General Interaction Picture from Action Principle for Mechanics

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In this paper we consider a general action principle for mechanics written by means of the elements of a Lie algebra. We study the physical reasons why we have to choose precisely a Lie algebra to write the action principle. By means of such an action principle we work out the equations of motion and a technique to evaluate perturbations in a general mechanics that is equivalent to a general interaction picture. Classical or quantum mechanics come out as particular cases when we make realizations of the Lie algebra by derivations into the algebra of products of functions or operators, respectively. Later on we develop in particular the applications of the action principle to classical and quantum mechanics, seeing that in this last case it agrees with Schwinger's action principle. The main contribution of this paper is to introduce a perturbation theory and an interaction picture of classical mechanics on the same footing as in quantum mechanics.

1. INTRODUCTION

We present in this paper a general action principle for mechanics, valid for classical or quantum problems. From such a principle the equations of motion may be derived, but its main application is the possibility of deducing an interaction picture, valid quite generally, from which perturbation expansions can be obtained. In particular, of course, we get a perturbation method for the two kinds of mechanics mentioned above.

We look for the "intersection" of the various dynamical structures in a common formalism. This common abstract mathematical structure is that of the realizations of a Lie algebra $\mathcal{L}$, by derivations in an associative linear algebra D. All dynamical theories can be unified in the above-mentioned manner, since they have enough features in common. We start from an initially very general presentation of the dynamical principle to obtain, later on, as realization of our principle, action principles for each one of the mentioned mechanics. But the main aim of this paper is the application of this technique to the evaluation of perturbations.[1] The elements of the Lie algebra are abstract mathematical entities isomorphically associated with the physical dynamical variables.

Let us examine the case for quantum mechanics. If we have only one irreducible representation of the "algebra of observables," all relevant information of the theory is contained in the algebraic structure alone. Hilbert space representations are not needed since they add nothing to our knowledge of the physical world: this is certainly the case when the number of degrees of freedom is finite. We may say, therefore, that for ordinary quantum mechanics, the purely algebraic approach should prevail. However, in quantum field theories we have infinitely many degrees of freedom, and it is well known that there exist, indeed, many inequivalent irreducible representations of the same algebra. Nevertheless, the differences between inequivalent representations of dynamics in quantum field theory are too fine and they do not have any physical importance. Any faithful representation of the algebra of observables will give the same physical results, and therefore, none of them is needed. Whether the number of degrees of freedom of quantum mechanics is finite or infinite, our discussion shows that the answer that we find is in favor of the purely algebraic approach. We conclude that all faithful representations are "physically" equivalent, even though they may be mathematically strong inequivalent, and conclude that none of them is needed.[2]

The vector space of the Lie algebra of the general dynamical structure of mechanics has a dual space whose elements are called states. The states determine the mapping of the Lie algebra $\mathcal{L}$ onto the field of real numbers, which are the elements that can be compared with the physical reality. They correspond to the expectation values of the observables for a state—a vector in Hilbert space—that are commonly used in quantum mechanics. The selection of a particular (faithful) representation is a matter of convenience without physical implications. It may provide a more or less handy analytical apparatus.

We can find many mappings of a Lie algebra into the field of real numbers. It is, therefore, possible to define states in many different ways, and so we can have many kinds of mechanics from the same dynamical Lie algebra structure. To obtain classical or quantum mechanics we have to specify clearly what kind of mapping has to be used for each case. However, a Lie algebra may have additional mappings,

[1] E. C. Sudarshan, *Lectures in Theoretical Physics, 1961 Brandeis Summer Institute* (W. A. Benjamin, Inc., New York, 1962), p. 144.

[2] R. Haag and D. Kastler, J. Math. Phys. 5, 848 (1964).

unexplored by physics as yet, into the field of real numbers, that eventually may generate another kind of mechanics. Of course, we can compare, and we here do so, the action principle presented in this paper only with action principles and perturbation methods for the two kinds of mechanics mentioned that are the ones used in physical problems. But we hope that the action principle presented is valid more generally, even though we are not able at the present time to check these further applications.

We do not study in this paper classical or quantum statistical mechanics, because we are essentially concerned with dynamics and they offer nothing new to the action principle that we present. Statistical mechanics differs from other kinds of mechanics not in the action principle but in the mapping of the elements of the Lie algebra into the field of real numbers; that is done by means of density operators or distribution functions, kinematical aspects to which we do not pay special attention here.

Dynamical variables and states are duals to each other. In a most general sense, states are the mappings of the Lie algebra onto the field of real numbers. Besides the action principle, which is purely dynamical, there is another aspect in all mechanics—namely, the choice of admissible states belonging to the dual space of the dynamical Lie algebra—a kinematical aspect that limits the mappings onto the field of real numbers which have physical meanings. Generally, there are additional requirements, most frequently imposed to preserve the meaning of probability, so that not every element of the vector dual space is an admissible physical state. The admissible states form a manifold that usually has to be convex, in order not to have negative probabilities. This manifold of states is in general not a subspace because the convexity conditions limit the number of admissible linear combinations that one may make. The natural determination of the admissible manifold of states imposes additional conditions to the Lie algebra $\mathfrak{L}$, or to its realizations into another linear associative algebra D, by means of derivations.

To determine the convex manifold of states, which is physically admissible, further additional information not included in the Lie algebra specifications is needed. The convex manifold of states must be so chosen that, in a Schrödinger-like picture of dynamics, the changes compatible with the action principle will not throw them out of the admissible manifold.

We do not study in this paper a Schrödinger-like picture of dynamics but rather a Heisenberg-like picture of dynamics deduced from the action principle that we here introduce.

There are dynamical theories which have to be Lorentz-covariant. Physically we have to require that for every element of the Poincaré group an automorphism of the algebra has to be introduced. The requirement that the Lorentz transformations be represented by unitary operators in Hilbert space for quantum mechanics is a very powerful restriction that may not be completely justified on physical grounds,[3] and in the same way, intimately connected with the action principle are questions about symmetry properties of the physical system. This means that a Lie algebra may have additional, unexplored structural features, the existence of which is inherent in the special form of its action element.

In Sec. 2 we present as a postulate the general action principle for a quite general mechanics without specifying whether it is classical or quantum mechanics. The action principle is written by means of the elements of an abstract algebra that is a Lie algebra. We examine immediately which is the physical meaning of all the properties of the Lie bracket multiplication. We apply the action principle to obtain the equations of motion and to arrive at an interaction picture in a general scheme of mechanics. Later on we examine the consistency requirements between both applications—for deduction of the equations of motion and for the evaluation of perturbations—of the action principle.

