1 = \sum_{j=1}^{\infty} h_j \psi_j \in C^\infty(\mathbb{D})$ , and  $f_2 = f - f_1$ . By (4.24),

$$G_{q, \frac{r}{2}}(f)^{\frac{pq}{p-q}}(z_j) \leq C \frac{1}{|D^{\frac{r}{2}}(z_j)|} \int_{D^{\frac{r}{2}}(z_j)} G_{q,r}(f)^{\frac{pq}{p-q}}(u) dA(u).$$

From (4.11) we have

$$\begin{aligned} [\rho(z) |\bar{\partial} f_1(z)|]^{\frac{pq}{p-q}} &\leq C \sum_{j \in J_z} G_{q, \frac{r}{2}}(f)^{\frac{pq}{p-q}}(z_j) \\ &\leq \frac{C}{|D^r(z)|} \sum_{j \in J_z} \int_{D^r(z_j)} G_{q,r}(f)^{\frac{pq}{p-q}}(u) dA(u) \\ &\leq \frac{C}{|D^r(z)|} \int_{D^{2r}(z)} G_{q,r}(f)^{\frac{pq}{p-q}}(u) dA(u). \end{aligned}$$

Integrating both sides on  $\mathbb{D}$  against the measure  $dA$ , and applying Fubini's theorem, one gets

$$\begin{aligned} (4.26) \quad &\int_{\mathbb{D}} [\rho(z) |\bar{\partial} f_1(z)|]^{\frac{pq}{p-q}} dA(z) \\ &\leq C \int_{\mathbb{D}} \frac{1}{|D^r(z)|} dA(z) \int_{\mathbb{D}} \chi_{D^{2r}(z)}(u) G_{q,r}(f)^{\frac{pq}{p-q}}(u) dA(u) \\ &\leq C \int_{\mathbb{D}} G_{q,r}(f)^{\frac{pq}{p-q}}(u) dA(u). \end{aligned}$$

Notice that  $\frac{1}{s} > 1$ . By (4.13) and (4.24) we obtain

$$\begin{aligned} M_{q,r}(f_2)(z) &\leq C \sum_{j=1}^{\infty} \left( \frac{1}{|D^r(z)|} \int_{D^r(z) \cap D^{r/2}(z_j)} |(f - h_j)|^q dA \right)^{\frac{1}{q}} \\ &\leq C \frac{1}{|D^r(z)|} \int_{D^{2r}(z)} G_{q,2r}(f)(\xi) dA(\xi) \\ &\leq C \left\{ \frac{1}{|D^r(z)|} \int_{D^{2r}(z)} G_{q,2r}^{\frac{1}{s}}(f)(\xi) dA(\xi) \right\}^s. \end{aligned}$$

This and Fubini's theorem turn out

$$(4.27) \quad \|M_{q,r}(f_2)\|_{L^{\frac{1}{s}}} \leq C \|G_{q,2r}(f)\|_{L^{\frac{1}{s}}}.$$

And it is trivial that the condition  $M_{q,r}(f_2) \in L^{\frac{1}{s}}$  is independent of  $r$ . We see that (4.26) and (4.27) give the statement (D).

Now we prove (D) $\Rightarrow$ (B). First, we claim that both  $f_1$  and  $\rho|\bar{\partial}f_1| \in \mathcal{S}$ . In fact, apply Lemma 3.3 from [10] to get

$$\begin{aligned} & \int_{\mathbb{D}} |f_2(\xi)K_z(\xi)|e^{-\varphi(\xi)}dA(\xi) \\ & \leq C \int_{\mathbb{D}} |f_2(\xi)| \left( \frac{1}{\rho(\xi)^2} \int_{D^{r/2}(\xi)} |K_z(\zeta)|e^{-\varphi(\zeta)}dA(\zeta) \right) dA(\xi) \\ & = C \int_{\mathbb{D}} |K_z(\zeta)|e^{-\varphi(\zeta)} \int_{\mathbb{D}} \chi_{D^{r/2}(\xi)}(\zeta)|f_2(\xi)|\frac{1}{\rho(\xi)^2} dA(\xi) dA(\zeta) \\ & \leq C \int_{\mathbb{D}} M_{1,r}(|f_2|)(\zeta)|K_z(\zeta)|e^{-\varphi(\zeta)}dA(\zeta) \\ & \leq C \int_{\mathbb{D}} M_{q,r}(|f_2|)(\zeta)|K_z(\zeta)|e^{-\varphi(\zeta)}dA(\zeta). \end{aligned}$$

