Noether's theorem and gauge transformations: Application to the bosonic string and CP_2^{n-1} model

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New results on the theory of constrained systems are applied to characterize the generators of Noether's symmetry transformations. As a byproduct, an algorithm to construct gauge transformations in Hamiltonian formalism is derived. This is illustrated with two relevant examples.

I. INTRODUCTION

It is superfluous to emphasize the relevance of gauge theories in modern physics. In spite of this, many aspects of the classical theory of constrained systems—those which have elbow room for gauge transformations—are not completely developed. The aim of this paper is to clarify the role of the Lagrangian Noether theorem in obtaining the generators of Hamiltonian gauge transformations. This is achieved by applying some results recently obtained concerning the relationship between the Hamiltonian and Lagrangian formalisms.[1–3] These new results apply to general constrained systems, with first- and second-class constraints, under the only regularity conditions of Ref. 2.

The paper is organized as follows. In Sec. II we set the notation and summarize some of the results of Refs. 1–3; they are used in Sec. III to characterize the Hamiltonian generators of a general symmetry Noether transformation. In Sec. IV the specific case of gauge transformations is considered. Section V is devoted to some relevant applications: the bosonic string and the CP_2^{n-1} model.

All structures are supposed to be C^∞. Indices of coordinates will be omitted.

II. PRELIMINARY RESULTS

Here we state some of the results needed in Sec. III. For more details see Refs. 2 and 3. Minor changes of notation have been done.

A configuration space Q and a Lagrangian L are given. We shall always work with natural coordinates such as (q,v) in $T(Q)$ and (q,p) in $T(Q)^*$.

Then the Euler–Lagrange equations for a curve $(q(t),p(t))$ in $T(Q)$ can be written as

$$\dot{q} = v, \tag{2.1}$$

$$W\dot{v} = \alpha, \tag{2.2}$$

where we have introduced the Hessian matrix

$$W := \frac{\partial^2 L}{\partial v\, \partial v} \tag{2.3}$$

and

$$\alpha := \frac{\partial L}{\partial q} - v\frac{\partial^2 L}{\partial q\, \partial v}. \tag{2.4}$$

The Legendre transformation $FL: T(Q) \to T(Q)^*$, with the local expression

$$FL(q,v) = \left(q, \frac{\partial L}{\partial v}\right), \tag{2.5}$$

has the image $M_0 \subset T(Q)^*$, which is assumed to be a submanifold (locally) defined by the m_0 primary Hamiltonian constraints ϕ_μ^0 $(1 \leqslant \mu \leqslant m_0)$.

The vertical vector fields

$$\Gamma_\mu := \gamma_\mu \frac{\partial}{\partial v} \tag{2.6}$$

constitute a frame for the sub-bundle $\operatorname{Ker} T(FL) \subset T(V)$, where

$$\gamma_\mu := FL^*\left(\frac{\partial \phi_\mu^0}{\partial p}\right) \tag{2.7}$$

are a basis for the null vectors of W.

An outstanding object in our development is the operator K^2, which is now understood[4] as a vector field along FL, that is to say, it is a mapping that makes the following diagram commutative:

$$\begin{tikzcd} & T(T(Q)^*) \arrow[d, "o_{T(Q)^*}"] \\ T(Q) \arrow[ur, "K"] \arrow[r, "FL"'] & T(Q)^* \end{tikzcd}$$

Its local expression is

$$K(q,v) = v\frac{\partial}{\partial q} + \frac{\partial L}{\partial q}\frac{\partial}{\partial p}.$$

In fact, we shall need K in the time-dependent case, so that we shall add $\partial/\partial t$ to it:

$$K(q,v,t) = v\frac{\partial}{\partial q} + \frac{\partial L}{\partial q}\frac{\partial}{\partial p} + \frac{\partial}{\partial t}. \tag{2.8}$$

Now K can be regarded as a differential operator as follows. If f is a function in $T(Q)^* \times \mathbf{R}$,

$$K\cdot f = v\, FL^*\left(\frac{\partial f}{\partial q}\right) + \frac{\partial L}{\partial q} FL^*\left(\frac{\partial f}{\partial p}\right) + FL^*\left(\frac{\partial f}{\partial t}\right) \tag{2.9}$$

is a function in $T(Q) \times \mathbf{R}$.

