Lee Hwa Chung theorem for presymplectic manifolds. Canonical transformations for constrained systems

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We generalize the analogous of Lee Hwa Chung's theorem to the case of presymplectic manifolds. As an application, we study the canonical transformations of a canonical system (M, S, Ω). The role of Dirac brackets as a test of canonicity is clarified.

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1. INTRODUCTION

Degenerate or singular Hamiltonian systems were introduced in mathematical physics by Dirac.[1,2] Later, the interest in this formalism has increased[3,4] mainly because it provides the suitable framework to deal with many physical theories, either in the infinite-dimensional case[2,5] (electromagnetic, gravitational, and Yang–Mills fields) or in the finite-dimensional one (relativistic systems of directly interacting particles).[6]

The usual way of dealing with these systems (we shall restrict ourselves to the finite-dimensional case) starts from a phase space M with a symplectic structure Ω, i.e., a nondegenerate Poisson bracket. Then, some constraints are introduced in order to define a submanifold S representing the set of all possible states of the physical system (e.g., the mass shell constraints in a system of relativistic particles). According to the mutual Poisson brackets, these constraints are classified in first and second class; as is commonly accepted, the latter correspond to spurious degrees of freedom and the former can be considered as generating functions for gauge motions of the system. It has been noticed by Shanmugadhasan[7] that an adapted canonical set of variables can be selected such that the constraints assume their simplest form, i.e., the submanifold S is obtained by making some coordinates and momenta equal to zero. The symplectic form Ω on M then induces a presymplectic structure ω on S, which is degenerate in most cases.

The invariance of the Poincaré–Cartan integral[8] turns out to be a sound principle to establish the features of nondegenerate Hamiltonian systems. Likewise, for degenerate systems, an analogous formalism can be set forth.[9,10] This formulation also stresses the close parallelism between the above-mentioned Shanmugadhasan transformation[7] and the Hamilton–Jacobi method when the canonical Hamiltonian vanishes.[10]

In dealing with Hamiltonian systems, either degenerate or not, it is of great interest to have a precise characterization of canonical transformations. In the nondegenerate case, this characterization can be presented in a very clear and elegant way by requiring the invariance of the Poincaré–Cartan integral under these transformations.[11] To show this result, it is necessary to use Lee Hwa Chung's theorem,[12] which states the only absolute integral invariants under every Hamiltonian system are the symplectic form and the exterior product of this form with itself any number of times.

However, a fully satisfactory definition of canonical transformations has not been attained in the singular case yet. A previous attempt[13] requires canonical transformation to preserve the elementary Poisson brackets (that is, the symplectic form Ω) on the constraint submanifold S. This condition is too strong because the initial phase space M and its symplectic structure Ω must not be considered as the physically relevant objects of the theory, but merely as the starting point to build up the submanifold S and the induced presymplectic structure ω, in terms of which the physical system is represented.

Our standpoint is that, in the degenerate Hamiltonian formalism, canonical transformations must be characterized as those preserving the physically significant submanifold S and its presymplectic structure ω. Thus, a result generalizing Lee Hwa Chung's theorem will a helpful tool in dealing with canonical transformations in the degenerate case. This is the purpose of the present paper.

The language used throughout this work is geometrical[14–18] where concepts, which classically lead to more or less involved equations in terms of coordinates and momenta, can be formulated and handled in a more clear and concise way.

In Sec. 2 we generalize to presymplectic manifolds some well-established results in the symplectic case, such as the concepts of the Hamiltonian vector field, Hamiltonian functions, and the Poincaré–Cartan integral. In Sec. 3, we generalize the above mentioned Lee Hwa Chung's theorem to the presymplectic case. In Sec. 4, previous results are applied to the particular case that the presymplectic manifold has been obtained by imposing a set of constraints on a bigger symplectic space, as is common in the physical applications of the degenerate Hamiltonian formalism. Section 5 is devoted to study the canonical transformations of a constrained Hamiltonian system.

2. HAMILTONIAN FIELDS AND THE POINCARÉ INTEGRAL IN A PRESYMPLECTIC MANIFOLD

Let us consider a manifold S and a presymplectic form on S (i.e., ω is a closed differential 2-form of constant class on S), so that the couple (S, ω) is called a presymplectic manifold.

The 2-form ω defines a differentiable linear map from the tangent vector fields $D(S)$ onto the differential 1-forms $\Lambda^1(S)$, given by

$$D(S) \to \Lambda^1(S),$$
$$\mathbf{X} \rightsquigarrow i_{\mathbf{X}}\omega, \tag{2.1}$$

where $i_{\mathbf{X}}$ denotes the inner product by $\mathbf{X}$. This mapping can-

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not be inverted, in general. That is, there are some 1-forms in $\Lambda^1(S)$ which are not in correspondence with any vector field in $D(S)$.

Definition 2.1: A function $h \in \Lambda^0(S)$ is called a *Hamiltonian function* relative to the presymplectic structure defined by ω iff there exists a vector field $\mathbf{X} \in D(S)$ such that

$$i_{\mathbf{X}}\omega = dh. \tag{2.2}$$

In other words, h is a Hamiltonian function iff the linear system (2.2) admits a nonempty family of solutions $\mathbf{X} \in D(S)$. Notice that if $\mathbf{X}_h$ is a particular solution of (2.2), then the general solution $K_h \subset D(S)$ is given by

$$K_h = \mathbf{X}_h + \mathscr{A}(\omega), \tag{2.3a}$$

where $\mathscr{A}(\omega)$ is the submodulus of $D(S)$ defined by

$$\mathscr{A}(\omega) = \{\mathbf{X} \in D(S) / i_{\mathbf{X}}\omega = 0\}. \tag{2.3b}$$

Definition 2.2: A vector field $\mathbf{X} \in D(S)$ is called a *Hamiltonian vector field* relatively to ω iff there exists a function $h \in \Lambda^0(S)$ such that

$$i_{\mathbf{X}}\omega = dh. \tag{2.4}$$

Any Hamiltonian field $\mathbf{X}$ satisfies

$$di_{\mathbf{X}}\omega = 0. \tag{2.5}$$

The converse is not true, since no closed differential form is exact. However, the Poincaré lemma[19] ensures that if Eq. (2.5) is fulfilled, then, for any $p \in S$, there exists an open neighborhood of p, $U \subset S$, and a function $f \in \Lambda^0(U)$ such that

$$i_{\mathbf{X}}\omega - df = 0 \quad \text{on } U. \tag{2.6}$$

This is the reason we introduce the concept of a local Hamiltonian vector field.

Definition 2.3: A vector field $\mathbf{X} \in D(S)$ is called *locally Hamiltonian* relatively to ω iff $i_{\mathbf{X}}\omega$ is closed [i.e., it is a solution of Eq. (2.5)].

Proposition 2.4: $\mathbf{X} \in D(S)$ is locally Hamiltonian relatively to ω iff $L_{\mathbf{X}}\omega = 0$, where L denotes the Lie derivatives.

The proof is immediate after recalling that $L_{\mathbf{X}} = i_{\mathbf{X}}d + di_{\mathbf{X}}$ and that ω is a closed 2-form.

Expressing the class of locally Hamiltonian vector fields relative to ω by

$$D_\omega(S) \equiv \{\mathbf{X} \in D(S) / L_{\mathbf{X}}\omega = 0\}, \tag{2.7}$$

we have the following interesting results:

Proposition 2.5: $D_\omega(S)$ is closed under the Lie bracket.

The proof is immediate since

$$L_{[\mathbf{X},\mathbf{Y}]} = L_{\mathbf{X}}L_{\mathbf{Y}} - L_{\mathbf{Y}}L_{\mathbf{X}}$$

(see, for example, Ref. 19).