In Sec. 3 we make concrete the realization of the action principle into the algebra that is proper for classical mechanics. A perturbation theory valid for classical mechanics is presented as deduced from our action principle. In Sec. 4 we do the same for quantum mechanics; in particular we observe how Schwinger's action principle can be deduced from the action principle postulated here.

We conclude this paper in Sec. 5 with a discussion of the possibility of extending the application of the present action principle to other mechanics that may eventually be derived.

The main contribution of this paper is to introduce an interaction picture, and from it a perturbation theory of classical mechanics on the same footing as in quantum mechanics.

2. ACTION PRINCIPLE

We plan to introduce the action principle as a postulate by means of the elements of a Lie algebra, which we designate by $\mathfrak{L}$. For any three elements A, B, C, such that

$$A, B, C \in \mathfrak{L}$$

[3] R. Haag, *Lectures in Theoretical Physics. 1964* (University of Colorado Press, Boulder, Colorado), p. 107.

of the Lie algebra, the distributive and nonassociative product of any two of the elements of $\mathfrak{L}$, which we write down as $[A, B]$, has to satisfy the following properties to generate a Lie algebra:

$$[A, B] = -[B, A], \tag{2.1}$$

that is, the antisymmetry condition, and

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0, \tag{2.2}$$

called the Jacobi identity.

Later on we examine which is the physical meaning of these two conditions on the elements of the algebra. Such a study gives us the reasons why we choose precisely a Lie algebra as the mathematical structure most fit to postulate the action principle. (The symbol [,] is called a Lie bracket.)

For any element A of the algebra, the action principle that we postulate is written as

$$\delta A = dA - \partial A = [\delta W, A], \tag{2.3}$$

where δA is the total infinitesimal variation of the element A of the algebra in relation to a certain parameter λ of a certain class of parameters that we study later. We should write more carefully as follows:

$$\delta A \equiv \delta_\lambda A, \tag{2.4}$$

notation that we use as it is needed. From the total variation dA, we have to subtract ∂A, which is the change in A associated with the explicit appearance in A of the parameter λ, since the latter cannot be produced by any action principle, but can be deduced immediately once we are given the explicit dependence of A on the parameter λ: it corresponds to the partial derivative of A with respect to the parameter λ. Without loss of generality and in order not to complicate the equations, we always suppose that the elements A of the Lie algebra do not depend explicitly on the parameters λ, so that

$$\partial A = 0. \tag{2.5}$$

The difference $\delta A = dA - \partial A$ is always the dynamical variation of the element A of the algebra, equal to the total variation less the explicit variation.

W is also an element of the algebra which plays a very special role and which we call Action. We study the general properties of W for any mechanics. The concrete specification of W depends on the kind of mechanics that we are considering and, more specifically, on the problem that we study. We call δW the variations of W in relation to a parameter λ of a class of parameters, some examples of which are presented later on. The elements δW are such that

$$\delta W \in \mathfrak{L}. \tag{2.6}$$

As we saw, we designate by [,] the combination or multiplication law for any ordered pair of elements of the algebra. The actual nature of the bracket [,] has to be specified for each kind of mechanics, as we see later on.

The requirement that a Lie bracket, the multiplication of the elements of the Lie algebra, be always expressible as a linear combination of the elements of the Lie algebra by means of the structure constants ensures, according to the action principle presented above, that the variations of any dynamical symbol are a linear combination of these same elements. Therefore, structure constants govern the dynamics.

We should present now the reasons for choosing Lie algebras to express the general action principle. This Lie algebra contains two elements W and $\mathcal{H}$ (so that $W, \mathcal{H} \in \mathfrak{L}$), called respectively Action and Hamiltonian of the system. The time variation of the Action yields the Hamiltonian, whose Lie bracket with any element of the algebra provides us with the dynamical time derivative, since the explicit time derivative of an element of the algebra cannot be generated by a Lie algebra bracket. Generally any element of the algebra A generates a certain dynamical variation of all the other elements of the algebra in relation to a certain parameter. We impose the physical condition that no element can produce a dynamical variation of itself, a condition that implies that the Lie bracket of an element A with itself is always zero:

$$[A, A] = 0. \tag{2.7}$$

If A and B are elements of $\mathfrak{L}$, then since an algebra is a vector space, $A + B$ will also be an element of $\mathfrak{L}$. From the above result and the fact that the combination relation of any algebra is distributive, we have

$$\begin{aligned} 0 &= [A + B, A + B] \\ &= [A, A] + [A, B] + [B, A] + [B, B] = 0, \end{aligned}$$

which gives

$$[A, B] = -[B, A]. \tag{2.8}$$

Therefore, the antisymmetry requirement of the Lie-bracket multiplication is equivalent to the physical condition that no element can produce the dynamical variations of itself.

Next, let us see where the Jacobi identity comes from. We should indeed require that the bracket-composition law be consistent with the dynamical variations of the elements of our algebra in relation to any parameter. This requirement is equivalent to the statement that any functional relationship, such as $C = [A, B]$, existing between any three elements of our algebra A, B, C for a certain value of the parameters that determine the dynamical variations, should be preserved for any other value of these parameters. For the sake of concreteness, we consider the dynamical time evolution of the system produced by $\mathcal{H}$, the Hamiltonian of the system, which is an element of the algebra we are considering. Then the elements A, B, C, $\mathcal{H}$ are considered at an instant of time t, and they satisfy

$$C = [A, B] \tag{2.9}$$

at time instant t. We would like to find out the requirements which our algebra has to satisfy in order that this relation be also valid at another time instant $t + dt$, infinitesimally different from t. The element A becomes $A + dt[A, \mathcal{H}]$ as deduced from an action principle as we see later. We would have similar expressions for the changes of B and C. In particular, C becomes $C + dt[C, \mathcal{H}]$. But from the relationship between C and the bracket $[A, B]$ that we want to preserve for the time instant $t + dt$, we should have

$$\begin{aligned} C + dt[C, \mathcal{H}] &= [A, B] + dt[[A, B], \mathcal{H}] \\ &= [A + dt[A, \mathcal{H}]], [B + dt[B, \mathcal{H}]]. \end{aligned} \tag{2.10}$$

If we keep only terms linear in dt and use the antisymmetry of the brackets already assumed in our algebra, then the Jacobi identity between A, B, and $\mathcal{H}$ immediately follows. If we had considered other kinds of dynamical variations, we would have obtained in a similar manner the Jacobi identity among any three elements of our algebra. This result is completely general, since the Lie-bracket multiplication is the only combination law to obtain from any pair of elements A, B, and a third element C.

We consider, therefore, that the Jacobi identity expresses the consistency between the algebra whose elements describe the physical system and the action principle that we have presented; i.e., the Jacobi identity guarantees that the variations of the elements of the algebra compatible with the dynamical action principle do not throw these elements out of the algebra.