Now let H be a Hamiltonian function, that is, $FL^*(H) = E_L$, where

$$E_L = v\frac{\partial L}{\partial v} - L \tag{2.10}$$

is the Lagrangian energy.

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One can prove[2,5] that there exist m_0 functions λ^μ in $T(Q)$ such that

$$v = FL^*\{q,H\} + \lambda^\mu FL^*\{q,\phi_\mu^0\}, \tag{2.11}$$

$$\frac{\partial L}{\partial q} = FL^*\{p,\dot{H}\} + \lambda^\mu FL^*\{p,\phi_\mu^0\}. \tag{2.12}$$

These functions λ^μ are not FL projectable since $\Gamma_\mu \cdot \lambda^\nu = \delta_\mu^\nu$. Then, it is easy to obtain

$$K\cdot f = FL^*\{f,H\} + \lambda^\mu FL^*\{f,\phi_\mu^0\} + FL^*\left(\frac{\partial f}{\partial t}\right) \tag{2.13}$$

and

$$\Gamma_\mu \cdot (K\cdot f) = FL^*\{f,\phi_\mu^0\}. \tag{2.14}$$

A careful analysis[2] of Eqs. (2.11) and (2.12) lets us write the Hamilton–Dirac equations as

$$\dot{f} \underset{M_0}{=} \{f,H\} + \eta^\mu\{f,\phi_\mu^0\} + \frac{\partial f}{\partial t}, \tag{2.15}$$

where η^μ are arbitrary functions of time.

Now derivation of (2.11) with respect to v expresses the identity matrix as

$$I = MW + \gamma_\mu \otimes \frac{\partial \lambda^\mu}{\partial v}, \tag{2.16}$$

where

$$M = FL^*\left(\frac{\partial^2 H}{\partial p\, \partial p}\right) + \lambda^\mu FL^*\left(\frac{\partial^2 \phi_\mu^0}{\partial p\, \partial p}\right). \tag{2.17}$$

Application of (2.16) to (2.1) and (2.2) leads to the introduction of time-evolution fields in $T(Q)$:

$$D_u := D_0 + u^\mu \Gamma_\mu, \tag{2.18}$$

where u^μ are arbitrary functions of time and

$$D_0 := v\frac{\partial}{\partial q} + \alpha M \frac{\partial}{\partial v} + \frac{\partial}{\partial t}. \tag{2.19}$$

Then the Euler–Lagrange equations also read

$$\dot{g} \underset{S_1}{=} D_u \cdot g, \tag{2.20}$$

where $S_1 \subset T(Q)$ is the submanifold defined by the primary Lagrangian constraints

$$\chi_\mu^1 = \alpha\gamma_\mu = K\cdot\phi_\mu^0. \tag{2.21}$$

Bearing all these relations in mind one can prove that

$$K\cdot f = D_u \cdot FL^*(f) + \chi_\mu^1 (Y^\mu \cdot f), \tag{2.22}$$

where we have introduced m_0 vector fields along FL:

$$Y^\mu(q,v) = \frac{\partial \lambda^\mu}{\partial v}\frac{\partial}{\partial p}. \tag{2.23}$$

Finally, we want to point out that at the present time most of these objects and relations can be defined or written intrinsically: not only (2.5) and (2.10), which are well-known,[6] but also (2.1)–(2.2), (2.8), and (2.9)[4]; (2.6)[7]; (2.11)–(2.12), (2.13), (2.22), and (2.23)[5]; and (2.21).[1,2]

III. CHARACTERIZATION OF NOETHER TRANSFORMATIONS

In the following it will be useful to enlarge our space with a third set of independent coordinates, the accelerations a; that is to say, we shall work in the second tangent bundle $T^2(Q)$.