Proposition 2.6: Given any point $p \in S$ and any tangent vector at p, $\mathbf{V} \in T_p(S)$, there is at least one locally Hamiltonian vector field $\mathbf{X} \in D_\omega(S)$ such that $\mathbf{X}_p = \mathbf{V}$.

Proof: Using a well-known result in differential geometry,[19] a local chart $(y^1, \ldots, y^s)$ in a neighborhood of $p \in S$ exists such that the presymplectic form ω can be written as

$$\omega_3 = dy^1 \wedge dy^{R+1} + \cdots + dy^R \wedge dy^{2R}, \quad 2R \leqslant s.$$

It is obvious that the vector fields $\partial/\partial y^1, \ldots, \partial/\partial y^s$ are locally Hamiltonian.

For a given $\mathbf{V} \in T_p(S)$, we expand it in the coordinate basis $\{(\partial/\partial y^i)_p\}_{p=1,\ldots,s}$, thus obtaining

$$\mathbf{V} = \sum_{i=1}^{s} V^i \left(\frac{\partial}{\partial y^i}\right)_p, \quad V^i \in \mathbb{R}.$$

Then, a solution for our problem is

$$\mathbf{X} = \sum_{i=1}^{s} V^i \frac{\partial}{\partial y^i}. \qquad \boxtimes$$

At this point it must be noted that any function $f \in \Lambda^0(S)$ depending on any of the variables y^i, $i > 2R$ (i.e., $\partial f/\partial y^i \neq 0$), cannot be a Hamiltonian function.

We are now going to generalize the Poincaré integral theorem for presymplectic manifolds. With this purpose we introduce the concept of a local one-parameter group of diffeomorphisms which consists of a differentiable mapping

$$\Phi\colon W \to S,$$

$$(t, p) \rightsquigarrow \varphi_t(p),$$

where $W \subset R \times S$ is an open neighborhood of $\{0\} \times S$, having the following properties:

(i) $\forall p \in S$, $(R \times \{p\}) \cap S$ is connected;

(ii) $\forall p \in S$, $\varphi_0(p) = p$;

(iii) If (t', p), $(t + t', p)$, and $(t, \varphi_{t'}(p))$ belong to W, then $\varphi_{t+t'}(p) = \varphi_t(\varphi_{t'}(p))$.

It is also known[19] that the integral orbits of a given differentiable vector field $\mathbf{X} \in D(S)$ permit to define the so-called local one-parameter group generated by $\mathbf{X}$.

Let c be a singular 2-cube on S.[20] Since $|c|$, the support of c, is a compact set in S, there exists a real positive number $\mathscr{E}_c$ such that $[-\mathscr{E}_c, \mathscr{E}_c] \times |c| \subset W$. Therefore, for any $t \in [-\mathscr{E}_c, \mathscr{E}_c]$, the map $\varphi_t\colon |c| \to S$ defines a diffeomorphism between $|c|$ and $\varphi_t|c|$, and $\varphi_t \circ c$ also is a singular 2-cube on S.

Definition 2.7: Let c be a singular 2-cube on S, $\mathbf{X} \in D(S)$, a differentiable vector field and Φ the local one-parameter group generated by $\mathbf{X}$. For any $t \in [-\mathscr{E}_c, \mathscr{E}_c]$, the Poincaré integral is defined as

$$I(t;c,\mathbf{X}) \equiv \int_{\varphi_t \circ c} \omega.$$

Theorem 2.8: The following conditions are equivalent:

(i) $\mathbf{X} \in D_\omega(S)$ (i.e., it is a locally Hamiltonian field);

(ii) $[dI(t;c,\mathbf{X})/dt]_{t=0} = 0$ for any singular 2-cube on S.

Proof: We have that

$$I(t;c,\mathbf{X}) = \int_{\varphi_t \circ c} \omega = \int_c \varphi_t^* \omega,$$

where φ_t^* is the pullback map associated with φ_t. Deriving then both sides with respect to t and taking $t = 0$, we have

$$\left[\frac{dI(t;c,X)}{dT}\right]_{t=0} = \int_c \lim_{t \to 0}\left(\frac{\varphi_T^* \omega - \omega}{t}\right),$$

but the limit on the right-hand side is nothing but $L_{\mathbf{X}}\omega$. Hence we can write

$$\left[\frac{dI(t;c,X)}{dt}\right]_{t=0} = \int_c L_{\mathbf{X}}\omega,$$

and the theorem follows immediately. $\boxtimes$

3. LEE HWA CHUNG'S THEOREM FOR PRESYMPLECTIC MANIFOLDS

In the case of symplectic manifolds, Lee Hwa Chung's theorem[12] fixes the class of differential forms which are in-

variant under any locally Hamiltonian vector fields. In this section, we arrive at a similar result for presymplectic manifolds.

Theorem 3.1: Let S be a s-manifold, $\omega\in\Lambda^2(S)$ a presymplectic form with constant class $2R$ on S, and α a differential p-form on S, $\alpha\in\Lambda^p(S)$. If $L_{\mathbf{X}}\alpha=0$, for every locally Hamiltonian vector field $\mathbf{X}\in D_\omega(S)$, then

(a) $\alpha=0$, if $p>2R$;

(b) $\alpha=0$, if $p=2l+1$, $l>R$;

(c) $\alpha=f\cdot\omega\wedge\overset{l}{\cdots}\wedge\omega$, if $p=2l$, $l\leqslant R$, where $f\in\Lambda^0(S)$, and is constant on any connected components of S.

Proof: Recalling Eq. (2.3b) and that $L_{\mathbf{X}}=di_{\mathbf{X}}+i_{\mathbf{X}}d$, we have that $\mathscr{A}(\omega)\subset D_\omega(S)$. The hypothesis of the theorem therefore implies that

$$L_{\mathbf{X}}\alpha=0,\quad \forall\mathbf{X}\in\mathscr{A}(\omega). \tag{3.1}$$

Moreover, since $\mathscr{A}(\omega)$ is closed under product by any $f\in\Lambda^0(S)$, we also have that

$$L_{f\mathbf{X}}\alpha=0,\quad \forall\mathbf{X}\in\mathscr{A}(\omega)\ \text{and}\ \forall f\in\Lambda^0(S). \tag{3.2}$$

Comparing both Eqs. (3.1) and (3.2), we arrive at

$$df\wedge i_{\mathbf{X}}\alpha=0,\quad \forall f\in\Lambda^0(S)\ \text{and}\ \forall\mathbf{X}\in\mathscr{A}(\omega). \tag{3.3}$$

For the sake of simplicity, we shall handle Eq. (3.3) at each point $z\in S$ [recall that $T_z(S)$ is a finite-dimensional real vector space, whereas $D(S)$ is a modulus on $\Lambda^0(S)$]. Applying Corollary A.4 of the Appendix, we have

$$i_{\mathbf{X}_z}\alpha_z=0,\quad \forall\mathbf{X}_z\in\mathscr{A}(\omega_z),\ \forall z\in S, \tag{3.4}$$

where $\mathscr{A}(\omega_z)=\{\mathbf{X}_z\in T_z(S)/i_{\mathbf{X}_z}\omega_z=0\}$.

For $p>2R$, using Corollary A.2, we obtain

$$\alpha_z=0,\quad \forall z\in S, \tag{3.5}$$

and, equivalently, $\alpha=0$, so that statement (a) is proved.

