A GEOMETRIC CHARACTERIZATION OF INTERPOLATION IN $\hat{E}'(\mathbb{R})$

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Abstract. We give a geometric description of the interpolating varieties for the algebra of Fourier transforms of distributions (or Beurling ultradistributions) with compact support on the real line.

1. Introduction

Let $\mathcal{E}(\mathbb{R})$ be the space of smooth functions in $\mathbb{R}$ and let $\mathcal{E}'(\mathbb{R})$ be its dual, the space of distributions with compact support on $\mathbb{R}$. It is well known that the space $\hat{\mathcal{E}}'(\mathbb{R})$ of Fourier transforms of distributions in $\mathcal{E}'(\mathbb{R})$ coincides with the algebra of entire functions $f$ such that

$$|f(z)| \leq C(1 + |z|)^A e^{B|\text{Im } z|},$$

where $A, B, C > 0$ may depend on $f$ (see [BrGa95, Theorem 1.4.15]).

A discrete sequence $\Lambda \subset \mathbb{C}$ is called $\hat{\mathcal{E}}'(\mathbb{R})$-interpolating when the interpolation problem $f(\lambda) = v_\lambda$, $\lambda \in \Lambda$, has a solution $f \in \hat{\mathcal{E}}'(\mathbb{R})$ for every sequence of complex values $\{v_\lambda\}_{\lambda \in \Lambda}$ having the characteristic growth of $\hat{\mathcal{E}}'(\mathbb{R})$ on $\Lambda$ (see the precise definition in Section 2).

The origin of the interest in $\hat{\mathcal{E}}'(\mathbb{R})$-interpolation lies in its relationship with convolution equations and, in particular, with the density of exponential families $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ in the space of solutions $g \in \mathcal{E}(\mathbb{R})$ of equations of type $\mu * g = 0$, $\mu \in \mathcal{E}'(\mathbb{R})$. Any solution $g$ to the convolution equation is the limit of linear combinations of $\{e^{i\lambda x}\}_{\lambda \in \Lambda}$ where $\Lambda$ is the zero set of $\hat{\mu}$. If, moreover, the sequence $\Lambda$ is $\hat{\mathcal{E}}'(\mathbb{R})$-interpolating, then the series that represents $g$ enjoys better convergence properties. For more on this relationship see [EhMa74] or [BrGa95, Chapter 6] (in particular, Theorem 6.1.11).

The space $\hat{\mathcal{E}}'(\mathbb{R})$ is a particular case of the algebras

$$A_p = \{f \in H(\mathbb{C}) : \log |f(z)| \leq A + Bp(z) \text{ for some } A, B > 0\}$$

associated to positive measurable weights $p$, obtained by taking

$$p(z) = |\text{Im } z| + \log(1 + |z|^2).$$

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There exists an analytic characterization of interpolating sequences for general $A_p$ spaces when $p$ is subharmonic (see Theorem A below). However, a complete geometric description exists only for subharmonic weights $p$ which are both radial ($p(z) = p(|z|)$) and doubling ($p(2z) \leq C p(z)$ for some $C > 0$); see [BrL95, Corollary 4.8].

For the weight (1) Ehrenpreis and Malliavin gave a necessary geometric condition which turns out to be sufficient provided that $\Lambda$ is a zero sequence of a slowly decreasing function (see [EhMa74, Theorem 4] and its proof). Later Squires, probably unaware of Ehrenpreis and Malliavin’s result (which was stated in terms of solutions to convolution equations), proved the same result [Sq83, Theorem 2].

In this paper we give a geometric characterization for $\hat{E}'(\mathbb{R})$-interpolating sequences (Theorem 1). The characterization shows in particular that the geometric condition given by Ehrenpreis & Malliavin and Squires is also sufficient whenever the sequence is contained in the region

$$|\text{Im } z| \leq C \log(1 + |z|^2).$$

In general, however, their condition alone is not sufficient.

A similar characterization is obtained for the more general Beurling weights. These weights appear naturally in the context of convolution equations when one replaces distributions with compact support with Beurling-Björck ultradistributions of compact support (see [Bj66]). They are not necessarily subharmonic, but we will prove that they are equivalent to a subharmonic weight (see Lemma 8).

The paper is structured as follows. In Section 2 we give the precise definition of interpolating variety, discuss the background for the problem and state the main result. In Section 3 we prove that the geometric conditions of Theorem 1 are necessary, while in Section 4 we show that they are also sufficient.

A final remark about notation. $C$ will always denote a positive constant and its actual value may change from one occurrence to the next. $A = O(B)$ and $A \lesssim B$ mean that $A \leq cB$ for some $c > 0$, and $A \simeq B$ is $A \lesssim B \lesssim A$.

2. Preliminaries

For the following definition and general background on the problem we refer to [BrGa95, Chapter 2].

A measurable function $p : \mathbb{C} \to \mathbb{R}_+$ is called a weight if for some $C, D > 0$:

(a) $\log(1 + |z|^2) \leq C p(z)$ for all $z \in \mathbb{C}$.

(b) $p(\zeta) \leq C p(z) + D$ if $|\zeta - z| \leq 1$.

The importance of these properties lies in their consequences for the ring $A_p$ defined in the introduction: (a) implies that $A_p$ contains all polynomials, and (b) that $A_p$ is closed under differentiation.

The algebra $A_p$ can be thought of as the union of the Hilbert spaces

$$A^2_{p,\alpha} := \left\{ f \in H(\mathbb{C}) : \|f\|_{A^2_{p,\alpha}}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\alpha p(z)} dm(z) < \infty \right\}$$

for $\alpha > 0$, as well as the union of the Banach spaces

$$A^\infty_{p,\alpha} := \left\{ f \in H(\mathbb{C}) : \|f\|_{A^\infty_{p,\alpha}} = \sup_{z \in \mathbb{C}} |f(z)| e^{-\alpha p(z)} < \infty \right\}.$$