We shall consider the operator [which maps functions in $T(Q)\times\mathbf{R}$ to functions in $T^2(Q)\times\mathbf{R}$]

$$\frac{d}{dt} = v\frac{\partial}{\partial q} + a\frac{\partial}{\partial v} + \frac{\partial}{\partial t}. \tag{3.1}$$

Then the Euler–Lagrange equations can be written as

$$[L]_{(q,\dot{q},\ddot{q})} = 0, \tag{3.2}$$

where we have defined

$$[L] := \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial v}\right) = \alpha - aW. \tag{3.3}$$

Noether's theorems yield a sufficient condition for a $\delta q(q,v,t)$ to be a *dynamical symmetry transformation* (DST) of L, that is to say, to map solutions into solutions. This condition can be written as[8–10]

$$[L]\delta q + \frac{dG}{dt} = 0 \tag{3.4}$$

for certain $G(q,v,t)$. We call such a δq a *Noether transformation*. The acceleration appears linearly in (3.4), so that it splits into two relations[10–12]:

$$\alpha\,\delta q + v\frac{\partial G}{\partial q} + \frac{\partial G}{\partial t} = 0, \tag{3.5}$$

$$\frac{\partial G}{\partial v} - W\,\delta q = 0. \tag{3.6}$$

An immediate result from (3.6) is that G is an FL-projectable function since $\Gamma_\mu \cdot G = \gamma_\mu(\partial G/\partial v) = \gamma_\mu W\,\delta q = 0$. Therefore, there exists $G_h(q,p,t)$ (up to primary constraints) such that

$$G = FL^*(G_h). \tag{3.7}$$

Now we apply the operator K to G_h, bearing (2.22), (2.19), and (2.16) in mind, under the only condition (3.7). The result is

$$\begin{aligned} K\cdot G_h = \chi_\mu^1 \frac{\partial \lambda^\mu}{\partial v}&\left(FL^*\left(\frac{\partial G_h}{\partial p}\right) - \delta q\right) \\ &+ \left(\alpha\,\delta q + v\frac{\partial G}{\partial q} + \frac{\partial G}{\partial t}\right) \\ &+ \alpha M\left(\frac{\partial G}{\partial v} - W\,\delta q\right). \end{aligned} \tag{3.8}$$

If G corresponds to a Noether transformation, (3.5) and (3.6) set the last two terms to zero. Moreover, assume $\delta q(q,v,t)$ to be *FL projectable*. There is $\delta q_h(q,p,t)$ (up to primary constraints) such that

$$\delta q = FL^*(\delta q_h). \tag{3.9}$$

Moreover,

$$\begin{aligned} 0 = \frac{\partial}{\partial v}(FL^*(G_h) - G) &= W\,FL^*\{q,G_h\} - \frac{\partial G}{\partial v} \\ &= W\,FL^*(\{q,G_h\} - \delta q). \end{aligned}$$

Thus there are functions $h^\mu(q,p,t)$ such that

$$\{q,G_h\} \underset{M_0}{=} \delta q_h + h^\mu \frac{\partial \phi_\mu^0}{\partial p}. \tag{3.10}$$

Redefining $G_h := G_h - h^\mu \phi_\mu^0$ we have $\{q,G_h\} \underset{M_0}{=} \delta q_h$, so that we can assume G_h and δq_h chosen in order that

$$\delta q_h = \{q,G_h\}. \tag{3.11}$$

Therefore, we conclude from (3.9) and (3.11) that (3.8) becomes

$$K\cdot G_h = 0. \tag{3.12}$$

Conversely, suppose we have $G_h(q,p,t)$ satisfying relation (3.12) and define δq_h, δq, and G as in (3.11), (3.9), and (3.7). Then we have $\partial G/\partial v = W FL^*(\partial G_h/\partial p) = W\delta q$, which is (3.6), and the identity for $K\cdot G_h$ [(3.8)] shows that (3.5) also holds; that is to say, (3.4) is satisfied. We have proven the following theorem.

Theorem 1: An infinitesimal projectable function $\delta q(q,v,t)$ is a Noether transformation if there exists $G_h(q,p,t)$ such that $K\cdot G_h = 0$ and $\delta q = FL^*\{q, G_h\}$.

Now we make use of this Lagrangian result to derive a sufficient condition for a $G_h(q,p,t)$ to generate a Hamiltonian DST in the sense that

$$\delta f = \{f,G_h\}. \tag{3.13}$$

Theorem 2: An infinitesimal function $G_h(q,p,t)$ satisfying $K\cdot G_h = 0$ generates a Hamiltonian DST.