Let us now take two locally Hamiltonian vector fields $\mathbf{X},\mathbf{Y}\in D_\omega(S)$. The Poincaré lemma guarantees that, $\forall z\in S$, there exists an open neighborhood of z, $U\subset S$, and two functions, $f,g\in\Lambda^0(U)$, such that

$$i_{\mathbf{X}_U}\omega_U=df\quad\text{and}\quad i_{\mathbf{Y}_U}\omega_U=dg, \tag{3.6}$$

where $\mathbf{X}_U\equiv\mathbf{X}\circ j_U\in D(U)$, $\mathbf{Y}_U\equiv Y\circ j_U\in D(U)$, $\omega_U\equiv\omega\circ j_U\in\Lambda^2(U)$, and j_U: $U\to S$ is the natural injection.

Note that $\mathbf{P}\equiv g\mathbf{X}_U+f\mathbf{Y}_U$ is a Hamiltonian vector field on U relative to ω_U and its corresponding Hamiltonian function is $fg\in\Lambda^0(U)$. Hence, by hypothesis, the differential form $\alpha_U=\alpha\circ j_U\in\Lambda^P(U)$ is invariant under $\mathbf{X}_U$, $\mathbf{Y}_U$, and $\mathbf{P}$, i.e.,

$$L_{\mathbf{X}_U}\alpha_U=L_{\mathbf{Y}_U}\alpha_U=L_{\mathbf{P}}\alpha_U=0, \tag{3.7}$$

which implies

$$df\wedge i_{\mathbf{Y}_U}\alpha_U+dg\wedge i_{\mathbf{X}_U}\alpha_U=0. \tag{3.8}$$

Using (3.6), specializing (3.8) at the point $z\in U$, and taking Proposition 2.6 into account, we can finally write

$$i_{\mathbf{X}_z}\omega_z\wedge i_{\mathbf{Y}_z}\alpha_z+i_{\mathbf{Y}_z}\omega_z\wedge i_{\mathbf{X}_z}\alpha_z=0,\quad \forall z\in S\ \text{and}\ \forall\mathbf{X}_z,\ \mathbf{Y}_z\in T_z(S). \tag{3.9}$$

At this point, the following result is needed:

Lemma 3.2: Let α_z be a p-form on $T_z(S)$ [i.e., $\alpha_z\in\Lambda^p_z(S)$], $p\leqslant 2R$, such that

(i) $\forall\mathbf{X}_z\in\mathscr{A}(\omega_z)$, $\quad i_{\mathbf{X}_z}\alpha_z=0$, (3.4)

(ii) $\forall\mathbf{X}_z,\mathbf{Y}_z\in T_z(S)$,

$$i_{\mathbf{X}_z}\omega_z\wedge i_{\mathbf{Y}_z}\alpha_z+i_{\mathbf{Y}_z}\omega_z\wedge i_{\mathbf{X}_z}\alpha_z=0. \tag{3.9}$$

Then,

—if $p=1$, then $\alpha_z=0$.

—if $p>1$, then there exists $\alpha_z^{(1)}\in\Lambda^{p-2}_z(S)$, fulfilling conditions (3.4) and (3.9) and also

$$i_{\mathbf{X}_z}\alpha_z=i_{\mathbf{X}_z}\omega_z\wedge\alpha_z^{(1)},\quad \forall\mathbf{X}_z\in T_z(S). \tag{3.10}$$

(The proof is given in the Appendix.)

This lemma provides a recursive algorithm to prove statements (b) and (c) of Theorem 3.1.

(c) Indeed, let us consider the case $p=2l$, $l\leqslant R$.

Starting from $\alpha_z^{(p)}\equiv\alpha_z\in\Lambda^p_z(S)$ and by iterative application of Lemma 3.2, we obtain a sequence of alternated forms:

$$\alpha_z^{(i)}\in\Lambda^{2(l-i)}_z(S),\quad i=0,\ldots,l,$$

such that

$$i_{\mathbf{X}_z}\alpha_z^{(i)}=i_{\mathbf{X}_z}\omega_z\wedge\alpha_z^{(i+1)},\quad \forall\mathbf{X}_z\in T_z(S)$$

whence, by induction, it follows immediately that

$$\alpha_z^{(i)}=[1/(l+i)!]\alpha_z^{(l)}\omega_z\wedge\overset{l-i}{\cdots}\wedge\omega_z, \tag{3.11}$$

where $\alpha_z^{(l)}\in\Lambda^0_z(S)$ is a real number.

Therefore, the differential p-form $\alpha\in\Lambda^p(S)$ can be written as

$$\alpha=f\cdot\omega\wedge\overset{l}{\cdots}\wedge\omega, \tag{3.12}$$

where $f\in\Lambda^0(S)$.

Finally, since α and ω are invariant under any locally Hamiltonian vector field, we have

$$\mathbf{X}f=0,\quad \forall\mathbf{X}\in D_\omega(S),$$

which, by Proposition 2.6, implies that

$$\mathbf{X}_z f=0,\quad \forall\mathbf{X}_z\in T_z(S)\ \text{and}\ \forall z\in S.$$

Hence, f is constant on any connected component of S, and the proof of the statement (c) is over.

(b) For the case $p=2l+1$, $l<R$, starting from $\alpha_z^{(0)}\equiv\alpha_z$, Lemma 3.2 yields again a sequence of alternated forms:

$$\alpha_z^{(i)}\in\Lambda^{2(l-i)+1}_z(S),\quad i=0,\ldots,l,$$

such that

$$i_{\mathbf{X}_z}\alpha_z^{(i)}=i_{\mathbf{X}_z}\omega_z\wedge\alpha_z^{(i+1)},\quad \forall\mathbf{X}_z\in T_z(S). \tag{3.13}$$

Since each one of these $\alpha^{(i)}$ fulfills the hypothesis of the Lemma 3.2, and $\alpha_z^{(l)}\in\Lambda^1_Z(S)$, it follows that

$$\alpha_z^{(l)}=0,\quad \forall z\in S,$$

which, by virtue of (3.13), implies $\alpha_z^{(i)}=0$, $i=1,\ldots,l$ and particularly, $\alpha_z=0$, $\forall z\in S$.

Thus, the proof is concluded.

4. CANONICAL SYSTEMS

A canonical system is characterized by a triplet (M,S,Ω), (M,Ω) being a symplectic $2n$-manifold and S an s-submanifold of M. We shall denote the natural injection by j_S: $S\to M$ and the corresponding pullback mapping by j_S^*: $\Lambda(M)\to\Lambda(S)$. We shall assume the closed differential 2-form

$$\omega\equiv j_S^*\Omega\in\Lambda^2(S)$$

has constant rank: rank $\omega = 2R$.

The relationships between $2n$, s, and $2R$ determine the class of the submanifold: namely,

(i) S is said to be a first class submanifold when $2n - s \neq 2R$,

(ii) S is second class when $s = 2R$,

(iii) otherwise S is a mixed submanifold.

This classification is equivalent to the usual one, which is formulated in terms of the Poisson brackets among the constraints defining S.

In this section we are going to particularize the results obtained in the preceeding sections to the case of canonical systems.

Definition 4.1: Let α be a differential p-form on M, $\alpha \in \Lambda^p(M)$. We shall refer to the p-form $j_\alpha^* \in \Lambda^p(S)$ as the *specialization* of α onto S.

Definition 4.2: Two p-forms $\alpha, \beta \in \Lambda^p(M)$ are said to be *weakly equal* on S iff $\alpha \circ j_S = \beta \circ j_S$. Weak equality shall be denoted hereafter by $\alpha \underset{S}{=}$ or $\alpha \approx \beta$.