Also, $A_p = \bigcup_{\alpha > 0} A^\infty_{p,\alpha}$ has the structure of an (LF)-space with the topology of the inductive limit.
Definition. Let \( \Lambda \) be a discrete sequence in \( \mathbb{C} \) and let \( \{m_\lambda\}_{\lambda \in \Lambda} \) be a sequence of natural numbers. The pair \( X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda} \) is called an interpolating variety for the space \( A_p \) if for every sequence of values \( \{v_\lambda^l\}_{\lambda, l} \), \( \lambda \in \Lambda, \ l = 0, \ldots, m_\lambda - 1 \), with

\[
\sup_{\lambda \in \Lambda} \left( \sum_{l=0}^{m_\lambda-1} |v_\lambda^l| \right) e^{-\alpha p(\lambda)} < \infty
\]

for some \( \alpha > 0 \), there exists \( f \in A_p \) with

\[
\frac{f^{(l)}(\lambda)}{l!} = v_\lambda^l, \quad \lambda \in \Lambda; \ l = 0, \ldots, m_\lambda - 1.
\]

The choice of condition (2) on the values to be interpolated reflects the fact that for every \( f \in A_p \) there exists \( \alpha > 0 \) such that

\[
\sup_{z \in \mathbb{C}} \left( \sum_{l=0}^{\infty} \frac{f^{(l)}(z)}{l!} \right) e^{-\alpha p(z)} < \infty.
\]

Thus, denoting by \( A_p(X) \) the space of sequences \( \{v_\lambda^l\}_{\lambda, l} \) satisfying (2) for some \( \alpha > 0 \), we can equivalently define interpolating varieties \( X \) as those such that the restriction operator

\[
\mathcal{R}_X : A_p(\mathbb{C}) \rightarrow A_p(X)
\]

\[
f \mapsto \left\{ \frac{f^{(l)}(\lambda)}{l!} \right\}_{\lambda, l}
\]

is onto.

There exists an analytic characterization of \( A_p \)-interpolating varieties for general subharmonic weights \( p \) (see [BrL95, Corollary 3.5]). Results for \( p(z) = |z| \) and the weight \( \Pi \) were previously obtained respectively by Leont’ev [Le72] and Squires [Sq81].

Given a holomorphic function \( f \) let \( Z(f) \) denote its zero variety, i.e., the set of pairs \( (z, m_z) \in \mathbb{C} \times \mathbb{N} \) such that \( f(z) = 0 \) with multiplicity \( m_z \).

Theorem A. A variety \( X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda} \) is \( A_p \)-interpolating if and only if there exists \( f \in A_p \) such that \( X \subset Z(f) \) and for some constants \( \delta, C > 0 \),

\[
\left| \frac{f^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda.
\]

We would like to give a geometric description of \( A_p \)-interpolating varieties for the non-isotropic Beurling weights

\[
p(z) = |\text{Im} \ z| + \omega(|z|),
\]

where \( \omega(t) \) is a subadditive increasing continuous function, normalized with \( \omega(0) = 0 \) and such that:

(c) \( \log(1 + t) \lesssim \omega(t) \) for \( t > 1 \).

(d) \( \int_0^\infty \frac{\omega(t)}{1 + t^2} dt < \infty \).

Canonical examples of such weights are given by \( \omega(t) = \log(1 + t^2) \) and \( \omega(t) = t^\gamma \), \( \gamma \in (0, 1) \).

Beurling weights satisfy the following additional properties:

(e) For every \( c > 0 \) there exists \( C > 0 \) such that \( p(\zeta) \leq Cp(z) \) if \( \zeta \in D(z, cp(z)) \).
\begin{enumerate}[label=(\Alph*)]
\item For $\varepsilon > 0$ small enough, there exists $C(\varepsilon) > 0$ such that if $z \in D(\zeta, \varepsilon p(\zeta))$, then $p(\zeta) \leq C(\varepsilon)p(z)$. Also, $C(\varepsilon)$ tends to 1 as $\varepsilon$ goes to 0.
\item For $x \in \mathbb{R}^+$ big enough, the function $\omega(x)$ does not oscillate too much. More precisely, for fixed $C > 0$, if $y \in (x - C\omega(x), x + C\omega(x))$, then $1/2 \leq \omega(y)/\omega(x) \leq 2$ for $x$ big enough.
\end{enumerate}

Properties (e) and (f) follow easily from the subadditivity of $\omega$. Property (g) follows from the subadditivity and the fact that $\omega(x) = o(|x|/\log |x|)$ (see [Bj60 Lemma 1.2.8]): for any $x \in (x - C\omega(x), x + C\omega(x))$,
\[
\omega(x - C\omega(x)) \leq \omega(y) \leq \omega(x + C\omega(x)) \leq \omega(x - C\omega(x)) + \omega(2C\omega(x)) \leq \omega(x - C\omega(x)) + \omega(2C|x|/\log x) \leq 2\omega(x - C\omega(x)).
\]

In order to state the geometric conditions on a variety $X$ as above, we consider the counting function $n(z, r) = \sum_{\lambda \in D(z, r)} m_\lambda$ and the integrated version
\[
N(z, r) = \int_0^r \frac{n(z, t) - n(z, 0)}{t} \, dt + n(z, 0) \log r.
\]

In case we want to specify the variety $X$ to which the functions $n$ and $N$ refer, we will use the notation $n(z, r, X)$ and $N(z, r, X)$ respectively.

We are ready to state our main result.

**Theorem 1.** A variety $X = \{(\lambda, m_\lambda)\}_{\lambda \in \Lambda}$ is $A_p$-interpolating if and only if:
\begin{enumerate}[label=(\roman*)]
\item There is $C > 0$ such that
\[
N(\lambda, p(\lambda), X) \leq Cp(\lambda) \quad \forall \lambda \in \Lambda.
\]
\item The following Carleson-type condition holds:
\[
\sup_{x \in \mathbb{R}^+} \sum_{\lambda : |\text{Im}\lambda| > \omega(|\lambda|)} m_\lambda \frac{|\text{Im}\lambda|}{|x - \lambda|^2} < \infty.
\]
\end{enumerate}

Since the Poisson kernel at $\lambda$ in the corresponding half-plane (upper half-plane if $\text{Im}\lambda > 0$ and lower half-plane when $\text{Im}\lambda < 0$) is $P(\lambda, x) = \frac{\text{Im}\lambda}{|x - \lambda|^2}$, a restatement of condition (ii) is that the measure $\sum_{\lambda : |\text{Im}\lambda| > \omega(|\lambda|)} m_\lambda \delta_\lambda$ has bounded Poisson balayage.