By Hölder's inequality with exponent  $\frac{1}{s} = \frac{pq}{p-q}$  and its conjugate exponent denoted by  $t$ , notice also that  $\|K_z\|_{L_\varphi^t} < \infty$ ,

$$\int_{\mathbb{D}} |f_2(\xi)K_z(\xi)|e^{-\varphi(\xi)}dA(\xi) \leq C \|M_{q,r}(f_2)\|_{L^{\frac{1}{s}}} \cdot \|K_z\|_{L_\varphi^t} < \infty.$$

This implies  $f_2 \in \mathcal{S}$ , and  $f_1 = f - f_2 \in \mathcal{S}$ . For  $\rho|\bar{\partial}f_1|$ , notice that  $\rho|\bar{\partial}f_1| \in L^{\frac{1}{s}}$  with  $\frac{1}{s} = \frac{pq}{p-q} > 1$ . Then

$$\int_{\mathbb{D}} \rho(\xi)|\bar{\partial}f_1(\xi)K_z(\xi)|e^{-\varphi(\xi)}dA(\xi) \leq \left\{ \int_{\mathbb{D}} |\rho(\xi)|\bar{\partial}f_1(\xi)|^{\frac{1}{s}} dA(\xi) \right\}^s \|K_z\|_{L_\varphi^t} < \infty.$$

It follows that  $\rho|\bar{\partial}f_1| \in \mathcal{S}$ .

As before, write  $d\nu = [\rho|\bar{\partial}f_1|]^q dA$ . Applying Hölder's inequality with exponent  $\frac{p}{p-q}$  and its conjugate  $p/q$ , we get

$$\begin{aligned} \|\widehat{\nu}_r\|_{L^{\frac{p}{p-q}}}^{\frac{p}{p-q}} & = \int_{\mathbb{D}} \left\{ \frac{\int_{D^r(\xi)} [\rho|\bar{\partial}f_1(\zeta)]^q dA(\zeta)}{|D^r(\xi)|} \right\}^{\frac{p}{p-q}} dA(\xi) \\ & \leq C \int_{\mathbb{D}} \left\{ \int_{D^r(\xi)} [\rho(\zeta)|\bar{\partial}f_1(\zeta)]^{\frac{pq}{p-q}} dA(\zeta) \right\} \frac{1}{\rho(\xi)^2} dA(\xi) \\ & \simeq C \int_{\mathbb{D}} [\rho(\zeta)|\bar{\partial}f_1(\zeta)]^{\frac{1}{s}} dA(\zeta) < \infty. \end{aligned}$$

Lemma 2.6 tells us that  $\nu$  is a  $q$ -Carleson measure for  $A_\varphi^p$ . Equivalently, the embedding  $\text{Id} : A_\varphi^p \hookrightarrow L^q(\mathbb{D}, e^{-q\varphi}d\nu)$  is compact with

$$\|\text{Id}\|_{A_\varphi^p \hookrightarrow L^q(\mathbb{D}, e^{-q\varphi}d\nu)}^q \leq C \|\widehat{\nu}_r\|_{L^{\frac{p}{p-q}}} \leq C \|\rho|\bar{\partial}f_1|\|_{L^{\frac{1}{s}}}^q < \infty.$$

Meanwhile, since both  $f_1$  and  $\rho|\bar{\partial}f_1|$  are in  $\mathcal{S}$ , for  $g \in \Gamma$ , as in (4.17), we have

$$\|H_{f_1}g\|_{L_\varphi^q} \leq C \|g(\rho|\bar{\partial}f_1)\|_{L_\varphi^q} = C \|\text{Id}(g)\|_{L^q(\mathbb{D}, e^{-q\varphi}d\nu)}.$$

