Exact temporal evolution for some nonlinear diffusion processes

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(Received 24 April 1984; accepted for publication 9 August 1984)

Exact solutions to Fokker–Planck equations with nonlinear drift are considered. Applications of these exact solutions for concrete models are studied. We arrive at the conclusion that for certain drifts we obtain divergent moments (and infinite relaxation time) if the diffusion process can be extended without any obstacle to the whole space. But if we introduce a potential barrier that limits the diffusion process, moments converge with a finite relaxation time.

I. INTRODUCTION

A time-dependent Fokker–Planck equation (FPE) describes the dynamical evolution of the diffusion processes. Nevertheless, when the dynamics of the process is nonlinear it is very difficult to obtain exact or even approximate solutions of such FPE's. Since at the present time nonlinear processes are of highest interest (instabilities, phase transitions, etc.)[1] many people have tried to find exactly soluble nonlinear models.[2–7]

The most common technique used to solve exactly a FPE consists in separating the temporal from the spatial dependence; this latter one is solved by means of an eigenfunction expansion in the same way as occurs with the Schrödinger equation.[2–4]

Another more direct although more skillful technique is initiated in Ref. 6 and continued in Ref. 8. It consists in separating the part which is most related to the potential of the process (which causes the nonlinearity) from the probability density $P(q,t)$; then, by means of convenient assumptions, the remaining part of $P(q,t)$ is solved separately assuming that it is Gaussian. Concretely in Ref. 8 we have found that the N-dimensional FPE

$$\dot{P}(q,t) = -\partial_\mu [f^\mu(q)\cdot P(q,t)] + \tfrac{1}{2}\partial_\mu \partial^\mu P(q,t), \tag{1.1}$$

when $\partial_\mu \equiv \partial/\partial q^\mu$ (sum over repeated Greek indices is assumed) has an exact solution, with the usual initial condition

$$P(q,0) = \delta^n(q - q_0) \tag{1.2}$$

if the drift $f^\mu(q) = f^\mu(q_1,q_2,...,q_N)$ has the form

$$f^\mu(q) = -aq^\mu + \partial_\mu \phi(q)/\phi(q), \tag{1.3}$$

where

$$\phi(q) = \prod_{k=1}^{N} \left| \alpha_k F\left(l_k \left| \frac{1}{z} \right| aq_k^2\right) + \beta_k q_k F\left(l_k + \frac{1}{2} \left| \frac{3}{2} \right| aq_k^2 \right) \right|, \tag{1.4}$$

the α_K and β_k being arbitrary constants and the l_k arbitrary parameters. The function $F(l_k|\frac{1}{2}|aq_k^2)$ is the hypergeometric confluent function. In this case the exact solution to the FPE (1.1) is

$$P(q,t/q_0) = \left(\frac{a}{2\pi}\right)^{N/2} \frac{\phi(q)}{\phi(q_0)} \times \frac{\exp\{-(b/2)t - a\,\Sigma_{k=1}^N (q_k - \beta_k(t))^2/\eta(t)\}}{(\sinh at)^{N/2}}, \tag{1.5}$$

with

$$b \equiv \left[4\left(\sum_{k=1}^{N} l_k\right) - N\right]a, \tag{1.6}$$

$$\beta_k(t) \equiv q_{0k} e^{-at}, \tag{1.7}$$

$$\eta(t) \equiv 1 - e^{-2at}. \tag{1.8}$$

In Ref. 8 we have also found solutions of Eqs. (1.1) for a drift with spherical symmetry of the form

$$\mathbf{f}(r) = \left[-ar + \frac{d\phi(r)|dr}{\phi(r)}\right]\frac{\mathbf{r}}{r} \tag{1.9}$$

with

$$\phi(r) = |\alpha F(l\,|N/2|ar^2)| \tag{1.10}$$

for whatever value of the dimension N of the phase space, and

$$\phi(r) = \left|\alpha F\left(l \left| \frac{N}{2} \right| ar^2\right) + \beta r^{2-N} \times F\left(l + 1 - \frac{N}{2} \left| 2 - \frac{N}{2} \right| ar^2\right)\right| \tag{1.11}$$

if N is odd. In these cases the normalized solution of the FPE is given by (1.5) with $\phi(q)$ given by (1.10) or (1.11) and

$$b \equiv (4l - N)a. \tag{1.12}$$

In this article we intend to study some of the applications of these solutions for concrete models.