We have, therefore, to write the action principle between elements of a Lie algebra in order that the dynamical evolution of the system produce new elements within the same algebra. We remember that we have used a Heisenberg-like picture of the dynamics of a system to arrive at these conclusions.

Besides the general form of the action principle as a Lie bracket, the practical basis for the applications of this dynamical principle is the fact that there exists a class of parameters λ such that the variations $\delta_\lambda W$ are obtained by appropriate variation of a single element W of the Lie algebra. The action principle must be complemented by the explicit specification of such a class.

Of the whole class of variation parameters that can be considered, we study here only the instances when λ is the time t of the system and, secondly, the case in which λ is the coupling parameter g between two systems. Variations with respect to the time yield the equations of motion, while when we change the coupling parameter infinitesimally we get a perturbation expansion that, as we said, is the main aim of this paper.

Let us consider the temporal evolution of the system. We designate by $A(t)$ any element of the algebra at instant t. There is an automorphism between the set of elements $A(t)$ and those of $A(t^1)$ considered at another instant t^1 of time. The action principle, in the form that we have presented it, implies that the dynamical time evolution of any element of the algebra is obtained by multiplying such an element by another δW of the same algebra, i.e., by an element δW evaluated at the same instant of time t. An element δW evaluated at another instant of time t^1 cannot generate according to (2.8) the time evolution at instant t. This deduction from the action principle (2.3) is equivalent to the principle of stationary action. It states that W, whose meaning is $\delta(W) = \delta W$, must be stationary with respect to variations at another time instant t^1, $t^1 \neq t$, since δW can only contain elements of the algebra associated with instant t. Therefore, we write

$$\delta_t W = \mathcal{H}(t)\delta t, \tag{2.11}$$

where $\mathcal{H}(t)$ is called the Hamiltonian of the system. The fact that the dynamical temporal variations of the elements of the algebra at an instant t can only be generated by an element of the algebra evaluated at the same time instant t, implies the existence of equations of the motion. The general equation of motion is

$$\delta A/\delta t = [A, \mathcal{H}]. \tag{2.12}$$

Since $[\mathcal{H}, A]$ is a linear combination of elements of the Lie algebra $\mathfrak{L}$, determined by the structure constants, the same equation of motion can be applied to $\delta A/\delta t$. We get

$$\delta^2 A/\delta t^2 = [[A, \mathcal{H}], \mathcal{H}], \tag{2.13}$$

and, in general,

$$\delta^n A/\delta t^n = [\cdots, [A, \mathcal{H}], \mathcal{H}, \cdots, \mathcal{H}] \tag{2.14}$$

with n multiplication brackets.

Applying Taylor's theorem, we can write

$$A(t) = \{e^{-t\mathcal{H}} A(0) e^{t\mathcal{H}}\}, \tag{2.15}$$

where the braces indicate that the expression is only symbolic in the sense that its only meaning is

$$A(t) = A(0) + \frac{t}{1!}[A(0), \mathcal{H}] + \frac{t^2}{2!}[[A(0), \mathcal{H}], \mathcal{H}] + \cdots. \tag{2.16}$$

Indeed, the exponentials of $\mathcal{H}$ in (2.15) are not defined, since the only multiplication that we have introduced in the algebra is the Lie-bracket multiplication according to which the powers of any element of the algebra are identically zero, given the antisymmetry of the brackets required by physical conditions. Indeed, for instance, so far

$$\mathcal{H}^2 \equiv [\mathcal{H}, \mathcal{H}] = 0. \tag{2.17}$$

The fact that the expression (2.15) is only symbolic is a serious inconvenience for the practical applications of the action principle, since we do not have an analytical apparatus to use in our calculations. This is the reason why we have to introduce realizations of the Lie algebra defining a new algebra and a new product (,) that, since it does not enter into the action principle, does not have to be antisymmetric as it is required on physical grounds for the Lie-bracket product [,]. Then, powers of an element are defined by means of this new kind of product. This is the reason why the dynamical Lie algebras are realized by means of derivations though, evidently, these realizations are not required by the physical content of the theory; they simply are convenient ways of performing the calculations that appear in the action principle and of mapping the Lie algebra into the field of real numbers.

Let us consider the introduction of a general interaction picture to study perturbations. We consider that the action element W can be divided into two parts coupled by the parameter g, so that

$$W = W_0 + gW_1, \tag{2.18}$$

and we want to obtain the change of any element of the algebra when the coupling parameter changes from g to $g + \delta g$. Action principle (2.3) yields

$$\delta_g A = [\delta_g W, A] = [W_1, A]\delta g, \tag{2.19}$$

where the Lie bracket $[A, W_1]$ has to be evaluated for the value $\lambda = g$ of the coupling parameter. W_0 is the unperturbed action; it corresponds to $g = 0$. The action corresponding to $g = 1$, $W = W_0 + W_1$, is the fully perturbed action, since we consider W_1 to be the perturbation.

We need to study the boundary conditions for the application of the perturbation. Undoubtedly the action W should contain two labels to indicate when the interaction begins and when it ends. So

$$W \equiv W(t, t_0), \tag{2.20}$$

where t_0 is the instant when the perturbation starts and t the final moment of action of the perturbation. The physical consistency requirement implies that

$$W(t_0, t_0) = 0 \tag{2.21}$$

and that

$$W(t, t_0) + W(t_0, t_1) = W(t, t_1), \tag{2.22}$$

from which we deduce

$$W(t, t_0) = -W(t_0, t). \tag{2.23}$$

The action element $W(t, t_0)$ evidently possesses the form

$$W(t, t_0) = \int_{t_0}^{t_1} dt_1 L(t_1), \tag{2.24}$$

where $L(t_1)$ is the Lagrangian. As we see later, the action $W(t, t_0)$ has to be varied in relation to the upper limit t in order to obtain the Hamiltonian at the instant, i.e.,

$$\delta_t W(t, t_0) = -\mathcal{H}(t)\delta t. \tag{2.25}$$

Taking the action principle (2.3) to evaluate perturbations, we deduce that

$$\delta^n A/\delta g^n = 0, \quad \text{if} \quad t = t_0 \quad \text{for any } n, \tag{2.26}$$

since $W(t_0, t_0) = 0$. From here we deduce that, as assumed before, t_0 is the instant when the perturbation starts to act and that, therefore, the perturbation acts during the interval $t - t_0$.

