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SHERALI–ADAMS RELAXATIONS AND INDISTINGUISHABILITY IN COUNTING LOGICS*

ALBERT ATSERIAS† AND ELITZA MANEVA‡

Abstract. Two graphs with adjacency matrices $\mathbf{A}$ and $\mathbf{B}$ are isomorphic if there exists a permutation matrix $\mathbf{P}$ for which the identity $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{B}$ holds. Multiplying through by $\mathbf{P}$ and relaxing the permutation matrix to a doubly stochastic matrix leads to the linear programming relaxation known as fractional isomorphism. We show that the levels of the Sherali–Adams (SA) hierarchy of linear programming relaxations applied to fractional isomorphism interleave in power with the levels of a well-known color-refinement heuristic for graph isomorphism called the Weisfeiler–Lehman algorithm, or, equivalently, with the levels of indistinguishability in a logic with counting quantifiers and a bounded number of variables. This tight connection has quite striking consequences. For example, it follows immediately from a deep result of Grohe in the context of logics with counting quantifiers that a fixed number of levels of SA suffice to determine isomorphism of planar and minor-free graphs. We also offer applications in both finite model theory and polyhedral combinatorics. First, we show that certain properties of graphs, such as that of having a flow circulation of a prescribed value, are definable in the infinitary logic with counting with a bounded number of variables. Second, we exploit a lower bound construction due to Cai, Fürer, and Immerman in the context of counting logics to give simple explicit instances that show that the SA relaxations of the vertex-cover and cut polytopes do not reach their integer hulls for up to $\Omega(n)$ levels, where n is the number of vertices in the graph.

Key words. first-order logic, counting quantifiers, linear programming, Weisfeiler–Lehman algorithm, graph isomorphism, combinatorial optimization

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1. Introduction.

Let $\mathbf{A}$ and $\mathbf{B}$ be the adjacency matrices of two labeled graphs on $\{1, \dots, n\}$. The two graphs being isomorphic is equivalent to the existence of a permutation matrix $\mathbf{P}$ for which the relation $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{B}$ holds. Multiplying both sides by $\mathbf{P}$ gives the equivalent condition $\mathbf{AP} = \mathbf{PB}$. At this point a linear programming relaxation suggests itself: relax the condition that $\mathbf{P}$ is a permutation matrix to a doubly stochastic matrix. How much coarser is this than actual isomorphism?

The concept of fractional isomorphism as defined in the preceding paragraph falls within the framework of linear programming relaxations of combinatorial problems. Other types of relaxations of isomorphism include the color-refinement method called the Weisfeiler–Lehman (WL) algorithm. In this algorithm the vertices of the graphs are classified according to their degree, then according to the multiset of degrees of their neighbors, and so on until a fixed point is achieved. If the two graphs get partitions with different parameters, the graphs are definitely not isomorphic. As it turns out, fractional isomorphism and color refinement yield one and the same relaxation: it was shown by Ramana, Scheinerman, and Ullman [31] that two graphs

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†Universitat Politècnica de Catalunya, 08034 Barcelona, Spain (atserias@lsi.upc.edu).

‡Universitat de Barcelona, 08007 Barcelona, Spain (elitza.maneva@gmail.com). This author's research was supported in part by MICINN Ramon y Cajal.

are fractionally isomorphic if and only if they are not distinguished by the color-refinement algorithm.

Despite its simplicity, the color-refinement algorithm is known to behave very well in practice and is in fact one of the most commonly used heuristics for isomorphism testing. An example adding support to this claim is a classical result of Babai, Erdős, and Selkow [4] showing that the color-refinement algorithm will end up distinguishing a randomly chosen graph from any other graph with high probability. That said, one obvious limitation of the method is that it will fail badly on regular graphs, as in such a case the algorithm cannot even start. To remedy this, the WL algorithm has been extended to refinement of colorings of k-tuples of vertices (the k-WL algorithm) for $k = 1, 2, 3, \dots$, thus yielding a hierarchy of increasingly powerful relaxations of isomorphism. The power of the resulting algorithms has also been studied in depth. For example, Kucera [22] shows that the algorithm for $k = 2$ decides isomorphism almost surely on random regular graphs. Another example of a quite different nature is the result of Grohe showing that there exists a fixed constant k for which the k-WL algorithm is able to distinguish any pair of nonisomorphic planar graphs [13]. This was extended recently to any nontrivial minor-closed class of graphs [15].

Hierarchies of relaxations such as the k-WL algorithm can also be considered in the context of fractional isomorphism through linear programming. The theory of lift-and-project methods in the mathematical programming literature provides such a framework. These are methods by which an initial relaxation P of an integral polytope $P^{\mathbb{Z}}$ is tightened into sharper and sharper polytopes, thus forming a hierarchy of relaxations:

$$P = P^1 \supseteq P^2 \supseteq \cdots \supseteq P^{\mathbb{Z}}.$$

Examples of these include the hierarchy of linear programming relaxations proposed by Sherali and Adams [35], that by Lovász and Schrijver [27], and their semidefinite programming versions, including the hierarchy of Lasserre [23]. See [24] for a survey and comparison. These have been applied for studying classical polytopes of combinatorial optimization such as the stable-set polytope, the cut polytope, and the matching polytope, among others [27, 24, 36, 28].

In this paper we show that for $k \geq 2$, the kth level of the Sherali–Adams (SA) hierarchy relaxation of graph isomorphism is sandwiched between the $(k-1)$-tuple version of the WL algorithm and its k-tuple version. What this means is that if two graphs are distinguishable by the $(k-1)$-WL algorithm, then the kth level of SA vanishes, and that if the kth level of SA vanishes, then they are distinguishable by the k-WL algorithm. Thus, the k-WL algorithm provides a combinatorial characterization of the power of this lift-and-project method applied to graph isomorphism. We call this sandwiching property the Transfer Lemma.

1.1. Consequences. The Transfer Lemma, in combination with the abovementioned strong results about the power of the WL algorithm, already has consequences for the graph isomorphism problem itself. For example, it follows directly from Grohe's results that there exists a fixed level of SA relaxations that becomes empty on any pair of nonisomorphic planar graphs. Quite remarkably, the proof of Grohe's result relies on the interpretation of the k-WL algorithm as deciding indistinguishability in a certain counting logic called $\mathrm{C}^{k+1}_{\infty\omega}$, which does not seem to be related to linear programming relaxations.

Less immediate applications of the Transfer Lemma arise from the link it sets between two different areas: polyhedral combinatorics through lift-and-project methods and finite model theory through the counting logics $\mathrm{C}^k_{\infty\omega}$. We offer applications going in both directions.

First, we exploit known results in polyhedral combinatorics to show that several properties of graphs are definable in the logic $\mathrm{C}^k_{\infty\omega}$, the infinitary logic with counting quantifiers and k variables, for an appropriate constant k. These properties include "having a matching of a given size in bipartite graphs" and "having an st-flow of a given value in networks with unit capacities." While the definability of the first follows also from a result by Blass, Gurevich, and Shelah [6] and is not strictly new, the second strengthens it and is new; see the section on related work for more on this.

As a second application we export the inexpressibility results due to Cai, Fürer, and Immerman [7] in the context of counting logics to get instances with fractional solutions in the context of SA relaxations. From the existence of two nonisomorphic n-vertex graphs of bounded degree that remain indistinguishable by the logic $\mathrm{C}^k_{\infty\omega}$ up to $k = \Omega(n)$, we get explicit instances of the max-cut and vertex-cover problems whose linear programming relaxations do not reach their integer hulls after $\Omega(n)$ levels of SA. Let us note that in both cases stronger results are known: Schoenebeck [33] proved that a nontrivial integrality gap for vertex-cover resists $\Omega(n)$ levels of the Lasserre hierarchy, and hence of the SA hierarchy, and similar techniques would apply to max-cut. At any rate, the point we are trying to make is not to get the strongest possible results, but to illustrate the power that the Transfer Lemma gives for exporting methods from one field into the other.

Both these applications of the Transfer Lemma make use of a general statement we prove about the preservation of solutions between *k-local linear programs*: if two graphs have a nonempty k-level SA polytope of fractional isomorphisms, our result implies that solutions to the linear program of one graph translate to solutions of the linear program of the other. As it turns out, the relaxations of vertex-cover and max-cut and their SA levels are all local in our sense.

1.2. Related work. For the origins of fractional isomorphism see the references in the monograph [32]. The connection between fractional isomorphism and the color-refinement algorithm for vertices was made in [31]. The extension to the levels of the SA hierarchy and to the tuple version of the WL algorithm and the logic with counting quantifiers is, to our knowledge, new. A subsequent work of Grohe and Otto [16] improves our result by showing, among other things, that a mixture of the kth and $(k+1)$st levels exactly captures the k-WL algorithm, and that neither of the pure levels does. This shows that the gap in our Transfer Lemma cannot be avoided.