Hence  $H_{f_1}$  is bounded from  $A_\varphi^p$  to  $L_\varphi^q$  with the norm estimate

$$(4.28) \quad \|H_{f_1}\|_{A_\varphi^p \rightarrow L_\varphi^q} \leq C \|\rho|\bar{\partial}f_1|\|_{L^{\frac{1}{s}}}.$$

We claim that  $H_{f_1}$  is compact as well. To see this, let  $\{g_m\}_{m=1}^\infty$  be any bounded sequence in  $A_\varphi^p$  with the property that  $\lim_{m \rightarrow \infty} \sup_{z \in K} |g_m(z)| = 0$  on any compact subset  $K \subset \mathbb{D}$ .

We are going to prove  $H_{f_1}(g_m) \rightarrow 0$  in  $L^q_\varphi$  as  $m \rightarrow \infty$ . For this purpose, for each  $m$  pick some  $h_m \in \Gamma$  so that  $\|g_m - h_m\|_{L^p_\varphi} < \frac{1}{m}$ . Set

$$u_m(z) = \sum_{j=1}^{\infty} K_{z_j}(z) \int_{\mathbb{D}} \frac{\phi_j(\xi)}{(\xi - z)K_{z_j}(\xi)} h_m(\xi) \bar{\partial} f_1(\xi) dA(\xi).$$

Then,  $\bar{\partial} u_m = h_m \bar{\partial} f_1$  and

$$\|u_m\|_{L^q_\varphi} \leq C \|h_m(\rho \bar{\partial} f_1)\|_{L^q_\varphi} = C \|h_m\|_{L^q(\mathbb{D}, e^{-q\varphi} d\nu)}.$$

Notice that  $\text{Id} : A^p_\varphi \hookrightarrow L^q(\mathbb{D}, e^{-q\varphi} d\nu)$  is compact, so  $\lim_{m \rightarrow \infty} \|h_m\|_{L^q(\mathbb{D}, e^{-q\varphi} d\nu)} = 0$ , showing that

$$\lim_{m \rightarrow \infty} \|u_m\|_{L^q_\varphi} = 0.$$

Then, as  $H_{f_1}(h_m) = u_m - P(u_m)$ , we get

$$(4.29) \quad \lim_{m \rightarrow \infty} \|H_{f_1}(h_m)\|_{L^q_\varphi} \leq (1 + \|P\|_{L^q_\varphi \rightarrow L^q_\varphi}) \lim_{m \rightarrow \infty} \|u_m\|_{L^q_\varphi} = 0.$$

On the other hand, by (4.28),

$$\lim_{m \rightarrow \infty} \|H_{f_1}(g_m - h_m)\|_{L^q_\varphi} \leq \|H_{f_1}\|_{A^p_\varphi \rightarrow L^q_\varphi} \lim_{m \rightarrow \infty} \|g_m - h_m\|_{L^p_\varphi} = 0.$$

This, together with (4.29), implies

$$\lim_{m \rightarrow \infty} \|H_{f_1}(g_m)\|_{L^q_\varphi} \leq \lim_{m \rightarrow \infty} \{ \|H_{f_1}(g_m - h_m)\|_{L^q_\varphi} + \|H_{f_1}(h_m)\|_{L^q_\varphi} \} = 0,$$

which gives the compactness of  $H_{f_1}$  from  $A^p_\varphi$  to  $L^q_\varphi$ .

Finally, we consider the compactness of  $H_{f_2}$ . Similarly,  $d\mu = |f_2|^q dA$  is a vanishing  $q$ -Carleson measure for  $A^p_\varphi$ . Equivalently,  $\text{Id} : A^p_\varphi \rightarrow L^q(\mathbb{D}, e^{-q\varphi} d\mu)$  is compact. By

$$(4.30) \quad \|H_{f_2}(g)\|_{L^q_\varphi} \leq C \|f_2 g\|_{L^q_\varphi} = C \|\text{Id}(g)\|_{L^q(\mathbb{D}, e^{-q\varphi} d\mu)},$$

with the similar approach for  $H_{f_1}$  above we know  $H_{f_2}$  is compact from  $A^p_\varphi$  to  $L^q_\varphi$  as well. This finishes the proof of implication (D) $\Rightarrow$ (B).