If we write the time labels explicitly, Eq. (2.19) has the following form:

$$\frac{\delta A(t)}{\delta g} = [W_1(t, t_0), A(t)] = \int_{t_0}^{t} dt_1 [L(t_1), A(t)]. \tag{2.27}$$

From here we also have, as before,

$$\frac{\delta^2 A(t)}{\delta g^2} = \int_{t_0}^{t} dt_1 \left[\frac{\delta L(t_1)}{\delta g}, A(t)\right] + \int_{t_0}^{t} dt_1 \left[L(t_1), \frac{\delta A(t)}{\delta g}\right], \tag{2.28}$$

which is a procedure that can be continued so as to evaluate $[\delta^n A(t)]/(\delta g^n)$ for any value of n.

The explicit expression for the element A for $g = 1$, i.e., fully perturbed, is obtained from the same element A for $g = 0$, i.e., from the unperturbed element, by means of a Taylor's expansion in powers of $\Delta g = 1$, and so

$$A(t)\big|_{g=1} = A(t)\big|_{g=0} + \frac{1}{1!}\frac{\delta A(t)}{\delta g}\bigg|_{g=0} + \cdots + \frac{1}{n!}\frac{\delta^n A(t)}{\delta g^n}\bigg|_{g=0} + \cdots, \quad (2.29)$$

which is an expansion that can also be written in a symbolic way by means of exponentials of W_1. This is the general interaction picture, since here all the successive Lie brackets have to be evaluated for $g = 0$, i.e., for the elements of the Lie algebra calculated for the unperturbed motion generated by W_0.

To apply the above formula to a perturbation expansion we have to suppose that the motion of the system generated by the unperturbed action W_0 has been solved exactly. Then we can calculate exactly the different successive Lie brackets that appear in (2.29). The term in this formula that contains n times the perturbing action W_1 is the nth perturbation. Indeed, to apply expression (2.29) to a concrete perturbation problem, we have to define for each kind of mechanics the Lie bracket. But, undoubtedly, we have written a general perturbation expansion.

The combination or multiplication law of any two elements A and B of the Lie algebra $\mathfrak{L}$, by means of which the action principle is introduced, is written as $[A, B]$. We see in the applications that, as a matter of fact, such a product becomes the Poisson bracket or the commutator between any two elements, respectively, in each one of the mechanics in which the action principle is applied.

In the vector space of the elements of the abstract Lie algebra $\mathfrak{L}$ we define a second combination law of the two elements, that we design by (,), which maps pairs of elements of the dynamical abstract Lie algebra, A and B, into another such element (A, B), under which the vector space becomes an associative algebra D. This implies that the new product (A, B) is also distributive. We also further require that the two product operations satisfy

$$[(A, B), C] = (A, [B, C]) + ([A, C], B) \quad (2.30)$$

for any three elements of the Lie algebra.[4] This property is referred to by saying that the Lie bracket is a derivation in a linear associative algebra with the product (A, B).

In a linear associative algebra, powers of an element are uniquely defined. The associative and distributive product (,) is often referred to as the ordinary product. As a matter of fact, the product (,) is either the ordinary product of analytic functions in classical mechanics or the ordinary product of operators in quantum mechanics.

By virtue of the derivation property of the Lie bracket, it follows that algebraic relations among the elements of the Lie algebra, involving either the ordinary product (,) or the Lie bracket [,], are preserved by infinitesimal transformations.

A Lie algebra provides only an abstract framework for the dynamical properties of a physical system, and even if this framework is supplemented by the dual space of physical states, it is not enough for the complete and practical specification of the physical situation. We have to introduce also the additional structure of an associative algebra D, and an explicit realization of the Lie algebra $\mathfrak{L}$, by derivations in this associative algebra D. And so, as we see, for classical mechanics we use analytic functions where (,) is the ordinary product of the same and [,] is the Poisson bracket; but for quantum mechanics we introduce operators in Hilbert space where (,) is now the ordinary product of the same operators, while [,] is proportional to the commutator. For instance, powers of the dynamical variables will, in general, have a meaning in the explicit realization of the algebra not being defined in the algebra $\mathfrak{L}$ itself.

Our dynamical scheme is as follows. We have an abstract Lie algebra $\mathfrak{L}$, whose elements constitute the dynamical variables, and a concrete linear associative algebra D, which furnishes a realization of $\mathfrak{L}$ by derivations.

We note in passing that classical and quantum mechanics, in order to be discussed, fall within this characterization. As a matter of fact, in Sec. 3 of this paper we examine the case for classical mechanics while in Sec. 4 we study, from the viewpoint of this paper, quantum mechanics.

The most important point that we want to make clear in this paper is that considering all different mechanics as different realizations of one and the same algebra $\mathfrak{L}$, we obtain a unified apparatus to formulate the *dynamical* properties of all mechanical systems, to introduce a general *interaction picture* for dynamics, and to deduce a general method for evaluating *perturbations* in all kinds of mechanical systems.

The entities that form the associative algebra D, in which we obtain realizations of the Lie algebra $\mathfrak{L}$ by derivations, have composition laws of their own, only part of which will reflect, in a homomorphic manner, the composition table of $\mathfrak{L}$. In general, we are able to define functions of the representatives of the elements, additional relations that it may not be possible to define in the original algebra $\mathfrak{L}$, and that give rise to elements that do not belong to the algebra

[4] J. P. Serre, *Lie Algebras and Lie Groups* (W. A. Benjamin, Inc., New York, 1965).

$\mathfrak{L}$. This additional content of the realization has always a definite physical meaning, allowing the introduction of physical degrees of freedom, and so it is not only a matter of mathematical freedom.

A canonical transformation is a mapping which leaves Lie-bracket relations invariant; they are essentially automorphisms of the algebra. The dynamical evolutions of the elements of the algebra are canonical transformations. The set of all automorphisms of a given Lie algebra constitutes the corresponding Lie group, which, accordingly, consists of sets of canonical transformations.

All these Lie algebras contain an identity I, which is an element whose Lie bracket with any other element of the algebra is zero. Normalization of the states is achieved, requiring that the identity be mapped into the real number 1. In this paper we do not study the mappings of the Lie-algebra elements into the field of real numbers, since we are mostly concerned with dynamical questions and not with the states.

We want to add, however, that the states can be characterized in classical and quantum mechanics in a very similar manner. From the physical viewpoint, the possibility to obtain this lies in the form that Ehrenfest gave to the principle of correspondence: the expectation values in quantum mechanics of dynamical operators obey the same equations of motion as the corresponding classical dynamical variables.

3. CLASSICAL MECHANICS

The action principle that we have established as a postulate is

$$\delta A = [\delta W, A], \tag{3.1}$$

where by [,] we indicate the Lie-bracket multiplication.