The logics $\mathrm{C}^k_{\infty\omega}$ are well studied in finite model theory [9, 26]. The connection between indistinguishability in these logics and the tuple version of the WL algorithm is from [18]. Despite the negative results from [7], the expressive power of these logics is still the object of study. Somewhat unexpectedly, it was shown in [6] that the property of having a perfect matching in bipartite graphs is expressible in the uniform version of $\mathrm{C}^k_{\infty\omega}$ called $\mathrm{IFP} + \mathrm{C}$. Here we revisit matchings in bipartite graphs and consider the more general problems of st-flows in networks with unit capacities. Our results show that the existence of such flows with prescribed values is expressible in $\mathrm{C}^3_{\infty\omega}$. Our techniques and those in [6] are completely different. The open problem from [6] about the definability of perfect matchings in general graphs in $\mathrm{C}^k_{\infty\omega}$, for some $k \geq 0$, stays open.

Lift-and-project methods for combinatorial optimization problems have been the object of intense study. An optimal integrality gap of 2 for vertex-cover was shown to resist $\Omega(\log n)$ levels of the Lovász–Schrijver (LS) hierarchy in [1]. This was later improved in [37, 34, 12] to more levels and to the semidefinite version LS+. For the SA hierarchy, it was shown in [8] that optimal gaps of 2 for vertex-cover and max-cut resist $n^{\Omega(1)}$ levels. For vertex-cover, a gap of 7/6 resists $\Omega(n)$ levels of Lasserre and hence of SA [33], and a gap of 1.36 resists $n^{\Omega(1)}$ levels of Lasserre [38]. For max-cut, we could not find any published lower bound on the SA rank, but Schoenebeck informs us that his methods would yield a nontrivial gap for up to $\Omega(n)$ levels of Lasserre and hence SA. See also [25] for related results.

2. Preliminaries. In this section we define SA relaxations of 0-1 integer linear programs and the basic definitions about counting logics and their corresponding pebble games.

2.1. SA relaxations and fractional isomorphism. Let $P \subseteq [0,1]^n$ be a polytope of the form

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{1}\}$$

for a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a column vector $\mathbf{b} \in \mathbb{R}^m$. We write $P^{\mathbb{Z}}$ for the convex hull of the 0-1 vectors in P. The sequence of SA relaxations of $P^{\mathbb{Z}}$ is a sequence of polytopes $P^1 \supseteq P^2 \supseteq \cdots$ starting at $P^1 = P$ and each containing $P^{\mathbb{Z}}$. The kth polytope P^k is defined in three steps.

In the first step, each defining inequality $\mathbf{a}^{\mathrm{T}}\mathbf{x} \geq b$ of P is multiplied by all possible terms of the form

$$\prod_{i \in I} x_i \prod_{j \in J} (1 - x_j),$$

where I and J are subsets of $[n]$ such that $|I \cup J| \leq k-1$ and $I \cap J = \emptyset$. This leaves a system of polynomial inequalities, each of degree at most k. In the second step, the system is *linearized* and hence relaxed: each square x_i^2 is replaced by x_i, each resulting monomial of the form $\prod_{i \in K} x_i$ is replaced by a new variable y_K, and the constraint $y_\emptyset = 1$ is added. The result is a system of linear inequalities defining a polytope P_L^k in $\mathbb{R}^{n_k}$ for $n_k = \sum_{i=0}^{k} \binom{n}{i}$. In the third step, the polytope is projected back to n dimensions by defining

$$P^k := \{\mathbf{x} \in \mathbb{R}^n : \text{ there exists } \mathbf{y} \in P_L^k \text{ such that } y_{\{i\}} = x_i \text{ for every } i \in [n]\}.$$

The polytope P^k is called the kth level SA relaxation of $P^{\mathbb{Z}}$. It is not hard to see that $P^k \supseteq P^{\mathbb{Z}}$. Indeed, the integer hull of P is achieved not later than after n steps [35]:

$$P = P^1 \supseteq P^2 \supseteq \cdots \supseteq P^n = P^{\mathbb{Z}}.$$

Thus, the SA hierarchy provides a sequence of tighter and tighter relaxations of the integral polytope $P^{\mathbb{Z}}$. The smallest k for which $P^k = P^{\mathbb{Z}}$ is called the SA *rank* of the polytope P.

Let us specialize this construction to the polytope defining fractional isomorphisms. Let $\mathbf{A} = (A, E^{\mathbf{A}}, C_1^{\mathbf{A}}, \ldots, C_r^{\mathbf{A}})$ and $\mathbf{B} = (B, E^{\mathbf{B}}, C_1^{\mathbf{B}}, \ldots, C_r^{\mathbf{B}})$ be colored directed graphs, i.e., $E^{\mathbf{A}} \subseteq A^2$ and $C_i^{\mathbf{A}} \subseteq A$ for $i \in [r]$, and the same for $\mathbf{B}$. Although it is not a very important point, note that we do not require the color classes given by $C_1, \ldots, C_r$ to be disjoint. Let $(A_{a,a'})_{a,a' \in A}$ and $(B_{b,b'})_{b,b' \in B}$ be the adjacency matrices of $\mathbf{A}$ and $\mathbf{B}$, which we also denote by $\mathbf{A}$ and $\mathbf{B}$. Let $(C_{a,c})_{a \in A, c \in [r]}$

and $(D_{b,d})_{b\in B,d\in[r]}$ be the 0-1 matrices that encode the colors, which we write as $\mathbf{C}$ and $\mathbf{D}$. For every pair $(a,b)\in A\times B$, let $X_{a,b}$ be a variable. Let $\mathbf{X}$ be the $|A|\times|B|$ matrix $(X_{a,b})_{a\in A,b\in B}$. The fractional relaxation of isomorphism is the following system of linear equalities and inequalities:

$$
\begin{array}{ll}
\mathbf{AX}=\mathbf{XB}, & \mathbf{C}=\mathbf{XD},\\
\mathbf{BX}^{\mathrm{T}}=\mathbf{X}^{\mathrm{T}}\mathbf{A}, & \mathbf{D}=\mathbf{X}^{\mathrm{T}}\mathbf{C},\\
\mathbf{Xe}=1, & \mathbf{X}^{\mathrm{T}}\mathbf{e}=1,\\
\multicolumn{2}{c}{\mathbf{X}\geq 0.}
\end{array}
\tag{1}
$$

We write $F(\mathbf{A},\mathbf{B})$ for this linear program. Note that if $\mathbf{A}$ and $\mathbf{B}$ are undirected graphs, then $\mathbf{A}=\mathbf{A}^{\mathrm{T}}$ and $\mathbf{B}=\mathbf{B}^{\mathrm{T}}$, and the equations $\mathbf{AX}=\mathbf{XB}$ and $\mathbf{BX}^{\mathrm{T}}=\mathbf{X}^{\mathrm{T}}\mathbf{A}$ in $F(\mathbf{A},\mathbf{B})$ become equivalent. In the general case of colored directed graphs, the equation $\mathbf{BX}^{\mathrm{T}}=\mathbf{X}^{\mathrm{T}}\mathbf{A}$ is added for symmetry purposes.

For every integer $k\geq 0$, let R_k denote the collection of all subsets $p\subseteq A\times B$ with $|p|\leq k$. For $p\in R_k$ and $(a,b)\in A\times B$, we use the notation $p\cup ab$ as an abbreviation for $p\cup\{(a,b)\}$. For every $p\in R_k$, let X_p be a variable. If $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_n\}$, the kth level of SA applied on $F(\mathbf{A},\mathbf{B})$ is equivalent to the system given by the following equalities and inequalities:

$$A_{a,a_1}X_{q\cup a_1b}+\cdots+A_{a,a_n}X_{q\cup a_nb}=X_{q\cup ab_1}B_{b_1,b}+\cdots+X_{q\cup ab_n}B_{b_n,b}, \tag{2}$$

$$B_{b,b_1}X_{q\cup ab_1}+\cdots+B_{b,b_n}X_{q\cup ab_n}=X_{q\cup a_1b}A_{a_1,a}+\cdots+X_{q\cup a_nb}A_{a_n,a} \tag{3}$$

together with

$$X_{q\cup ab_1}D_{b_1,c}+\cdots+X_{q\cup ab_n}D_{b_n,c}=X_qC_{a,c}, \tag{4}$$

$$X_{q\cup a_1b}C_{a_1,c}+\cdots+X_{q\cup a_nb}C_{a_n,c}=X_qD_{b,c}, \tag{5}$$

$$X_{q\cup ab_1}+\cdots+X_{q\cup ab_n}=X_q, \tag{6}$$

$$X_{q\cup a_1b}+\cdots+X_{q\cup a_nb}=X_q, \tag{7}$$

$$X_{q\cup ab}\geq 0, \tag{8}$$

$$X_\emptyset=1 \tag{9}$$

for $a\in A$, $b\in B$, $c\in[r]$, and q an element of R_{k-1}. We obtained these inequalities by multiplying each equation in (1) by a term of the form $\prod_{ab\in I}X_{ab}$ for $I\subseteq A\times B$ with $|I|\leq k-1$ and linearizing. Note that the factors of the form $\prod_{ab\in J}(1-X_{ab})$ that appear to be missing here are really implicit, as the resulting equations can be obtained as linear combinations of those given. This holds in this special case since all relevant constraints are equalities instead of inequalities. We write $F_k(\mathbf{A},\mathbf{B})$ for this system. Note that $k=1$ gives $F(\mathbf{A},\mathbf{B})$. If $F_k(\mathbf{A},\mathbf{B})$ is satisfiable, we write $\mathbf{A}\equiv^k_{\mathrm{SA}}\mathbf{B}$.

