Poincaré–Cartan integral invariant and canonical transformations for singular Lagrangians: An addendum

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The results of a previous work, concerning a method for performing the canonical formalism for constrained systems, are extended when the canonical transformation proposed in that paper is explicitly time dependent.

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In a previous paper we discussed in the framework of the Poincaré–Cartan integral invariant, a method for performing the canonical formalism for constrained systems. The basic idea consists of considering a canonical transformation which brings the constraints into a subset of the canonical variables. Thus the physical variables can be easily obtained by means of a reduction of the phase space. Our method is different from the path-integral approach of Faddeev (see also Ref. 3), which use in addition a set of gauge-fixing conditions, one for each first-class constraint. Two applications of our procedure concerning action-at-a-distance relativistic models have been recently studied. In this note we extend the method by considering a time-dependent general canonical transformation, such that all the constraints acquire an explicit time dependence.

Let us consider a dynamical system described in terms of degrees of freedom in the phase space \( q_a, \phi_a \, (s = 1,...,n) \) and constrained to the hypersurface \( S \) defined by

\[
\Omega_\alpha(q,\phi) = 0 \quad (\alpha = 1,...,T - W),
\]

\[
\Omega_\beta(q,\phi) = 0 \quad (\beta = T - W + 1,...,T),
\]

where \( \Omega_\alpha \) are \( T - W \) first-class \(^5\) and \( \Omega_\beta \) \( W \) second-class constraints. In order to guarantee the stability of \( S \) during the evolution, the \( \Omega_\alpha \) are required to satisfy

\[
[\Omega_\alpha, H_\alpha] \simeq 0,
\]

where \( H_\alpha \) is the canonical Hamiltonian. The notation \( \simeq \) means equality on the hypersurface \( S \) (“weak” equality).

Now, given the set (2), according to some theorems on function groups \(^6\) and involutory systems \(^7\) it is possible, at least locally, to find a canonical transformation

\[
[q_a, \phi_a, \, s = 1,...,n] \rightarrow [Q^{s}, P^{s}, \, s = 1,...,n],
\]

such that the equations

\[
Q^{s}_{;f} = P^{s}_{;f} = 0 \quad (f = n_2 + 1,...,n_2),
\]

define the same surface as Eqs. (2) and the following equations,

\[
[Q^{s}, P^{s}_{;f}] = \delta^{s}_{f},
\]

\[
[Q^{s}, Q^{s}_{;f}] = [P^{s}, P^{s}_{;f}] = 0,
\]

are identically (and not only “weakly”) satisfied.

If we denote the generating function by \( F \), defined as

\[
p_{;f} \delta q_{a} - H_{;\beta} dt = P_{;f} \delta Q^{s}_{;f} - K_{;\beta} dt - \delta F,
\]

the Hamilton equations for the new variables are given by

\[
Q^{s}_{;f} \simeq [Q^{s}, K(Q^{s}, P^{s}_{;f}, t)],
\]

\[
P_{;f} \simeq [P^{s}, K(Q^{s}, P^{s}_{;f}, t)]
\]

where \( K \),

\[
K = K_{e} + l_{\alpha} \tilde{\Omega}_{\alpha} - \tilde{\Omega}_{\beta} \tilde{C}_{\beta} \left[ \tilde{\Omega}_{\beta}, K_{e} \right] + \frac{\partial \tilde{\Omega}_{\beta}}{\partial t},
\]

is the extended Hamiltonian with \( l_{\alpha} \) arbitrary functions. \( \tilde{\Omega}_{\beta} \) are obtained from Eqs. (1) and (2) by substitution of variables, and \( \tilde{C}_{\beta} \) is defined by

\[
\tilde{C}_{\beta} [\tilde{\Omega}_{\beta}, \tilde{\Omega}_{\alpha}] \simeq \delta_{\beta\alpha}.
\]