We call such a DST a Hamiltonian Noether transformation. We have shown that $\delta q := FL^*\{q,G_h\}$ is a Lagrangian DST. Taking into account the equivalence of both formalisms,[2] we only need show $\delta(\partial L/\partial v) = FL^*\{p,G_h\}$. To this end we write the following identity, which can be obtained using (2.9) and the chain's rule:

$$\frac{\partial}{\partial v}(K\cdot G_h) = -FL^*\{p,G_h\} + \delta\frac{\partial L}{\partial v} + [L]\,\frac{\partial\,\delta q}{\partial v} + \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial q}\right)\left(\frac{\partial G}{\partial v} - W\delta q\right). \tag{3.14}$$

The last term in (3.14) vanishes because $\partial G/\partial v = W(\partial G_h/\partial p)$ and, since we transform solutions of the Euler–Lagrange equations, $[L] = 0$. Then $K\cdot G_h = 0$ implies

$$\delta\frac{\partial L}{\partial v} = FL^*\{p,G_h\}, \tag{3.15}$$

so that Theorem 2 is proven.

Let us observe that if $K\cdot G_1 = K\cdot G_2 = 0$, then $K\cdot\{G_1,G_2\} = 0$. Therefore, generators of Hamiltonian Noether transformations close under the Poisson bracket.

Finally, we want to express (3.12) in an equivalent way, which will prove to be useful in the case of gauge transformations. Application of (2.14) to (3.12) shows that $FL^*\{G_h,\phi_\mu^0\} = 0$, that is to say,

$$\{G_h,\phi_\mu^0\} \underset{M_0}{=} 0. \tag{3.16}$$

Now (2.13) leads to $FL^*(\{G_h,H\} + \partial G_h/\partial t) = 0$, which implies

$$\{G_h,H\} + \frac{\partial G_h}{\partial t} \underset{M_0}{=} 0. \tag{3.17}$$

Conversely, by (2.13), (3.16) and (3.17) imply (3.12). Therefore, the following theorem holds.

Theorem 3: A function $G_h(q,p,t)$ satisfying (3.16) and (3.17) generates a Hamiltonian DST.

It can be shown that these sufficient conditions [(3.16) and (3.17)] are in fact very close to those that are necessary.[13] Notice, also, from (3.17) that G_h is a constant of motion. Moreover, in a constrained system G_h is a first class function because it must be tangent to the final constraint manifold.

IV. HAMILTONIAN GAUGE TRANSFORMATIONS

The preceding results apply to DST in general dynamical systems. Now we consider the specific case of gauge transformations, that is to say, *DST depending on arbitrary functions and their derivatives.* Thus we are necessarily dealing with a constrained system. We will write a generator $G(q,p,t)$ of a gauge transformation in the form

$$G(q,p,t) = \sum_{k\geqslant 0} \epsilon^{(-k}(t) G_k(q,p), \tag{4.1}$$

where ϵ is an arbitrary function of time and $\epsilon^{(-k}(t)$ is a primitive of order k. As a result of the arbitrariness of ϵ, conditions (3.16) and (3.17) split into

$$\{\phi_\mu^0,G_k\} \underset{M_0}{=} 0, \tag{4.2}$$

$$G_0 \underset{M_0}{=} 0, \tag{4.3}$$

$$G_{k+1} + \{G_k,H\} \underset{M_0}{=} 0. \tag{4.4}$$

Relations (4.3) and (4.4) can be seen as a mechanism to construct a gauge transformation. Since G is first class, the G_k are also first class. To be precise, the G_k are first-class constraints: Let us prove this inductively; it is obvious for G_0[(4.3)]. Suppose we have chosen H to be first class (which is always possible; for instance, the $H^{(f+1)}$ reached in Ref. 2). Then if G_k is a first-class constraint, $\{G_k,H\}$ is as well. Therefore, (4.4) implies that G_{k+1} is also a first-class constraint. Notice, also, that $G_{k+1} + \{G_k,H\}$ is a primary first-class constraint.

The algorithm can be applied in the following way (see, also, Ref. 14, which proposes an algorithm to construct the gauge generator when no second class constraints are present): H is a first-class Hamiltonian and

$$G_0 = \text{primary first-class constraint}, \tag{4.5}$$

$$G_{k+1} = -\{G_k,H\} + \text{primary first-class constraints}. \tag{4.6}$$

One must play with this indeterminacy in order to let the test (4.2) hold. It is worth observing that the simpler form of a primary first-class constraint may not be suitable to begin (4.5).

