**Petrov type D perfect-fluid solutions in generalized Kerr–Schl"{o}d form**

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Generalized Kerr–Schl"{o}d space-times for a perfect-fluid source are investigated. New Petrov type D perfect fluid solutions are obtained starting from conformally flat perfect-fluid metrics.

I. INTRODUCTION

This work is concerned with perfect-fluid solutions of Einstein’s equations for a metric in generalized Kerr–Schl"{o}d form. Since the original Kerr–Schl"{o}d paper, a lot of generalizations of the Kerr–Schl"{o}d ansatz have appeared. Bilge and G"{u}rses have shown how the Newman–Penrose spin coefficients, trace-free Ricci, Ricci scalar, and Weyl spinors transform under the most general Kerr–Schl"{o}d transformation. In this paper we treat generalized Kerr–Schl"{o}d metrics of the form

\[ \tilde{g}_{a\beta} = g_{a\beta} + 2H_l l^\beta, \]  

(1.1)

where \( g_{a\beta} \) is the metric of any space-time, \( l^\beta \) is a null, geodesic vector field for the metric \( g_{a\beta} \), and \( H \) is a scalar field.

As far as we know, no perfect-fluid solution of the Kerr–Schl"{o}d type is known. All the solutions we obtain are of Petrov type D, and most of these are new since the velocity of the fluid does not lie in the two-space defined by the principal null directions of the Weyl tensor.

In Sec. II we obtain the Riemann, Ricci, and Weyl tensors of the metric \( \tilde{g} \) as functions of the Riemann, Ricci, and Weyl tensors of the metric \( g \) and the spin coefficients defined by a null tetrad associated with \( l^\beta \). Our notation and calculations are quite close to those of Taub (Ref. 2). Section III is devoted to writing down the equations in the case where \( g \) is a conformally flat solution of Einstein’s equations for a perfect fluid. It is shown easily that the geodesic (shear-free) null vector fields in a conformally flat space-time are the geodesic (shear-free) null vector fields in flat space-time. Since the most general vector field of this kind is already known,\(^5\) one has great freedom in choosing the vector field \( l^\beta \). Two cases appear depending on whether \( l_a \) is shear-free or not. They are studied in Secs. IV and V. Finally, in Sec. VI we give some examples of how the method works and some explicit solutions.

II. THE RIEMANN,RICCI, AND WEYL TENSORS OF GENERALIZED KERR–SCHL"{O}D METRICS

It is easily shown that

\[ \tilde{g}^{a\beta} = g^{a\beta} - 2H^a l^\beta, \]  

(2.1)

\[ \tilde{l}^a = l^a, \quad \tilde{l}_a = 0. \]  

(2.2)

Then, we obtain for the Christoffel symbols

\[ \tilde{\Gamma}^a_{\beta\lambda} = \Gamma^a_{\beta\lambda} + A^a_{\beta\lambda} + 2H^a l^\beta l_\lambda l^\gamma \nabla_\gamma H, \]  

(2.3)

where the \( \Gamma^a_{\beta\lambda} \) are the Christoffel symbols for the metric \( g \) and

\[ A^a_{\beta\lambda} = \nabla_\beta (H l^a l_\lambda) + \nabla_\lambda (H l^a l_\beta) - \nabla_\alpha (H l^\alpha l_\beta), \]  

(2.4)

or

\[ A^a_{\beta\lambda} = H l^a l_\lambda + l_\gamma (l^a \nabla_\lambda H + H \nabla_\lambda l^a) \]  

\[ + l_\lambda (l^a \nabla_\beta H + H \nabla_\beta l^a) - l^\gamma l_\lambda \nabla_\gamma H, \]  

(2.5)

with

\[ S_{a\beta} = \nabla_\beta l_\alpha - \nabla_\alpha l_\beta, \quad A_{a\beta} = \nabla_\alpha l_\beta - \nabla_\beta l_\alpha. \]  

(2.6)

Next, we compute the Riemann tensor from the expression (2.3) and we find the following:

\[ \tilde{\mathbf{R}}_{\alpha\beta\mu} = R_{\alpha\beta\mu} + l_\lambda \nabla_\lambda A_{\alpha\beta} - \nabla_\mu A_{\alpha\beta}, \]  

(2.7)

where \( \tilde{\mathbf{R}}_{\alpha\beta\mu} \) is the Riemann tensor for the metric \( g \). Then, for the Ricci tensor we obtain

\[ \tilde{\mathbf{R}}_{\alpha\beta} = R_{\alpha\beta} + l_\lambda \nabla_\lambda A_{\alpha\beta} + 2H^\alpha l^\mu \nabla_\lambda A^\lambda_{\mu\beta}, \]  