In classical mechanics we have to introduce the sets of canonical conjugate variables q_k, p_k, where $k = 1, 2, 3, \cdots, n$, by means of which we define the Poisson bracket between two analytical functions U and V of the sets q_k, p_k that we designate by $[U, V]_c$

$$[U, V]_c = \sum_k \left(\frac{\partial U}{\partial q_k}\frac{\partial V}{\partial p_k} - \frac{\partial U}{\partial p_k}\frac{\partial V}{\partial q_k}\right), \tag{3.2}$$

where the subscript c comes from classical mechanics.

Our action principle is translated into classical mechanics when the Lie-bracket multiplication is the Poisson bracket between analytic functions of the canonical set of conjugate variables, as follows:

$$\delta A = [\delta W, A] = [\delta W, A]_c \tag{3.3}$$

that, when A represents the variables q_k and p_k, yield

$$\delta q_k(t) = \frac{-\partial \delta W(t, t_0)}{\partial p_k(t)}, \quad \delta p_k(t) = \frac{\partial \delta W(t, t_0)}{\partial q_k(t)}, \tag{3.4}$$

which are equations of motion already obtained before.[5]

To simplify notation, when we deal with classical mechanics, the notation indicating Lie bracket will denote the Poisson bracket, i.e., we do not write from now on a special sign to specify that in classical mechanics the Lie bracket is interpreted always as the Poisson bracket. We see that the elements of the Lie algebra for classical mechanics are represented by analytic functions where the multiplication (,) is simply ordinary multiplication of functions and the Lie-bracket multiplication [,] is the Poisson bracket between the elements that are multiplied, with the notation $(u, v) = uv$. We see quite easily that relation (2.30) is satisfied between these two kinds of products.

As has been shown,[5] the temporal evolution of the physical system is obtained from action principle (3.1) or its equivalent (3.4). Indeed, to see this fact we study complete variations of the action integral that corresponds to an intrinsic variation $\Delta q(t)$ of the dynamical variables and to a change of the upper limit of the action integral

$$\delta W(t, t_0) = \int_{t_0}^{t+\delta t} L(\bar{q}(t_1), \dot{\bar{q}}(t_1))\, dt_1 - \int_{t_0}^{t} L(q(t_1), \dot{q}(t_1))\, dt_1. \tag{3.5}$$

The intrinsic variation

$$\Delta q(t_1) = \bar{q}(t_1) - q(t_1) \tag{3.6}$$

is supposed to be zero at $t_1 = t_0$. To evaluate the complete variation of the dynamical variable at time t, we have to add to $\Delta q(t)$ the variation due to the shift $\bar{t} = t + \delta t$ of the upper limit

$$q(\bar{t}) = q(t) + \delta t \dot{q}(t), \tag{3.7}$$

so that its complete variation is

$$\delta q(t) = \Delta q(t) + \delta t \dot{q}(t). \tag{3.8}$$

Now the evaluation of $\delta W(t, t_0)$ is straightforward. We get

$$\delta W(t, t_0) = -\delta t(\mathcal{H} q(t), p(t)) + p(t)\delta q(t), \tag{3.9}$$

where the Hamiltonian $\mathcal{H}(q, p)$ is defined, as is usually done, by

$$-\mathcal{H} = L - p\dot{q}. \tag{3.10}$$

[5] L. M. Garrido, J. Math. Anal. Appl. 3, 295 (1961).

The time evaluation corresponds to

$$\delta_t W(t, t_0) = -\delta t \mathcal{H}(q(t), p(t)), \tag{3.11}$$

the case in which the action principle yields

$$\dot{q} = \frac{\delta_t q(t)}{\delta t} = \frac{\partial \mathcal{H}}{\partial p}, \quad -\dot{p} = -\frac{\delta_t p(t)}{\delta t} = \frac{\partial \mathcal{H}}{\partial q}, \tag{3.12}$$

which are Hamilton's equations of motion.

The variation

$$\delta_q W(t, t_0) = p(t)\delta q(t) \tag{3.13}$$

gives the kinematic independence of $q(t)$ and $p(t)$, since then

$$\delta_q q(t) = \frac{\partial[p(t)\delta q(t)]}{\partial p(t)} = \delta q(t),$$
$$-\delta_q p(t) = \frac{\partial p(t)}{\partial q(t)}\,\delta q(t) = 0. \tag{3.14}$$

So far we have done nothing new. The preceding formulas are well known and so the postulated action principle appears as a different way of writing the equations of motion. Such a postulate is only meaningful if we can also obtain from it other results beyond the equations of motion. This is the case, since our action principle yields also perturbation theory and provides a means of writing an interaction picture for classical mechanics. And this is what we do next.

We would like to study the system whose action suffers the effect of a perturbing Lagrangian so that the new action becomes

$$W(t, t_0) = W^{(g)}(t, t_0) = W_0(t, t_0) + gW(t, t_0)$$
$$= \int_{t_0}^{t} L^{(g)}(q_g(t'), p_g(t'))\, dt', \tag{3.15}$$

where

$$L^{(g)} = L_0 + gL_1. \tag{3.16}$$

Here t_0 is the time instant when the perturbation starts out. For the study of perturbations we fix t_0 and t, but change the coupling parameter g from g to $g + \delta g$, so that

$$\delta W^{(g)}(t, t_0) = \delta g W^{(g)}(t, t_0), \tag{3.17}$$

where

$$W_1^{(g)}(t, t_0) = \int_{t_0}^{t} L_1(q_g(t'), p_g(t'))\, dt_1, \tag{3.18}$$

since in the evaluation of $W_1^{(g)}(t, t_0)$ we have to use the canonical conjugate variables evaluated at the value g of the coupling parameter q_g and p_g in order to calculate the Poisson bracket that appears in classical mechanics. Usually, however, the Lagrangian is not given as presented above but in terms of a set of generalized coordinates q and the time derivatives of the same $\dot{q}$. Using the definition of generalized momenta canonically conjugate to a given generalized coordinate, we can eliminate the time derivative of the coordinate $\dot{q}$ and write the Lagrangian as a function of sets of canonical conjugate variables q and p. If this variation is applied to the action principle, we obtain

$$\delta_g A_g = \delta g[W_1^{(g)}(t, t_0), A_g] \tag{3.19}$$

where now $A \equiv A_g$, i.e., the element of the Lie algebra depends on the coupling parameter. To have a clear idea of how to calculate the Poisson bracket that appears here we work out two examples later on.