**Remark 2.** Notice that for sequences $\Lambda$ within the region $|\text{Im}\lambda| \leq \omega(|\lambda|)$, condition (i) (shown to be necessary by Ehrenpreis & Malliavin and Squires) provides a complete characterization. However, this is not the case in general, i.e. condition (ii) does not follow from (i), as it is shown in the following example. Take the sequence $\Lambda$ contained in the sector $\mathcal{A} = \{z \in \mathbb{C} : |\text{Re}z| < |\text{Im}z|\}$ and having in each segment $\{\text{Im}z = 2^n\} \cap \mathcal{A}$ exactly $2^n$ equispaced points. Then $\Lambda$ satisfies condition (i) (basically $n(\lambda, t) \leq t$ for $t \leq p(\lambda)$), but it does not satisfy (ii) (it is not even a Blaschke sequence).

3. Necessary conditions

A standard feature of the spaces $A_p$ is that the interpolation can be performed in a stable way. This consequence of the open mapping theorem for $(\mathcal{L}\Phi)$-spaces applied to the restriction mapping $R_X$ defined in Section 2 (see [BrGa95 Lemma 2.2.6]) is stated precisely in the following lemma.
Lemma 3. If $X$ is an interpolating variety, there exist $C > 0$, $M \in \mathbb{N}$ such that for every $\lambda \in \Lambda$ there are functions $f_{\lambda}, g_{\lambda} \in A_p$ with bounded norms $\|f_{\lambda}\|_{A_{p,M}^\infty}, \|g_{\lambda}\|_{A_{p,M}^\infty} \leq C$ and

$$f_{\lambda}^{(l)}(\lambda')/l! = \delta_{\lambda\lambda'} \delta_{l0},$$

$$g_{\lambda}^{(l)}(\lambda')/l! = \delta_{\lambda\lambda'} \delta_{l(m_{\lambda}-1)} \quad \forall \lambda, \lambda' \in \Lambda, \ 0 \leq l \leq m_{\lambda}.$$

An application of Jensen’s Formula to the functions $f_{\lambda}$, $g_{\lambda}$ in the disk $D(\lambda, p(\lambda))$ gives the following result (see [EhMa74, Theorem 4] or [Sq83, Theorem 1]).

Theorem 4. If $X$ is $A_p$-interpolating, then condition (i) of Theorem 1 holds.

The necessity of condition (ii) is an immediate consequence of the following result. Assume that $\Lambda \cap \mathbb{R} = \emptyset$; otherwise move the horizontal line so that it does not touch any of the points in $\Lambda$. Let $\mathbb{H}$ denote the upper half-plane.

Proposition 5. Let $X$ be $A_p$-interpolating. There exist $C > 0$ such that

$$\sum_{\lambda' \in \Lambda' \cap \mathbb{H}, \lambda' \neq \lambda} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \lambda'} \right|^{-1} \leq Cp(\lambda) \quad \forall \lambda \in \Lambda \cap \mathbb{H}.$$

Of course, an analogous result could be given for any upper ($\{z : \text{Im} z > a\}$) or lower ($\{z : \text{Im} z < a\}$) half-plane.

Before giving the proof of Proposition 5 we show that it implies condition (ii) of Theorem 1. Define $\Lambda_+ = \Lambda \cap \{z, \text{Im} z > \omega(|z|)\}$. Given $x \in \mathbb{R}$ consider $\lambda \in \Lambda_+$ such that $|x - \lambda| = \inf_{\Lambda_+} |x - \lambda|$. Then

$$|\lambda - \lambda'| \leq |\lambda - x| + |x - \lambda'| = |\lambda - x| + |x - \lambda'| \leq 2|x - \lambda'|,$$

and therefore

$$\sum_{\lambda' \in \Lambda_+, \lambda' \neq \lambda} m_{\lambda'} \frac{|\text{Im} \lambda'|}{|x - \lambda'|^2} \leq 2 \sum_{\lambda' \in \Lambda_+} m_{\lambda'} \frac{|\text{Im} \lambda'|}{|\lambda - \lambda'|^2}.$$

The estimate $\log t^{-1} \geq 1 - t$ for $t \in (0, 1)$ shows that

$$\sum_{\lambda' \in \Lambda_+, \lambda' \neq \lambda} m_{\lambda'} \frac{|\text{Im} \lambda||\text{Im} \lambda'|}{|\lambda - \lambda'|^2} \leq \sum_{\lambda' \in \Lambda_+, \lambda' \neq \lambda} m_{\lambda'} \log \left| \frac{\lambda - \lambda'}{\lambda - \lambda'} \right|^{-1}.$$

Since $p(\lambda) \simeq |\text{Im} \lambda|$ for $\lambda \in \Lambda_+$, it is clear that this implies condition (ii) of Theorem 1.

Remark 6. The necessary condition of Proposition 5 can be seen as a Carleson-type condition; it can be rewritten as

$$|B_{\lambda}(\lambda)| \geq \delta e^{-Cp(\lambda)}, \quad \lambda \in \Lambda \cap \mathbb{H},$$

where $B$ denotes the Blaschke product in $\mathbb{H}$ of $\{\lambda, (m_{\lambda})\}_{\lambda \in \Lambda \cap \mathbb{H}}$, and

$$B_{\lambda}(z) = B(z) \left( \frac{z - \bar{\lambda}}{z - \lambda} \right)^{m_{\lambda}}.$$

It can also be seen as density conditions for the counting function associated to the hyperbolic metric in the half-plane. Letting $\nu = \sum_{\lambda \in \Lambda \cap \mathbb{H}} m_{\lambda} \delta_{\lambda}$ and using the distribution function we have

$$\sum_{\lambda \in \Lambda \cap \mathbb{H}} m_{\lambda} \log \left| \frac{z - \lambda}{z - \lambda} \right|^{-1} = \int_{\mathbb{H}} \log \left| \frac{z - \zeta}{z - \zeta} \right|^{-1} d\nu(\zeta) = \int_0^1 n_{\mathbb{H}}(z, t) \frac{dt}{t},$$

where $n_{\mathbb{H}}(z, t)$ is the hyperbolic metric in $\mathbb{H}$.
where
\[ D_H(z, t) = \{ \zeta : \frac{|z - \zeta|}{|z|} < t \}, \quad \text{and} \quad n_H(z, t) := \nu(D_H(z, t)) \]
is the number of points of \( \Lambda \) in the pseudohyperbolic disk of “center” \( z \) and “radius” \( t \) (actually the true disk of center \( \text{Re} z + \frac{1}{2} \omega \) and radius \( \frac{t}{1 + \omega} \)).