(2.8)

where \( \tilde{\mathbf{R}}_{\alpha\beta} \) is the Ricci tensor for the metric \( g \). After a long calculation it may be shown that the Weyl tensor \( \Phi_{\alpha} \) is given, the Einstein equations

\[ \Gamma = (\rho + \rho_0) \nabla_\alpha (H l^\gamma) - 2H (\rho \tilde{\nabla} + \rho_0 \tilde{\nabla} + \phi_{\alpha}), \]  

(2.10)

\[ \Omega = 2\rho \nabla_\alpha (H l^\gamma) - 2H (\psi_0 + 2 \rho_0), \]  

(2.11)

\[ \Sigma = \nabla_\alpha \psi (\beta + \alpha + \gamma) \nabla_\gamma (H l^\alpha) \]  

\[ - 2H \left[ \psi_1 - \rho (\beta + \alpha) - \alpha (\beta + \alpha) - \phi_{\alpha1} \right], \]  

\[ \Phi_{\alpha} = \nabla_\alpha \psi (H l^\alpha) - H \nabla_\alpha \nabla_\beta \nabla_\gamma R_{\alpha\beta}, \]  

\[ \Omega = - g^{\rho\sigma} \nabla_\rho \nabla_\sigma H + 2 \nabla_\rho \nabla_\sigma (H l^\alpha), \]  

(2.12)

\[ - 2H (k^\rho \nabla_\rho l_\alpha) - 2k^\rho \nabla_\rho (H l_\alpha), \]  

\[ - 2k^\rho \nabla_\rho \nabla_\alpha (H l^\rho) + 4 \nabla_\rho (H l_\alpha), \]  

(2.13)

Therefore, if the metric \( g \) is given, the Einstein equations

\[ \tilde{\mathbf{R}}_{\alpha\beta} = \mathcal{F} (\tilde{T}_{\alpha\beta} - \frac{1}{2} \tilde{g}_{\alpha\beta} \tilde{T}) \]  

(2.15)

are defined by (2.8) and (2.9). In particular, from (2.8), (2.9),

}\]
and (2.15) we obtain the interesting relation
\[ \chi^l a T_{au} = l^a R_{au} + l_a (l^p \Phi_p + (\gamma/2) \tilde{T}). \]  
(2.16)

We distinguish two cases.

1. If \( R_{au} = a l_u \). In this case \( l_a \) is an eigenvector of \( T_{ab} \) and then perfect-fluid solutions cannot exist.

2. If \( R_{au} \neq a l_u \). In this case \( T_{ab} \) can be the energy-momentum tensor of a perfect fluid. It is the purpose of this paper to study this case when both \( T_{ab} \) and \( T_{ab} \) are perfect-fluid energy-momentum tensors.

Let \( \{ T^a, k^a, m^a, \tilde{m}^a \} \) be a null tetrad for the metric \( g \).

Then
\[ \tilde{\gamma}_0 = \gamma_0, \quad \tilde{\gamma}_1 = \gamma_1, \quad \tilde{\gamma}_2 = \gamma_2 - 2H \phi_0 - \frac{1}{2} \nabla_\alpha [l^a \nabla_\alpha (H^a)] \]
\[ + 3 \phi \{ D H - (H - \rho \beta), \quad \tilde{\gamma}_3 = \gamma_3 - H \bar{\psi}_1 + \frac{1}{2} \nabla_\alpha (H^a) \}
\]
\[ - 2 \phi \tilde{\gamma}_0 - H \bar{\psi}_1 \nabla_\alpha (H^a) + 2H \tilde{\gamma}_0 + 2H \tilde{\gamma}_1 + 2H \tilde{\gamma}_2 \]
\[ + 4 \phi \tilde{\gamma}_0 + 2H \tilde{\gamma}_1 + 2H \tilde{\gamma}_2 - 2H \tilde{\gamma}_3 \]
\[ + (\bar{\beta} + 3 \alpha - 2 \phi) \tilde{\gamma}_0 - 2H \tilde{\gamma}_1 - 2H \tilde{\gamma}_2 \].
(3.17)

III. THE EINSTEIN EQUATIONS FOR A CONFORMALLY FLAT PERFECT-MEDIUM FLUID \( g_{ab} \)

Henceforth, we choose the metric \( g \) to be conformally flat; that is,
\[ \psi_0 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 0 \Leftrightarrow g_{a_b} = \phi^2 g_{a_b}, \]
(3.1)
where \( \phi^2 \) is a positive function of the coordinates and \( g_{a_b} \) is the metric of flat space-time. Moreover, we assume that \( g_{a_b} \) is a solution of Einstein’s equations for a perfect-fluid energy-momentum tensor; that is to say
\[ R_{a_b} = \chi (T_{a_b} - \frac{1}{2} \tilde{g}_{a_b} T), \]
(3.2)
\[ T_{a_b} = (q + p) a_b + \tilde{p}_{a_b}, \quad g_{a_b} u_a u_b = -1. \]
(3.3)

All metrics of this kind are known: they are either generalized interior Schwarzschild solutions or generalized Friedmann solutions (Ref. 4).