Definition 4.3: Two p-forms $\alpha, \beta \in \Lambda^p(M)$ are said to be *strongly equal* on S iff $\alpha \underset{S}{=} \beta$ and $d\alpha \underset{S}{=} d\beta$. Throughout this paper we shall write $\alpha \equiv \beta$ for strong equality.

Proposition 4.4: Let ζ^ν, $\nu = 1,\ldots,n-s$, be a set of independent constraint functions defining S, and let $\alpha \in \Lambda^p(M)$. Then, the specialization of α onto S vanishes ($j_S^* \alpha = 0$) if, and only if, there exist $n - s$ differential $(p-1)$-forms $\eta_0 \in \Lambda^{p-1}(M)$, $\nu = 1,\ldots,n-s$, such that

$$\alpha \underset{S}{=} \sum_{\nu=1}^{n-s} \eta_\nu \wedge d\zeta^\nu.$$

Proof: See, for example, Ref. 15.

Definition 4.5: Let $\mathbf{X} \in D(M)$ be a vector field on M. $\mathbf{X}$ is said to be *tangent* to the submanifold S iff

$$\mathbf{X}\zeta \underset{S}{=} 0 \tag{4.1}$$

for any given $\zeta \in \Lambda^0(M)$ such that $\zeta \underset{S}{=} 0$.

We shall denote by $\overline{D(S)}$ the set of those vector fields which are tangent to S.

Condition (4.1) is equivalent to

$$\mathbf{X} \circ j_S \in j_S^T(D(S)).$$

Hence, given any $X \in \overline{D(S)}$, there is a vector field on $D(S)$, which we shall denote by $\mathbf{X}^S$, such that

$$\mathbf{X} \circ j_S = j_{S*}(\mathbf{X}^S), \tag{4.2}$$

i.e., $\forall f \in \Lambda^0(M)$, $\mathbf{X}^S(j_S^* f) = j_S^*(\mathbf{X}f)$, where j_{S*} denotes the Jacobian map.

Proposition 4.6: Let $\mathbf{X} \in \overline{D(S)}$, $\alpha \in \Lambda^p(M)$, then

(i) $j_S^* i_{\mathbf{X}} \alpha = i_{\mathbf{X}^S}(j_S^* \alpha)$,

(ii) $j_S^* L_{\mathbf{X}} \alpha = L_{\mathbf{X}^S}(j_S^* \alpha)$.

The proof follows immediately using very well-known results of differential geometry (see, for example, Ref. 19).

Let $\mathbf{H} \in \overline{D(S)}$ and $\mathbf{H}^S \in D(S)$ be the vector field associated with $\mathbf{H}$ by Eq. (4.2). According to Definition 2.2, $\mathbf{H}^S$ will be Hamiltonian relative to $j_S^* \Omega$ iff a function $f \in \Lambda^0(S)$ exists such that

$$i_{\mathbf{H}^S}(j_S^* \Omega) = df. \tag{4.3}$$

Let us now pick up a function $h_c \in \Lambda^0(M)$ such that $f = j_S^* h_C$ (there is a large class of them) and rewrite (4.3) as

$$j_S^*(i_{\mathbf{H}} \Omega - dh_c) = 0. \tag{4.4}$$

Definition 4.7: A vector field $\mathbf{H} \in D(M)$ is said to be *weakly Hamiltonian* relatively to the canonical structure (M, S, Ω) iff

(i) $\mathbf{H} \in \overline{D(S)}$,

(ii) there exists $h_c \in \Lambda^0(M)$ such that Eq. (4.4) holds.

The function h_c is commonly called a *canonical Hamiltonian* corresponding to $\mathbf{H}$.

We are now going to express Eq. (4.4) in terms of a given set of $2n - s$ constraints defining the submanifold S. These constraints can be always arranged in such a way that:

(i) Some of them $\{\xi^\rho, \rho = 1,\ldots,l\}$, which are said to be *first class*, have weakly vanishing Poisson brackets with any other constraint, and

(ii) the remaining $2n - s - l$ $\{\chi^\alpha, \alpha = 1,\ldots,2n - s - l\}$, which are called *second class* and must be in even number, satisfy the inequality

$$\det(\{\chi^\alpha, \chi^\beta\})_{\alpha,\beta = 1,\ldots,2n-s-l} \neq 0. \tag{4.5}$$

Using Proposition 4.4, one can immediate see that condition (4.4) is equivalent to require that $2n - s$ functions a_ρ, $b_\alpha \in \Lambda^0(S)$ exist ($\rho = 1,\ldots,l$; $\alpha = 1,\ldots,2n - s - l$), such that

$$i_{\mathbf{H}} \Omega \approx dh_c + a_\rho \, d\xi^\rho + b_\alpha \, d\chi^\alpha, \tag{4.6}$$

where the sume convention has been used. Realize that the role played by the a_ρ's and b_α's is quite similar to the Lagrange multipliers in many geometrical problems involving the specialization of a differential form onto a submanifold.

Furthermore, condition (i) in definition 4.7 must be also be taken into account. In terms of the constraints, this condition reads:

$$\mathbf{h}\xi^\rho \approx 0 \quad \text{and} \quad \mathbf{H}\chi^\alpha \approx 0,$$

which, according to (4.6), can also be written as

$$\{h_c, \xi^\rho\} \approx 0, \quad \rho = 1,\ldots,l, \tag{4.7}$$

$$\{h_c, \chi^\xi\} + b_\beta \{\chi^\beta, \chi^\alpha\} \approx 0, \quad \alpha, \beta = 1,\ldots,2n - s - l. \tag{4.8}$$

Hence, Eq. (4.7) delimitates the domain of canonical Hamiltonians, and Cramer's linear system (4.8)—remember Eq. (4.5)—determines the Lagrange multipliers b_α, $\alpha = 1,\ldots,2n - s - l$, which are associated with the second-class constraints

$$b_\alpha \approx -c_{\beta\alpha}\{\chi^\alpha, h_c\}, \tag{4.9}$$

where $c_{\beta\alpha}$ is the inverse matrix of $\{\chi^\alpha, \chi^\gamma\}$, i.e.,

$$c_{\beta\alpha}\{\chi^\alpha, \chi^\gamma\} = \delta_\beta{}^\gamma.$$

Substituting (4.9) into (4.6), we arrive at

$$i_{\mathbf{H}} \Omega \approx dh_c + a_\rho \, d\xi^\rho - \{h_c, \chi^\alpha\} c_{\alpha\beta} \, d\chi^\beta, \tag{4.10}$$

where the indeterminacy associated with the first-class constraints appear manifestly.

Summarizing, given a function $h_c \in \Lambda^0(M)$:

(i) The linear system (4.4) will have a nonempty solution if, and only if, condition (4.7) is fulfilled.

(ii) The indeterminacy of the solution $\mathbf{H}$ is related to the arbitrary Lagrange multipliers associated with the first-class constraints.

An explicit expression for $\mathbf{H}$ is

$$\mathbf{H} \approx \{\cdot, h_c^*\} + \{\cdot, a_\rho \xi^\rho\}, \tag{4.11}$$

where the "star function" h_c^* means[21]

$$h_c^* = h_c - \{h_c, \chi^\alpha\} c_{\alpha\beta} \chi^\beta.$$

Definition 4.8: A vector field $\mathbf{H} \in D(M)$ is said to be *locally weakly Hamiltonian* relative to (M, S, Ω) iff

(i) $\mathbf{H} \in \overline{D(S)}$,

(ii) $j_S^*(i_{\mathbf{H}} \Omega)$ is a closed differential form on S.