II. A FIRST MODEL OF NONLINEAR DIFFUSION

By means of an adequate selection of the constants α_k and β_k that appear in (1.4) we can write

$$\phi(q) = \left| \prod_{k=1}^{N} \{e^{(a|2)q_k^2} D_{-2l_k}(\sqrt{2a}q_k)\} \right|, \tag{2.1}$$

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where $D_{-l_k}(\sqrt{2a}q_k)$ is the parabolic cylindric function defined by[9]

$$D_\nu(z) = z^{\nu/2} e^{-z^2/4} \left\{ \frac{\Gamma(1/2)}{\Gamma((1-\nu)/2)} F\left(\frac{-\nu}{2}\middle|\frac{1}{2}\middle|\frac{z^2}{2}\right) + \frac{z}{\sqrt{2}} \frac{\Gamma(-1/2)}{\Gamma(-\nu/2)} F\left(\frac{1-\nu}{2}\middle|\frac{3}{2}\middle|\frac{z^2}{2}\right)\right\}. \quad (2.2)$$

For $l>0$ the functions $D_{-l_k}(\sqrt{2a}q_k)$ do not have real zeros and instead of (2.1) we can write

$$\phi(q) = \prod_{k=1}^{N} \{ e^{(a/2)q_k^2} D_{-2l_k}(\sqrt{2a}q_k)\}. \quad (2.3)$$

The characteristic function $\Theta(\mu_1,\ldots,\mu_N)$ associated to a density of probability $P(q,t)$ is given by[1]

$$\theta(\mu_1,\ldots,\mu_N) \equiv \int dq_1 \cdots dq_N \times \exp\{i(\mu_1 q_1 + \cdots + \mu_N q_N)\} P(q,t). \quad (2.4)$$

Substituting in (2.4) the probability density given by Eq. (1.5) with $\phi(q)$ given by (2.3) we have

$$\Theta(\mu_1,\ldots,\mu_N) = \left(\frac{a}{2\pi}\right)^{N/2} \frac{e^{-(b/2)t}}{\phi(q_0)(\sinh at)^{N/2}} \times \prod_{k=1}^{N} \int_{-\infty}^{+\infty} dq_k D_{-2l_k}(\sqrt{2a}q_k) \exp\left\{ i\mu_k q_k + \frac{1}{2} a q_k^2 - \frac{a(q_k - \beta_k(t))^2}{\eta(t)} \right\}, \quad (2.5)$$

since[9]

$$D_{-2l_k}(\sqrt{2a}q_k) = \frac{e^{-(1/2)aq_k^2}}{\Gamma(2l_k)} \int_0^\infty \exp(-\sqrt{2a}q_k s - \tfrac{1}{2}s^2) s^{2l_k - 1}\, ds \quad (l_k > 0). \quad (2.6)$$

Substituting (2.6) in (2.5) we finally arrive at

$$\theta(\mu_1,\ldots,\mu_N) = \frac{1}{\phi(q_0)} \cdot \prod_{k=1}^{N} \left[\exp\left\{ i\mu_k \beta_k(t) - \frac{1}{4a}\eta(t) u_k^2 \right\} \times e^{-(1/2)\rho_k^2} D_{-2l_k}(\rho_k) \right], \quad (2.7)$$

where

$$\rho_k \equiv \sqrt{2a} q_{0k} + i(z/\sqrt{2a})(\sinh at)\mu_k. \quad (2.8)$$

Once we have evaluated the characteristic function, moments follow easily[1]:

$$\langle q_1,\ldots,q_r \rangle = \frac{1}{i^r} \frac{\partial^r \theta(\mu_1 \cdots \mu_r)}{\partial \mu_1 \cdots \partial \mu_r}\bigg|_{\mu_1 = \cdots = \mu_r = 0}. \quad (2.9)$$

In our case

$$\langle q_k(t) \rangle = q_{0k} e^{-at} - \frac{4l_k}{\sqrt{2a}} (\sinh at) \frac{D_{-2l_k-1}(\sqrt{2a}q_k)}{D_{-2l_k}(\sqrt{2a}q_k)} \quad (l_k > 0) \quad (2.10)$$

and

$$\langle [q_k(t)]^2 \rangle = q_{0k}^2 e^{-2at} + \frac{1 - e^{-2at}}{2a} - \frac{8l_k}{\sqrt{2a}} q_{0_k} e^{-at} (\sinh at) \frac{D_{-2l_k-1}(\sqrt{2a}q_{0k})}{D_{-2l_k}(\sqrt{2a}q_{0k})} + \frac{4l_k(l_k+1)}{a} (\sinh^2 at) \frac{D_{-2l_k-2}(\sqrt{2a}q_k)}{D_{-2l_k}(\sqrt{2a}q_{0k})} \quad (l_k > 0). \quad (2.11)$$