Since the perturbation is analytic in the coupling parameter g, we make use of Taylor's expansion

$$A_1 \equiv A_g\big|_{g=1} = A_0 + \frac{1}{1!}\frac{\delta_g A_g}{\delta g}\bigg|_{g=0} + \frac{1}{2!}\frac{\delta_g^2 A_g}{\delta g^2}\bigg|_{g=0}$$
$$+ \cdots + \frac{1}{n!}\frac{\delta_g^n A_g}{\delta g^n}\bigg|_{g=0} + \cdots, \tag{3.20}$$

where the unperturbed system is obtained for $g = 0$ and the fully perturbed motion corresponds to $g = 1$. We have to evaluate $(\delta_g^n A_g)/\delta g^n$ from our action principle starting from

$$\frac{\delta A}{\delta g} = [W_1, A]. \tag{3.21}$$

In this way we obtain the general interaction picture for classical mechanics and a procedure to evaluate perturbations in classical mechanics to any order in the perturbing action.

The perturbation method that results from the action principle (3.3) in classical mechanics does not follow the same steps as the technique deduced before for Hamiltonian's equations of motion.[6,7]

We now clarify most of our ideas, working out some examples that indicate how the perturbation techniques deduced from the action principle postulated above can be applied to classical mechanics. The action integral is defined as follows for the unperturbed motion:

$$W_0(t, t_0) = \int_{t_0}^{t} L_0(q_0, \dot{q}_0, t_1)\, dt_1. \tag{3.22}$$

The Lagrangian may depend on time explicitly. For simplicity, we limit ourselves to the case when the motion of the system is properly described by a single generalized coordinate and its time derivative, but the principle is valid for motions with any fixed number of generalized coordinates.

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[6] L. M. Garrido, Proc. Phys. Soc. (London) **76**, 33 (1960).

[7] L. M. Garrido and F. Gascon, Proc. Phys. Soc. (London) **81**, 1115 (1963).

If we call $p_0 \equiv \partial L_0(q_0, \dot{q}_0)/\partial \dot{q}_0$ the canonical conjugate momentum, we can eliminate the time derivative $\dot{q}_0$ of the coordinate and write the Lagrangian as a function of q_0 and p_0. Using the same symbol for the new function, we get

$$L_0(q_0, p_0) = L_0(q_0, \dot{q}_0). \tag{3.23}$$

The action integral becomes

$$W_0(t, t_0) = \int_{t_0}^{t} L_0(q_0, p_0, t_1)\, dt_1, \tag{3.24}$$

which cannot be evaluated until the equations of motion are solved. Let us suppose that we have solved such equations exactly, equations corresponding to the unperturbed motion, and have written their solutions in terms of the boundary values at the time origin

$$\begin{aligned} q_0 &\equiv q_0(t) = q_0(q_0(0), p_0(0), t), \\ p_0 &\equiv p_0(t) = p_0(q_0(0), p_0(0), t). \end{aligned} \tag{3.25}$$

By means of these expressions we can evaluate q_0 and p_0 at any time $t = t'$, and consider $q_0(t')$ and $p_0(t')$ as boundary values, so that we are able to write the solutions of the unperturbed equations of motion as

$$\begin{aligned} q_0 &\equiv q_0(t) = q_0(q_0(t'), p_0(t'), t - t'), \\ p_0 &\equiv p_0(t) = p_0(q_0(t'), p_0(t'), t - t'). \end{aligned} \tag{3.26}$$

Now we define derivatives with respect to the boundary values at any instant, and evaluate expressions like $[\partial p_0(t)]/[\partial p_0(t')]$, which is the derivative of a function with respect to the same function at any other time.

Evidently

$$\begin{aligned} \frac{\partial p_0(t)}{\partial p_0(t)} &= 1, \quad \frac{\partial p_0(t)}{\partial q_0(t)} = 0, \\ \frac{\partial q_0(t)}{\partial q_0(t)} &= 1, \quad \frac{\partial q_0(t)}{\partial p_0(t)} = 0. \end{aligned} \tag{3.27}$$

To be more concrete, we evaluate derivatives with respect to boundary values when the unperturbed system is the harmonic oscillator, whose Lagrangian is

$$L_0(q_0, p_0) = \tfrac{1}{2}p_0^2 - \omega^2 \tfrac{1}{2} q_0^2. \tag{3.28}$$

The equation of motion yield the following solutions:

$$\begin{aligned} q_0(t) &= q_0(0) \cos \omega t + \frac{p_0(0)}{\omega} \sin \omega t, \\ p_0(t) &= -q_0(0)\omega \sin \omega t + p_0(0) \cos \omega t, \end{aligned} \tag{3.29}$$

and therefore

$$\begin{aligned} \frac{\partial q_0(t)}{\partial q_0(t')} &= \cos \omega(t - t'), & \frac{\partial p_0(t)}{\partial q_0(t)} &= -\omega \sin \omega(t - t'), \\ \frac{\partial q_0(t)}{\partial p_0(t')} &= \frac{1}{\omega} \sin \omega(t - t'), & \frac{\partial p_0(t)}{\partial p_0(t')} &= \cos \omega(t - t'). \end{aligned} \tag{3.30}$$

Making use of the above calculations, let us evaluate the first term of the perturbation series of the harmonic oscillator perturbed by the Lagrangian

$$L_1(q, p) = -\mu^2 \tfrac{1}{2} q^2, \tag{3.31}$$

when $t_0 = 0$.

With the help of expansions obtained before, we get

$$\begin{aligned} q(t) &= q_0(t) + \frac{\mu^2}{2} \int_{t_0}^{t} \frac{\partial q_0^2(t_1)}{\partial p_0(t)}\, dt_1 + \cdots \\ &= q_0(t) + \frac{\mu^2}{\omega} \int_0^t q_0(t_1) \sin \omega(t_1 - t)\, dt_1 + \cdots \\ &= q_0(t) - \frac{\mu^2}{\omega} t q_0(0) \sin \omega t + \frac{\mu^2}{2\omega^2} t p_0(0) \cos \omega t \\ &\quad - \frac{\mu^2}{2\omega^3} p_0(0) \sin \omega t + \cdots, \\ p(t) &= p_0(t) - \frac{\mu^2}{2} \int_0^t \frac{\partial q_0^2(t_1)}{\partial q_0(t)}\, dt_1 + \cdots \\ &= p_0(t) - \mu^2 \int_0^t q_0(t_1) \cos \omega(t_1 - t)\, dt_1 + \cdots \\ &= p_0(t) - \frac{\mu^2}{2} t q_0(0) \cos \omega t - \frac{\mu^2}{2\omega} t p_0(0) \sin \omega t \\ &\quad - \frac{\mu^2}{2\omega} q_0(0) \sin \omega t + \cdots. \end{aligned} \tag{3.32}$$

As a final application, we deduce from our action principle the definition of the Poisson bracket introduced by Peierls[8] for the nonrelativistic case. To define the Poisson bracket between $A(q, p)$ and $B(q, p)$ at time $t = T$, Peierls introduces a perturbing Lagrangian

$$L_1 = A(q(t), p(t))\delta(t - T), \tag{3.33}$$

where $\delta(t - T)$ is Dirac's delta, and considers an infinitesimal variation of the coupling parameter around $g = 0$. He then evaluates variations corresponding to two-boundary conditions, $t_0 = \infty$ and $t_0 = -\infty$, called advanced and retarded perturbations, respectively. Correspondingly, we have to calculate the changes induced in $B(q, p)$ that we call respectively $\delta^- B$ and $\delta^+ B$. Peierls' definition of the

[8] R. Peierls, Proc. Roy. Soc. (London) **A214**, 143 (1952).