Furthermore, from (4.28), (4.30) and (4.26), (4.27) we have

$$\|H_f\|_{A^p_\varphi \rightarrow L^q_\varphi} \leq C \inf \{ \|H_{f_1}\|_{A^p_\varphi \rightarrow L^q_\varphi} + \|H_{f_2}\|_{A^p_\varphi \rightarrow L^q_\varphi} \} \leq C \|G_{q,r}(f)\|_{L^{\frac{1}{s}}},$$

where the "inf" is taken over all decomposition  $f = f_1 + f_2$  as (4.22). This and (4.25) imply (4.23). The proof is completed.  $\square$

## 5. SIMULTANEOUSLY BOUNDEDNESS OF $H_f$ AND $H_{\bar{f}}$

For  $f \in L^q_{loc}(\mathbb{D})$  with  $1 \leq q < \infty$  and  $0 < r < \alpha$ , set  $f_{D^r(z)} = \frac{1}{|D^r(z)|} \int_{D^r(z)} f dA$ ,

$$MO_{q,r}(f)(z) = \left\{ \frac{1}{|D^r(z)|} \int_{D^r(z)} |f - f_{D^r(z)}|^q dA \right\}^{\frac{1}{q}}$$

and

$$\text{Osc}_r(f)(z) = \sup_{\xi \in B(z,r)} |f(\xi) - f(z)|.$$

**Lemma 5.1.** *Let  $1 \leq q < \infty$ ,  $0 < s \leq \infty$ ,  $-\infty < \gamma < \infty$ , and let  $f \in L^q_{loc}(\mathbb{D})$ . Then the following statements are equivalent:*

- (A) *For some (or any)  $0 < r \leq \alpha$ , both  $\rho^\gamma G_{q,r}(f)$  and  $\rho^\gamma G_{q,r}(\bar{f})$  are in  $L^s$ ;*
- (B) *For some (or any)  $0 < r \leq \alpha$ , one has  $\rho^\gamma MO_{q,r}(f) \in L^s$ ;*
- (C)  *$f = f_1 + f_2$  with  $f_1 \in C(\mathbb{D})$ , and for some (or any)  $0 < r \leq \alpha$*

$$(5.1) \quad \rho^\gamma \text{Osc}_r(f_1) \in L^s \quad \text{and} \quad \rho^\gamma M_{q,r}(f_2) \in L^s.$$

Furthermore,

$$(5.2) \quad \|\rho^\gamma G_{q,r}(f)\|_{L^s} + \|\rho^\gamma G_{q,r}(\bar{f})\|_{L^s} \simeq \|\rho^\gamma MO_{q,r}(f)\|_{L^s}.$$

*Proof.* By definition, we know

$$(5.3) \quad G_{q,r}(f)(z) \leq MO_{q,r}(f)(z) \quad \text{and} \quad G_{q,r}(\bar{f})(z) \leq MO_{q,r}(f)(z)$$

which gives the implication (A)  $\Rightarrow$  (B).

Similar to the estimate (2.7) in [11], for fixed  $r > 0$ , we have some constant  $C$  independent of  $z$  such that

$$\|u - u(z)\|_{L^q(D^r(z), dA)} \leq C \|v\|_{L^q(D^r(z), dA)}$$

for all real valued functions  $u$  and  $v$  so that  $u + iv \in H(D^r(z))$ . From this, as done in Proposition 2.5 in [11], we know

$$(5.4) \quad MO_{q,r}(f)(z) \leq C \{G_{q,r}(f)(z) + G_{q,r}(\bar{f})(z)\}.$$

This means that (B) implies (A).