In this model the drift may be written in the form

$$f^\mu(q) = -aq^\mu - 2l_\mu \sqrt{a} \left[D_{-2l_\mu - 1}(\sqrt{2a}q) / D_{-2l_\mu}(\sqrt{2a}q) \right]. \quad (2.12)$$

The first moment (2.10) as a function of the drift is

$$\langle q_k(t) \rangle = q_{0k} e^{-at} + (\sinh at)[q_{0k} + f_k(q_0)/a]. \quad (2.13)$$

This first moment presents a "boomerang" effect since the average velocity

$$\frac{d\langle q_k(t) \rangle}{dt} = -a \left\{ q_{0k} e^{-at} - (\cosh at) \left[q_0 + \frac{f_k(q_0)}{a} \right] \right\} \quad (2.14)$$

becomes equal to zero and changes the sign if $0 < l_k < \frac{1}{4}$ and $-q_{1k} < q_{0k} < q_{1k}$ ($\pm q_{1k}$ are the positions of the maxima of the potential of the drift) for a time t_b (see Ref. 6):

$$(t_b)_k = \frac{1}{2a} \ln \left[\frac{aq_{0_k} - f_k(q_0)}{aq_{0k} + f_k(q_0)} \right]. \quad (2.15)$$

We easily observe that Eqs. (2.10) and (2.11) give

$$\lim_{t\to\infty} |\langle q_k(t) \rangle| = \lim_{t\to\infty} |\langle q_k(t)^2 \rangle| = \infty, \quad (2.16)$$

except for $l_k = 0$ that corresponds to the case of linear drift [see Eq. (2.12)].

We can also consider the model with spherical symmetry such that, when $q_0 = 0$, the probability density (1.5) can be written as

$$P(r,t|0) = \left(\frac{q}{2\pi}\right)^{N/2} \frac{\phi(r)}{\phi(0)} \frac{\exp\{-(b/2)t - ar^2/\eta(t)\}}{(\sinh at)^{N/2}}, \quad (2.17)$$

where $r \equiv (\sum_{k=1}^N q_k^2)^{1/2}$ and $\phi(r)$ is given by (1.10). For $l>0$ the function $F(1|N/2|ar^2)$ has no zeros and since $F(a|c|0) = 1$, we have

$$\phi(r)/\phi(0) = F(l|N/2|ar^2) \quad (l>0). \quad (2.18)$$

For this model the potential $V(r)$ of the drift (1.3) is

$$V(r) = \tfrac{1}{2}ar^2 - \ln F(l|N/2|ar^2) \quad (l>0).$$

In Fig. 1 we represent this potential when $N=1$ and $a=1$, for the cases (a) $l=0$, (b) $l=0.1$, and (c) $l=0.5$. The moments $\langle [r(t)]^m \rangle$ are given by

$$\langle [r(t)]^m \rangle = \frac{(2\pi)^{N/2}}{\Gamma(N/2)} \int_0^\infty r^{N+m-1} P(r,t|0) dr \quad (m = 1,2,\ldots). \quad (2.19)$$

Substituting in (2.19) the probability density given by (2.17) and since[9]

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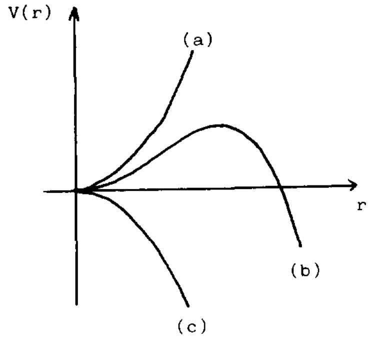

FIG. 1. Representation of the potential $V(r)$ when $N=1$ and $a=1$ for the cases (a) $l=0$, (b) $l=0.1$, and (c) $l=0.5$.

$$\int_0^\infty e^{-\alpha z} z^{V-1} F(a|c|z) dz$$
$$= \alpha^{-V}\Gamma(V) F\left(a, V|c|\frac{k}{\alpha}\right), \qquad (2.20)$$

for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\alpha > \operatorname{Re} k$, $\operatorname{Re} V > 0$, we arrive easily at

$$\langle [r(t)]^m\rangle = \frac{\Gamma((m+N)/2)}{2^{1-N/2}a^{m|2}\Gamma(N/2)} e^{-2lat}[\eta(t)]^{m/2} \times F\left(l, \frac{m+N}{2}\left|\frac{N}{2}\right|\eta(t)\right). \qquad (2.21)$$