There is no guarantee that this algorithm has a solution; however, it is reached in usual computations. Moreover, in these cases one can choose $G_k = 0$ for $k \geqslant f+1$ (if the stabilization algorithm ends at the fth step). For this reason the generator is usually written as

$$G = \sum_{k=0}^{f} \epsilon^{(k} G_{f-k}. \tag{4.7}$$

V. APPLICATIONS

A. The Polyakov string

The Lagrangian density of the Polyakov string is given by[15,16]

$$\mathscr{L} = (\sqrt{-g}/2) g^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu$$
$$= (-1/2\sqrt{-g})(g_{11}\dot{x}^2 - 2g_{01}(\dot{x}x') + g_{00}x'^2). \quad (5.1)$$

The canonical momenta are

$$p_\mu = \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} = \frac{-1}{\sqrt{-g}}(g_{11}\dot{x}_\mu - g_{01}x'_\mu),$$
$$\Pi^{\alpha\beta} = \frac{\partial \mathscr{L}}{\partial \dot{g}_{\alpha\beta}} = 0, \quad (5.2)$$

so that we obtain the canonical Hamiltonian density

$$\mathscr{H}_c = -(\sqrt{-g}/g_{11})H + (g_{01}/g_{11})T, \quad (5.3)$$

where $H = \frac{1}{2}(p^2 + x'^2)$ and $T = (px')$. We also obtain the primary constraints

$$\Pi_{00} = \Pi_{01} = \Pi_{11} = 0 \quad (5.4)$$

whose stability gives

$$\dot{\Pi}_{00} = \{\Pi_{00}, H_c\} = (-1/2\sqrt{-g})H,$$
$$\dot{\Pi}_{01} = \{\Pi_{01}, H_c\} = (g_{01}/\sqrt{-g}g_{11})H - (1/g_{11})T, \quad (5.5)$$
$$\dot{\Pi}_{11} = \{\Pi_{11}, H_c\} = \frac{-1}{\sqrt{-g}}\left(\frac{g_{00}}{2g_{11}} - \frac{g}{g_{11}^2}\right)H + \frac{g_{01}}{g_{11}^2}T.$$

Thus H and T are independent secondary constraints. As a result of the algebra

$$\{H(\sigma), H(\sigma')\} = T(\sigma)\partial_\sigma \delta(\sigma - \sigma') - T(\sigma')\partial_{\sigma'} \times \delta(\sigma - \sigma'),$$
$$\{H(\sigma), T(\sigma')\} = H(\sigma)\partial_\sigma \delta(\sigma - \sigma') - H(\sigma')\partial_{\sigma'} \times \delta(\sigma - \sigma'), \quad (5.6)$$
$$\{T(\sigma), T(\sigma')\} = T(\sigma)\partial_\sigma \delta(\sigma - \sigma') - T(\sigma')\partial_{\sigma'} (\sigma - \sigma'),$$

no tertiary constraints appear and we are left with five (Π_{00}, Π_{01}, Π_{11}, H, and T) first-class constraints.

We have three primary first-class constraints, so we expect three independent gauge transformations. The algorithm for constructing a canonical gauge generator starts by selecting a combination of primary first-class constraints. In order to simplify the expressions and taking into account that the three primary constraints give only two secondary constraints, let us consider the following combinations:

$$\varphi_W = g_{00}\Pi_{00} + g_{01}\Pi_{01} + g_{11}\Pi_{11},$$
$$\varphi_1 = (2\sqrt{-g}g_{01}/g_{00})\Pi_{01} + (2\sqrt{-g}g_{11}/g_{00})\Pi_{11}, \quad (5.7)$$
$$\varphi_2 = [(2g_{01}^2 - g_{00}g_{11})/g_{00}]\Pi_{01} + (2g_{01}g_{11}/g_{00})\Pi_{11},$$

which are such that

$$\dot{\varphi}_W = \{\varphi_W, H_c\} = 0,$$
$$\dot{\varphi}_1 = \{\varphi_1, H_c\} = H, \quad \dot{\varphi}_2 = \{\varphi_2, H_c\} = T. \quad (5.8)$$

Thus we see that the generator starting with φ_W has only one piece:

$$G_W = \int d\sigma\, \epsilon_W(\sigma)(g_{00}\Pi_{00} + g_{01}\Pi_{01} + g_{11}\Pi_{11}). \quad (5.9)$$