The latter condition is equivalent to

$$L_{\mathbf{H}^S}(j_S^* \Omega) = 0. \tag{4.12}$$

The proof is similar to the one of Proposition 2.1.

A result analogous to the Poincaré integral theorem holds as well for locally weakly Hamiltonian systems:

Theorem 4.9: Let φ_T be the local one-parameter group generated by a given $\mathbf{H} \in \overline{D(S)}$. Thus, $\mathbf{H}$ is locally weakly Hamiltonian if, and only if,

$$\left[\frac{d}{dt} \int_{\varphi_t \circ c} j_S^* \Omega\right]_{t=0} = 0 \tag{4.13}$$

for any singular 2-cube on $S \subset M$.

We finally have that the generalization of Lee Hwa Chung's theorem proved in the Sec. 3 leads to the following:

Theorem 4.10: Let (M, S, Ω) be a canonical system and $\alpha \in \Lambda^p(M)$ such that

$$L_{\mathbf{H}^S}(j_S^* \alpha) = 0$$

for every locally weakly Hamiltonian field $\mathbf{H}$. Thus we have that:

(i) If either $p > 2R$ or $p = 2l + 1$, $l < R$, then $j_S^* \alpha = 0$.

(ii) If $p = 2l$, $l < R$, then a function $f \in \Lambda^0(M)$ exists such that

$$j_S^*(\alpha - f\Omega \wedge \overset{l}{\cdots} \wedge \Omega) = 0$$

and that $f \circ j$ is a constant on any connected component component of S.

5. CANONICAL TRANSFORMATIONS

Throughout this section, (M, S, Ω) and (M', S', Ω') will denote two given canonical systems and $j_S: S \hookrightarrow M$ and $j_{S'}: S' \hookrightarrow M'$, the natural injections. We shall also assume that $j_S^* \Omega$ is a form of constant rank on S.

Definition 5.1: The map $\Phi: M \to M'$ is said to be a *canonical transformation* of (M, S, Ω) into (M', S', Ω') iff:

(i) Φ is a diffeomorphism;

(ii) $\Phi(S) = S'$;

(iii) the Jacobian map $\Phi_*: T(M) \to T(M')$ maps every locally weakly Hamiltonian field relative to (M, S, Ω) into a locally weakly Hamiltonian field relative to (M', S', Ω').

Thanks to condition (ii), the map Φ induces a diffeomorphism $\varphi: S \to S'$ by merely taking

$$\forall x \in S, \quad \varphi(x) = \Phi(x) \in S'.$$

(We denote this mapping by other symbol than Φ because its domain is not M but S.) Since φ has been purposely defined, we have that the diagram

$$\begin{tikzcd} M \arrow[r, "\Phi"] & M' \\ S \arrow[u, "j_S"] \arrow[r, "\varphi"'] & S' \arrow[u, "j_{S'}"'] \end{tikzcd} \tag{5.1}$$

is commutative.

The aim of this section is to give several characterizations of canonical transformations between constrained systems.

Lemma 5.2: If $\mathbf{H} \in \overline{D(S)}$, then $\Phi_* \mathbf{H} \in \overline{D(S')}$.

Proof: $\forall \xi' \in \Lambda^0(M')$, such that $j_{S'}^* \xi' = 0$, we have, taking the commutativity of diagram (5.1) into account, that

$$j_S^*(\Phi^* \xi') = (j_S^* \circ \Phi^*)(\xi') = \varphi^*(j_{S'}^* \xi') = 0. \tag{5.2}$$

If we now make $\Phi_* \mathbf{H}$ act on any function $\xi' \in \Lambda^0(M')$, such that $j_{S'}^* \xi' = 0$, we obtain

$$\Phi_* \mathbf{H}(\xi') = \mathbf{H}(\Phi^* \xi') = 0$$

because $\mathbf{H} \in \overline{D(S)}$ and $\Phi^* \xi'$ vanishes on S. Hence $\Phi_* \mathbf{H} \in \overline{D(S')}$. ⊠

Theorem 5.3: If conditions (i) and (ii) are satisfied by a given map $\Phi: M \to M'$, then Φ is a canonical transformation if, and only if,

$$\Phi^*(j_{S'}^* \Omega') = c\, j_S^* \Omega, \tag{5.3}$$

where $c \in \Lambda^0(S)$ is a locally constant function.

Proof: $\Phi^*((j_{S'}^*, \Omega'))$ is a differential 2-form in the submanifold S which, at its turn, is endowed with the presymplectic structure given by $j_S^* \Omega$.

Let $\mathbf{H}$ be locally weakly Hamiltonian relatively to (M, S, Ω). According to Lemma 5.2, since $\mathbf{H} \in \overline{D(S)}$, then $\Phi_* \mathbf{H}$ will belong to $\overline{D(S')}$ as well. Furthermore, using Eq. (5.1), it can be seen in an obvious way that the vector field $(\Phi_* \mathbf{H})^{S'}$ associated with $\Phi_* \mathbf{H}$ by Eq. (4.2) is none but $\varphi_*(\mathbf{H}^S)$. Thus, for every $\mathbf{H} \in \overline{D(S)}$, we can write

$$\begin{aligned} L_{\mathbf{H}^S}(\varphi^*(j_{S'}^*, \Omega')) &= \varphi^*\{L_{\varphi_*(\mathbf{H}^S)}(j_{S'}^*, \Omega')\} \\ &= \varphi^*\{L_{(\Phi_* \mathbf{H})^{S'}}(j_{S'}^*, \Omega')\}. \end{aligned} \tag{5.4}$$

(a) If Φ is a canonical transformation, then $\Phi_* \mathbf{H}$ is a locally weakly Hamiltonian field relatively to (M', S', Ω'). Therefore, we have, by Eq. (4.12), that the right-hand side of (5.4) vanishes. Whence the hypothesis of Theorem 4.10 is fulfilled and, consequently, Eq. (5.3) follows.

(b) Conversely, if Eq. (5.3) holds, we then have that

$$L_{\mathbf{H}^S}(\Phi^*(j_{S'}^* \Omega')) = c L_{\mathbf{H}^S}(j_S^* \Omega) = 0. \tag{5.5}$$

That is, the left-hand side of Eq. (5.4) vanishes. Therefore, since φ is a diffeomorphism, we arrive at

$$L_{(\Phi_* \mathbf{H})^{S'}}(j_{S'}^* \Omega') = 0,$$

which means that $\Phi_* \mathbf{H}$ is locally weakly Hamiltonian relatively to (M', S', Ω').

In most cases occurring in analytical dynamics, the symplectic forms Ω and Ω' are not only closed forms but exact as well and can be derived from the respective Liouville forms, $\theta \in \Lambda^1(M)$ and $\theta' \in \Lambda^1(M')$, i.e., $\Omega = d\theta$ and $\Omega' = d\theta'$. Then there follows immediately:

Corollary 5.4: A map $\Phi: M \to M'$, fulfilling conditions (i) and (ii) of Definition 5.1, is a canonical transformation if, and only if, for every $z \in S \subset M$, there exists an open neighborhood $U \subset M$, a function $F \in \Lambda^0(U)$, and a constant c such that

$$j_S^*(\Phi^* \theta' - c\theta - dF) = 0. \tag{5.6}$$

Lemma 5.5: Let $\mathbf{H} \in D(M)$ be a weakly Hamiltonian field and $h \in \Lambda^0(M)$ a Hamiltonian function for $\mathbf{H}$. If $\Phi: M \to M'$ is a canonical transformation, then $h' \equiv (1/c)\Phi^{*-1} h$ is a Hamiltonian function for $\mathbf{H}' \equiv \Phi_* \mathbf{H} \in D(M')$.