2.2. Counting logics and pebble games. A *counting quantifier* has the form $\exists^{\geq m}x\phi$, where m is a nonnegative integer. The meaning is that "there exist at least m distinct x satisfying ϕ." For example, the formula

$$\forall x(\exists^{\geq d}y(E(x,y))\wedge\neg\exists^{\geq d+1}y(E(x,y)))$$

says of a graph that it is d-regular, and it does so using exactly two variables. For the rest of the paper, let $\mathrm{C}^k_{\infty\omega}$ denote the collection of all formulas made from atomic formulas and equalities by means of finitary and infinitary conjunctions, negations, and standard and counting quantifiers, using at most k different variables. For more background on $\mathrm{C}^k_{\infty\omega}$ we refer the reader to [29, 7, 9].

An essential concept from logic is that of indistinguishability by the formulas of a logical language. We say that two structures $\mathbf{A}$ and $\mathbf{B}$ are $\mathrm{C}^k_{\infty\omega}$-indistinguishable if every $\mathrm{C}^k_{\infty\omega}$-sentence that is true in $\mathbf{A}$ is also true in $\mathbf{B}$, and vice versa. This defines an equivalence relation on the class of structures that we write as $\mathbf{A} \equiv^k_{\mathrm{C}} \mathbf{B}$. This indistinguishability relation has an alternative interpretation in terms of a two-player game. For first-order logic these sorts of games go back to Ehrenfeucht [10] and Fraïssé [11]. For the logic $\mathrm{C}^k_{\infty\omega}$ we follow [7, 17], but see also [20].

We define the game for $\equiv^k_{\mathrm{C}}$. Let $\mathbf{A}$ and $\mathbf{B}$ be colored directed graphs as before. Let $(\mathbf{a}, \mathbf{b})$ be a pair of k-tuples, where $\mathbf{a} = (a_1, \ldots, a_k)$ has $a_i \in A \cup \{\star\}$, and $\mathbf{b} = (b_1, \ldots, b_k)$ has $b_i \in B \cup \{\star\}$. We say that $(\mathbf{a}, \mathbf{b})$ defines a partial k-isomorphism from A to B if the following conditions hold for every $i \in [k]$, every $j \in [k]$, and every $c \in [r]$:

1. $a_i = \star$ if and only if $b_i = \star$;
2. $a_i = a_j$ if and only if $b_i = b_j$;
3. $(a_i, a_j) \in E^{\mathbf{A}}$ if and only if $(b_i, b_j) \in E^{\mathbf{B}}$;
4. $a_i \in C^{\mathbf{A}}_c$ if and only if $b_i \in C^{\mathbf{B}}_c$.

The game is played by two players: Spoiler and Duplicator. The goal of Spoiler is to show a difference between $\mathbf{A}$ and $\mathbf{B}$. The goal of Duplicator is to hide such a difference. There are $2k$ pebbles matched in pairs, initially off the board. In each round, Spoiler can remove a pair of matched pebbles off the board or choose such a pair of pebbles to play. Let us say he chooses the ith pair to play. Then he chooses a structure, $\mathbf{A}$ or $\mathbf{B}$, and a subset X of the universe of that structure. In response, Duplicator must choose a subset Y of the universe of the other structure such that $|Y| = |X|$; if she cannot even do that, she loses immediately. To complete the round, Spoiler places one of the pebbles of the ith pair over an element of Y, and in response Duplicator places the other pebble of the ith pair over an element of X. At the end of the round the sets X and Y are forgotten, but the pebbles are retained on the board. Spoiler wins if at any round the correspondence between pebbles $a_i \mapsto b_i$ for $i = 1, \ldots, k$ is not a partial isomorphism between $\mathbf{A}$ and $\mathbf{B}$ (if the pair i is off the board, then $a_i = b_i = \star$). We say that Duplicator has a winning strategy if she has a strategy to keep playing forever.

Formally, winning strategies are defined through back-and-forth systems as follows. For a k-tuple $\mathbf{a} = (a_1, \ldots, a_k)$, an index $i \in [k]$, and an element a, we write $\mathbf{a}[i/a]$ for the result of replacing the ith component of $\mathbf{a}$ by a. A *winning strategy for the Duplicator in the k-pebble game on* $\mathbf{A}$ *and* $\mathbf{B}$ is a nonempty $\mathcal{F} \subseteq (A \cup \{\star\})^k \times (B \cup \{\star\})^k$ such that every $(\mathbf{a}, \mathbf{b})$ in $\mathcal{F}$ defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{B}$ and for every $i \in [k]$ the following properties are satisfied:

1. $(\mathbf{a}[i/\star], \mathbf{b}[i/\star])$ belongs to $\mathcal{F}$;
2. for every $X \subseteq A$ there exists $Y \subseteq B$ with $|Y| = |X|$ such that for every $b \in Y$ there exists $a \in X$ such that $(\mathbf{a}[i/a], \mathbf{b}[i/b])$ belongs to $\mathcal{F}$;
3. for every $Y \subseteq B$ there exists $X \subseteq A$ with $|X| = |Y|$ such that for every $a \in X$ there exists $b \in Y$ such that $(\mathbf{a}[i/a], \mathbf{b}[i/b])$ belongs to $\mathcal{F}$.

The first is called the *subtuple property*, the second is the *forth property*, and the third is the *back property*. If there exists such a strategy, we write $\mathbf{A} \equiv^k_{\mathrm{C}} \mathbf{B}$. It is a theorem that this notion agrees with indistinguishability by the logic $\mathrm{C}^k_{\infty\omega}$ [7]. A way to decide whether such a strategy exists is by running the k-dimensional WL algorithm (see the next section), which runs in time polynomial in $|A|^k + |B|^k$. For the statement and a proof of correctness see [7, 30].

2.3. WL algorithm. One way to determine whether $\mathbf{A} \equiv_{\mathrm{C}}^{k+1} \mathbf{B}$ is by running the k-tuple WL algorithm on each structure and checking whether the resulting parameters match. Let us now give the details of the algorithm. This exposition follows [7].

The k-WL algorithm run on $\mathbf{A}$ starts with all k-tuples of elements of $\mathbf{A}$ classified into bags labeled by the isomorphism type that the tuples induce on $\mathbf{A}$, where the isomorphism type induced by a k-tuple $(a_1, \ldots, a_k)$ is the collection of all atomic formulas on the variables $x_1, \ldots, x_k$ that are satisfied by the assignment $x_1 = a_1, \ldots, x_k = a_k$. (The atomic formulas are the formulas of the form $x_i = x_j$ or $E(x_i, x_j)$ or $C_c(x_i)$ for some $c \in [r]$.) At each iteration, the algorithm cycles through all possible k-tuples $(a_1, \ldots, a_k)$ and counts, for each isomorphism type of $(k+1)$-tuples T and each k-tuple of bags $(B_1, \ldots, B_k)$, the number of $a \in A$ for which the $(k+1)$-tuple $(a_1, \ldots, a_k, a)$ induces on $\mathbf{A}$ a substructure of isomorphism type T, and the k-tuple $(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_k)$ belongs to the bag B_i for every $i \in [k]$. Once these counts are over, it refines each bag of tuples into subbags labeled by the outcomes of these counts. When no further splitting is possible, the algorithm stops. To avoid the size of the labels increasing exponentially, after each iteration the bags are ordered in some standard way (lexicographically by their labels, say) and relabeled by their position in this order. The parameters of the output are the counts that result at the final collection of bags. Note, by the way, that the splitting process must finish after no more than $|A|^k$ iterations since whenever a bag contains a single tuple it cannot split any further. When the k-WL algorithm is run on both $\mathbf{A}$ and $\mathbf{B}$, we say that the parameters match if the parameters of their outputs are the same. The claim is that for $k \geq 1$, it holds that $\mathbf{A} \equiv_{\mathrm{C}}^{k+1} \mathbf{B}$ if and only if the parameters match when the k-WL algorithm is run on $\mathbf{A}$ and $\mathbf{B}$. For a proof see [7, 30].

There is one subtle difference between our definition of the k-WL algorithm and the definition in [7] that is nonetheless relevant only if $k = 1$. The difference is that we introduce isomorphism types of $(k+1)$-tuples into the counts. In the case $k \geq 2$ it can be seen that these counts are redundant since the maximum arity of the relations in $\mathbf{A}$ is 2. The good news is that our definition unifies the algorithm and its proof of correctness for the cases $k = 1$ and $k > 1$. In contrast the definition in [7] required splitting into cases. Also the generality of working with isomorphism types is necessary for dealing with directed graphs (in the case $k = 1$). Our definition of k-WL appeared first in [14].