**Proof of Proposition 5.** Let \( z = x + iy \) and consider the Poisson transform of \( \omega(|t|) \):

\[ u(z) := P[\omega](z) = \int_{\mathbb{R}} \frac{y\omega(|t|)}{(x-t)^2 + y^2} \, dt, \]

which converges by (d). Define \( H = \exp(u + i\bar{u}) \), where \( \bar{u} \) is a harmonic conjugate of \( u \).

Given \( \lambda \in \Lambda \cap \mathbb{H} \), take the function \( f_\lambda \) given by Lemma 3 and define

\[ h_\lambda(z) = \frac{f_\lambda(z)e^{iM_1z}}{(H(z))^{M_2}}, \]

with \( M_1, M_2 \) to be chosen. It is clear that \( h_\lambda \) is holomorphic in \( \mathbb{H} \). On the other hand, for all \( z \) in the upper half-plane \( |\log |H(z)| - \omega(|\text{Re} z|)| \leq A + B|\text{Im} z| \); see [B65] Lemma 1.3.11. Moreover, \( |\omega(|\text{Re} z|) - \omega(|z|)| \leq \omega(|\text{Im} z|) \leq A + B|\text{Im} z| \), thus \( |\log |H(z)| - \omega(|z|)| \leq A + B|\text{Im} z| \). Therefore, if \( M_1 \) and \( M_2 \) are big enough, \( h_\lambda \) is bounded in \( \mathbb{H} \) by a constant which does not depend on \( \lambda \):

\[ |h_\lambda(z)| \leq Ce^{M_1|\text{Im} z| - M_2 \log |H(z)|} < 1. \]

Also,

\[ |h_\lambda(\lambda)| = e^{-M_1 \text{Im} \lambda - M_2 \log |H(\lambda)|} \geq e^{-Cp(\lambda)}. \]

Now apply Jensen’s formula in the half-plane to the function \( h_\lambda \):

\[ \log |h_\lambda(\lambda)| = \int_{\mathbb{R}} P(\lambda, x) \log |h_\lambda(x)| \, dx - \int_{\mathbb{H}} G(\lambda, \zeta) \Delta \log |h_\lambda(\zeta)|, \]

where \( P(\lambda, x) \) denotes the Poisson kernel and \( G(\lambda, \zeta) = \log \left| \frac{\lambda - \zeta}{\lambda - \zeta} \right|^{-1} \) is the Green function in \( \mathbb{H} \) with pole in \( \lambda \).

Since \( h_\lambda \) vanishes on \( \Lambda \setminus \{ \lambda \} \), Jensen’s Formula and the estimates above yield

\[ \sum_{\lambda \in \Lambda \cap \mathbb{H} \setminus \lambda} m_\lambda \log \left| \frac{\lambda - \lambda'}{\lambda - \lambda''} \right|^{-1} \leq \sup_{R} \log |h_\lambda| - \log |h_\lambda(\lambda)| \lesssim p(\lambda). \]

\[ \square \]

4. Sufficient conditions

We split the sequence into three pieces, according to the non-isotropy of the weight \( p \). Consider the regions

\[ \Omega_0 = \{ z \in \mathbb{C} : |\text{Im} z| \leq \omega(|z|) \}, \]
\[ \Omega_+ = \{ z \in \mathbb{C} : |\text{Im} z| > \omega(|z|) \}, \]
\[ \Omega_- = \{ z \in \mathbb{C} : |\text{Im} z| > -\omega(|z|) \}, \]

and define \( \Lambda_0 = \Lambda \cap \Omega_0 \), \( \Lambda_+ = \Lambda \cap \Omega_+ \) and \( \Lambda_- = \Lambda \cap \Omega_- \). Let also \( X_0 = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda_0}, X_+ = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda_+} \) and \( X_- = \{ (\lambda, m_\lambda) \}_{\lambda \in \Lambda_-} \).

It is enough to prove that each piece \( X_+, X_-, X_0 \) of the variety \( X \) is \( A_p \)-interpolating. This is so because \( X \) is weakly separated (see Lemma 7(i) below), and
a weakly separated union of a finite number of $A_p$-interpolating varieties is also $A_p$-interpolating \cite[Theorem II.1]{O}. It is also clear that the varieties $X^+$ and $X^-$ can be dealt with similarly.

We start with some easy consequences of condition (i) of Theorem \[\text{(i) If there exists $\varepsilon, C > 0$ such that $n(z, \varepsilon p(z), X) \leq C p(z), \forall z \in \mathbb{C}$.}

\textbf{Proof.} (i) If there exists $\lambda'$ such that $|\lambda' - \lambda| < 1$, then

$$N(\lambda, p(\lambda), X) \geq \int_{|\lambda' - \lambda|}^{1} \frac{n(\lambda, t) - m_{\lambda}}{t} dt \geq \int_{|\lambda' - \lambda|}^{1} \frac{m_{\lambda}}{t} dt = \log \left( \frac{1}{|\lambda' - \lambda|} \right)^{m_{\lambda}}.$$\]

Using condition (i) in Theorem \[\text{and reversing the roles of $\lambda$ and $\lambda'$ we obtain the desired estimate.}

(iii) When $z = \lambda \in \Lambda$, this is immediate from the estimate

$$\int_{1/2p(\lambda)}^{p(\lambda)} \frac{n(\lambda, 1/2p(\lambda)) - 1}{t} dt \leq N(\lambda, p(\lambda)).$$\]

When $z \notin \Lambda$, then let $\varepsilon > 0$ be such that $\zeta \in D(z, \varepsilon p(z))$ implies

$$D(z, \varepsilon p(z)) \subset D(\zeta, 1/2p(\zeta)),$$\]

which exists by property (f) of the weight. Take $\lambda \in D(z, \varepsilon p(z))$ (if there is no such $\lambda$ the estimate is obviously true). Then, by the previous case and property (e) of the weight

$$n(z, \varepsilon p(z)) \leq n(\lambda, 1/2p(\lambda)) \lesssim p(\lambda) \lesssim p(z).$$\]

\[\square\]

\[\text{4.1 Case $A_0$. We would like to prove that $X_0 = \{(\lambda, m_{\lambda})\}_{\lambda \in A_0}$ is $A_p$-interpolating using a $\bar{\partial}$-scheme. This is easier if we can regularize the weight in the following way.}

\textbf{Lemma 8. There exists } \tilde{p} \text{ subharmonic in } \mathbb{C} \text{ such that } p(z) \simeq \tilde{p}(z) \text{ and}

\[1/\tilde{p}(z) \lesssim \Delta \tilde{p}(z) \quad \text{if } |\text{Im} z| \leq 2\omega(|z|).\]

The fact that $p \simeq \tilde{p}$ clearly implies that $A_p = A_{\tilde{p}}$ and the interpolating varieties for $A_p$ and $A_{\tilde{p}}$ are the same.