In this case the velocity of the moments is given by

$$\frac{d\langle [r(t)]^m\rangle}{dt} = \frac{2^{N|2}\Gamma((m+r)/2)}{a^{m/2-1}\Gamma(N|z)} e^{-2lat}[\eta(t)]^{m/2-1} \times\left\{\left[\frac{m}{2}(1-\eta(t)) + l\eta(t)\right] \times F\left(l, \frac{m+N}{2}\left|\frac{N}{2}\right|\eta(t)\right) + \frac{l(N+m)}{N}\eta(t)(1-\eta(t))F\left(1+l, 1 + \frac{m+N}{2}\left|1+\frac{N}{2}\right|\eta(t)\right)\right\}. \qquad (2.22)$$

Now we investigate the behavior of the moments $\langle [r(t)]^m\rangle$ for large times ($t \gg 1/2a$). In this case

$$\eta(t) \to 1 \quad (t \gg 1/2a) \qquad (2.23)$$

and since[9]

$$F(a,b|c|z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)} \times F(a,b|a+b-c+1/1-z) + (1-z)^{(-a-b)}\frac{\Gamma(a)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \times F(c-a, c-b|c-a-b+1|1-z) \qquad (2.24)$$

and $F(a,b|c|1-n) \simeq 1$ $(t \gg 1/2a)$, we arrive at

$$\langle [r(t)]^m\rangle \simeq \frac{\Gamma(l+m/2)}{2^{1-N/2}a^{m/2}\Gamma(l)} e^{mat} \quad \left(t \gg \frac{1}{2a} \quad \text{and} \quad l \neq 0\right). \qquad (2.25)$$

This expression diverges when $t \to \infty$.

The case $l=0$ corresponds to linear drift and, therefore, presents no difficulty.

We come to the conclusion that both models presented in this section could not be valid for the study of the temporal evolution of physical systems towards equilibrium. In the following section we find a mechanism that yields exactly soluble models that relax towards equilibrium with a finite relaxation time.

III. ONE-DIMENSIONAL MODEL WITH POTENTIAL BARRIER

For a one-dimensional system the potential $V(x)$ of the drift (1.3) is

$$V(x) = \tfrac{1}{2}ax^2 - \ln\phi(x), \qquad (3.1)$$

where $\phi(x)$ is given by Eq. (1.4). By means of an adequate choice of the constants α_k and β_k, we can write

$$\phi(x) = U(l\,|\tfrac{1}{2}|ax^2), \qquad (3.2)$$

where $U(l\,|\tfrac{1}{2}|ax^2)$ is a function of Kummer.[9] Let us suppose now, that, for a certain value of $x_1 < x_0$, there exists a potential barrier, that is,

$$V(x) = \begin{cases} \tfrac{1}{2}ax^2 - \ln\phi(x), & x \geqslant x_1, \\ \infty, & x < x_1, \end{cases} \qquad (3.3)$$

which is equivalent to the following expression for $\phi(x)$

$$\phi(x) = \begin{cases} U(l\,|\tfrac{1}{2}|ax^2), & x \geqslant x_1, \\ 0, & x < x_1. \end{cases} \qquad (3.4)$$

This is possible since $\phi(x) = 0$ is also a solution of the differential equation that satisfies the function (3.2) (see Ref. 8).

In this case, and supposing that $x_0 = 0$, the probability density is

$$P(x,t\,|0) = \left(\frac{a}{2\pi}\right)^{1/2}\frac{\phi(x)}{\phi(0)}\frac{\exp\{-(b/2)t - ax^2/\eta(t)\}}{(\sinh at)^{1/2}}. \qquad (3.5)$$