3. Transfer Lemma. At this point we have provided all the necessary background to state the main result.

THEOREM 1 (Transfer Lemma). *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be colored directed graphs, and let* $k \geq 1$. *Then*

$$\mathbf{A} \equiv_{\mathrm{SA}}^{k+1} \mathbf{B} \quad \Longrightarrow \quad \mathbf{A} \equiv_{\mathrm{C}}^{k+1} \mathbf{B} \quad \Longrightarrow \quad \mathbf{A} \equiv_{\mathrm{SA}}^{k} \mathbf{B}.$$

For $k = 1$ the second implication can be reversed; that is, $\mathbf{A} \equiv_{\mathrm{SA}}^{1} \mathbf{B}$ is equivalent to $\mathbf{A} \equiv_{\mathrm{C}}^{2} \mathbf{B}$. Indeed, $\equiv_{\mathrm{SA}}^{1}$ is just fractional isomorphism as discussed in the introduction and $\equiv_{\mathrm{C}}^{2}$ is known to be equivalent to 1-WL (see [18]). Thus, the equivalence between $\equiv_{\mathrm{SA}}^{1}$ and $\equiv_{\mathrm{C}}^{2}$ is the result from [31], which was the starting point for our work. For $k > 1$ we cannot show such an equivalence, and indeed subsequent work [16] shows that, in general, it does not hold.

The proof will proceed by showing a longer chain of implications that involves two more notions of indistinguishability: $\equiv_{\mathrm{EP}}^{k}$ is an equivalence relation that extends the combinatorial notion known as "equitable partitions" (see [32]) to k-tuples, while $\equiv_{\mathrm{CS}}^{k}$

is another pebble game that we call the sliding game. The complete statement that we will prove is the following:

$$\mathbf{A} \equiv_{\mathrm{SA}}^{k+1} \mathbf{B} \Longrightarrow \mathbf{A} \equiv_{\mathrm{C}}^{k+1} \mathbf{B} \Longrightarrow \mathbf{A} \equiv_{\mathrm{CS}}^{k} \mathbf{B} \Longrightarrow \mathbf{A} \equiv_{\mathrm{EP}}^{k} \mathbf{B} \Longrightarrow \mathbf{A} \equiv_{\mathrm{SA}}^{k} \mathbf{B}.$$

We define $\equiv_{\mathrm{CS}}^{k}$ and $\equiv_{\mathrm{EP}}^{k}$ in the beginning of the next section and then proceed to the proof.

4. Proof of the Transfer Lemma.

For this section, let $\mathbf{A}$ and $\mathbf{B}$ be colored directed graphs, and let $k \geq 1$ be a natural number. To prove the Transfer Lemma we will prove the longer chain of implications referred to at the end of the previous section. Before that, we need to define the two new notions of indistinguishability.

4.1. Formal definition of the sliding game.

Intuitively, the sliding game is a variant of the pebble game in which the Spoiler is allowed to slide pebbles forward or backward along the edges of one of the directed graphs, and the Duplicator is required to slide the corresponding matched pebble in the same direction along the edges of the other graph. To formalize this we need some notation.

For a in $A \cup \{\star\}$, define $N^+(a)$ and $N^-(a)$ as follows:

1. $N^+(a) = \{a' \in A : (a, a') \in E^{\mathbf{A}}\}$ if $a \neq \star$;
2. $N^-(a) = \{a' \in A : (a', a) \in E^{\mathbf{A}}\}$ if $a \neq \star$;
3. $N^+(a) = N^-(a) = A$ if $a = \star$.

For b in $B \cup \{\star\}$, define $N^+(b)$ and $N^-(b)$ analogously.

A *winning strategy for the Duplicator in the k-pebble sliding game on* $\mathbf{A}$ *and* $\mathbf{B}$ is a nonempty $\mathcal{F} \subseteq (A \cup \{\star\})^k \times (B \cup \{\star\})^k$ such that every $(\mathbf{a}, \mathbf{b})$ in $\mathcal{F}$ defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{B}$ and for every $i \in [k]$ and every $o \in \{+, -\}$, the following properties are satisfied:

1. $(\mathbf{a}[i/\star], \mathbf{b}[i/\star])$ belongs to $\mathcal{F}$;
2. for every $X \subseteq N^o(a_i)$ there exists $Y \subseteq N^o(b_i)$ with $|Y| = |X|$ such that for every $b \in Y$ there exists $a \in X$ such that $(\mathbf{a}[i/a], \mathbf{b}[i/b])$ belongs to $\mathcal{F}$;
3. for every $Y \subseteq N^o(b_i)$ there exists $X \subseteq N^o(a_i)$ with $|X| = |Y|$ such that for every $a \in X$ there exists $b \in Y$ such that $(\mathbf{a}[i/a], \mathbf{b}[i/b])$ belongs to $\mathcal{F}$.

If there exists such a strategy, we write $\mathbf{A} \equiv_{\mathrm{CS}}^{k} \mathbf{B}$.

4.2. Analogue of equitable partition for tuples.

For an integer $k \geq 1$, we write S_k for the set of all permutations on $[k]$. For a permutation $\pi \in S_k$, we write $\mathbf{a} \circ \pi$ for the tuple $(a_{\pi(1)}, \ldots, a_{\pi(k)})$.

Let $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{a}' = (a'_1, \ldots, a'_k)$ be tuples in $(A \cup \{\star\})^k$. For every $i \in [k]$ and $o \in \{+, -\}$, define

$$d_i^o(\mathbf{a}, \mathbf{a}') = \begin{cases} 1 & \text{if } \mathbf{a} \neq \mathbf{a}' \text{ and there exists } a \in N^o(a_i) \cup \{\star\} \text{ such that } \mathbf{a}' = \mathbf{a}[i/a], \\ 0 & \text{otherwise.} \end{cases}$$

Note that $d_i^+(\mathbf{a}, \mathbf{a}') = d_i^-(\mathbf{a}', \mathbf{a})$. Let S and T be subsets of $(A \cup \{\star\})^k$. For every $i \in [k]$ and $o \in \{+, -\}$, define

$$d_i^o(S, T) = \sum_{\mathbf{a} \in S} \sum_{\mathbf{a}' \in T} d_i^o(\mathbf{a}, \mathbf{a}').$$

Note that $d_i^+(S, T) = d_i^-(T, S)$. If S is a singleton $\{\mathbf{a}\}$, we write $d_i^o(\mathbf{a}, T)$ instead of $d_i^o(\{\mathbf{a}\}, T)$. We call $d_i^+(\mathbf{a}, T)$ the *out-degree of* $\mathbf{a}$ *in* T *on its* i*th component* and $d_i^-(\mathbf{a}, T)$ the *in-degree of* $\mathbf{a}$ *in* T *on its* i*th component*.

Let $(P_1,\dots,P_s)$ be a partition of $(A\cup\{\star\})^k$ into nonempty parts. For every $\mathbf{a}\in(A\cup\{\star\})^k$, let $c(\mathbf{a})$ be the unique $m\in[s]$ such that $\mathbf{a}$ belongs to P_m. The partition $(P_1,\dots,P_s)$ is called a k-equitable partition of $\mathbf{A}$ if for every $m\in[s]$ and every $\mathbf{a},\mathbf{a}'\in P_m$, the following conditions hold:

1. $(\mathbf{a},\mathbf{a}')$ defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{A}$;
2. $c(\mathbf{a}[i/\star])=c(\mathbf{a}'[i/\star])$ for every $i\in[k]$;
3. $d_i^o(\mathbf{a},P_n)=d_i^o(\mathbf{a}',P_n)$ for every $i\in[k]$, $o\in\{+,-\}$, and $n\in[s]$;
4. $|P_{c(\mathbf{a})}|=|P_{c(\mathbf{a}\circ\pi)}|$ for every permutation $\pi\in S_k$;
5. $c(\mathbf{a}\circ\pi)=c(\mathbf{a}'\circ\pi)$ for every permutation $\pi\in S_k$.

By 3, we note that the following identity holds for every $m,n\in[s]$, $\mathbf{a}\in P_m$, $\mathbf{a}'\in P_n$, and $i\in[k]$:

$$|P_m|d_i^+(\mathbf{a},P_n)=d_i^+(P_m,P_n)=d_i^-(P_n,P_m)=|P_n|d_i^-(\mathbf{a}',P_m). \tag{10}$$

We say that $\mathbf{A}$ and $\mathbf{B}$ have a common k-equitable partition if there exist a k-equitable partition $(P_1,\dots,P_s)$ of $\mathbf{A}$ and a k-equitable partition $(Q_1,\dots,Q_t)$ of $\mathbf{B}$ such that the following conditions are satisfied:

1. $s=t$ and $|P_m|=|Q_m|$ for every $m\in[s]$;

and, for every $m\in[s]$, $\mathbf{a}\in P_m$, and $\mathbf{b}\in Q_m$, the following hold:

2. $(\mathbf{a},\mathbf{b})$ defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{B}$;
3. $c(\mathbf{a}[i/\star])=c(\mathbf{b}[i/\star])$ for every $i\in[k]$;
4. $d_i^o(\mathbf{a},P_n)=d_i^o(\mathbf{b},Q_n)$ for every $i\in[k]$, $o\in\{+,-\}$, and $n\in[s]$;
5. $c(\mathbf{a}\circ\pi)=c(\mathbf{b}\circ\pi)$ for every permutation $\pi\in S_k$.