It may easily be verified that if \( g_{a_b} \) is a conformally flat space-time, and if \( l_a \) is a null geodesic (shear-free) vector field for \( g_{a_b} \) then it is also a null geodesic (shear-free) vector field for flat space-time. But the general solution for vector fields of this kind in flat space-time is known and is given by
\[ l = du + Y d\zeta + Y d\zeta + Y d\zeta, \]
(3.4)
where \( Y \) is a complex function of the coordinates \( u, v, \zeta, \bar{\zeta} \) verifying
\[ Y \frac{\partial Y}{\partial \zeta} + \bar{\bar{\zeta}} \frac{\partial Y}{\partial v} - \frac{\partial Y}{\partial Y} - Y \frac{\partial Y}{\partial v} = 0. \]
(3.5)
When \( l_a \) is also shear-free, \( Y \) is defined implicitly by
\[ F(Y, \bar{\zeta} Y + u, v Y + \zeta) = 0, \]  
(3.6)
where \( F \) is an arbitrary analytic function of three complex variables. The coordinates \( u, v, \zeta, \bar{\zeta} \) are related with the usual coordinates of the Minkowski space-time by
\[ \sqrt{2} u = t - z, \quad \sqrt{2} v = t + z, \quad \sqrt{2} \zeta = x + iy, \]
(3.7)
and the metric \( g_{a_b} \) may be written in these coordinates as
\[ g_{a_b} d\zeta^a d\zeta^b = 2(d^a - du + d\zeta^a d^b). \]
(3.8)

Now, we choose the null tetrad associated with \( l_a \) as follows:
\[ l, \quad k = \phi^2 dv, \quad m = \phi(d\zeta^a + Y dv). \]
(3.9)

Then, after a straightforward calculation, we obtain the spin coefficients
\[ \pi = -\alpha = -\phi^{-1} m^a \nabla_a \phi, \quad \gamma = -\mu = \phi^{-1} k^a \nabla_a \phi, \quad \rho = \phi^{-2} [\rho_M + \phi \tilde{a}^a \nabla_a \phi], \quad \rho_M = \frac{\partial Y}{\partial \zeta}, \]
(3.10)
\[ \tau = \phi^{-1} \tau_M + \tilde{a}^{-1}, \quad \tau_M = -\frac{\partial Y}{\partial \zeta}, \]
\[ \sigma = \phi^{-2} \sigma_M, \quad \sigma_M = -\frac{\partial Y}{\partial \zeta} - Y \frac{\partial Y}{\partial \zeta}, \]
\[ \kappa = \varepsilon = \lambda = \beta = -\sigma = 0. \]

Moreover, it is well known that the null tetrad is defined up to a transformation of the form
\[ l' = l, \quad m' = e^{\xi}(m + 2l), \]
(3.11)
\[ k' = k + Zm + 2l + 2Zl \]

We make such a transformation choosing
\[ Z = -m^a u_a / l^a u_a \]
(3.12)
so that
\[ m^a u_a = 0. \]
(3.13)

After this change of null tetrad, the new spin coefficients are
\[ \pi' = \pi - DZ, \quad \kappa' = \varepsilon = 0, \quad \rho' = \rho, \]
(3.14)
\[ \sigma' = \sigma, \quad \beta' = \frac{\partial Y}{\partial \zeta}, \quad \alpha' = \alpha + Z \rho, \]
\[ \lambda' = \lambda + \frac{\partial Y}{\partial \zeta}, \quad \gamma' = \gamma + Z \rho + Z \bar{\zeta}, \quad \rho' = \rho + Z \bar{\zeta}, \]
\[ \sigma' = \sigma + Z \bar{\zeta}, \quad \beta' = \frac{\partial Y}{\partial \zeta} + Z \bar{\zeta}, \quad \gamma' = \gamma + Z \rho + Z \bar{\zeta}, \quad \rho' = \rho + Z \bar{\zeta}, \]
\[ + 2Z \bar{\zeta}, \quad \lambda' = \lambda + \frac{\partial Y}{\partial \zeta}. \]

Hereafter, we shall drop the primes.

We search for solutions \( g_{a_b} \) of Einstein’s equations for a perfect-fluid energy-momentum tensor
\[ T_{a_b} = (q + p) a_b + \tilde{p}_{a_b}, \]
(3.15)
\[ \tilde{g}_{a_b} u_a u_b = -1. \]
(3.16)