In this model the moments are evaluated by

$$\langle [x(t)]^m\rangle = \int_{x_1}^\infty x^m P(x,t\,|0)dx \quad (m = 1,2,...), \qquad (3.6)$$

following the procedure described in the Appendix. Expression (3.6) becomes

$$\langle [x(t)]^m\rangle = \frac{\Gamma(l+\frac{1}{2})}{2\pi a^{m/2}}(\eta(t))^{m/2-l} \times \exp[-2lat + ax_1^2/\eta(t)] \times \sum_{n=0}^\infty \psi_n^{(m)}(e, x_1; \eta(t)), \qquad (3.7)$$

where

$$\psi_n^{(1)}(l, x_1; \eta(t)) = \frac{(l)_n(l+\frac{1}{2})_n}{n!}\left(1 - \frac{1}{\eta(t)}\right)^n \times U\left(l+n\left|-\frac{1}{2}\right|\frac{ax_1^2}{\eta(t)}\right), \qquad (3.8a)$$

$$\psi_n^{(2)}(l, x_1; \eta(t)) \equiv \frac{(l)_n(l+\frac{1}{2})_n}{n!}\left(1 - \frac{1}{\eta(t)}\right)\left[\frac{ax_1^2}{\eta(t)}U\left(l+n+\frac{1}{2}\left|\frac{1}{2}\right|\frac{ax_1^2}{\eta(t)}\right) + U\left(l+n+\frac{1}{2}\left|-\frac{1}{2}\right|\frac{ax_1^2}{\eta(t)}\right)\right]. \qquad (3.8b)$$

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When $t \ll 1/2a$, then $\eta(t) \simeq 2at \ll 1$ and the main term of Eq. (3.7) is $\exp(-ax_1^2/\eta(t))$. Therefore,

$$\langle [x(t)]^m \rangle \sim e^{-x_1^2/2t} \quad (t \ll 1/2a). \tag{3.9}$$

When $t \gg 1/2a$, then $\eta(t) \simeq 1$ and the main term of Eq. (3.7) is $\exp(-2lat)$, i.e.,

$$\langle [x(t)]^m \rangle \sim e^{-2lat} \quad (t \gg 1/2a), \tag{3.10}$$

that yields a relaxation time

$$\tau_{\text{relax}} = 1/2la. \tag{3.11}$$

The moments, for small times, are growing functions of time; however, for large times, they are decreasing functions of time. Therefore, in the cases in which (3.7) are continuous functions of time, the moments pass through a maximum for $t \sim 1/2a$ and we again have the "boomerang" effect.

We will finish this section studying the case when the potential barrier is very far away from the origin (that is, our initial state), i.e., when

$$ax_1^2 \gg 1. \tag{3.12}$$

In such a case, as in Ref. 10,

$$U(a|c|z) \sim z^{-a} \quad \text{for} \quad z \to \infty \quad (\text{Re}\, a > 0), \tag{3.13}$$

the functions $\psi_n^{(m)}(l,x_1;\eta(t))$, defined by (3.8), may be written

$$\psi_n^{(m)}(l,x_1;\eta(t)) \simeq \left(\frac{\eta(t)}{ax_1^2}\right)^{l+(1-m)/2} \times \left\{\frac{(l)_n(l+\frac{1}{2})_n}{n!}\left(\frac{\eta(t)-1}{ax_1^2}\right)^n\right\}$$

$$(m = 1,2).$$

Taking into consideration[9]

$${}_2F_0(a,b;z) \equiv \sum_{n=0}^{\infty} (a)_n (b)_n \frac{z^n}{n!},$$

$${}_2F_0(a,b;z^{-1}) = z^a U(a|a-b+1|z),$$

we arrive at

$$\sum_{n=0}^{\infty} \psi_n^{(m)}(l,x;\eta(t)) \simeq \left(\frac{\eta(t)}{ax_1^2}\right)^{1-m/2}\left(\frac{\eta(t)}{1-\eta(t)}\right)^l \times U\left(l\left|\frac{1}{2}\right|\frac{ax_1^2}{1-\eta(t)}\right) \quad (ax_1^2 \gg 1), \tag{3.14}$$

for $m = 1,2$. Substituting (3.14) into Eq. (3.7) we finally get

$$\langle [x(t)]^m \rangle \simeq \frac{\Gamma(l+\frac{1}{2})}{2a^{m/2}\pi}\frac{e^{-ax_1^2/\eta(t)}}{(ax_1^2)^{1-m/2}}e^{-4lat} \times U\left(l\left|\frac{1}{2}\right|\frac{ax_1^2}{\eta(t)}\right) \quad (ax_1^2 \gg 1), \tag{3.15}$$

for $m = 1,2$.