Now let us consider φ_1 and apply the algorithm

$$G_0(\sigma) = \varphi_1(\sigma),$$
$$G_1(\sigma) + \{G_0(\sigma), H_c\} = \int d\sigma' (\alpha\varphi_W(\sigma') + \beta\varphi_1(\sigma') + \gamma\varphi_2(\sigma')),$$

where $\alpha = \alpha(\sigma,\sigma')$, etc. Then

$$G_1(\sigma) = -H(\sigma) + \int d\sigma' (\alpha\varphi_W(\sigma') + \beta\varphi_1(\sigma') + \gamma\varphi_2(\sigma')).$$

The next step is

$$G_2(\sigma) + \{G_1(\sigma), H_c\} = \text{primary first-class constraints}$$

or

$$G_2(\sigma) = \{H(\sigma), H_c\} - \int d\sigma' (\beta H(\sigma') + \gamma T(\sigma')) + \text{primary first-class constraints}.$$

We need to compute

$$\{H(\sigma), H_c\} = -2T\partial_\sigma\left(\frac{\sqrt{-g}}{g_{11}}\right) + 2H\partial_\sigma\left(\frac{g_{01}}{g_{11}}\right) - \frac{\sqrt{-g}}{g_{11}}\partial_\sigma T + \frac{g_{01}}{g_{11}}\partial_\sigma H.$$

Thus we realize that we can finish the algorithm with the choice

$$\beta(\sigma,\sigma') = 2\partial_\sigma(g_{01}/g_{11})\delta(\sigma - \sigma') + (g_{01}/g_{11})(\sigma)\partial_\sigma \delta(\sigma - \sigma'),$$
$$\gamma(\sigma,\sigma') = -2\partial_\sigma(\sqrt{-g}/g_{11})\delta(\sigma - \sigma') - (\sqrt{-g}/g_{11})(\sigma)\partial_\sigma \delta(\sigma - \sigma'),$$
$$\alpha(\sigma,\sigma') = 0.$$

Then the generator has two pieces and after integrating by parts can be written as

$$G_A = \int d\sigma\Big[\dot{\epsilon}_A\varphi_1 + \epsilon_A\left(-H + \partial_\sigma\left(\frac{g_{01}}{g_{11}}\right)\varphi_1 - \partial_\sigma\left(\frac{\sqrt{-g}}{g_{11}}\right)\varphi_2\right) + \epsilon'_A\left(-\frac{g_{01}}{g_{11}}\varphi_1 + \frac{\sqrt{-g}}{g_{11}}\varphi_2\right)\Big]. \quad (5.10)$$

As a result of our choice (5.7), the consistency condition (4.2) is trivially satisfied because H, T do not depend on g's. Starting with φ_2 we could have constructed

$$G_B = \int d\sigma'\Big[\dot{\epsilon}_B\varphi_2 + \epsilon_B\left(-T - \partial_\sigma\left(\frac{\sqrt{-g}}{g_{11}}\right)\varphi_1 + \partial_\sigma\left(\frac{g_{01}}{g_{11}}\right)\varphi_1\right) + \epsilon'_B\left(\frac{\sqrt{-g}}{g_{11}}\varphi_1 - \frac{g_{01}}{g_{11}}\varphi_2\right)\Big]. \quad (5.11)$$

The action of the three generators G_W, G_A, and G_B on the fields $g_{\alpha\beta}(\sigma)$, $x_\mu(\sigma)$ yields

$$\delta g_{00} = \epsilon_W g_{00},$$

$$\delta g_{01} = \epsilon_W g_{01} + \dot{\epsilon}_A \frac{2\sqrt{-g}g_{01}}{g_{00}} + \epsilon_A\left(\frac{2g_{01}^2 - g_{00}g_{11}}{2g_{00}\sqrt{-g}} g'_{00} - g'_{01}\frac{g_{01}}{\sqrt{-g}} + g'_{11}\frac{g_{00}}{2\sqrt{-g}}\right) - \epsilon'_A\sqrt{-g} + \dot{\epsilon}_B \frac{2g_{01}^2 - g_{00}g_{11}}{g_{00}} + \epsilon_B\left(\frac{g_{01}}{g_{00}} g'_{00} - g'_{11}\right) - \epsilon'_B g_{01},$$