Taking into account all previous assumptions and results, the Einstein equations (2.15), (2.8), and (2.9)—once they are projected onto the null tetrad—lead us to the following set of equations:
\[ m^a u_a = 0, \]
(3.17)
\[ \Omega = 0. \]
(3.18)
\[
\begin{align*}
(q + \bar{p})(l^\alpha \bar{u}_\alpha)^2 &= (q + p)(l^\alpha u_\alpha)^2, \\
\nabla_\alpha [l^\alpha \nabla_\mu (H^{\mu\nu})] &= \chi [\frac{1}{2} (q - \bar{p}) - \frac{1}{2} (q - 3p) \\
&+ H (q + p)(l^\alpha u_\alpha)^2], \\
\Gamma &= (\chi/2)[q - \bar{p} - q + p], \\
N &= \chi \left[\frac{\bar{q} + \bar{p}}{4(l^\alpha u_\alpha)^2} - \frac{q + p}{4(l^\alpha u_\alpha)^2}\right] \\
&- H^2 (q + p)(l^\alpha u_\alpha)^2 - 2H p, \\
\Sigma &= 0. \\
\n\text{Starting from (2.11), Eq. (3.18) becomes} \\
\sigma [DH - H (\rho - \bar{p})] &= 0. \\
\end{align*}
\]

We can consider two cases.

(A) \( \sigma \neq 0 \). Then Eq. (3.18') implies that we must have
\[
\rho = \bar{p}, \quad DH = 0. \\
\]
This case is studied in the following section.

(B) \( \sigma = 0 \). Then (3.18') is automatically satisfied. This case is studied in Sec. V.

IV. THE CASE \( \sigma \neq 0 \)

In this section we try to solve Eqs. (3.17)–(3.23) with the assumption \( \sigma \neq 0 \).

Throughout this and the next sections we shall use repeatedly (but not explicitly) the Bianchi identities and the Newman–Penrose equations for the metric \( g_{\alpha \beta} \) (the Bianchi identities are given in the Appendix). Whenever we make some assumption or specialization we must restrict these equations in the appropriate fashion. The details are omitted.

First of all, from (3.20), (3.21), and (3.24) we obtain \( \bar{q} \) and \( \bar{p} \) as functions of \( q, p, \) and \( H \),
\[
\begin{align*}
\chi \bar{p} &= \chi p + 2H \phi_{\infty}, \\
\chi \bar{q} &= \chi q + 4H (\rho^2 - \sigma \bar{\sigma}) - 2H \phi_{\infty}.
\end{align*}
\]
Furthermore, from (3.19), (3.17), and (3.16), we get \( \bar{u}_\alpha \) :
\[
(l^\alpha \bar{u}_\alpha)^2 = \chi [q + p](l^\alpha u_\alpha)^2 + \chi [q + p] + 4H (\rho^2 - \sigma \bar{\sigma})^{-1},
\]
\[
m^\alpha u_\alpha = 0. \\
\]
Then, we only must solve Eqs. (3.22) and (3.23).

Starting from (2.12) and (2.13), making use of (3.24), and after some standard calculations, we obtain for (3.23)
\[
\delta H = 2H (\alpha + \bar{\beta}) - H \bar{\tau} - H \rho (\tau/\sigma). \\
\]
In the same way, it follows from (2.14), (3.24), and (4.1)–(4.4) that Eq. (3.22) becomes
\[
\rho \Delta H = H \left[ \rho (\mu + \gamma + \bar{\gamma}) + \sigma \lambda + \delta (\rho (\tau/\sigma)) \\
+ \rho (\tau/\sigma)(\bar{\sigma} - \beta) - \Delta \rho - \rho^2 (\tau/\sigma) \bar{\sigma} \right] \\
- 4(\phi_{11}/\phi_{\infty}) (\rho^2 - \sigma \bar{\sigma})^2 - 2A \right].
\]
From (2.18) and (2.19) with (3.24), (4.5), and (4.4) we get \( \psi_0 = 0 \) for the sake of brevity \( \psi_1 \) is not written here
\[
\begin{align*}
\psi_0 &= 0, \\
\psi_1 &= H \left( \tau/\sigma \right)(\sigma - \rho^2).
\end{align*}
\]

The question now is the following: Are Eqs. (3.24), (4.4), and (4.5) compatible? Since Eqs. (3.24) and (4.4) are linear in \( H \), their compatibility with (4.5) (which is nonlinear in \( H \)) gives us an expression for \( H \) which is not, in general, a solution of Eqs. (3.24), (4.4), and (4.5). In order to proceed we assume
\[
\rho^2 - \sigma \bar{\sigma} = \phi_{\infty},
\]
so that Eq. (4.5) becomes linear as well.

Keeping this in mind, the compatibility condition of (3.24) and (4.4) is simply
\[
\tau = 0 \\
\]
and the compatibility condition of (4.4) with its complex conjugate is the reality condition
\[
\delta (\bar{\sigma} + \sigma (\beta - \bar{\sigma})) = \sigma (\delta \bar{\tau} + \bar{\tau} (\bar{\beta} - \alpha)).
\]
Furthermore, Eq. (4.5) now becomes
\[
\Delta H = H \left[ 2 (\mu + \gamma + \bar{\gamma}) \\
- 2 \rho \frac{\lambda}{\sigma} - \frac{1}{2} \left( \frac{\tau^2}{\sigma} + \bar{\tau}^2 \right) \right],
\]
which is compatible with (3.24). Finally, a new integrability condition arises from (4.4) and (4.10):
\[
\Delta \tau + 3 \bar{\lambda} \bar{\tau} + \sigma (\mu + \gamma - \bar{\gamma} + \tau/\sigma) \bar{\tau} + 2p \bar{\nu} = 0.
\]
It is easily shown that this condition is compatible with the Newman–Penrose equations.