Poisson bracket $[A, B]$ is

$$[A, B] = \lim_{\delta g \to 0} \frac{1}{\delta g} (\delta^+ - \delta^-)B. \tag{3.34}$$

To show the validity of Peierls' definition is quite easy if we utilize the action principle that we have postulated, since then

$$\begin{aligned}
&\lim_{\delta_g \to 0} \frac{1}{\delta g} (\delta^+ - \delta^-) q(T) \\
&\quad = -\left[\partial \int_{-\infty}^{\infty} A(q(t_1), p(t_1))\delta(t_1 - T)\, dt_1\right] \Big/ [\partial p(T)] \\
&\quad = -\frac{\partial A(q(T), p(T))}{\partial p(T)}, \\
&\lim_{\delta g \to 0} \frac{1}{\delta g} (\delta^+ - \delta^-) p(T) = \frac{\partial A(q(T), p(T))}{\partial q(T)},
\end{aligned} \tag{3.35}$$

and therefore

$$\begin{aligned}
&\lim_{\delta g \to 0} \frac{1}{\delta g} (\delta^+ - \delta^-) B(q(T), p(T)) \\
&\qquad = \frac{\partial B}{\partial p}\frac{\partial A}{\partial q} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p}\bigg|_{t=T}.
\end{aligned} \tag{3.36}$$

That justifies Peierls' statement.

4. QUANTUM MECHANICS

The equations of motion of quantum mechanics can be formulated in a form which is isomorphic to the Poisson-bracket formulation of classical mechanics, with quantities proportional to commutators taking the place of Poisson brackets. The quantum scheme introduces the associated algebra by means of all "analytic functions" of operators in Hilbert space. The expression "analytic function" is here understood with the meaning of convergent symmetrized power series.

We have stated our action principle as follows:

$$\delta A = [\delta W, A], \tag{4.1}$$

where [,] is the Lie-bracket multiplication. In quantum mechanics the Lie-algebra multiplication is realized by means of commutation of operators in Hilbert space as follows:

$$[\delta W, A] \equiv [\delta W, A]_q = (1/i\hbar)(\delta W A - A \delta W). \tag{4.2}$$

Here the associative algebra D has the product equal to the ordinary product of operators in Hilbert space. The Poisson-bracket multiplication is equal to the factor $(1/i\hbar)$ multiplied by the commutator of operators as specified in (4.2). The ordinary product (,) is in quantum mechanics

$$(\delta W, A) = \delta W A, \tag{4.3}$$

where δW and A are Hilbert space operators. With these two definitions of [,] and of (,), we can see quite easily that the relation (2.30) is satisfied.

As a particular case, we can consider the time evolution by means of the relation $\delta_t W = -\mathcal{H}\delta t$ that yields

$$i\hbar \frac{\delta A}{\partial t} = A\mathcal{H} - \mathcal{H}A, \tag{4.4}$$

which is the well-known Heisenberg equation of motion in Heisenberg picture. Usually we would have to specify the time limits in the variation of the action as done in (2.28), the case when the action principle becomes

$$\delta A(t) = [\delta W(t, t_0), A(t)], \tag{4.5}$$

where t_0 is the instant when the perturbation starts to act.

We have stated the action principle by means of variations of the elements of the Lie algebra, that in quantum mechanics are operators of Hilbert space. In this way such a principle is applicable both to classical and to quantum mechanics. But in quantum mechanics only, we want to transform this action principle to another, written by means of the variation of the transformation function as it was done by Schwinger.[9]

The quantities that in quantum mechanics are related to the physical reality are the matrix elements. To transfer from a Heisenberg-like picture in which we stated the action principle, to a Schrödinger-like picture as Schwinger stated it for quantum mechanics, we have to remark that the infinitesimal unitary transformation of the observables given by

$$\delta A(t) = \bar{A}(t) - A(t), \tag{4.6}$$

$$\bar{A}(t) = \left(1 + \frac{1}{i\hbar}\delta W(t, t_0)\right) A(t) \left(1 - \frac{1}{i\hbar}\delta W(t, t_0)\right) \tag{4.7}$$

$$= A(t) + [\delta W(t, t_0), A(t)] \tag{4.8}$$

induces in the eigenstates a transformation from $|a\rangle$ to $|\bar{a}\rangle$ given by

$$\langle \bar{a}| \bar{A} |\bar{a}\rangle = \langle a| A |a\rangle, \tag{4.9}$$

$$\delta |a\rangle = |\bar{a}\rangle - |a\rangle \equiv |\delta a\rangle, \tag{4.10}$$

$$\delta |a\rangle = \frac{\delta W(t_a, t_0)}{i\hbar} |a\rangle, \tag{4.11}$$

where t_a is the time instant at which the state vector $|a\rangle$ is evaluated.

[9] J. Schwinger, Phys. Rev. **91**, 713 (1953).

The variation of the transformation function is given by the following expression if the two eigenvectors of the transformation function are varied independently:

$$\begin{aligned}\delta\langle a \mid b\rangle &= \langle \delta a \mid b\rangle + \langle a \mid \delta b\rangle \\ &= i\frac{1}{\hbar}\langle a|\, \delta W(t_a, t_0) - \delta W(t_b, t_0)\, |b\rangle \\ &= i\frac{1}{\hbar}\langle a|\, \delta W(t_a, t_b)\, |b\rangle, \qquad (4.12)\end{aligned}$$

which is the action principle for quantum mechanics as stated by Schwinger.[9]

Schwinger considers (4.12) as the definition of the infinitesimal operator $\delta W(t_a, t_b)$, from which he deduces that the requirement that any infinitesimal variation maintains the multiplicative composition law of transformation functions implies the additive composition law for the infinitesimal operators (2.25). The fundamental dynamical principle is contained in the postulate that there exists a class of transformation-function alterations for which the characterizing operators δW are obtained by appropriate variation of a single operator W given by (2.27).