$$\delta g_{11} = \epsilon_W g_{11} + \dot{\epsilon}_A \frac{2\sqrt{-g}g_{11}}{g_{00}} + \epsilon_A\left(\frac{g_{01}g_{11}}{g_{00}\sqrt{-g}} g'_{00} - \frac{2g_{11}}{\sqrt{-g}} g'_{01} + \frac{g_{01}}{\sqrt{-g}} g'_{11}\right) + \dot{\epsilon}_B \frac{2g_{01}g_{11}}{g_{00}} + \epsilon_B\left(\frac{g_{11}}{g_{00}} g'_{00} - g'_{11}\right) - 2\epsilon'_B g_{11}, \tag{5.12}$$

$$\delta x_\mu = \epsilon_A p_\mu - \epsilon_B x'_\mu.$$

Our canonical gauge transformations do not have the nice, well-known form

$$\delta x_\mu = \epsilon^\alpha \delta_\alpha x_\mu,$$
$$\delta g_{\alpha\beta} = \Lambda g_{\alpha\beta} + \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma}. \tag{5.13}$$

This fact was really expected because Eq. (5.13) are not *FL* projectable: They contain the velocities $\dot{g}_{\alpha\beta}$. However, a change in the arbitrary parameters can always make the connection. In our case the change is given by

$$\epsilon_A = (\sqrt{-g}/g_{11})\epsilon^0, \quad \epsilon_B = -\epsilon^1 - (g_{01}/g_{11})\epsilon^0,$$
$$\epsilon_W = \Lambda + \epsilon^0(\dot{g}_{00}/g_{00}) + 2\dot{\epsilon}^0 + \epsilon^1(g'_{00}/g_{00}) + 2\dot{\epsilon}^1(g_{01}/g_{00}). \tag{5.14}$$

Substitution of (5.14) in (5.12) gives the covariant form (5.13). Notice that relations (5.14) involve non-*FL*-projectable functions, as must occur. Also, notice the fact that the first-class constraints T, H satisfy a nontrivial algebra, making the first-class primary constraints φ_1, φ_2 enter the generator in a definite way and giving the correct gauge transformations, so that the canonical gauge generator is not simply an arbitrary combination of first-class constraints. The first class Hamiltonian we have used is simply the canonical Hamiltonian because no second-class constraints are present. The procedure is less trivial in the example in Sec. V B.

B. The CP_2^{n-1} model.

The Lagrangian density is[17]

$$\mathscr{L} = (D_\mu Z_\alpha)^*(D^\mu Z_\alpha) - \lambda(Z_\alpha^* Z_\alpha - n/2g), \tag{5.15}$$

where $D_\mu = \partial_\mu + iA_\mu$ and $g^{\mu\sigma} = \mathrm{diag}(\pm)$. Here A_μ is a two-dimensional auxiliary gauge field and λ is a field which enforces the condition $Z_\alpha^* Z_\alpha = n/2g$ on the n complex fields Z_α.

The infinitesimal gauge invariance of the theory is given by

$$\delta Z_\alpha = -i\theta Z_\alpha, \quad \delta Z_\alpha^* = i\theta Z_\alpha^*,$$
$$\delta A_\mu = \partial_\mu \theta, \quad \delta\dot{\lambda} = 0, \tag{5.16}$$

where $\theta = \theta(x^0, x^1) = \theta(\tau,\sigma)$ is an arbitrary parameter. The canonical momenta are

$$\Pi_\mu = \frac{\partial\mathscr{L}}{\partial\dot{A}^\mu} = 0, \quad \Pi_\lambda = \frac{\partial\mathscr{L}}{\partial\dot{\lambda}} = 0,$$
$$\Pi_\alpha = \frac{\partial\mathscr{L}}{\partial\dot{Z}_\alpha} = \dot{Z}_\alpha^* - iZ_\alpha^* A_0, \quad \Pi_\alpha^* = \frac{\partial\mathscr{L}}{\partial\dot{Z}_\alpha^*} = \dot{Z}_\alpha + iZ_\alpha A_0. \tag{5.17}$$