\textbf{Proof.} We will construct $\tilde{p}(z) = |\text{Im} z| + r(z)$, where $r$ satisfies the following properties:

(i) $r \geq 0$ and $\tilde{p}$ is subharmonic in $\mathbb{C}$.

(ii) $r(z) = 0$ if $|\text{Im} z| \geq 10\omega(|z|)$.

(iii) $1/p(z) \lesssim \Delta \tilde{p}(z)$ and $r(z) \simeq \omega(|z|)$ if $|\text{Im} z| \leq 2\omega(|z|)$.
In order to construct \( r \), we partition the real line into intervals \( I_n \) defined in the following way.

Let \( x_1 > 1 \), \( x_{n+1} = x_n + \omega(x_n) \) for \( n \geq 1 \) and \( x_n = -x_{-n} \) for \( n \leq -1 \). Set \( I_0 = [x_{-1}, x_1] \), \( I_n = [x_n, x_{n+1}] \) for \( n \geq 1 \) and \( I_n = [x_{n-1}, x_n] \) for \( n \leq -1 \). Denote by \( \omega_n \) the length of \( I_n \).

We consider two measures in \( \mathbb{R} \). The first one is the usual length measure \( d\nu \) in \( \mathbb{R} \), which we split \( d\nu = \sum_n d\nu_n \), with \( d\nu_n = dx|_{I_n} \). The second one is defined as a sum of convolutions of the \( d\nu_n \)’s: let

\[
d\mu_n(z) = \frac{1}{100\pi\omega_n^2} \int_{I_n} \chi D_n(z-x) dx \, dm(z),
\]

where \( D_n = D(0, 10\omega_n) \), and define \( d\mu = \sum_n d\mu_n \).

Notice that when \( z \) is at a distance of \( I_n \) smaller than \( 2\omega_n \), we can use property \((g)\) of the Beurling weights to deduce that \( d\mu(z) \approx 1/\omega(|z|) \approx 1/p(z) \). Hence \( d\mu(z) \approx dm(z)/p(z) \).

Define

\[
r(z) = \int_{\mathbb{C}} \log |z-w|(d\mu(w) - d\nu(w)).
\]

Since \( \Delta |\text{Im} z| = d\nu \), we have \( \Delta \rho = d\mu \geq 0 \).

Let \( S_n \) denote the support of \( \mu_n \). Let

\[
r_n(z) := \int_{\mathbb{C}} \log |z-w|(d\mu_n(w) - d\nu_n(w)) = \int_{S_n} \log |z-w|d\mu_n(w) - \int_{I_n} \log |z-x| dx.
\]

Using the definition of \( \mu_n \) and reversing the order of integration we get

\[
r_n(z) = \int_{I_n} M(x) dx,
\]

where

\[
M(x) = \frac{1}{100\pi\omega_n^2} \int_{D(x, 10\omega_n)} \log |z-w| dm(w) - \log |z-x| 
\]

\( \geq 0 \).

In particular, \( r \) is non-negative in \( \mathbb{C} \).

If \( z \notin S_n \) and \( x \in I_n \), \( \log |z-w| \) is harmonic in \( D(x, 10\omega_n) \), hence \( r_n(z) = 0 \).

Suppose now \( z \in D(x_n, 3\omega_n) \). Then, for each \( x \in I_n \), \( |z-x| \leq 4\omega_n \) and

\[
M(x) \geq \frac{1}{100\pi\omega_n^2} \int_{[w-x] \leq 10\omega_n} \log \frac{|z-w|}{|z-x|} dm(w) \gtrsim 1.
\]

Thus, \( r_n(z) \gtrsim \omega_n \gtrsim \omega(|z|) \).

If \( z \in S_n \), using that \( \mu_n \) and \( \nu_n \) have the same mass \( \omega(x_n) \), we obtain

\[
\int_{\mathbb{C}} \log |z-w|(d\mu_n(w) - d\nu_n(w)) \leq \int_{\mathbb{C}} \log \frac{|z-w|}{\omega(x_n)} (d\mu_n(w) + d\nu_n(w))
\]

\[\leq \int_{\mathbb{C}} \frac{|x_n-w|}{\omega(x_n)} (d\mu_n(w) + d\nu_n(w)) \lesssim \omega(|z|).
\]

Since \( |\text{Im} z| \leq 2\omega(|z|) \), \( z \) belongs at most to a finite number of \( S_n \)’s and at least to one \( D(x_n, 10\omega_n) \), by property \((g)\) of the Beurling weights, we are done.

Let us prove now that \( X_0 \) is \( A_p \)-interpolating. In view of Lemma\[S\] we assume that \( 1/p \lesssim \Delta \rho \) on \( |\text{Im} z| \leq 2\omega(|z|) \).

Consider the separation radii \( \delta_\lambda := \delta e^{-CP(\lambda)} \) given by Lemma\[Z\](i).
Given a sequence of values \( \{v^i_\ell\}_{\lambda,\ell} \) satisfying (2), define the smooth interpolating function
\[
F(z) = \sum_{\lambda \in \Lambda_0} p_\lambda(z) \mathcal{X}\left(\frac{|z - \lambda|^2}{\delta^2}\right),
\]
where \( p_\lambda(z) = \sum_{l=0}^{m_\lambda-1} v^l_\lambda(z - \lambda)^l \) and \( \mathcal{X} \) is a smooth cut-off function with \( |\mathcal{X}'| \leq 1 \), \( \mathcal{X}(x) = 1 \) if \( |x| \leq 1 \) and \( \mathcal{X}(x) = 0 \) if \( |x| \geq 2 \).