We can summarize our results as follows: Let us choose the conformally flat perfect-fluid metric \( g_{\alpha \beta} \) and the null geodesic vector field \( l_\alpha \) such that they verify \( \rho = \bar{p}, \) \( (4.7), \) \( (4.8), \) \( (4.9), \) and \( (4.11) \). Then, let us solve the integrable system of equations for \( H \) given by (3.24), (4.4), and (4.10). The new Kerr–Schild metric \( g_{\alpha \beta} \) is a solution of Einstein’s equations for a perfect-fluid energy-momentum tensor (3.15), where \( \bar{q}, \bar{p}, \) and \( \bar{u}_\alpha \) are given by (4.1), (4.2), and (4.3). The Weyl tensor of these new solutions is \( (\tau \) never vanishes)
\[
\begin{align*}
\bar{\psi}_0 &= \psi_1 = 0, \\
3 \psi_2 &= -2H \phi_{\infty}, \\
\psi_3 &= -H (\tau/\sigma) \phi_{\infty}, \\
\psi_4 &= -H (\tau^2/\sigma^2) \phi_{\infty},
\end{align*}
\]
so that we have \( 3 \bar{\psi}_0 \bar{\psi}_2 = 2 \phi_{\infty}^2 \), and therefore they are of Petrov type D. Since \( \psi_0 \) and \( \psi_3 \) do not vanish, the vector field \( k^\alpha \) is not a multiple null eigenvector of the Weyl tensor, but \( l^\alpha \) certainly is. From (4.4) we have
\[
\bar{u}^\alpha = a l^\alpha + b k^\alpha
\]
and then \( \bar{u}^\alpha \) does not lie in the preferred two-space spanned by the two multiple null eigenvectors of the Weyl tensor. Excepting the Wahlquist solution, no solutions of this kind were known up to now.

V. THE CASE \( \sigma = 0 \)

Now, we assume \( \sigma = 0 \) so that the function \( Y \) of (3.4) is defined by (3.6) and also we have
\[
\beta = \lambda = 0. \\
\]
We define in this section
\[
\begin{align*}
U &= \delta H - 2H \bar{\alpha}, \\
V &= DH - 2H \rho.
\end{align*}
\]
From (3.20), (3.21), (3.19), (3.17), and using the same procedure of the previous section we get
\( \chi \tilde{p} = \chi q - DV - V(3\rho + \tilde{p}) - 3H\rho^2 - 4H\rho \tilde{p} \\
+ H\tilde{p}^2 + 4H\phi_{\infty} \)  
(5.3)

\( \chi \tilde{q} = \chi q + V(\tilde{p} - \rho) - DV + 3H(\rho^2 + \tilde{p}^2), \)
(5.4)

\( (\rho^2 - \tilde{p})^2 = 2\phi_{\infty}[\chi \{ q + \rho \} + 4H\phi_{\infty} \\
- 2DV + 2V\rho + 2H\tilde{p}(\rho - \tilde{p})]^{-1}. \)  
(5.5)

Equations (3.23) and (3.22) become, respectively,
\[
\delta V + (\rho + \tilde{p})U + (\tau - \tilde{q})V \\
+ H[\tilde{p}\tilde{q} + \tilde{p}^2 + 2\tilde{q}(\rho - \tilde{p})] = 0,
\]
(5.6)

\[ \rho \Delta H = -H\Delta(\rho + \tilde{p}) - \delta\tilde{U} + \tilde{q}U - \tau\tilde{U} - \mu V \\
- 4H\phi_{\infty} + H(\gamma + \tilde{q})(\rho - \tilde{p}) - (1/4\phi_{\infty}) \\
\times [DV + 2V\rho + 2H\tilde{p}(\rho - \tilde{p})] \\
\times [DV + 2V\rho + 2H\tilde{p}(\rho - \tilde{p})] \\
- 4H\phi_{\infty} - \delta\phi_{11}. \]  
(5.7)

In order to make compatible \( U \) and \( V \) we must have
\[ DU + (2\rho - \rho U) + (\tau + \tilde{p})V \\
+ H[\tilde{p}\tilde{q} + \tilde{p}^2 + 2\tilde{q}(\rho - \tilde{p})] = 0. \]  
(5.8)

Also, \( U \) must verify the reality condition
\[ \tilde{D}U + \tilde{p}\Delta H + \tilde{p}\tilde{V} - U\alpha + 2H\rho(\gamma + \tilde{q}) = \text{c.c.}. \]  
(5.9)