If there exists a common k-equitable partition, we write $\mathbf{A}\equiv^k_{\mathrm{EP}}\mathbf{B}$.

4.3. From SA to pebble game. We show the first implication in the Transfer Lemma.

LEMMA 1. *Let $k\geq 2$. If $\mathbf{A}\equiv^k_{\mathrm{SA}}\mathbf{B}$, then $\mathbf{A}\equiv^k_{\mathrm{C}}\mathbf{B}$.*

Proof. Let $(X_p)_{p\in R_k}$ be a feasible solution for $F_k(\mathbf{A},\mathbf{B})$. Let $\mathcal{F}$ be the collection of all pairs of k-tuples $(\mathbf{a},\mathbf{b})\in(A\cup\{\star\})^k\times(B\cup\{\star\})^k$ for which the following two conditions are satisfied:

1. $a_i=\star$ if and only if $b_i=\star$ for every $i\in[k]$;
2. $p=\{(a_i,b_i):i\in[k],\ a_i\neq\star,\ b_i\neq\star\}$ satisfies $X_p\neq 0$.

Note that $\mathcal{F}$ is nonempty as the pair of k-tuples $(\star^k,\star^k)$ satisfies the two conditions since in this case $p=\emptyset$ and $X_\emptyset\neq 0$ by (9). We proceed to show that each $(\mathbf{a},\mathbf{b})$ in $\mathcal{F}$ defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{B}$ and that the subtuple and back-and-forth properties are satisfied. We start with the subtuple property.

CLAIM 1. *Let $p,q\in R_k$. If $q\subseteq p$, then $X_p\leq X_q$.*

Proof. Assume $q\subseteq p$. We proceed by induction on the cardinality of the difference $|p-q|$. If $|p-q|=0$, then $p=q$ and we are done. Assume $|p-q|>0$. Let $(a,b)\in p-q$, and define $p'=p-\{(a,b)\}$. Then $q\subseteq p'$ and $|p'-q|<|p-q|$. By (6) with $p'\in R_{k-1}$ we have

$$X_{p'}=\sum_{b'\in B}X_{p'\cup ab'}.$$

Since each term in the sum is nonnegative by (8), we get $X_{p'\cup ab}\leq X_{p'}$. Since $p'\cup ab=p$, the inequality $X_p\leq X_q$ follows from the induction hypothesis $X_{p'}\leq X_q$. ∎

Before we continue we need a definition. Let $p=\{(a_1,b_1),\dots,(a_s,b_s)\}\in R_k$, with $s\leq k$ and $(a_h,b_h)\neq(a_\ell,b_\ell)$ for every $h,\ell\in[s]$, $h\neq\ell$. We say that p is a partial (k-)isomorphism from $\mathbf{A}$ to $\mathbf{B}$ if the following conditions are satisfied for every $h, \ell \in [s]$ and $c \in [r]$:

1. if $a_h = a_\ell$, then $b_h = b_\ell$ (and hence $h = \ell$);
2. if $b_h = b_\ell$, then $a_h = a_\ell$ (and hence $h = \ell$);
3. if $A_{a_h,a_\ell} = 1$, then $B_{b_h,b_\ell} = 1$;
4. if $B_{b_h,b_\ell} = 1$, then $A_{a_h,a_\ell} = 1$;
5. if $C_{a_h,c} = 1$, then $D_{b_h,c} = 1$;
6. if $D_{b_h,c} = 1$, then $C_{a_h,c} = 1$.

With this definition we are ready to state the second property of the solutions to $F_k(\mathbf{A}, \mathbf{B})$.

CLAIM 2. *Let $p \in R_k$. If $X_p \neq 0$, then p is a partial isomorphism from $\mathbf{A}$ to $\mathbf{B}$.*

Proof. Assume $X_p \neq 0$. Let $p = \{(a_1, b_1), \ldots, (a_s, b_s)\}$, with $s \leq k$ and $(a_h, b_h) \neq (a_\ell, b_\ell)$ for every $h, \ell \in [s]$, $h \neq \ell$. We need to check all six conditions in the definition of partial k-isomorphism above.

For 1, assume for contradiction that $a_h = a_\ell$ and $b_h \neq b_\ell$. Let $q = p - \{(a_\ell, b_\ell)\}$, and note that $q \in R_{k-1}$. From (6) for this q and $a = a_h$ we get

$$X_{q \cup a_h b_h} = X_q - \sum_{\substack{b \in B \\ b \neq b_h}}^{n} X_{q \cup a_h b}.$$

Since (a_h, b_h) belongs to q, we have $q \cup a_h b_h = q$ and therefore

$$\sum_{\substack{b \in B \\ b \neq b_h}}^{n} X_{q \cup a_h b} = 0.$$

Each term in the sum is nonnegative by (8), and hence each is 0. In particular, either $h = \ell$ and then we are done, or $X_{q \cup a_h b_\ell} = 0$. But $a_h = a_\ell$ and $q \cup a_\ell b_\ell = p$, and hence $X_p = 0$, a contradiction.

For 2, argue as in 1 using (7) for $q = p - \{(a_h, b_h)\}$ and $b = b_\ell$.

For 3, assume for contradiction that $A_{a_h,a_\ell} = 1$ and $B_{b_h,b_\ell} = 0$. Let $q = \{(a_h, b_h)\}$. Note that $q \in R_1 \subseteq R_{k-1}$ since $k \geq 2$. From (2) for this q, $a = a_h$, and $b = b_\ell$ we get

$$X_{q \cup a_\ell b_\ell} = \sum_{b \in B} X_{q \cup a_h b} B_{b,b_\ell} - \sum_{\substack{a \in A \\ a \neq a_\ell}} A_{a_h,a} X_{q \cup a b_\ell}. \tag{11}$$

Since (a_h, b_h) belongs to q, by part 1 of this lemma we have $X_{q \cup a_h b} = 0$ whenever $b \neq b_h$. Moreover, whenever $b = j_h$ we have $B_{b,b_\ell} = 0$ by assumption. Both things together mean that the first sum in (11) vanishes. Since every term in the second sum in that same equation is nonnegative by (8), we get $X_{q \cup a_\ell b_\ell} \leq 0$. Since $q \cup a_\ell b_\ell \subseteq p$, by Claim 1 we get $X_p \leq 0$. But also $X_p \geq 0$ by (8), so $X_p = 0$, a contradiction.

For 4, argue as in 3 using part 2 of this lemma.

For 5, assume for contradiction that $C_{a_h,c} = 1$ and $D_{b_h,c} = 0$. Let $q = p - \{(a_h, b_h)\}$. Note that $q \in R_{k-1}$. From (4) for this q and $a = a_h$ we get

$$X_q C_{a_h,c} = \sum_{b \in B} X_{q \cup a_h b} D_{b,c} \leq X_{q \cup a_h b_h} D_{b_h,c}. \tag{12}$$

But then the conditions $C_{a_h,c} = 1$ and $D_{b_h,c} = 0$ imply that $X_q \leq 0$. Since $q \subseteq p$, we get $X_p \leq 0$ from Claim 1, and hence $X_p = 0$, a contradiction.

For 6, argue as in 5 using (5) for the same q and $b = b_h$. ∎

The next claim states the forth property.

CLAIM 3. *Let $q \in R_{k-1}$. If $X_q \neq 0$, then for every $X \subseteq A$ there exists $Y \subseteq B$ with $|Y| = |X|$ such that for every $b \in Y$ there exists $a \in X$ such that $X_{q \cup ab} \neq 0$.*

Proof. Assume $X_q \neq 0$. For every $(a, b) \in A \times B$, define $Y_{a,b} = X_{q\cup ab}/X_q$ and let $\mathbf{Y}$ be the $|A| \times |B|$ matrix $(Y_{a,b})_{a\in A,b\in B}$. Equations (6), (8), and (9) imply that $\mathbf{Y}$ is a doubly stochastic matrix. Therefore, by the Birkhoff–von Neumann theorem, $\mathbf{Y}$ is the convex combination of one or more permutation matrices: $\mathbf{Y} = \sum_{t=1}^{r} \alpha_t \Pi_t$, with $r \geq 1$ and $\alpha_t > 0$ for every $t \in \{1, \ldots, r\}$. Let π be the permutation underlying Π_1 interpreted like a bijection from A to B. For every $X \subseteq A$, define $Y = \pi(X)$. Obviously $|Y| = |X|$. Moreover, for every $b \in Y$, choose $a = \pi^{-1}(b) \in X$ and check that

$$Y_{a,b} = \sum_{t=1}^{r} \alpha_t \Pi_t(a, b) \geq \alpha_1 \Pi_1(a, b) = \alpha_1 > 0.$$

This implies $X_{q\cup ab} \neq 0$, and we are done. ∎

The final claim states the back property.