It is clear that \( F^{(i)}(\lambda)/i! = v^i_\lambda \), and that \( F \) has the characteristic growth of \( A_p \) functions; the support of \( F \) is contained in \( \bigcup_\lambda D_\lambda \) and for \( z \in D_\lambda \),
\[
|F(z)| \leq \sum_{l=0}^{m_\lambda-1} |v^l_\lambda| \leq C e^{\alpha p(\lambda)} \lesssim e^{Kp(z)}.
\]

There is also a good estimate on \( \partial \bar{\partial} F \). Its support is the union of the annuli
\[
C_\lambda = \{ z \in \mathbb{C} : \delta_\lambda \leq |z - \lambda| \leq 2\delta_\lambda \},
\]
and for \( z \in C_\lambda \),
\[
\left| \frac{\partial F}{\partial \bar{\varepsilon}}(z) \right| \lesssim \sum_{l=0}^{m_\lambda-1} |v^l_\lambda| |\mathcal{X}'| \frac{1}{\delta_\lambda} \lesssim e^{Cp(\lambda)} \lesssim e^{Kp(z)}
\]
for \( K \) big enough.

Altogether, there exists \( \gamma > 0 \) such that
\[
\int_C |F(z)| e^{-\gamma p(z)} < \infty, \quad \int_C |\partial \bar{\partial} F(z)| e^{-\gamma p(z)} < \infty.
\]
Now, when looking for a holomorphic interpolating function of the form \( f = F - u \), we are led to the \( \partial \bar{\partial} \)-problem
\[
\partial \bar{\partial} u = \partial \bar{\partial} F,
\]
which we solve using Hörmander’s theorem [Ho94, Theorem 4.2.1]: given a (pluri)-subharmonic function \( \psi \) in \( \mathbb{C} \), there exists a solution \( u \) to the above equation such that
\[
2 \int_C |u|^2 e^{-\psi} \frac{1}{(1 + |z|^2)^2} dm \leq \int_C |\partial \bar{\partial} F|^2 e^{-\psi} dm.
\]
We apply Hörmander’s theorem with
\[
\psi_\beta(z) = \beta p(z) + v(z),
\]
where \( \beta > 0 \) will be chosen later on and
\[
v(z) = \sum_{\lambda \in \Lambda_0} m_\lambda \left[ \log |z - \lambda|^2 - \frac{1}{\pi \varepsilon^p p^2(\lambda)} \int_{D(\lambda, \varepsilon p(\lambda))} \log |z - \zeta|^2 dm(\zeta) \right].
\]
Here \( \varepsilon \) is a fixed small constant to be determined later on.

Integrating by parts the equality
\[
\int_0^{2\pi} \log |a - re^{i\theta}|^2 \frac{d\theta}{2\pi} = \begin{cases} \log |a|^2 & \text{if } |a| > r, \\ \log r^2 & \text{if } |a| \leq r, \end{cases}
\]
one sees that for \( a \in \mathbb{C} \) and \( r > 0 \),
\[
\log |a|^2 - \frac{1}{\pi r^2} \int_{D(a, r)} \log |\zeta|^2 dm(\zeta) = \begin{cases} \log |\frac{a}{r}|^2 + 1 & \text{if } |a| \leq r, \\ 0 & \text{if } |a| > r. \end{cases}
\]
Thus
\[ v(z) = \sum_{\lambda:|\lambda-z|\leq \varepsilon p(\lambda)} m_\lambda \left[ \log \frac{|z-\lambda|^2}{\varepsilon^2 p^2(\lambda)} + 1 - \frac{|z-\lambda|^2}{\varepsilon^2 p^2(\lambda)} \right]. \]

In particular, \( v \leq 0 \) and \( \Delta v(z) = 0 \) if \( z \not\in \bigcup \lambda D(\lambda, \varepsilon p(\lambda)) \). For \( z \in \bigcup \lambda D(\lambda, \varepsilon p(\lambda)) \) we have \( \Im z \leq 2 \varepsilon \omega(|z|) \) and
\[ \Delta v(z) \geq \sum_{\lambda:|\lambda-z|\leq \varepsilon p(\lambda)} \frac{-m_\lambda}{\varepsilon^2 p^2(\lambda)} + \sum_{\lambda:|\lambda-z|\leq C(\varepsilon)p(\lambda)} \frac{-m_\lambda}{p^2(z)} = - \frac{n(z, C(z)p(z))}{p^2(z)}. \]

As observed in Lemma 7(ii), with \( \varepsilon \) small enough \( n(z, C(z)p(z)) \lesssim p(z) \), thus \( \Delta v(z) \gtrsim -1/p(z) \). This and (3) show that \( \psi_\beta \) is subharmonic if \( \beta \) is chosen big enough.

Also, we deduce from (c) that for any \( \beta' > \beta \),
\[ \int_\mathbb{C} |u|^2 e^{-\beta'p} dm \lesssim \int_\mathbb{C} |u|^2 \frac{e^{-\psi_\beta}}{(1 + |z|^2)^2} dm \lesssim \int_\mathbb{C} |\partial F|^2 e^{-\psi_\beta} dm. \]

We need to control \( \psi_\beta \) on the support of \( \partial F \). For \( z \in C_\lambda \),
\[ |\psi_\beta(z) - \beta p(z)| \leq \sum_{\lambda:|\lambda-z|\leq \varepsilon p(\lambda)} m_\lambda \log \frac{\varepsilon^2 p^2(\lambda)}{|z-\lambda|^2} + \sum_{\lambda':|\lambda'-z|\leq \varepsilon p(\lambda')} m_{\lambda'} \log \frac{\varepsilon^2 p^2(\lambda')}{|z-\lambda'|^2} \lesssim p(\lambda) + \sum_{\lambda':|\lambda'-z|\leq C(\varepsilon)p(\lambda)} m_{\lambda'} \log \frac{C(\varepsilon)^2 p^2(z)}{|z-\lambda'|^2} \lesssim p(z) + N(z, C(\varepsilon)p(z)). \]

Claim 9. For \( \varepsilon \) small enough \( N(z, C(\varepsilon)p(z)) \lesssim p(z) \) for all \( z \in \text{supp}(\partial F) \).