Proof: According to the hypothesis and Eq. (4.4), we have that

$$j_{S'}^*(i_{\mathbf{H}}\Omega - dh) = 0. \tag{5.7}$$

Since Φ is canonical, Theorem 5.3 can be applied to express (5.7) as

$$\frac{i}{c}\, i_{\varphi_*^{-1}(\mathbf{H}',S')}\{\varphi^*(j_{S'}^*\,\Omega') - (\varphi^*\circ j_{S'}^*)d(\Phi^{*-1}h)\} = 0$$

or

$$(\varphi^*\circ j_{S'}^*)[i_{(1/c)\mathbf{H}'}\Omega' - d(\Phi^{*-1}h)] = 0.$$

Finally, as φ^* is a diffeomorphism, we arrive at

$$j_{S'}^*[i_{(1/c)\mathbf{H}'}\Omega' - d(\phi^{*-1}h)] = 0, \tag{5.8}$$

which proves the lemma.

Using the Poincaré lemma, a similar result can be proved to hold for locally weakly Hamiltonian fields. Taking Eq. (4.11) into account, we can then use Lemma 5.5 to write $\Phi_*\mathbf{H}$ as

$$\Phi_*\mathbf{H} \underset{S}{=} \{\cdot, c\tilde{h}\}'^* + \{\cdot, \tilde{a}_\alpha\tilde{\xi}^\alpha\}', \tag{5.9}$$

here $\{\ ,\ \}'$ and $\{\ ,\ \}'^*$ are, respectively, the Poisson and Dirac brackets corresponding to (M', S', Ω'), and

$$\tilde{h} \equiv \Phi^{*-1}h, \quad \tilde{a}_\alpha = \Phi^{*-1}a_\alpha, \quad \tilde{\xi}^\alpha = \Phi^{*-1}\xi^\alpha.$$

If we now apply $\mathbf{H}$ to any Hamiltonian function g, we obtain, taking (5.9) and (4.7) into account, that

$$(\mathbf{H}g) \underset{S}{=} \{g, h\}^* \tag{5.10}$$

and, for $\Phi_*\mathbf{H}$ acting on $\tilde{g}$, we also have

$$(\Phi_*\mathbf{H})\tilde{g} \underset{S}{=} c\{\tilde{g}, \tilde{h}\}'^*, \tag{5.11}$$

where $\tilde{g} \equiv \Phi^{*-1}g$.

Comparison of both equations, (5.10) and (5.11), suggest to us the following:

Theorem 5.6: A map $\Phi: M \to M'$ fulfilling conditions (i) and (ii) definition 5.1 is a canonical transformation if, and only if,

(a) For every couple of functions g and h, which are Hamiltonian relative to (M, S, Ω),

$$\{\tilde{g}, \tilde{h}\}'^*\circ\varphi \approx (1/c)\{g, h\}^*. \tag{5.12}$$

(b) If $g \in \Lambda^0(M)$ is Hamiltonian relative to (M, S, Ω), then $\tilde{g} \equiv \Phi^{*-1}g$ is Hamiltonian relative to (M', S', Ω') as well.

Proof: Provided that Φ is a canonical transformation and taking Eqs. (5.10), (5.11) and the commutativity of (5.1) into account, it follows that

$$\begin{aligned}\{g, h\}^* &= j_S^*(\mathbf{H}g) = (\varphi^*\circ j_{S'}^*\circ\Phi^{*-1})(\mathbf{H}g)\\ &= \varphi^*\circ j_{S'}^*[(\Phi_*\mathbf{H})(\tilde{g})]\\ &= c\{\tilde{g}, \tilde{h}\}'^*,\end{aligned}$$

and (a) is proved.

Statement (b) follows immediately applying Lemma 5.5.

Let us now conversely assume (a) and (b) and try to prove Φ is a canonical transformation. According to Eq. (5.12), for every weakly Hamiltonian vector field $\mathbf{H} \in D(M)$ with the associated Hamiltonian function $h \in \Lambda^0(M)$, and every Hamiltonian function $g \in \Lambda^0(M)$, we have that

$$[(\Phi_*\mathbf{H} - c\mathbf{X}_{\tilde{h}})\tilde{g}]\circ j_{S'}^* = 0, \tag{5.13}$$

where $\mathbf{X}_{\tilde{h}} \in \overline{D(S')}$ is a solution of

$$j_{S'}^*(i_{\mathbf{X}}\Omega' - d\tilde{h}) = 0. \tag{4.4'}$$

The assumption (b) guarantees the existence of such a solution $\mathbf{X}_{\tilde{h}}$. It is also this assumption which ensures that $\tilde{g} \in \Lambda^0(M)$ will be Hamiltonian relative to (M', S', Ω'). From Eq. (5.13) we can therefore state that

$$[(\Phi_*\mathbf{H} - c\mathbf{X}_{\tilde{h}})f]\circ j_{S'} = 0. \tag{5.14}$$

Finally, taking Proposition 2.6 into account, Eq. (5.14) implies immediately that

$$(\Phi_*\mathbf{H} - c\mathbf{X}_{\tilde{h}})\circ j_{S'} \in \mathscr{A}(j_{S'}^*\Omega')$$

or

$$\Phi_*\mathbf{H} \underset{S'}{=} c\mathbf{X}_{\tilde{h}} + \mathscr{A}(j_S^*\Omega')$$

which, by inner product with $j_{S'}^*\Omega'$, yields

$$j_{S'}^*(i_{\Phi_*\mathbf{H}}\Omega') = c\,j_{S'}^*(i_{\mathbf{X}_{\tilde{h}}}\Omega') = c\,j_{S'}^*\,d\tilde{h} = j_{S'}^*\,d(c\tilde{h}),$$

whence $\Phi_*\mathbf{H} \in D(M')$ is a weakly Hamiltonian field relatively to (M', S', Ω') and $c\tilde{h}$ is an associated Hamiltonian function. ⊠

Theorem 5.6 will be a useful tool to characterize canonical transformations of constrained Hamiltonian systems in terms of the Dirac brackets. Indeed, let us consider a Shanmugadhasan set of coordinates and momenta $q_a, p_B; Q_f^1, P_g^1; Q_k^2, P_h^2$ $(a, b = 1,...,r; f, g = 1,...,l; k, h = 1,...,t)$ defined on some open domain $U \subset M$. In terms of this coordinates the submanifold S is defined by

$$\begin{array}{lll}\text{(first class)} & P_g^1 \approx 0, & g = 1,...,l,\\ \text{(second class)} & Q_k^2 \approx P_h^2 \approx 0, & k, h = 1,...,t\end{array}$$

(r and t are related to the dimension $2n$ and s by $2r = s - l$ and $2t = 2n - s - l$), the differential 2-forms Ω and $j_S^*\Omega$ can be written as

$$\Omega = \sum_{a=1}^{r} dq_a \wedge dp_a + \sum_{f=1}^{l} dQ_f^1 \wedge dP_f^1 + \sum_{k=1}^{t} dQ_k^2 \wedge dP_k^2$$

and

$$j_S^*\Omega = \sum_{a=1}^{r} dq_a \wedge dp_a$$

and, according to (4.7), Hamiltonian functions are characterized by

$$\left(\frac{\partial F}{\partial Q_g^1}\right)_{(q,p;Q^1,0,0,0)} = 0, \quad g = 1,...,l,$$

that is, the elementary Hamiltonian functions are q_a, p_B, P_f^1 $(a, b = 1,...,r; f = 1,...,l)$. (Realize that, although the second-class constraints Q_h^2, P_k^2 also satisfy the latter relations, they are not relevant since their Dirac brackets with any other function vanish.)