CLAIM 4. *Let $q \in R_{k-1}$. If $X_q \neq 0$, then for every $Y \subseteq B$ there exists $X \subseteq A$ with $|X| = |Y|$ such that for every $a \in X$ there exists $b \in Y$ such that $x_{q\cup ab} \neq 0$.*

Proof. This proof is as in Claim 3; reverse the roles of X and Y, and a and b. ∎

These claims complete the proof of the lemma. ∎

4.4. From pebble game to sliding game. We show that if the Duplicator has a winning strategy in the nonsliding game with $k + 1$ pebbles, then she also has a winning strategy in the sliding game with k pebbles. Intuitively, the idea is that the Duplicator can use her strategy in the nonsliding game to simulate the moves of the sliding game by pretending that the Spoiler makes restricted use of pebble $k + 1$.

More precisely, if Spoiler slides pebble $i \in [k]$ from a to a' in the sliding game, then Duplicator pretends that Spoiler actually does the following: place pebble $k + 1$ on a' to force the sliding condition on the Duplicator side, then move pebble i from a to a' to actually get the move done, and finally remove pebble $k + 1$ off the board to leave it free for the next move. We make this argument formal in the next lemma.

LEMMA 2. *Let $k \geq 1$. If $\mathbf{A} \equiv_{\mathrm{C}}^{k+1} \mathbf{B}$, then $\mathbf{A} \equiv_{\mathrm{CS}}^{k} \mathbf{B}$.*

Proof. Let $\mathcal{F}$ be a winning strategy witnessing that $\mathbf{A} \equiv_{\mathrm{C}}^{k+1} \mathbf{B}$. Let $\mathcal{H}$ be the collection of all pairs of k-tuples $(\mathbf{a}', \mathbf{b}')$, with $\mathbf{a}' = (a'_1, \ldots, a'_k) \in (A \cup \{\star\})^k$ and $\mathbf{b}' = (b'_1, \ldots, b'_k) \in (B \cup \{\star\})^k$, for which there exists $(\mathbf{a}, \mathbf{b})$ in $\mathcal{F}$, with $\mathbf{a} = (a_1, \ldots, a_{k+1})$ and $\mathbf{b} = (b_1, \ldots, b_{k+1})$, such that $a_i = a'_i$ and $b_i = b'_i$ for every $i \in [k]$. In words, $\mathbf{a}'$ and $\mathbf{b}'$ are the projections on the first k components of some pair of tuples $(\mathbf{a}, \mathbf{b})$ that belongs to $\mathcal{F}$. We claim that $\mathcal{H}$ is a winning strategy in the k-pebble sliding game.

First, $\mathcal{H}$ is nonempty since $\mathcal{F}$ is nonempty. Second, every $(\mathbf{a}', \mathbf{b}')$ in $\mathcal{H}$ is a partial k-isomorphism since the corresponding $(\mathbf{a}, \mathbf{b})$ in $\mathcal{F}$ is a partial $(k + 1)$-isomorphism. Third, for every $(\mathbf{a}', \mathbf{b}')$ in $\mathcal{H}$ and every $i \in [k]$, the pair $(\mathbf{a}'[i/\star], \mathbf{b}'[i/\star])$ belongs to $\mathcal{H}$ by the closure under subtuples of $\mathcal{F}$. Next we argue that the back and forth properties are satisfied. By symmetry, it suffices to check the forth property with +-orientation.

Fix $(\mathbf{a}', \mathbf{b}')$ in $\mathcal{H}$, with $\mathbf{a}' = (a'_1, \ldots, a'_k)$ and $\mathbf{b}' = (b'_1, \ldots, b'_k)$. Let the corresponding pair of tuples in $\mathcal{F}$ be $\mathbf{a} = (a'_1, \ldots, a'_k, a_{k+1})$ and $\mathbf{b} = (b'_1, \ldots, b'_k, b_{k+1})$. Fix $i \in [k]$ and $X \subseteq N^+(a'_i)$. By the closure under subtuples of $\mathcal{F}$ and the forth property of $\mathcal{F}$ applied to the pair $(\mathbf{a}[k+1/\star], \mathbf{b}[k+1/\star])$, component $k+1$, and set $X \subseteq N^+(\star) = A$, there exists $Y \subseteq N^+(\star) = B$ with $|Y| = |X|$ such that for every $b \in Y$ there exists $a \in X$ such that $(\mathbf{a}[k+1/a], \mathbf{b}[k+1/b])$ belongs to $\mathcal{F}$. Now let us show the following.

CLAIM 5. $Y \subseteq N^+(b'_i)$.

Proof. If $b'_i = \star$, there is nothing to show since in that case $N^+(b'_i) = B$ and it is obvious that $Y \subseteq B$. Assume then that $b'_i \neq \star$. Fix an arbitrary element $b \in Y$. We want to show that (b'_i, b) is an edge in $\mathbf{B}$. By the choice of Y, there exists $a \in X$ such that $(\mathbf{a}[k+1/a], \mathbf{b}[k+1/b])$ belongs to $\mathcal{F}$. In particular, $(\mathbf{a}[k+1/a], \mathbf{b}[k+1/b])$ is a partial $(k+1)$-isomorphism, and since (a'_i, a) is an edge in $\mathbf{A}$, (b'_i, b) must also be an edge in $\mathbf{B}$. This shows that $Y \subseteq N^+(b'_i)$. ∎

Next we show the following.

CLAIM 6. *For every $b \in Y$, there is $a \in X$ such that $(\mathbf{a}'[i/a], \mathbf{b}'[i/b])$ is in $\mathcal{H}$.*

Proof. In the proof of the previous claim we argued that for every $b \in Y$ there exists $a \in X$ such that $(\mathbf{a}[k+1/a], \mathbf{b}[k+1/b])$ belongs to $\mathcal{F}$. By the forth property of $\mathcal{F}$ applied to the pair of tuples $(\mathbf{a}[k+1/a], \mathbf{b}[k+1/b])$, component i, and set $X' = \{a\} \subseteq N^+(a'_i)$, there exists $Y' \subseteq N^+(b'_i)$ with $|Y'| = |X'|$ such that for every $b' \in Y'$ there exists $a' \in X'$ such that $(\mathbf{a}[k+1/a, i/a'], \mathbf{b}[k+1/b, i/b'])$ belongs to $\mathcal{F}$. But since the members of $\mathcal{F}$ define partial $(k+1)$-isomorphisms and the only a' in X' is a, necessarily $Y' = \{b\}$ since otherwise the components i and $k+1$ would be equal in $\mathbf{a}[k+1/a, i/a']$ and different in $\mathbf{a}[k+1/b, i/b']$.

The previous paragraph shows that for every $b \in Y$ there exists $a \in X$ such that the pair $(\mathbf{a}[k+1/a, i/a], \mathbf{b}[k+1/b, i/b])$ belongs to $\mathcal{F}$. Since $(\mathbf{a}'[i/a], \mathbf{b}'[i/b])$ is precisely the pair of projections on the first k components of the tuples in $(\mathbf{a}[k+1/a, i/a], \mathbf{b}[k+1/b, i/b])$, this shows that for every $b \in Y$ there exists $a \in X$ such that $(\mathbf{a}'[i/a], \mathbf{b}'[i/b])$ belongs to $\mathcal{H}$. ∎

The forth property of $\mathcal{H}$ is proved, which proves the lemma. ∎

4.5. From sliding game to common equitable partition. Let $\mathbf{a}$ and $\mathbf{b}$ be k-tuples in $(A \cup \{\star\})^k$ and $(B \cup \{\star\})^k$, respectively. Define $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}, \mathbf{B})$ if $(\mathbf{a}, \mathbf{b})$ belongs to some winning strategy for the Duplicator in the k-pebble sliding game on $\mathbf{A}$ and $\mathbf{B}$.