Assuming the claim we have \( |\psi_\beta(z) - \beta p(z)| \leq Kp(z) \) on \( \text{supp}(\partial F) \). Therefore, for \( \beta \) big enough
\[ \int_\mathbb{C} |u|^2 e^{-\beta'p} dm \lesssim \int_\mathbb{C} |\partial F|^2 e^{-\psi_\beta} dm \lesssim \int_\mathbb{C} |\partial F|^2 e^{-\gamma p} dm < \infty. \]

This shows that \( f := F - u \in A_p \). Since \( e^{-\psi_\beta} \simeq |z-\lambda|^{-2m_\lambda} \) around each \( \lambda \), also \( u^{(l)}(\lambda) = 0 \) for all \( \lambda \in \Lambda, \ l = 0, \ldots, m_\lambda - 1 \), and therefore \( f^{(l)}(\lambda)/l! = f^{(l)}(\lambda)/l! = v_l \), as required.

Proof of the claim. Assume \( z \in C_\lambda \) and observe that \( n(z, t) = 0 \) for \( t < \delta_\lambda \) and that \( n(z, t) \leq m_\lambda \) for \( \delta_\lambda \leq t < 2\delta_\lambda \). Since \( D(z, t) \subset D(\lambda, t + 2\delta_\lambda) \) and \( |z| < |\lambda| + 2\delta_\lambda \), we have (changing into \( s = t + 2\delta_\lambda \))
\[ N(z, C(\varepsilon)p(z)) \leq \int_{\delta_\lambda}^{2\delta_\lambda} \frac{m_\lambda}{t} dt + \int_{2\delta_\lambda}^{C(\varepsilon)p(z)} \frac{n(z, t) - m_\lambda}{t} dt \leq p(\lambda) + \int_{2\delta_\lambda}^{C(\varepsilon)p(z) + 2\delta_\lambda} \frac{n(\lambda, s) - m_\lambda}{s - 2\delta_\lambda} ds \lesssim p(\lambda) + N(\lambda, C'(\varepsilon)p(\lambda)). \]
From the properties of the weight and the hypothesis we have finally that for \( \varepsilon \) small \( N(z, C(\varepsilon)p(z)) \lesssim p(\lambda) \lesssim p(z) \).

4.2. Case \( \Lambda^+ \). According to Theorem A, it is enough to construct a function \( G \in A_p \) such that \( X_+ \subset Z(G) \) and

\[
\frac{|G^{(m_\lambda)}(\lambda)|}{m_\lambda!} \geq \varepsilon e^{-Kp(\lambda)}, \quad \lambda \in \Lambda_+,
\]

for some constants \( \varepsilon, k > 0 \). In fact, the hypotheses of Theorem A require the weight \( p \) to be subharmonic, and our weights are not necessarily so. Nevertheless, by Lemma \( \square \) there exists a subharmonic weight \( \tilde{p} \) equivalent to \( p \), and we may apply Theorem A to \( \tilde{p} \).

Take any entire function \( F \) such that \( Z(F) = X_+ \). Since the necessary conditions imply that \( X_+ \) satisfies the Blaschke condition in \( \mathbb{H} \), we can consider also the Blaschke product

\[
B(z) = \prod_{\lambda \in \Lambda_+} \left( \frac{z - \lambda}{z - \bar{\lambda}} \right)^{m_\lambda}, \quad z \in \mathbb{H}.
\]

Define

\[
\phi(z) = \begin{cases} 
\log \left| \frac{F(z)}{B(z)} \right|, & \text{Im } z > 0, \\
\log |F(z)|, & \text{Im } z \leq 0.
\end{cases}
\]

**Lemma 10.** \( \phi \) is harmonic outside the real axis, subharmonic on \( \mathbb{C} \) and its Laplacian is uniformly bounded.

**Proof.** It is clear, by definition, that \( \phi \) is harmonic on \( \mathbb{C} \setminus \mathbb{R} \). In order to prove that \( \phi \) is subharmonic on \( \mathbb{C} \), it is enough to check the mean inequality for \( x \in \mathbb{R} \). We have

\[
\phi(x) = \log |F(x)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(x + re^{i\theta})|d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(x + re^{i\theta})d\theta.
\]

Since \( \Delta \log |F| \equiv 0 \) around \( \mathbb{R} \), it is enough to compute the Laplacian of

\[
\psi(z) = \begin{cases} 
\log \frac{1}{|B(z)|}, & \text{Im } z > 0, \\
0, & \text{Im } z \leq 0.
\end{cases}
\]

Being

\[
\log \frac{1}{|B(z)|} = \frac{1}{2} \sum_{\lambda \in \Lambda_+} m_\lambda \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2,
\]

it will be enough to compute the Laplacian of each term

\[
\psi_\lambda(z) = \begin{cases} 
\log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2, & \text{Im } z > 0, \\
0, & \text{Im } z \leq 0.
\end{cases}
\]

It is clear that \( \partial \psi_\lambda / \partial x = 0 \) on \( \mathbb{R} \), hence \( \Delta \psi_\lambda = \partial^2 \psi_\lambda / \partial y^2 \). Since \( \psi_\lambda \) is continuous around \( \mathbb{R} \), this Laplacian has a magnitude equivalent to the jump of the first derivative of \( \psi_\lambda \). The derivative of the Green function on the half-plane with respect to the normal direction \( y \) is the Poisson kernel:

\[
\frac{\partial}{\partial y} \log \left| \frac{z - \bar{\lambda}}{z - \lambda} \right|^2 |_{y=0} = \frac{4\text{Im } \lambda}{|x - \lambda|^2}.
\]
Therefore,
\[ \Delta \phi(x) = 4 \sum_{\lambda \in \Lambda^+} m_\lambda \frac{\Im \lambda}{|x - \lambda|^2} \, dx, \]
which is bounded by the hypothesis.

Define
\[ \Psi(z) = N|\Im z| - \phi(z). \]

Observe that \( \Delta \Psi(z) = N \, dx - \Delta \phi(x) \, dx \), thus according to the previous lemma \( \Delta \Psi \approx dx \) when \( N \in \mathbb{N} \) is big enough. In this situation, according to [OrSe99, Lemma 3], there exists a multiplier associated to \( \Psi \), i.e., an entire function \( h \) such that:

(a) \( \mathcal{Z}(h) \) is a separated sequence contained in \( \mathbb{R} \).