When $t \gg 1/2a$, we have $\eta(t) \simeq 1$. With the approximation (3.13) we have

$$\langle [x(t)]^m \rangle \simeq \frac{\Gamma(l+\frac{1}{2})}{2\pi a^{m/2}}\frac{e^{-ax_1^2}}{(ax_1^2)^{1-m/2}}e^{-2lat}$$

$$(ax_1^2 \gg 1 \quad \text{and} \quad t \gg 1/2a). \tag{3.16}$$

When $t \ll 1/2a$ we can approximate

$$\eta(t) \simeq 2at \ll 1, \quad 1-\eta(t) \simeq 1, \quad e^{-4lat} \simeq 1$$

(for l moderate).

Remembering (3.13), Eq. (3.15) becomes

$$\langle [x(t)]^m \rangle \simeq \frac{\Gamma(l+\frac{1}{2})}{2\pi a^{m/2}}\frac{e^{-x_1^2/2t}}{(ax_1^2)^{l+(1-m)/2}}$$

$$(ax_1^2 \gg 1 \quad \text{and} \quad t \ll 1/2a). \tag{3.17}$$

Therefore, even when the potential barrier is very far away from the initial state, the evolution of the system depends on the position x_1 of the barrier.

If we compare Eq. (3.16) with Eq. (3.10), and Eq. (3.17) with Eq. (3.9), we observe that the asymptotic temporal evolution of the model is similar to the evolution of the general case.

For $x < x_1$ the potential (3.3) is a hard-core potential. This implies that the barrier is a reflecting barrier. Thus, the probability current $J(x,t)$ must be zero for $x \leqslant x_1$. In our case $J(x,t)$ is given by

$$J(x,t) = K(x,t)[a(1/\eta(t)-1)x\phi(x) + \tfrac{1}{2}\phi'(x)], \tag{3.18}$$

where

$$K(x,t) \equiv \left(\frac{a}{2\pi}\right)^{1/2}\frac{\exp\{-(b/2)t - ax^2/\eta(t)\}}{(\sinh at)^{1/2}}$$

and[9]

$$\phi'(x) = \begin{cases} -2l^2 x U(l+1|\frac{3}{2}|ax^2), & x \geqslant x_1, \\ 0, & x < x_1. \end{cases} \tag{3.19}$$

Thus $J(x,t)$ will be zero at the barrier if $\phi'(x)$ is a continuous function at $x = x_1$. This implies that the potential barrier must be located at the zeroes of the Kummer function.

If, instead of (3.4), we write

$$\phi(x) = \begin{cases} U(l|\frac{1}{2}|ax^2), & x \geqslant x_1, \\ \alpha, & x < x_1, \end{cases}$$

where $\alpha \ll 1$ (our potential is not completely hard core), we have

$$P(x,t/0) \cong 0 \quad \text{for } x \leqslant x_1 \tag{3.20}$$

[see Eq. (3.5)] and the barrier may be located anywhere.

In Fig. 2 we have a representation of the potential (3.3) in the case where $l = -0.5$.

IV. STATIONARY DISTRIBUTIONS

As is well known a one-dimensional FPE

$$\dot{P}(x,t) = -\frac{\partial}{\partial x}[f(x)P(x,t)] + \frac{1}{2}\frac{\partial^2 P(x,t)}{\partial x^2} \tag{4.1}$$

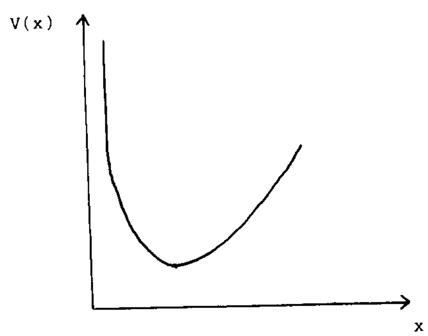

FIG. 2. Representation of the potential (3.3) in the case $l = -0.5$. The potential barrier is located at $x_1 = 0.25\, a^{1/2}$, and the minimum at $x_0 = a^{-1/2}$.

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has a stationary distribution of the form

$$P_{\rm st}(x) = N \exp\left\{2 \int^x f(x')dx'\right\}, \quad (4.2)$$

when the probability current

$$J(x,t) = f(x)P(x,t) - \frac{1}{2}\frac{\partial P(x,t)}{\partial x} \quad (4.3)$$

satisfies the boundary condition[10]

$$\lim_{x\to\pm\infty} J(x,t) = 0. \quad (4.4)$$

Both our general model represented by Eq. (1.4) as well as the models represented by Eqs. (2.1) and (3.2) satisfy these boundary conditions. Their stationary solution is