In the most cases of interest in mathematical physics, a canonical transformation Φ acts from (M, S, Ω) into itself. In terms of the above set of Shanmugadhasan variables, Φ is expressed by $2n$ functions $\tilde{z}_\alpha(z_\beta)$ $(\alpha, \beta = 1,...,2n)$, where z_β denotes generically the coordinates and momenta $q, p; Q^1, P^1; Q^2, P^2$.

Theorem 5.6 then states that Φ is a canonical transformation of (M, S, Ω) into itself if, and only if,

(i) $\tilde{Q}^2_k(q,p;Q^1,0;0,0) = 0,$

$$\tilde{P}^2_k(q,p;Q^1,0;0,0) = 0, \quad k,h = 1,\ldots,t, \tag{5.15}$$

$$\tilde{P}^1_g(q,p;Q^1,0;0,0) = 0, \quad g = 1,\ldots,l$$

that is, $\Phi(S) = S$.

(ii) Φ transforms Hamiltonian functions into Hamiltonian functions, that is,

$$\{\tilde{q}_a, P^1_g\} \approx 0, \quad \{\tilde{p}_b, P^1_g\} \approx 0, \quad \{\tilde{P}^1_f, P^1_g\} \approx 0. \tag{5.16}$$

(iii) There exists a constant $c \in R$ such that

$$\{\tilde{q}_a, \tilde{p}_b\}^* \approx c\delta_{ab} \tag{5.17}$$

and

$$\{\tilde{q}_a, \tilde{q}_b\}^* \approx \{\tilde{p}_a, \tilde{p}_b\}^* \approx \tilde{P}^1_f, \tilde{P}^1_g\}^*$$
$$\approx \{\tilde{q}_a, \tilde{P}^1_f\}^* \approx \{\tilde{p}_b, \tilde{P}^1_f\}^* \approx 0. \tag{5.18}$$

6. CONCLUSION AND OUTLOOK

In this paper we have studied in depth constrained Hamiltonian systems. In particular, we have introduced the concepts of Hamiltonian function and Hamiltonian vector field relative to a given constrained Hamiltonian system.

The Hamiltonian study of a dynamical system leads to the concept of canonical transformation in a natural way (i.e., that which preserves the canonical formalism). In the nondegenerate formalism, the invariant integral of Poincaré[8] and the theorem of Lee Hwa Chung[12] permit us to characterize canonical transformations as those which preserve the symplectic structure apart from a multiplicative constant ($\Phi^*\Omega' = c\Omega$, $c \in \mathbb{R}$) or, equivalently, as those preserving the Poisson brackets (except for a constant factor).

In a similar way, we have here derived a generalization of Lee Hwa Chung's theorem for constrained Hamiltonian systems which, insofar as the generalization of the Poincaré integral invariant has been already obtained, has allowed us to characterize canonical tranformations of a given constrained system into itself as those preserving: (i) the submanifold S and (ii) the induced presymplectic structure on S [i.e., $\Phi^*(j^*_S\Omega) = c\, j^*_S\Omega$; $c \in \mathbb{R}$].

We have also derived a formulation for the latter condition in terms of the elementary Dirac brackets.

Previous attempts to generalize the concept of canonical transformations for a constrained system (M, S, Ω) were based on the condition of preserving the Poisson bracket on the constraint submanifold S. In the present work we have intended to clarify that this condition is too restrictive. This is owing to the fact that the physical system is completely represented in the presymplectic manifold $(S, j^*_S\Omega)$ and the degrees of freedom beyond S are physically irrelevant. On the contrary, in order to avoid unphysical restrictions, our study of canonical tranformations starts from the most primary characteristic, that is, to transform weakly Hamiltonian vector fields into weakly Hamiltonian vector fields. Then, the results obtained in Secs. 2, 3, and 4 permit us to give more elaborated characterizations of canonical transformations for constrained systems, either in terms of the presymplectic form $j^*_S\Omega$ or in terms of the Dirac brackets and weakly Hamiltonian functions.

It is finally interesting to remark that a good definition for canonical transformations will be of great help to study the symmetries of a given constrained Hamiltonian system, and a forthcoming paper will be devoted to this problem.

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APPENDIX

Throughout this appendix $\mathscr{E}$ is a finite dimensional linear space ($\dim \mathscr{E} = s$) and $\omega \in \Lambda^2(\mathscr{E})$ is an alternated 2-form.

The 2-form ω defines the linear mapping

$$\mathscr{E} \to \mathscr{E}^* = \Lambda^1(\mathscr{E}), \tag{A1}$$
$$\mathbf{V} \rightsquigarrow i_{\mathbf{V}}\omega,$$

the kernel of which is $\mathscr{A}(\omega) = \{\mathbf{V} \in \mathscr{E} / i_{\mathbf{V}}\omega = 0\}$. Also, it can be easily proved that its image is

$$i_{\mathscr{A}}\omega = \mathscr{A}^\perp(\omega) \equiv \{\lambda \in \mathscr{E}^* / i_{\mathbf{V}}\lambda = 0, \ \forall \mathbf{V} \in \mathscr{A}(\omega)\} \tag{A2}$$

The rank of ω is defined by

$$\operatorname{rank} \omega = \dim \mathscr{A}^\perp(\omega), \tag{A3}$$

which is always an even number

$$\operatorname{rank} \omega = 2R \leqslant s. \tag{A4}$$

Let $\Lambda^p(\mathscr{E}/\mathscr{A}(\omega))$ be the space of alternated p-forms on the quotient space $\mathscr{E}/\mathscr{A}(\omega)$, and let us consider the mapping:

$$\Lambda^p(\mathscr{E}/\mathscr{A}(\omega)) \to \Lambda^p(\mathscr{E}),$$
$$\alpha \rightsquigarrow \tilde{\alpha}$$

such that

$$\forall \mathbf{V}_1,\ldots,\mathbf{V}_p \in \mathscr{E}, \quad \tilde{\alpha}(\mathbf{V}_1,\ldots,\mathbf{V}_p) \equiv \alpha([\mathbf{V}_1],\ldots,[\mathbf{V}_p]),$$

where $[\mathbf{V}_i] \in \mathscr{E}/\mathscr{A}(\omega)$ is the class of $\mathbf{V}_i \in \mathscr{E}$.

Proposition A.1: The mapping $\alpha \to \tilde{\alpha}$ defines an isomorphism between $\Lambda^p(\mathscr{E}/\mathscr{A}(\omega))$ and

$$\Lambda^{p\perp}(\omega) \equiv \{\gamma \in \Lambda^p(\xi) / i_{\mathbf{V}}\gamma = 0,$$
$$\forall \mathbf{V} \in \mathscr{A}(\omega)\} \subset \Lambda^p(\mathscr{E}). \tag{A5}$$

The latter result is an immediate consequence of proposition (I.1.12) of Ref. 19.

Since

$$\dim(\mathscr{E}/\mathscr{A}(\omega)) = \dim \mathscr{A}^\perp(\omega) = 2R, \tag{A6}$$

we have that $p > 2R \Rightarrow \Lambda^p(\mathscr{E}/\mathscr{A}(\omega)) = \{0\}$, which, by Proposition A.1, implies the following:

Corollary A.2: If $\alpha \in \Lambda^p(\mathscr{E})$, $p > 2R$ and $i_{\mathbf{V}}\alpha = 0$, for any $\mathbf{V} \in \mathscr{A}(\omega)$, then $\alpha = 0$.