LEMMA 3. $\equiv$ *is an equivalence relation.*

Proof. The symmetry of the relation follows from the symmetry of the game, and its reflexivity is clear. The only property that requires checking is transitivity. Assume $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}, \mathbf{B})$ and $(\mathbf{b}, \mathbf{B}) \equiv (\mathbf{c}, \mathbf{C})$. Let $\mathcal{F}$ and $\mathcal{F}'$ be the two winning strategies witnessing these facts. Let $\mathcal{G}$ be the collection of all pairs of k-tuples $(\mathbf{a}', \mathbf{c}')$ with $\mathbf{a}' \in (A \cup \{\star\})^k$ and $\mathbf{c}' \in (C \cup \{\star\})^k$ for which there exists a k-tuple $\mathbf{b}' \in (B \cup \{\star\})^k$ such that $(\mathbf{a}', \mathbf{b}')$ belongs to $\mathcal{F}$ and $(\mathbf{b}', \mathbf{c}')$ belongs to $\mathcal{F}'$. Clearly, each $(\mathbf{a}', \mathbf{c}')$ in $\mathcal{G}$ defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{C}$. Moreover, $(\mathbf{a}'[i/\star], \mathbf{c}'[i/\star])$ belongs to $\mathcal{G}$ by the closure under subtuples properties of $\mathcal{F}$ and $\mathcal{F}'$. Indeed, $(\mathbf{a}'[i/\star], \mathbf{b}'[i/\star])$ belongs to $\mathcal{F}$ and $(\mathbf{b}'[i/\star], \mathbf{c}'[i/\star])$ belongs to $\mathcal{F}'$ for the $\mathbf{b}'$ that witnesses that $(\mathbf{a}', \mathbf{c}')$ belongs to $\mathcal{G}$. The back and forth properties of $\mathcal{G}$ are also easily derived from the back and forth properties of $\mathcal{F}$ and $\mathcal{F}'$. Finally, $\mathcal{G}$ contains the pair $(\mathbf{a}, \mathbf{c})$ by construction, which means that it is nonempty and hence a winning strategy witnessing that $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{c}, \mathbf{C})$. ∎

In restriction to a single structure $\mathbf{A}$, the equivalence relation $\equiv$ can be thought as an equivalence relation on $(A \cup \{\star\})^k$.

LEMMA 4. *The sequence of equivalence classes of* $\equiv$ *on* $(A \cup \{\star\})^k$ *is a k-equitable partition of* $\mathbf{A}$.

Proof. Let $(P_1, \ldots, P_s)$ be the equivalence classes of $\equiv$ on $(A \cup \{\star\})^k$. This forms a partition of $(A \cup \{\star\})^k$. Fix an index $m \in [s]$, and fix tuples $\mathbf{a} = (a_1, \ldots, a_k)$

and $\mathbf{a}' = (a'_1, \ldots, a'_k)$ in P_m. Since $\mathbf{a} \equiv \mathbf{a}'$, the pair $(\mathbf{a}, \mathbf{a}')$ belongs to some winning strategy $\mathcal{F}$. In particular it defines a partial k-isomorphism from $\mathbf{A}$ to $\mathbf{A}$.

To argue that $c(\mathbf{a}[i/\star]) = c(\mathbf{a}'[i/\star])$ for every $i \in [k]$, note that $(\mathbf{a}[i/\star], \mathbf{a}'[i/\star])$ also belongs to $\mathcal{F}$ by the closure under subtuples property in the definition of winning strategy.

Next we want to show that $d_i^o(\mathbf{a}, P_n) = d_i^o(\mathbf{a}', P_n)$ for every $i \in [k]$, $o \in \{+, -\}$, and $n \in [s]$. First we consider the case that $\mathbf{a}[i/\star]$ lands in P_n. In this case $d_i^o(\mathbf{a}, P_n) = 1$ since every tuple in P_n must be equivalent to $\mathbf{a}[i/\star]$ and hence have $\star$ in the ith component, and $d_i^o(\mathbf{a}, P_n)$ is precisely the number of $a \in N^o(a_i) \cup \{\star\}$ such that $\mathbf{a}[i/a]$ belongs to P_n. Also $\mathbf{a}'[i/\star]$ lands in P_n by the previous paragraph, and hence $d_i^o(\mathbf{a}', P_n) = 1$ by the same argument.

Next we consider the case where $\mathbf{a}[i/\star]$ does not land in P_n. Let X be the set of all $a \in N^o(a_i)$ such that $\mathbf{a}[i/a]$ belongs to P_n. Then $|X| = d_i^o(\mathbf{a}, P_n)$. Similarly, let X' be the set of all $a' \in N^o(a'_i)$ such that $\mathbf{a}'[i/a']$ belongs to P_n. Since $\mathbf{a}'[i/\star]$ does not land in P_n either because $c(\mathbf{a}[i/\star]) = c(\mathbf{a}'[i/\star])$, we have $|X'| = d_i^o(\mathbf{a}', P_n)$. We show that $|X| = |X'|$.

Let $Y \subseteq N^o(a'_i)$ be the set guaranteed to exist by the forth property of $\mathcal{F}$ for the pair of tuples $(\mathbf{a}, \mathbf{a}')$, index i, and set X. Then $|Y| = |X|$. We claim that $Y \subseteq X'$. To show this, observe that for each $a' \in Y$ there exists some $a \in X$ such that $(\mathbf{a}[i/a], \mathbf{a}'[i/a'])$ belongs to $\mathcal{F}$. Hence $\mathbf{a}[i/a] \equiv \mathbf{a}'[i/a']$, which means that $\mathbf{a}'[i/a']$ belongs to the equivalence class P_n of $\mathbf{a}[i/a]$. This shows that $Y \subseteq X'$. Therefore $|Y| \le |X'|$ and hence $|X| \le |X'|$ because $|Y| = |X|$. The symmetric argument exchanging the roles of $\mathbf{a}$, X and $\mathbf{a}'$, X' would show that $|X'| \le |X|$. Thus $|X| = |X'|$, as was to be shown.

To argue that $c(\mathbf{a} \circ \pi) = c(\mathbf{a}' \circ \pi)$ for every permutation $\pi \in S_k$, note that $\mathcal{F} \circ \pi$ defined as $\{(\mathbf{c} \circ \pi, \mathbf{c}' \circ \pi) : (\mathbf{c}, \mathbf{c}') \in \mathcal{F}\}$ is also a winning strategy. The same argument shows that $|P_{c(\mathbf{a})}| = |P_{c(\mathbf{a}\circ\pi)}|$. ∎

LEMMA 5. *Let $k \ge 1$. If $\mathbf{A} \equiv_{\mathrm{CS}}^k \mathbf{B}$, then $\mathbf{A} \equiv_{\mathrm{EP}}^k \mathbf{B}$.*

Proof. Let $(P_1, \ldots, P_s)$ be the k-equitable partition given by $\equiv$ on $\mathbf{A}$. Similarly, let $(Q_1, \ldots, Q_t)$ be the k-equitable partition given by $\equiv$ on $\mathbf{B}$.

By hypothesis there exists a winning strategy for the Duplicator on $\mathbf{A}$ and $\mathbf{B}$. Let $\mathcal{F}$ be such a strategy. By the forth property of $\mathcal{F}$, for every $\mathbf{a}$ in A^k there exists $\mathbf{b} = \mathbf{b}(\mathbf{a})$ in B^k such that $(\mathbf{a}, \mathbf{b})$ belongs to $\mathcal{F}$, and therefore $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}(\mathbf{a}), \mathbf{B})$. Moreover, by the transitivity of the equivalence relation and the fact that $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}(\mathbf{a}), \mathbf{B})$ for every $\mathbf{a} \in (A \cup \{\star\})^k$ it follows that $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{a}', \mathbf{A})$ if and only if $(\mathbf{b}(\mathbf{a}), \mathbf{B}) \equiv (\mathbf{b}(\mathbf{a}'), \mathbf{B})$. This means that there exists a well-defined injective mapping $\alpha : \{1, \ldots, s\} \to \{1, \ldots, t\}$ that takes $m \in [s]$ to the unique $n \in [t]$ such that every $\mathbf{a}$ in P_m is equivalent to every $\mathbf{b}$ in Q_n.

CLAIM 7. $s = t$.

Proof. The injective mapping $\alpha : \{1, \ldots, s\} \to \{1, \ldots, t\}$ shows that $s \le t$. By symmetry we also get $t \le s$ and hence $s = t$. ∎

Since α is indeed a bijection, we may assume that it is the identity by rearranging the partitions. In other words, from now on we assume that $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}, \mathbf{B})$ if and only if $c(\mathbf{a}) = c(\mathbf{b})$.

CLAIM 8. $c(\mathbf{a}[i/\star]) = c(\mathbf{b}[i/\star])$ *for every* $i \in [k]$, $m \in [s]$, $\mathbf{a} \in P_m$, *and* $\mathbf{b} \in Q_m$.

Proof. Since $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}, \mathbf{B})$, the pair $(\mathbf{a}, \mathbf{b})$ belongs to some winning strategy $\mathcal{F}$, but then the pair $(\mathbf{a}[i/\star], \mathbf{b}[i/\star])$ also belongs to $\mathcal{F}$ by the closure under subtuples of winning strategies. This shows that $c(\mathbf{a}[i/\star]) = c(\mathbf{b}[i/\star])$. ∎

Next we show that the degrees are the same.

CLAIM 9. *$d_i^o(\mathbf{a}, P_n) = d_i^o(\mathbf{b}, Q_n)$ for every $i \in [k]$, $o \in \{+,-\}$, $m, n \in [s]$, $\mathbf{a} \in P_m$, and $\mathbf{b} \in Q_m$.*

Proof. Let $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_k)$. First we consider the case that $\mathbf{a}[i/\star]$ lands in P_n. In this case $d_i^o(\mathbf{a}, P_n) = 1$ since every tuple in P_n must be equivalent to $\mathbf{a}[i/\star]$ and hence have $\star$ in the ith component, and $d_i^o(\mathbf{a}, P_n)$ is precisely the number of $a \in N^o(a_i) \cup \{\star\}$ for which $\mathbf{a}[i/a]$ lands in P_n. By Claim 8, $\mathbf{b}[i/\star]$ also lands in Q_n. Hence $d_i^o(\mathbf{b}, Q_n) = 1$ by the same argument, which completes this case.