Now, the Weyl tensor is given by
\[ \psi_0 = \tilde{\psi}_1 = 0, \quad -6\psi_2 = [D - 2(\rho - \tilde{p})]V, \]
(5.10)

\[ \tilde{\psi}_3 = H[\rho \alpha + \tilde{\rho} - \tilde{q}(\rho - \tilde{p}) + (2\rho - \tilde{p})\tilde{U}, \]
(5.11)

\[ \tilde{\psi}_4 = -[\tilde{H}(z - 3\alpha - 2\tilde{p})]. \]
(5.12)

In this paper, we only solve these equations under the assumptions
\[ \rho = \tilde{p}, \quad V = -2H\rho, \]  
(5.13)

and so we have
\[ DH = 0. \]  
(5.14)

Then, Equations (5.6), (5.7), (5.8), and (5.9) become, respectively,
\[ \tau = 0, \]  
(5.15)

\[ \rho \Delta H = H[-2\Delta \rho + 2\mu \rho - 4\rho^2(\phi_{11}/\phi_{\infty})] - \delta\tilde{U} + \tilde{q}U \\
- H^2(\phi_{\infty})(\rho^2 - \phi_{11}), \]  
(5.16)

\[ DU + \rho U = 0, \]
(5.17)

\[ \tilde{D}U - \alpha U = \tilde{E}U - \alpha \tilde{U}. \]  
(5.18)

As in the previous section, in order to avoid nonlinear terms in \( \tilde{H} \) we assume
\[ \rho^2 = \phi_{\infty}, \]  
(5.19)

so that Eq. (5.14) may be written
\[ \rho \Delta H = 2H[\rho(\gamma + \tilde{q}) - 2(\phi_{11} + \Lambda)] - \delta\tilde{U} + \tilde{q}U. \]  
(5.20)

The compatibility of this equation with (5.12) leads us to
\[ \mu \rho + \phi_{11} + \Lambda = 0. \]  
(5.21)

This condition eliminates many candidates for \( g_{\alpha\beta} \) (i.e., all generalized Schwarzschild metrics). Now, the integrability condition of (5.18) with \( U \) is

\[ \rho \Delta U + 2\delta\tilde{U} - 2\alpha\tilde{U} - 2\alpha\tilde{U} \]

\[ + U[3a\phi - \rho(5\mu + 3\tilde{q} + \gamma)] = 0. \]  
(5.22)

For the Weyl tensor we have
\[ 3\tilde{\psi}_2 = -2H\rho^2, \quad \tilde{\psi}_3 = \rho\tilde{U}, \quad \tilde{\psi}_4 = -[\tilde{H} - 3\alpha] \tilde{U}. \]  
(5.23)

Consequently, if we want to obtain Petrov type \( D \) solutions, that is to say
\[ 3\tilde{\psi}_2 \tilde{\psi}_4 = 2\tilde{\psi}_3, \]  
we must have
\[ \delta U = 3\alpha U + U^2/H. \]  
(5.24)

We put
\[ f = U/H \]  
(5.25)

and then Eqs. (5.15), (5.16), and (5.22) are written as follows:
\[ Df = -\rho f, \]  
(5.26)

\[ \tilde{f}f + f(\alpha + f) = \tilde{f} + f(\alpha + f), \]  
(5.27)

\[ \delta f = \alpha f. \]  
(5.28)

On the other hand, bearing Eqs. (5.23)–(5.26) in mind, Eq. (5.20) becomes
\[ \rho \Delta f - \rho f(\mu + \gamma - \gamma) + f(\alpha \tilde{f} - \alpha f - 4\alpha \tilde{U}) \]

\[ + (\alpha + f)\tilde{f} = 0. \]  
(5.29)

Equations (5.24)–(5.27) are satisfied by choosing
\[ f = \alpha \tilde{f}, \]  
(5.30)

where \( A \) is an arbitrary real constant, and where two supplementary conditions remain:
\[ \delta \tilde{f} = \alpha, \quad \Delta \tilde{f} = \alpha \mu + \gamma - (\alpha \tilde{f} + \rho)(5 + \alpha). \]  
(5.31)

These conditions are compatible with the Newman–Penrose equations.

Now, we summarize our results in this section: Let us choose the conformally flat perfect-fluid metric \( g_{\alpha\beta} \) and the shear-free geodesic null vector field \( l_{\alpha} \) verifying \( \rho = \tilde{p} \), (5.13), (5.17), (5.19), and (5.28). Then we set \( U = \alpha H \tilde{f} \) and we solve Eqs. (5.12), (5.18), and \( \delta H = (2 + \alpha)H \tilde{f} \). These equations always have solutions. The new generalized Kerr–Schild metric \( \tilde{g}_{\alpha\beta} \) is a solution of the Einstein equations for a perfect-fluid energy-momentum tensor (3.15), where \( \tilde{g}_{\alpha \beta} \), \( \tilde{p} \), and \( \tilde{u}_{\alpha} \) are given by (5.3)–(5.5) (when they are conveniently restricted to the case we have studied). The Weyl tensor of the new metrics is Petrov type \( D \). Unless we have \( A = 0 \) or \( \alpha = 0 \), reasoning similar to that in the previous section leads us to solutions previously unknown, as \( \tilde{u}_{\alpha} \) does not lie in the preferred two-space spanned by the two multiple null eigenvectors of the Weyl tensor. In the cases \( A = 0 \) or \( \alpha = 0 \) the solutions may belong to the family given by Wainwright. 11

Obviously, we only have solved a very particular case in this section. Other more general cases remain for a subsequent paper.