Now, let $\mathscr{F}$ be a linear subspace of $\mathscr{E}$ ($\dim \mathscr{F} = m$).

Proposition A.3: If $\gamma \in \Lambda^p(\mathscr{F})$, $p < s - m$, and $\lambda \wedge \gamma = 0$, for any $\lambda \in \mathscr{F}^\perp$, then $\gamma = 0$.

Proof: Let us choose a basis $\epsilon^1,\ldots,\epsilon^{s-m}$ of $\mathscr{F}^\perp$ (i.e., the subspace of 1-forms annihilating $\mathscr{F}$). There will exist $s - m$ vectors

$$\mathbf{X}_1,\ldots,\mathbf{X}_{s-m} \in \mathscr{E}$$

such that

$$i_{\mathbf{X}_j}\epsilon^l = \delta^l_j, \quad l,j = 1,\ldots,s-m.$$

By the hypothesis $\epsilon^j \wedge \gamma = 0, j = 1,\ldots,s-m$, and taking into account that

$$i_{\mathbf{X}_j}(\epsilon^j \wedge \gamma) = \gamma - \epsilon^j \wedge i_{\mathbf{X}_j}\gamma, \quad j = 1,\ldots,s-m,$$

we conclude

$$\gamma = \epsilon^j \wedge i_{\mathbf{X}_j}\gamma, \quad j = 1,\ldots,s-m,$$

whence it immediately follows that

$$\gamma = \epsilon^p \wedge \epsilon^j \wedge (i_{\mathbf{X}_j} i_{\mathbf{X}_p}\gamma), \quad j,l = 1,\ldots,s-m.$$

A recursive application of this method leads to

$$\gamma = 0 \quad \text{if} \quad p < s-m$$

or

$$\gamma = \epsilon_1 \wedge \cdots \epsilon^{s-m} \wedge (i_{\mathbf{X}_{s-m}} \cdots i_{\mathbf{X}_1}\gamma) \quad \text{if } p \geqslant s-m.$$

Particularizing the latter proposition for the cases $\mathscr{F} = 0$ and $\mathscr{F} = \mathscr{A}(\omega)$ we have, respectively:

Corollary A.4: If $\gamma \in \Lambda^p(\mathscr{E})$, $p < s$, and $\gamma \wedge \gamma = 0$, for any $\lambda \in \mathscr{E}^*$, then $\gamma = 0$.

Corollary A.5: If $\gamma \in \Lambda^p(\mathscr{E})$, $p < 2R = \operatorname{rank} \omega$, and $\lambda \wedge \gamma = 0$ for any $\lambda \in \mathscr{A}^\perp(\omega)$, then $\gamma = 0$.

Finally, we are going to prove the main result of this appendix which is needed in the proof of Theorem 3.1.

Proposition A.6: If $\mathscr{A} \in \Lambda^p(\mathscr{E})$, $p \leqslant 2R$, satisfies

(i) $\forall \mathbf{V} \in \mathscr{A}(\omega), \quad i_{\mathbf{V}}\alpha = 0,$ (A7)

(ii) $\forall \mathbf{X}, \mathbf{Y} \in \mathscr{E}, \quad i_{\mathbf{X}}\omega \wedge i_{\mathbf{Y}}\alpha + i_{\mathbf{Y}}\omega \wedge i_{\mathbf{X}}\alpha = 0,$ (A8)

then, either $p = 1$, which implies $\alpha = 0$, or $p > 1$, which implies that there exists $\alpha^{(1)} \in \Lambda^{p-2}(\mathscr{E})$, fulfilling (A7) and (A8), such that

$$\forall \mathbf{X} \in \mathscr{E}, \quad i_{\mathbf{X}}\alpha = i_{\mathbf{X}}\omega \wedge \alpha^{(1)}.$$

Proof: In accordance with Eq. (A2), we have that, given any $\lambda \in \mathscr{A}^\perp(\omega)$, there exists $\mathbf{V} \in \mathscr{E}$ such that $\lambda = i_{\mathbf{V}}\omega$.

Condition (A6) and Corollary A.4 ensure that, given any two $\mathbf{V}_1, \mathbf{V}_2, \in \mathscr{E}$ such that $i_{\mathbf{V}_1}\omega = i_{\mathbf{V}_2}\omega = \lambda$; then

$$i_{\mathbf{V}_1}\alpha = i_{\mathbf{V}_2}\alpha. \tag{A9}$$

Therefore, there is no ambiguity in defining the mapping:

$$\begin{aligned} &\Phi: \mathscr{A}^\perp(\omega) \to \Lambda^{p-1}(\mathscr{E}), \\ &\lambda \rightsquigarrow \Phi(\lambda) = i_{\mathbf{V}}\alpha, \end{aligned} \tag{A10}$$

$\mathbf{V}$ being any vector such that $i_{\mathbf{V}}\omega = \lambda$. Then, taking $\mathbf{X} = \mathbf{Y}$ in Eq. (A8), we obtain

$$\forall \mathbf{X} \in \mathscr{E}, \quad i_{\mathbf{X}}\omega \wedge i_{\mathbf{X}}\alpha = 0,$$

which, due to Eq. (A2), is equivalent to

$$\lambda \wedge \Phi(\lambda) = 0, \quad \forall \lambda \in \mathscr{A}^\perp(\omega).$$

This implies that either (i) $\Phi(\lambda) = 0$ if $p = 1$ or (ii) there exists $\alpha^{(1)} \in \Lambda^{p-2}(\mathscr{E})$, such that

$$\Phi(\lambda) = \lambda \wedge \alpha^{(1)} \quad \text{if } p > 1.$$

Recalling now (A10), we see that this is equivalent to either (i) $\alpha = 0$, if $p = 1$, or (ii) there exists $\alpha^{(1)} \in \Lambda^{p-2}(\mathscr{E})$ such that

$$i_{\mathbf{X}}\alpha = i_{\mathbf{X}}\omega \wedge \alpha^{(1)}, \quad \forall \mathbf{X} \in \mathscr{E} \quad \text{if} \quad p > 1. \tag{A11}$$

To complete the proof, we must still check whether $\alpha^{(1)}$ fulfills conditions (A7) and (A8).

By inner product of both sides of Eq. (A11) with any $\mathbf{V} \in \mathscr{E}$, and, since $i_{\mathbf{X}}$ and $i_{\mathbf{V}}$ anticommute, we obtain that

$$i_{\mathbf{V}}\omega \wedge i_{\mathbf{X}}\alpha^{(1)} + i_{\mathbf{X}}\omega \wedge i_{\mathbf{V}}\alpha^{(1)} = 0, \quad \forall \mathbf{V}, \mathbf{X} \in \mathscr{E}. \tag{A12}$$

That is, condition (A8) holds good for $\alpha^{(1)}$ also.

Taking $\mathbf{V} \in \mathscr{A}(\omega)$ in Eq. (A12), it yields

$$i_{\mathbf{X}}\omega \wedge i_{\mathbf{V}}\alpha^{(1)} = 0, \quad \forall \mathbf{X} \in \mathscr{E}, \quad \forall \mathbf{V} \in \mathscr{A}(\omega),$$

which, by virtue of Eq. (A2), is equivalent to

$$\lambda \wedge i_{\mathbf{V}}\alpha^{(1)} = 0, \quad \forall \lambda \in \mathscr{A}^\perp(\omega), \ \forall \mathbf{V} \in \mathscr{A}(\omega);$$

applying corollary A.5, it implies that

$$i_{\mathbf{V}}\alpha^{(1)} = 0, \quad \forall \mathbf{V} \in \mathscr{A}(\omega),$$

and the proof is completed.

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