Suppose  $f = f_1 + f_2$  is as in statement (C). From

$$\begin{aligned} MO_{q,r}(f_1)(z) &= \left\{ \frac{1}{|D^r(z)|} \int_{D^r(z)} \left| \frac{1}{|D^r(z)|} \int_{D^r(z)} (f_1(\xi) - f_1(\zeta)) dA(\zeta) \right|^q dA(\xi) \right\}^{\frac{1}{q}} \\ &\leq 2Osc_r(f_1)(z) \end{aligned}$$

and  $MO_{q,r}(f_2)(z) \leq 2M_{q,r}(f_2)(z)$ , we know that  $f$  satisfies (A).

To prove the implication (B)  $\Rightarrow$  (C) we set  $f_1(z) = f_{D^r(z)}$  and  $f_2 = f - f_1$ . As in the proof of Lemma 8.3 in [18] we have

$$Osc_{r/2}(f_1)(z) \leq CMO_{q,r}(f)(z) \quad \text{and} \quad M_{q,r/2}(f_2)(z) \leq CMO_{q,r}(f)(z).$$

And it is easy to see that the condition (5.1) is independent of  $r \in (0, \alpha]$ . Then (C) follows from (B). The equivalence (5.2) comes from (5.3) and (5.4).  $\square$

**Lemma 5.2.** *Let  $1 \leq q < \infty$ ,  $0 < s \leq \infty$ ,  $-\infty < \gamma < \infty$ , and let  $f \in L^q_{loc}(\mathbb{D})$ . Then the following statements are equivalent:*

- (A) For some (or any)  $0 < r \leq \alpha$ ,  $\lim_{|z| \rightarrow 1} \{\rho(z)^\gamma G_{q,r}(f)(z) + \rho(z)^\gamma G_{q,r}(\bar{f})(z)\} = 0$ ;
- (B) For some (or any)  $0 < r \leq \alpha$ ,  $\lim_{|z| \rightarrow 1} \rho(z)^\gamma MO_{q,r}(f)(z) = 0$ ;
- (C)  $f = f_1 + f_2$  with  $f_1 \in C(D)$ , and for some (or any)  $0 < r \leq \alpha$

$$\lim_{|z| \rightarrow 1} \{\rho(z)^\gamma Osc_r(f_1)(z) + \rho(z)^\gamma M_{q,r}(f_2)(z)\} = 0.$$

The proof of this lemma can be carried out with the same approach as that of Lemma 5.1 and will be omitted here.

Here are three theorems for simultaneous boundedness (or compactness) of Hankel operators  $H_f$  and  $H_{\bar{f}}$  from  $A^p_\varphi$  to  $L^q_\varphi$ .

**Theorem 5.3.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and let  $1 \leq p \leq q < \infty$ . Set  $s = \frac{1}{q} - \frac{1}{p}$ . Then for  $f \in \mathcal{S}$ , the following statements are equivalent:*

- (A)  $H_f, H_{\bar{f}} : A^p_\varphi \rightarrow L^q_\varphi$  are simultaneously bounded;
- (B) For some (or any)  $0 < r \leq \alpha$ ,  $\rho^{2s} MO_{q,r}(f) \in L^\infty$ ;
- (C)  $f$  admits a decomposition  $f = f_1 + f_2$ , where  $f_1 \in C^1(\mathbb{D})$  satisfying

$$\rho^{2s} Osc_r(f_1) \in L^\infty, \quad \text{and} \quad \rho^{2s} M_{q,r}(f_2) \in L^\infty$$

for some (or any)  $0 < r \leq \alpha$ .

Furthermore,

$$(5.5) \quad \|H_f\|_{A^p_\varphi \rightarrow L^q_\varphi} + \|H_{\bar{f}}\|_{A^p_\varphi \rightarrow L^q_\varphi} \simeq \|\rho^{2s} MO_{q,r}(f)\|_{L^\infty}.$$

**Theorem 5.4.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and let  $1 \leq p \leq q < \infty$ . Set  $s = \frac{1}{q} - \frac{1}{p}$ . Then for  $f \in \mathcal{S}$ , the following statements are equivalent:*