Thus we obtain the canonical Hamiltonian density

$$\mathscr{H}_c = \Pi_\alpha\Pi_\alpha^* - iA_0(\Pi_\alpha Z_\alpha - \Pi_\alpha^* Z_\alpha^*) + \partial_1 Z_\alpha^* \partial_1 Z_\alpha + iA_1(Z_\alpha \partial_1 Z_\alpha^* - Z_\alpha^* \partial_1 Z_\alpha) + A_1^2 Z_\alpha^* Z_\alpha + \lambda(Z_\alpha^* Z_\alpha - n/2g) \tag{5.18}$$

and the primary constraints

$$\Pi_0 = 0, \quad \Pi_1 = 0, \quad \Pi_\lambda = 0. \tag{5.19}$$

Then the primary Hamiltonian density is

$$\mathscr{H}_p = \mathscr{H}_c + v_0\Pi_0 + v_1\Pi_1 + v_\lambda\Pi_\lambda, \tag{5.20}$$

where $v_\mu = \dot{A}_\mu$ and $v_\lambda = \dot{\lambda}$. After the stability algorithm is performed it turns out that the theory contains two first-class constraints,

$$\varphi_1 = \Pi_0,$$
$$\varphi_2 = i(\Pi_\alpha Z_\alpha - \Pi_\alpha^* Z_\alpha^*) - \partial_1 \Pi_1 \tag{5.21a}$$

and six second-class constraints,

$$\chi_1 = Z_\alpha\Pi_\alpha + Z_\alpha^*\Pi_\alpha^*,$$
$$\chi_2 = \lambda Z_\alpha^* Z_\alpha - \Pi_\alpha\Pi_\alpha^* + \partial_1 Z_\alpha^* \partial_1 Z_\alpha + iA_1(Z_\alpha^* \partial_1 Z_\alpha - Z_\alpha \partial_1 Z_\alpha^*) - A_1^2 Z_\alpha^* Z_\alpha, \tag{5.21b}$$

$$\chi_3 = \Pi_\lambda,$$
$$\chi_4 = Z_\alpha^* Z_\alpha - n/2g,$$
$$\chi_5 = \Pi_1,$$
$$\chi_6 = iZ_\alpha \partial_1 Z_\alpha^* - iZ_\alpha^* \partial_1 Z_\alpha + 2A_1 Z_\alpha^* Z_\alpha.$$

Because the primary constraints Π_1, Π_λ have become second class, the arbitrary functions $v_1 = \dot{A}_1$, $v_\lambda = \dot{\lambda}$ have been canonically determined:

$$v_1 = (g/n)(iZ_\alpha \partial_1\Pi_\alpha + i\Pi_\alpha^* \partial_1 Z_\alpha - iZ_\alpha^* \partial_1 \Pi_\alpha^* - i\Pi_\alpha \partial_1 Z_\alpha + 2A_1(Z_\alpha\Pi_\alpha + Z_\alpha^*\Pi_\alpha^*)) - \partial_1 A_0 \equiv f_1 \tag{5.22a}$$

and

$$v_\lambda = \frac{-2g}{n}\left(\{\chi_2, H_c\} + \int d\sigma' v_1(\sigma')\{\chi_2,\chi_5\}\right) \equiv f_\lambda. \tag{5.22b}$$

Thus the first-class Hamiltonian density is

$$\mathscr{H} = \mathscr{H}_c + f_1\Pi_1 + f_\lambda\Pi_\lambda, \tag{5.23}$$

which incorporates the second-class primary constraints in the correct way. Now we can begin the algorithm with

$$G_0(\sigma) = \Pi_0(\sigma)$$

and

$$G_1(\sigma) = -\{G_0(\sigma),H\} + \text{primary first-class constraints}$$
$$= -i(\Pi_\alpha Z_\alpha - \Pi^*_\alpha Z^*_\alpha)$$
$$+ \partial_1 \Pi_1 + \text{primary first-class constraints.}$$

It can be checked that $\{-i(\Pi_\alpha Z_\alpha - \Pi^*_\alpha Z^*_\alpha) + \partial_1 \Pi_1, H\} = 0$. Thus the algorithm ends at this stage and the gauge generator is

$$G = \int d\sigma[\dot{\theta}\Pi_0 - i\theta(\Pi_\alpha Z_\alpha - \Pi^*_\alpha Z^*_\alpha) - \partial_1 \theta \Pi_1], \tag{5.24}$$

which gives the correct gauge transformations. In this condition (4.2) is trivially satisfied in a natural way.

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