VI. EXPLICIT EXAMPLES

In this section we give some examples of how the equations may be solved explicitly. We can assume two different forms for the metric \( g_{\alpha\beta} \): the form manifestly conformally
flat as given in (3.8) and other forms in which the spin coefficients of the null tetrad are adapted to the conditions obtained in Secs. IV and V, even though we do not know the conformal factor explicitly. In the first case the conditions on the spin coefficients become equations for the function \( Y \) of (3.4). Once we have obtained the function \( Y \), we can solve the integrable equations for \( H \). In the second case, we have the advantage that we do not need the conformal factor, which is unknown in many metrics. Next, we give some examples for both cases.

(1) In this example, we choose the conformally flat metric given by Olesen\(^\text{12}\) in coordinates \([x^0, x^1, x^2, x^3] = [u, t, x, y] \) in the following form:

\[
\begin{align*}
\delta_{ab} dx^a dx^b &= t^{3/4}(dx - (2/\sqrt{t}) G_x du)^2 \\
&+ \sqrt{t} (dy + 2\sqrt{t} G_y du)^2 \\
&+ 2G dt du + 2G^2 M du^2,
\end{align*}
\]

\( x = \frac{\partial}{\partial x}, \quad M(t) = 2\sqrt{a^2 + b^2 t} \), \( a, b = \text{const.} \)

\[ G(x, y, u) = g(x)h(y)\eta(u), \quad g_{xx} + a^2 g = 0, \]

\[ h_{yy} + b^2 h = 0, \quad a, b = \text{const.} \]

Therefore, condition (4.7) is also automatically satisfied. Next, we obtain

\[
\begin{align*}
\mu &= -2b^2 \sqrt{t}, \quad \gamma - \bar{\gamma} = 2iG^{-2} G_x G_y, \\
G_{xx} - G^{-1} G_x G_x = 0, \quad G_{yy} - G^{-1} G_y G_y = 0.
\end{align*}
\]

Consequently, we must restrict the metric \( \delta_{ab} \) to the case in which

\[ G(x, y, u) = g(x)h(y)\eta(u), \]

where \( \eta(u) \) is an arbitrary function of the variable \( u \). Once this restriction is imposed we know that the equations for \( H \) are compatible. The integration of these equations \([3.24], (4.4), \) and \((4.10)\] is standard and we finally obtain

\[
H = ch(y)g(x), \quad c = \text{const.}
\]

From (4.1)-(4.3) we have

\[
[l^a u_a]^2 = [2|H + M|]^{-1}, \quad m^a u_a = 0,
\]

\[ \chi \bar{p} = \chi p + 3H/8t^2, \quad \chi \bar{q} = \chi (\bar{p} + 12b^2/\sqrt{t}).\]

The final form for the metric \( \delta_{ab} \) is the following:

\[
\delta_{ab} dx^a dx^b = g_{ab} dx^a dx^b + 2h(y)g^{-3}(x)G^2 du^2,
\]

\[ G(x, y, u) = g(x)h(y)\eta(u), \quad g_{xx} + a^2 g = 0, \]

\[ h_{yy} + b^2 h = 0, \quad a, b = \text{const.} \]

(2) The most simple metric \( g_{ab} \) we can choose is the “flat” Robertson-Walker metric, that is to say

\[
\begin{align*}
g_{ab} dx^a dx^b &= 2R^2 (-du dv + dx dy), \\
R &= R(u + v), \quad q = q(u + v), \quad p = p(u + v), \\
q &= -3(q + p), \quad R^2 = \frac{4qR^4}{3}, \quad \frac{\partial}{\partial u}, \\
t &= (1/\sqrt{2})(u + v), \quad l_a dx^a = du + \bar{\gamma} dv + \gamma dy,
\end{align*}
\]

\[ u_a dx^a = -(R/\sqrt{2})(du + dv).\]

Now, the function \( Z \) of (3.12) is given by

\[
Z = -\bar{\gamma} R/(1 + \gamma \bar{\gamma}),
\]

so that Eqs. (3.11) and (3.14) provide the null tetrad and the spin coefficients, respectively. The function \( Y \) is defined by (3.6).