The fact that the action element has to be the integral of the Lagrangian can be deduced in quantum mechanics from the requirement that any infinitesimal alteration of the transformation function maintains the multiplicative composition law of the same transformation functions. This conclusion, however, is a consequence of the fact that usually, in quantum mechanics, the dynamical Lie-algebra principle is realized by means of an algebra of operators in Hilbert space. We deduce immediately that the action principle (4.12) is also valid for the calculation of perturbations in quantum mechanics.

Indeed, if we use action principle (4.12) and apply Taylor's theorem

$$\langle a \mid b\rangle = \langle a \mid b\rangle|_{g=0} + \frac{\partial}{\partial g}\langle a \mid b\rangle|_{g=0} + \frac{\partial^2}{\partial g^2}\langle a \mid b\rangle|_{g=0} + \cdots, \qquad (4.13)$$

where g is the coupling parameter between the two parts of the Action

$$W(t_a, t_b) = \int_{t_b}^{t_a} dt\{L_0(t) + gL_1(t)\}. \qquad (4.14)$$

Action principle (4.12) yields immediately

$$\frac{\partial}{\partial g}\langle a \mid b\rangle = \frac{i}{\hbar}\langle a|\int_{t_b}^{t_a} dt L_1(t)\, |b\rangle, \qquad (4.15)$$

which is a relation valid for any value of the coupling constant g and, in particular, for $g = 0$. We have also

$$\frac{\partial^2}{\partial g^2}\langle a \mid b\rangle = \frac{i}{\hbar}\frac{\partial}{\partial g}\sum_{c,d}\int_{t_b}^{t_a} dt\langle a \mid c\rangle\langle c|\, L_1(t)\, |d\rangle\langle d \mid b\rangle \qquad (4.16)$$

with the restriction $t_c = t_d = t$, that implies

$$\frac{\partial}{\partial g}\langle c|\, L_1(t)\, |d\rangle = 0. \qquad (4.17)$$

Therefore, after applying again the result (4.15), we obtain

$$\begin{aligned}&\frac{\partial^2}{\partial g^2}\langle a \mid b\rangle \\ &= \langle a|\left(\frac{i}{\hbar}\right)^2\int_{t_b}^{t_a} dt\left\{\int_{t_b}^{t_a} dt' L_1(t')L_1(t) + \int_{t_b}^{t_a} dt' L_1(t)L_1(t')\right\}|b\rangle \\ &= \langle a|\left(\frac{i}{\hbar}\right)^2\int_{t_b}^{t_a} dt\int_{t_b}^{t_a} dt'[L_1(t)L_1(t')]_+\, |b\rangle, \qquad (4.18)\end{aligned}$$

where we have introduced the time-ordering operation $[\;]_+$, which has the property that in operating on a product of time-labeled operators, it rearranges them in the same order as the time sequence of their labels, the latest one in time occurring first in the product.

In general, we have

$$\begin{aligned}\frac{\partial^n}{\partial g^n}\langle a \mid b\rangle = \langle a|\left(\frac{i}{\hbar}\right)^n\int_{t_b}^{t_a} dt^{(1)}\int_{t_b}^{t_a} dt^{(2)}\cdots\int_{t_b}^{t_a} dt^{(n)} \\ \times [L_1(t^{(1)})L_1(t^{(2)})\cdots L_1(t^{(n)})]_+\, |b\rangle \quad (4.19)\end{aligned}$$

that has to be evaluated for $g = 0$ to obtain the expression for $\langle a \mid b\rangle$ in (4.13). So we have

$$\langle a \mid b\rangle = \langle a|\left(\exp\left\{\frac{i}{\hbar}\int_{t_b}^{t_a} L_1(t)\, dt\right\}\right)_+ |b\rangle|_{g=0}, \quad (4.20)$$

which is an expression from which we deduce the well-known formula for the evolution operator in the interaction picture with which we can calculate perturbations in quantum mechanics.

Indeed, if U is the evolution operator in the interaction picture we have

$$\langle a \mid b\rangle = \langle a|\, U(t_a, t_b)\, |b\rangle|_{g=0}, \qquad (4.21)$$

and, therefore,

$$U(t_a, t_b) = \left(\exp\left\{\frac{i}{\hbar}\int_{t_b}^{t_a} L_1(t)\, dt\right\}\right)_+, \qquad (4.22)$$

which is a very well-known expression.

5. CONCLUSION

The general structural features of dynamical theories that we have exhibited have a profound physical meaning. Classical and quantum dynamics require particular realizations (functions or operators) as the natural realizations of the Lie algebra of dynamics common to both of them. The realization that is important for kinematics and for the physical interpretation of the theory is not important for the dynamical structure analysis. That is why we can obtain, as we have done above, a general interaction picture valid for any kind of mechanics.

The relative importance of the selected realization of the kinematic part of any mechanics is illustrated, considering the possibility of a transcription of classical and quantum mechanics each into the natural realization of the other.[10] Doing so we see that many of the features of the formalisms of all kinds of mechanics become identical. From this point of view the main difference between the two mechanics is in the choice of the Lie bracket. The difference between classical and quantum mechanics resides mainly in the choice of realization for the dynamical group. But we see also that each mechanics is very awkward in the natural representation of the other.

As has been shown before,[10] there is a general form of a Lie bracket which includes the brackets of classical and quantum mechanics as special cases. This fact suggests the existence of more general mechanical formalisms.

After we have examined the validity of the action principle for classical and quantum mechanics, a question that arises quite naturally is whether there exists a superscheme beyond, and inclusive of, the two kinds of mechanics that we have specifically studied. The problem of the existence of a universal superscheme has to be answered affirmatively, so far as we know now; however, it has to be left open in this paper.

Such a superscheme will be obtained when the action principle is extended to yield also the variations of the Lie-algebra elements induced when we change the realization by derivations of the algebra. It can be seen that action principle (2.3) gives also these variations when δW is interpreted as the change in the action that is induced by the change of realization.

The different realizations of the algebra will be mapped isomorphically into a set of parameters that have continuous or discrete values. We obtain classical and quantum mechanics when we give to these sets of parameters a concrete, fixed set of values. Since the action principle is valid also for the variations of the elements of the algebra corresponding to the variations of these sets of parameters which determine the realization, we can obtain, in a form compatible with the action principle, continuously or discretely different kinds of mechanics. This process allows us to obtain, in a quite natural way, a semiclassical approximation to quantum mechanics, for instance.

The principal aim of this paper has been to write an action principle (2.3) from which an interaction picture valid for classical and quantum mechanics could be deduced, and from it to write down a general procedure to evaluate perturbations in both kinds of mechanics mentioned.

[10] T. F. Jordan and E. C. Sudarshan, Rev. Mod. Phys. 33, 515 (1961).