(b) Given any \( \varepsilon > 0 \), \( |h(z)| \approx \exp(\Psi(z)) \) for all points \( z \) such that \( d(z, \mathcal{Z}(h)) > \varepsilon \).

Define now \( G = hF \). It is clear that \( G \in A_p \):
\[ |G(z)| \lesssim e^{\Psi(z) + \log |F(z)|} \leq e^{\Psi(z) + \phi(z)} \leq e^{Np(z)}, \quad z \in \mathbb{C}. \]

It is also clear that \( X_+ \subset \mathcal{Z}(G) \), since \( X_+ \subset \mathcal{Z}(F) \).

In order to prove that there exist \( \varepsilon, C > 0 \) such that
\[ (5) \quad \left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| \geq \varepsilon e^{-Cp(\lambda)} \]
consider then the disjoint disks \( D_\lambda = D(\lambda, \delta_\lambda), \delta_\lambda = \delta e^{-C(G^{(m_\lambda)})} \) given by Lemma [7(i)].
Since \( \Lambda_+ \) is far from \( \mathcal{Z}(h) \), the estimate
\[ |G(z)| = |h(z)|e^{\phi(z)}|B(z)| \approx e^{N|\Im z||B(z)|}, \quad z \in \partial D_\lambda, \]
holds.

**Claim 11.** There exists \( C > 0 \) such that \( |B(z)| \geq \varepsilon e^{-Cp(z)}, \quad z \in \partial D_\lambda. \)

Assuming this we have \( |G(z)| \gtrsim e^{-Cp(z)} \) for all \( z \in \partial D_\lambda \). Define then \( g(z) = G(z)/(z - \lambda)^{m_\lambda} \). It is clear that \( g \) is holomorphic, non-vanishing in \( D_\lambda \), and \( |g(z)| \gtrsim e^{-cp(\lambda)} \) for \( z \in \partial D_\lambda \). By the minimum principle
\[ \left| \frac{G^{(m_\lambda)}(\lambda)}{m_\lambda!} \right| = |g(0)| \gtrsim e^{-cp(\lambda)}, \]
as desired.

**Proof of the claim.** As observed in Remark [3(b)], the estimate we want to prove is equivalent to
\[ \int_{0}^{1} \frac{n_{\delta}(z, t)}{t} \, dt \lesssim p(z), \quad z \in \partial D_\lambda. \]
This is proved like Claim [9] except we replace the Euclidean disks by the hyperbolic ones. We have
\[ \int_{0}^{1} \frac{n_{\delta}(z, t)}{t} \, dt \lesssim \int_{2\delta_\lambda}^{\delta_\lambda} \frac{m_\lambda}{t} \, dt + \int_{2\delta_\lambda}^{1} \frac{n_{\delta}(z, t) - m_\lambda}{t} \, dt. \]
The first term is controlled by $p(\lambda)$. In order to control the second term observe that $D_{\mathbb{H}}(z,t) \subset D_{\mathbb{H}}(\lambda, \frac{t+\delta e_{\lambda}}{1+\delta e_{\lambda}})$; hence changing the variable into $s = \frac{t+\delta e_{\lambda}}{1+\delta e_{\lambda}}$ we get
\[
\int_{2\delta}^{1} \frac{n_{\mathbb{H}}(z,t) - m_{\lambda}}{t} \, dt \leq \int_{\frac{3\delta}{1+2\delta}}^{1} \frac{n_{\mathbb{H}}(\lambda, s) - m_{\lambda}}{s - \delta_{\lambda}} \frac{1 - \delta_{\lambda}^{2}}{(1 - \delta_{\lambda})^{2}} \, ds.
\]
There is no restriction in assuming that $\delta_{\lambda} < 1/2$. Then $\frac{3\delta}{1+2\delta} > 2\delta_{\lambda}$ and therefore $s - \delta_{\lambda} > s/2$. With this and condition (ii) in Theorem I, we obtain
\[
\int_{0}^{1} \frac{n_{\mathbb{H}}(z,t)}{t} \, dt \lesssim p(\lambda) + \int_{0}^{1} \frac{n_{\mathbb{H}}(\lambda, s) - m_{\lambda}}{s} \, ds.
\]
Since $p(\lambda) \lesssim p(z)$, we will be done as soon as we prove that
\[
\int_{0}^{1} \frac{n_{\mathbb{H}}(\lambda, s) - m_{\lambda}}{s} \, ds \lesssim p(\lambda).
\]
There exists $\delta > 0$ (independent of $\lambda$) such that $D_{\mathbb{H}}(\lambda, \delta) \subset D(\lambda, p(\lambda))$. Then
\[
\int_{0}^{\delta} \frac{n_{\mathbb{H}}(\lambda, s) - m_{\lambda}}{s} \, ds = \sum_{0 < |\lambda - \lambda| < \delta} m_{\lambda'} \log \frac{\delta}{|\lambda - \lambda'|} \lesssim \sum_{0 < |\lambda - \lambda| < \delta} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \lesssim \sum_{0 < |\lambda - \lambda| < p(\lambda)} m_{\lambda'} \log \frac{p(\lambda)}{|\lambda - \lambda'|} \lesssim \sum_{0 < |\lambda - \lambda| < p(\lambda)} N(\lambda, p(\lambda)) \lesssim p(\lambda).
\]
For the remaining part we use condition (ii) in Theorem I and the estimate $\log t^{-1} \simeq 1 - t$ for $\delta < t < 1$. Taking $x = \Re \lambda$ we have
\[
\int_{\delta}^{1} \frac{n_{\mathbb{H}}(\lambda, s) - m_{\lambda}}{s} \, ds \lesssim \sum_{\lambda \neq \lambda'} \frac{|\Im \lambda||\Im \lambda'|}{|\lambda - \lambda'|^{2}} \lesssim \sum_{\lambda \neq \lambda'} \frac{|\Im \lambda||\Im \lambda'|}{|x - \lambda'|^{2}} \lesssim |\Im \lambda| \simeq p(\lambda).
\]

**References**


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