To satisfy Eqs. (5.13) and (5.19) it is necessary that

\[ Y = 0. \]

Then (5.28) is automatically verified. Finally, the condition (5.17) leads us to\(^{13}\)

\[ p = -\frac{1}{2} \gamma, \]

and therefore we must restrict the Robertson-Walker metric such that

\[ q = A^2 R^2, \quad R = R \pm C, \quad C = \sqrt[3]{3} A, \quad A, B = \text{const.} \]

Solving the integrable system of equations for \( H \) we easily obtain

\[ H = \text{const.} \]

Consequently, we obtain the following solution:

\[
\begin{align*}
\delta_{ab} dx^a dx^b &= g_{ab} dx^a dx^b + 2H du^2, \\
R &= R \pm C, \quad \bar{q} = A^2 R^2 \left(1 + \frac{1}{3R^2}\right), \quad \bar{p} = \frac{A^2}{R^2} \left(\frac{1}{3R^2} - 3\right), \\
(l^a u_a)^2 &= 1/2(H + R^2), \quad m^a u_a = 0, \quad A, B, H = \text{const.}
\end{align*}
\]

APPENDIX: BIANCHI IDENTITIES

Next, we list the Bianchi identities for a conformally flat perfect-fluid metric. We choose the null tetrad (3.11) such that

\[
\begin{align*}
\phi_{01} &= \phi_{02} = \phi_{12} = \phi_{22} = \kappa = \epsilon = \rho = -\bar{p} = 0, \\
\Lambda = (\chi/24)(q - 3p), \quad \phi_{00} = (\chi/2)(q + p)(l^a u_a)^2, \\
\phi_{11} = (\chi/8)(q + p), \quad \phi_{00} \phi_{22} = 4\delta^2, \quad
\end{align*}
\]
and then we have

\[ \phi_{00} \lambda = 2\phi_{11} \tilde{\sigma}, \quad \phi_{00} \tilde{\nu} = 2\phi_{11}(\tau + \tilde{\nu}), \]
\[ \delta(\phi_{11} + \Lambda) = 0, \quad \delta \phi_{11} = \tilde{\nu} \phi_{11}, \quad \mu = \tilde{\mu}, \]
\[ D(\phi_{11} + \Lambda) = \mu \phi_{00} - 2\rho \phi_{11}, \]
\[ \Delta(\phi_{11} + \Lambda) = 2\mu \phi_{11} - \rho \phi_{22}, \]
\[ \delta \phi_{00} = (\tilde{\sigma} - 2\tilde{\xi} - 2\beta) \phi_{00}, \]
\[ 2\Delta \phi_{11} - D \phi_{22} + \rho \phi_{22} - 2\mu \phi_{11} = 0, \]
\[ \Delta \phi_{00} - 2D \phi_{11} - 2\rho \phi_{11} + \phi_{00}(\mu + 2\gamma + 2\tilde{\gamma}) = 0. \]


6In this paper there are three kinds of objects related to the metrics \( \bar{\mathbf{g}}_{ab}, \mathbf{g}_{ab}, \) or \( \eta_{ab} \). We denote these, respectively, by a tilde (i.e., \( \bar{\phi}_a, \bar{k}^a, \) etc.), no label (i.e., \( k_a, \sigma_c, \rho, \) etc.), and by \( M \) (i.e., \( \sigma_{ab}, P_M, \) etc.). Consequently, we raise and lower indices of the tensors with \( \bar{\mathbf{g}}_{ab}, \mathbf{g}_{ab}, \) or \( \eta_{ab} \), respectively.

7We use standard notation in Newman-Penrose formalism. Our conventions coincide with those of Kramer et al.\(^4\) except for the spin coefficients where the sign is changed. The signature of the metric is \( -, +, +, + \).


9It is a consequence of the expression (4.6) for the Weyl tensor that solutions of Petrov type \( N \) cannot be obtained. This may be seen as follows. Oleson\(^12\) has shown that perfect-fluid solutions of Petrov type \( N \) with a geodesic principal null vector must satisfy \( \tilde{\sigma} \neq 0 \) and \( \phi_{00} = 3\tilde{\mu} \). It is easily shown that \( \tilde{\sigma} = \sigma \) and \( \phi_{00} = 3\tilde{\mu} \), therefore we must have \( \sigma \neq 0 \) and \( \phi_{00} = 3\tilde{\mu} \).

10Generalized Schwarzschild interior metrics verify \( \phi_{11} + \Lambda = \text{const} > 0 \) and \( \rho = 2(\mu + 3\tilde{\mu}) \). Therefore \( \mu > 0 \) and Eq. (5.19) is not possible.


12M. Oleson, J. Math. Phys. 12, 666 (1971). We take the metrics belonging to class I of Oleson’s paper with the assumption \( \sigma = \text{const} \) so that we have a conformally flat metric. In fact, these metrics belong to the Friedmann class of perfect-fluid space-times.

13Although \( \rho = -\bar{\rho} \) is not physically admissible we allow it since we can obtain an energy density \( \bar{\rho} \) and a pressure \( \bar{\mu} \) that are physically reasonable.