- (A)  $H_f, H_{\bar{f}} : A_\varphi^p \rightarrow L_\varphi^q$  are simultaneously compact;
- (B) For some (or any)  $0 < r \leq \alpha$ ,  $\lim_{|z| \rightarrow 1} \rho^{2s}(z) MO_{q,r}(f)(z) = 0$ ;
- (C)  $f$  admits a decomposition  $f = f_1 + f_2$ , where  $f_1 \in C^1(\mathbb{D})$  satisfying

$$\lim_{|z| \rightarrow 1} \rho^{2s}(z) Osc_r(f_1)(z) = 0, \text{ and } \lim_{|z| \rightarrow 1} \rho^{2s}(z) M_{q,r}(f_2)(z) = 0$$

for some (or any)  $0 < r \leq \alpha$ .

**Theorem 5.5.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and let  $1 \leq q < p < \infty$ . Set  $s = \frac{1}{q} - \frac{1}{p}$ . Then for  $f \in \mathcal{S}$ , the following statements are equivalent:*

- (A)  $H_f, H_{\bar{f}} : A_\varphi^p \rightarrow L_\varphi^q$  are bounded.
- (B)  $H_f, H_{\bar{f}} : A_\varphi^p \rightarrow L_\varphi^q$  are compact.
- (C) For some (or any)  $0 < r \leq \alpha$ ,  $MO_{q,r}(f) \in L^{\frac{1}{s}}$ .
- (D)  $f = f_1 + f_2$  with  $f_1 \in C(\mathbb{D})$ ,

$$Osc_r(f_1) \in L^{\frac{1}{s}}, \text{ and } M_{q,r}(f_2) \in L^{\frac{1}{s}}$$

for some (or any)  $0 < r < \alpha$ .

Furthermore,  $\|H_f\|_{A_\varphi^p \rightarrow L_\varphi^q} \simeq \|MO_{q,r}(f)\|_{L^{\frac{1}{s}}}$ .

*Proof.* The proof of Theorem 5.3, 5.4 and 5.5 are in the same approach, so we only write out the one for Theorem 5.3 here.

Theorem 4.2 tells us that the statement (A) is equivalent to

$$\rho^{2s} G_{q,r}(f) + \rho^{2s} G_{q,r}(\bar{f}) \in L^\infty.$$

And this, by Lemma 5.2, is equivalent to Statement (B). The equivalence between (B) and (C) comes from Lemma 5.2 as well.  $\square$

When  $f$  is holomorphic, it is trivial that  $H_f = 0$ . Furthermore, for fixed  $0 < r \leq \alpha$  there are two positive constant  $C_1$  and  $C_2$  such that

$$C_1 \rho(z) \sup_{\xi \in D^r(z)} |f'(\xi)| \leq M_{q,r}(f)(z) \leq C_2 \rho(z) \sup_{\xi \in D^r(z)} |f'(\xi)|.$$

Therefore we have the following theorem on Hankel operators with conjugate holomorphic symbols. The case  $\varphi \in \mathcal{BDK}$  and  $p = q = 2$ , was previously obtained in [9].

**Theorem 5.6.** *Let  $\varphi \in \mathcal{W}_0$  with  $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$ , and set  $s = \frac{1}{q} - \frac{1}{p}$  for  $1 \leq p, q < \infty$ . Then for  $f \in \mathcal{S} \cap H(\mathbb{D})$  the following statements are true.*

- (A) For  $p \leq q$ ,  $H_{\bar{f}}$  is bounded from  $A_\varphi^p$  to  $L_\varphi^q$  if and only if  $\rho^{2s+1} f' \in L^\infty$ ;  $H_{\bar{f}}$  is compact from  $A_\varphi^p$  to  $L_\varphi^q$  if and only if  $\lim_{|z| \rightarrow 1} \rho^{2s+1} f'(z) = 0$ .
- (B) For  $p > q$ ,  $H_{\bar{f}}$  is bounded from  $A_\varphi^p$  to  $L_\varphi^q$  if and only if  $H_{\bar{f}}$  is compact from  $A_\varphi^p$  to  $L_\varphi^q$  if and only if  $\rho f' \in L^{\frac{1}{s}}$

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