$$P_{\rm st}(x) = N[\phi_i(x)]^2 e^{-ax^2}, \quad (4.5)$$

where

$$\phi_1(x) = \alpha F(l\,|\tfrac{1}{2}|ax^2) + \beta x F(l+\tfrac{1}{2}|\tfrac{3}{2}|ax^2), \quad (4.6a)$$

$$\phi_2(x) = e^{(a/2)x^2} D_{-2l}(\sqrt{2a}x), \quad (4.6b)$$

$$\phi_3(x) = U(l\,|\tfrac{1}{2}|ax^2), \quad (4.6c)$$

and

$$N = \frac{a^{1/2}}{\pi}\left[\frac{\alpha^2 l\Gamma(-l)}{\Gamma(\frac{1}{2}-l)} + \frac{\beta^2(l+\frac{1}{2})\Gamma(-l-\frac{1}{2})}{4a\Gamma(1-l)}\right]^{-1} \quad (4.7)$$

[provided that the proper choice of the constants α and β extends this normalization to the models (4.6b) and (4.6c)].

Let us study the stability of these stationary distributions. Following the criterion given in Ref. 4 we can affirm that the stochastic process represented by Eq. (4.1) has a stable stationary solution, and all moments $\langle x^m \rangle$ up to the mth order exist if the following inequality is satisfied:

$$L \equiv \lim_{x\to\infty} \frac{-2\int^x f(x')dx'}{(m+1)\ln x} > 1. \quad (4.8)$$

In our case we have [see Eq. (1.3) and Ref. 9]

$$L_1 = -\infty, \quad L_2 = L_3 = +\infty, \quad (4.9)$$

whatever the values of m and l. We see therefore that the general model is completely unstable (let us remember that in this model, when $\beta = 0$, the model presented in Ref. 6 is included). As a matter of fact, both the general model (4.6a) as well as the model of Ref. 6 do not behave correctly at infinity since $P_{\rm st}(x) \to \infty$ when $x \to \pm\infty$.

Thus, we can affirm that the models presented in Secs. II and III are stable for any value of the parameter l.

V. CONCLUSIONS

Relating the results of Sec. II with those of Sec. III, we observe that the nonlinear diffusion process, represented in general form by the drift (1.3), yields divergent momenta (and infinite relaxation times) if the diffusion process can be extended to the whole physical space. Nevertheless when, due to the introduction of a potential barrier, the diffusion process takes place in a limited part of space, the moments converge with finite relaxation time given by

$$\tau_{\rm nonlinear} = 1/2la.$$

Comparing this relaxation time with the one that corresponds to a linear drift, $f^{\mu}(q) = aq^{\mu}$, that is,

$$\tau_{\rm linear} = 1/a,$$

we observe that this process of nonlinear diffusion relaxes quicker than the linear diffusion if

$$l > \tfrac{1}{2}.$$

Let us remark also that the nondivergent model studied in this paper can reach large parts of space since the asymptotic evolution of the process is the same no matter how far away the potential barrier is from the initial state.

The general model represented by Eq. (1.4) is unstable since its stationary distribution $P_{\rm st}(x)$ diverges. The models studied in Secs. II and III are stable.

APPENDIX: EVALUATION OF INTEGRALS

We have to evaluate the integral

$$I^{(m)} = \int_{x_1}^{\infty} x^m P\left(x, \frac{t}{0}\right)dx \quad (m = 1,2,...). \quad (A1)$$

By means of the change of variables $z = ax^2$, we get

$$I^{(m)} = K(t)[I_1^{(m)} + I_2^{(m)}], \quad (A2)$$

with

$$K(t) \equiv \frac{\Gamma(l+\frac{1}{2})}{2\pi a^{m/2}} e^{-2lat}(\eta(t))^{-1/2}, \quad (A3)$$

$$I_1^{(m)} \equiv \int_{z_1}^{0} z^{(m-1)/2} e^{-z/\eta(t)} U(l\,|\tfrac{1}{2}|z)dz, \quad (A4)$$

$$I_2^{(m)} \equiv \int_0^{\infty} z^{(m-1)/2} e^{-z/\eta(t)} U(l\,|\tfrac{1}{2}|z)dz. \quad (A5)$$

The evaluation of $I_2^{(m)}$ is immediate taking into account that[9]

$$\int_0^{\infty} e^{-sz} z^{k-1} U(a|c|z)dz = \frac{\Gamma(b)\Gamma(1+b-c)}{\Gamma(1+a+b-c)} s^{-b} \times F\left(a,b\,|1+a+b-c|1-\frac{1}{s}\right),$$

where Re $s > 1/2$ and $F(a,b\,|c|z)$ is the hypergeometric function.