In I we have shown that it is possible to write the equations of motion for the reduced set of variables \( R^{s} = [Q^{s}, P^{s}_{;f}, j = 1,...,n_{2}] \) which are free with respect to the second-class constraints (5) in a simple form

\[
Q^{s}_{;f} \simeq [Q^{s}, \tilde{K}^{s}_{;f}]_{R^{s}},
\]

\[
P_{;f} \simeq [P^{s}, \tilde{K}^{s}_{;f}]_{R^{s}},
\]

\[
\tilde{K}^{s} = \frac{\partial Q^{s}_{;f}}{\partial P^{s}_{;f}} = \tilde{K}^{s}(Q^{s}, P^{s}_{;f}, t) + l_{\alpha} \tilde{\Omega}_{\alpha}(Q^{s}, P^{s}_{;f}, t)
\]

where \( \cdot , \cdot \) \( R^{s} \) denote the Poisson brackets defined on the space \( R^{s} \) and \( \tilde{\Omega}_{\alpha} \) are obtained by setting equal to zero the variables \( Q^{s} \) and \( P^{s}_{;f} \) corresponding to the second-class constraints, in \( K \) and \( \tilde{\Omega}_{\alpha} \) of Eq. (9). As shown in I the \( \tilde{\Omega}_{\alpha} \) so obtained are first class, i.e.,

\[
[\tilde{\Omega}_{\alpha}, \tilde{\Omega}_{\beta}]_{R^{s}} \simeq 0
\]

and, as a consequence of \( d / dt \Omega_{\alpha} (q, \phi) \simeq 0 \), satisfy the stability condition

\[
\frac{d}{dt} \tilde{\Omega}_{\alpha} = \frac{\partial \tilde{\Omega}_{\alpha}}{\partial t} + [\tilde{\Omega}_{\alpha}, \tilde{K}]_{R^{s}} \simeq 0.
\]

In Eq. (14) we have now supposed the \( \tilde{\Omega}_{\alpha} \) explicitly time dependent, unlike what we did for the sake of simplicity in I.

A similar procedure of reduction of the phase space can be performed also for the first-class constraints. In fact, a theorem on involutory systems \(^7\) guarantees that it is possible, at least locally, to replace the \( \tilde{\Omega}_{\alpha} \) by an equivalent set of equations

\[
P_{;e} (Q^{s}, P^{s}_{;f}, t) = 0 \quad (e = n_{1} + 1,...,n_{1}),
\]

\( (n_{1} = n - T + W / 2) \), which are in involution. For instance, the set (15) can be obtained by solving the equations

\[

\frac{d}{dt} \tilde{\Omega}_{\alpha} = \frac{\partial \tilde{\Omega}_{\alpha}}{\partial t} + [\tilde{\Omega}_{\alpha}, \tilde{K}]_{R^{s}} \simeq 0.
\]
with respect to an equal number $n_2 - n_1$ of momenta. Without loss of generality we suppose Eq. (16) be solved with respect to $P_j^e (e = n_1 + 1, \ldots, n_2)$, or

$$ \frac{\partial \hat{\alpha}_a}{\partial P_j^e} \neq 0. $$

(17)

Let

$$ P_j^e = f_j (Q_j^e, Q_k^e, P_j^e, t) \quad (k = 1, \ldots, n_1) $$

(18)

be the expression of the equations in involution. The stability of the hypersurface (18) can be easily proved. In fact, from

$$ \hat{\alpha}_a (Q_j^e, Q_k^e, P_j^e, P_k^e, t) = 0 $$

(19)

we get

$$ \frac{\partial \hat{\alpha}_a}{\partial t} \approx - \frac{\partial \hat{\alpha}_a}{\partial P_j^e} \frac{\partial P_j^e}{\partial t} - \frac{\partial \hat{\alpha}_a}{\partial P_k^e} \frac{\partial P_k^e}{\partial t}, $$

(20)

$$ - \{ P_j^e, \hat{\alpha}_a \}_K^e \approx - \frac{\partial \hat{\alpha}_a}{\partial Q_j^e} \frac{\partial Q_j^e}{\partial t} - \frac{\partial \hat{\alpha}_a}{\partial P_j^e} \frac{\partial P_j^e}{\partial t}, $$

(21)

Therefore, from Eq. (14) we get

$$ \frac{\partial \hat{\alpha}_a}{\partial P_j^e} \frac{\partial P_j^e}{\partial t} \approx 0, $$

(22)

and using Eq. (17)

$$ \frac{\partial P_j^e}{\partial t} + \{ P_j^e, \hat{\alpha}_a \}_K^e \approx 0. $$

(23)

As a final step we make a transformation

$$ \{ Q_j^e, P_j^e \} = \delta_{e,1}, \quad \{ Q_j^e, P_k^e \} = \delta_{e,k}, $$

(24)

where part of the momenta are the set of functions in the involution (18) which are equivalent to the first-class constraints.