The final result is

$$I_2^{(m)} = \frac{\Gamma((1+m/2)/2)\Gamma(1+m/2)}{\Gamma(1+l+m/2)}(\eta(t))^{(1+m)/2} \times F(l,(1+m)/2|1+l+m/2|1-\eta(t)), \quad (A6)$$

valid for $m = 1,2,3,...$.

To evaluate $I_1^{(m)}$ we perform the change of variable $y = z/\eta(t)$. Using the multiplication theorem[9]

$$U(a|c|zz') = (z')^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(1+a-c)_n}{n!} \times \left(1-\frac{1}{z'}\right)^n U(a+n|c|z),$$

we have

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$$I_1^{(m)} = (\eta(t))^{(1+m)/2-l} \sum_{n=0}^{\infty} \frac{(l)_n (l+\frac{1}{2})_n}{n!} \left(1 - \frac{1}{\eta(t)}\right)^n \times \int_{y_1}^{0} y^{(m-1)/2} e^{-y} U(l+n|\tfrac{1}{2}|y) dy. \tag{A7}$$

For $m = 1$, and recalling that[9]

$$\int e^{-z} U(a|c|z) dz = -e^{-z} U(a|c-1|z) + C, \tag{A8}$$

the expression (A7) becomes

$$I_1^{(1)} = (\eta(t))^{1-l} \cdot \left\{ e^{-ax_1^2/\eta(t)} \left[\sum_{n=0}^{\infty} \frac{(l)_n (l+\frac{1}{2})_n}{n!} \left(1 - \frac{1}{\eta(t)}\right)^n \times U\left(l+n \,\middle|\, -\frac{1}{2} \,\middle|\, \frac{ax_1^2}{\eta(t)}\right)\right] - \frac{\Gamma(\frac{3}{2})}{\Gamma(l+\frac{3}{2})} F\left(l,l+\frac{1}{2} \,\middle|\, 1+\frac{3}{2} \,\middle|\, 1-\frac{1}{\eta(t)}\right)\right\}, \tag{A9}$$

since

$$U(l+n|-\tfrac{1}{2}|0) = \frac{\Gamma(\frac{3}{2})}{\Gamma(l+n+\frac{3}{2})},$$

$$\Gamma(l+n+\tfrac{3}{2}) = \Gamma(l+\tfrac{3}{2})(l+\tfrac{3}{2})_n,$$

$$F(a,b|c|z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}.$$

For $m = 2$, integrating by parts and taking into account (A8), Eq. (A7) becomes

$$I_1^{(2)} = (\eta(t))^{(3/2)-l} \left\{ e^{-ax_1^2/\eta(t)} \left[\sum_{n=0}^{\infty} \frac{(l)_n (l+\frac{1}{2})_n}{n!} \left(1 - \frac{1}{\eta(t)}\right)^n\right] \times \left[\frac{ax_1^2}{\eta(t)} U\left(l+n+\frac{1}{2} \,\middle|\, \frac{1}{2} \,\middle|\, \frac{ax_1^2}{\eta(t)}\right) + U\left(l+n+\frac{1}{2} \,\middle|\, -\frac{1}{2} \,\middle|\, \frac{ax_1^2}{\eta(t)}\right)\right] - \frac{\Gamma(\frac{3}{2})}{\Gamma(l+2)} F(l,l+\tfrac{1}{2}|l+2|1-\frac{1}{\eta(t)}\right\}. \tag{A10}$$

In general,

$$I_1^{(m)} = (\eta(t))^{(1+m)/2-l} \left\{ e^{-ax_1^2/\eta(t)} \times \left[\sum_{n=0}^{\infty} \psi_n^{(m)}(l,x_1;\eta(t))\right] - \frac{\Gamma(\frac{3}{2})}{\Gamma(1+l+m/2)} \times F\left(l,l+\frac{1}{2} \,\middle|\, 1+l+\frac{m}{2} \,\middle|\, 1-\frac{1}{\eta(t)}\right), \tag{A11}$$

with $\psi_n^{(m)}(l,x_1;\eta(t))$ given by (3.8). Expression (A11) is only valid for $m = 1,2$.

Substituting (A11), (A6), and (A3) in Eq. (A1) and with the help of Ref. 9,

$$F(a,b|c|z) = (1-z)^{-a} F(a,c-b|c|z/(z-1));$$

in this way we obtain Eq. (3.7).

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