If we denote the new canonical Hamiltonian by $K^e$, and the new expression for the constraints by

$$ \hat{\alpha}_a (Q_k^e, P_k^e, Q_e^e, P_e^e, t), $$

(25)

the Hamiltonian equations are given by

$$ \hat{\dot{Q}}_k \approx \{ Q_k^e, K^e \} + \{ Q_k^e, \hat{\alpha}_a \}_K^e, $$

(26)

$$ \hat{\dot{P}}_k \approx \{ P_k^e, K^e \} + \{ P_k^e, \hat{\alpha}_a \}_K^e, $$

(27)

$$ \hat{\dot{Q}}_e \approx \{ Q_e^e, K^e \} + \{ Q_e^e, \hat{\alpha}_a \}_K^e, $$

(28)

$$ \hat{\dot{P}}_e \approx \{ P_e^e, K^e \} + \{ P_e^e, \hat{\alpha}_a \}_K^e, $$

(29)

where now $\{ , \}_K^e$ denote the Poisson brackets with respect to the set

$$ R = \{ Q_k^e, P_k^e, Q_e^e, P_e^e, k = 1, \ldots, n_1, e = n_1 + 1, \ldots, n_2 \}. $$

With respect to the stability of the hypersurface $\hat{\alpha}_a = 0$, after the canonical transformations (24) we have

$$ \frac{\partial \hat{\alpha}_a}{\partial t} + \{ \hat{\alpha}_a, K^e \}_K \approx 0. $$

(30)

On the other hand, due to the equivalence between $\hat{\alpha}_a$ and $P_k^e$ we may write

$$ \hat{\alpha}_a (Q_k^e, P_k^e, Q_e^e, P_e^e, t) = g_{ae} (Q_k^e, P_k^e, Q_e^e, P_e^e, t) P_e^e, \quad \text{det} \ g \neq 0, $$

(31)

where we introduced the strong equality notation $\approx$ following Sudarshan and Mukunda.\(^8\)

Thus from Eq. (30) we have

$$ \frac{\partial \hat{\alpha}_a}{\partial t} \approx 0, $$

(32)

and using Eqs. (30) and (29) in Eq. (28), we get

$$ \hat{\dot{P}}_e \approx \{ P_e^e, K^e \} + \lambda_e, $$

(33)

where $\lambda_e = g_{ae} l_a$. Then we may write

$$ \hat{\dot{Q}}_e = \{ Q_e^e, K^e \} + \lambda_e, $$

(34)

Finally, the remaining equations (27) and (28) become

$$ \hat{\dot{Q}}_k \approx \{ Q_k^e, K^e \} + \{ Q_k^e, \hat{\alpha}_a \}_K^e, $$

(35)

$$ \hat{\dot{P}}_k \approx \{ P_k^e, K^e \} + \{ P_k^e, \hat{\alpha}_a \}_K^e, $$

(36)

where the $Q_e$ dependence disappears due to Eq. (32) and the $Q_k$'s are gauge-dependent variables.

In conclusion, we have isolated the set of the gauge-dependent variables $Q_e$ from a set of physical (gauge-independent) variables $Q_k, P_k$.\(^9\)

\(^1\)D. Dominici and J. Gomis, J. Math. Phys. 21, 2124 (1980); from now on we will call it I.


\(^5\)P. A. M. Dirac, Can. J. Math. 2, 129 (1950); see also "Lectures on Quantum Mechanics", Belfer Graduate School of Science, Yeshiva University, New York, 1954.

\(^6\)L. P. Eisenhart, Continuous Groups of Transformations (Dover, New York, 1961), Chap. VI.