Next we consider the case that $\mathbf{a}[i/\star]$ does not land in P_n. Let X be the set $\{a \in N^o(a_i) : \mathbf{a}[i/a] \in P_n\}$. Thus $|X| = d_i^o(\mathbf{a}, P_n)$. By the definition of winning strategy for the Duplicator there exists a set $Y \subseteq N^o(b_i)$ with $|Y| = |X|$ such that for every $b \in Y$ there exists $a \in X$ such that $(\mathbf{a}[i/a], \mathbf{b}[i/b])$ belongs to $\mathcal{F}$. Since this implies $(\mathbf{a}[i/a], \mathbf{A}) \equiv (\mathbf{b}[i/b], \mathbf{B})$, we can conclude that $\mathbf{b}[i/b] \in Q_n$ for every $b \in Y$. Thus $d_i^o(\mathbf{b}, Q_n) \geq |Y| = |X| = d_i^o(\mathbf{a}, P_n)$.

The symmetric condition for winning strategy implies the opposite inequality, and putting the two together we have $d_i^o(\mathbf{a}, P_n) = d_i^o(\mathbf{b}, Q_n)$. ∎

Next we show that the classes have the same sizes.

CLAIM 10. *$|P_m| = |Q_m|$ for every $m \in [s]$.*

Proof. First notice that the fact that there is a winning strategy for the Duplicator implies that $|A| = |B|$. To see this note first that the pair of k-tuples $(\star^k, \star^k)$ belongs to the winning strategy by the closure under subtuples of winning strategies, and that the forth property applied to this pair of tuples and any $i \in [k]$, $o \in \{+,-\}$ requires that for every $X \subseteq N^o(\star) = A$ there must exist a $Y \subseteq N^o(\star) = B$ such that $|Y| = |X|$, among other properties. In particular, choosing $X = A$ we get $|B| \geq |A|$. By the symmetric condition we also get $|A| \geq |B|$. Using the equality between the sizes of A and B the statement of this claim follows from the previous one, as we show next.

For every $m, n \in [s]$, $\mathbf{a} \in P_m$, $\mathbf{a}' \in P_n$, $\mathbf{b} \in Q_m$, and $\mathbf{b}' \in Q_n$ we have the identities

$$|P_m| d_i^+(\mathbf{a}, P_n) = d_i^+(P_m, P_n) = d_i^-(P_n, P_m) = |P_n| d_i^-(\mathbf{a}', P_m),$$
$$|Q_m| d_i^+(\mathbf{b}, Q_n) = d_i^+(Q_m, Q_n) = d_i^-(Q_n, Q_m) = |Q_n| d_i^-(\mathbf{b}', Q_m).$$

Therefore

$$\frac{|P_m|}{|P_n|} = \frac{d_i^-(\mathbf{a}', P_m)}{d_i^+(\mathbf{a}, P_n)} = \frac{d_i^-(\mathbf{b}', Q_m)}{d_i^+(\mathbf{b}, Q_n)} = \frac{|Q_m|}{|Q_n|},$$

where the second equality follows from the previous claim. This means that the ratio $r = |P_m|/|Q_m|$ does not depend on m, and since $|A| = \sum_{m=1}^s |P_m| = r \sum_{m=1}^s |Q_m| = r|B|$, it follows that $r = 1$. ∎

CLAIM 11. *$c(\mathbf{a} \circ \pi) = c(\mathbf{b} \circ \pi)$ for every permutation $\pi \in S_k$, $m \in [s]$, $\mathbf{a} \in P_m$, and $\mathbf{b} \in Q_m$.*

Proof. Since $(\mathbf{a}, \mathbf{A}) \equiv (\mathbf{b}, \mathbf{B})$, the pair $(\mathbf{a}, \mathbf{b})$ belongs to some winning strategy $\mathcal{F}$. But then the pair $(\mathbf{a} \circ \pi, \mathbf{b} \circ \pi)$ belongs to $\mathcal{F} \circ \pi$ defined by $\{(\mathbf{c} \circ \pi, \mathbf{c}' \circ \pi) : (\mathbf{c}, \mathbf{c}') \in \mathcal{F}\}$, which is again a winning strategy. This shows that $c(\mathbf{a} \circ \pi) = c(\mathbf{b} \circ \pi)$. ∎

These claims show that $(P_1, \ldots, P_s)$ and $(Q_1, \ldots, Q_s)$ witness that $\mathbf{A}$ and $\mathbf{B}$ have a common k-equitable partition. ∎

4.6. From common equitable partition to SA. We prove the last implication of the Transfer Lemma.

LEMMA 6. *Let $k \geq 1$. If $\mathbf{A} \equiv_{\mathrm{EP}}^k \mathbf{B}$, then $\mathbf{A} \equiv_{\mathrm{SA}}^k \mathbf{B}$.*

Proof. Let $(P_1,\dots,P_s)$ and $(Q_1,\dots,Q_s)$ be the common k-equitable partition of $\mathbf{A}$ and $\mathbf{B}$.

For every $q \subseteq A \times B$ with $|q| \leq k$, if q is not a partial mapping, define $X_q = 0$. If q is a partial mapping, define X_q as follows. Let $a_1,\dots,a_r$ be an enumeration without repetitions of $\mathrm{Dom}(q)$. In particular, $r \leq k$. Let $\mathbf{a} = (a_1,\dots,a_r,\star,\dots,\star)$ be the k-tuple that starts with $a_1,\dots,a_r$ and is padded to length k by adding stars. Let $\mathbf{b} = (b_1,\dots,b_k)$ be the k-tuple defined by $b_i = q(a_i)$ for every $i \in \{1,\dots,r\}$ and $b_i = \star$ for every $i \in \{r+1,\dots,k\}$. Let $m = c(\mathbf{a})$ and $n = c(\mathbf{b})$. If $m \neq n$, define $X_q = 0$. If $m = n$, define $X_q = 1/|P_m| = 1/|Q_m|$. Since $c(\mathbf{a}\circ\pi) = c(\mathbf{b}\circ\pi)$ and $|P_{c(\mathbf{a})}| = |P_{c(\mathbf{a}\circ\pi)}|$ hold for every permutation $\pi \in S_k$, this definition does not depend on the choice of the enumeration $a_1,\dots,a_r$ and is hence well defined.

CLAIM 12. *If $|q| < k$ and $a \in A$, then $X_q = \sum_{b\in B} X_{q\cup ab}$.*

Proof. If q is not a partial mapping, then $X_q = 0$ and $X_{q\cup ab} = 0$ for every $b \in B$, and the identity is obvious. Assume then that q is a partial mapping and that $|q| < k$. Let $a_1,\dots,a_r$ be an enumeration without repetitions of $\mathrm{Dom}(q)$. In particular, $r < k$. Let $\mathbf{a} = (a_1,\dots,a_r,\star,\dots,\star)$ be the k-tuple that starts with $a_1,\dots,a_r$ and is padded to length k by adding stars. Let $\mathbf{b} = (b_1,\dots,b_k)$ be the k-tuple defined by $b_i = q(a_i)$ for every $i \in \{1,\dots,r\}$ and $b_i = \star$ for every $i \in \{r+1,\dots,k\}$. Setting $i = r+1$ for the rest of the proof, in particular, $a_i = b_i = \star$.

Let $m = c(\mathbf{a})$ and $n = c(\mathbf{b})$. If $m \neq n$, we have $X_q = 0$ by definition, and also $X_{q\cup ab} = 0$ for every $b \in B$ since otherwise $c(\mathbf{a}[i/a]) = c(\mathbf{b}[i/b])$, which implies $c(\mathbf{a}) = c(\mathbf{b})$, and hence $m = n$, by the definition of common equitable partition. Since this makes the identity obvious, we may assume that $m = n$.

Recall $i = r+1$, and let $\mathbf{a}' = \mathbf{a}[i/a]$ and $m' = c(\mathbf{a}')$. Note that none of the tuples $\mathbf{b}'$ in $Q_{m'}$ can have $\star$ in the ith component since $(\mathbf{a}',\mathbf{b}')$ must define a partial k-isomorphism; $\mathbf{a}'$ does not have it. We claim that

$$\sum_{b\in B} X_{q\cup ab} = \frac{d_i^+(\mathbf{b},Q_{m'})}{|Q_{m'}|} = \frac{d_i^-(Q_{m'},Q_m)}{|Q_m||Q_{m'}|} = \frac{1}{|Q_m|} = X_q.$$

The first equality follows from the definition of $X_{q\cup ab}$ that sets $X_{q\cup ab} = 0$ if $\mathbf{b}[i/b]$ does not belong to $Q_{m'}$, and $X_{q\cup ab} = 1/|Q_{m'}|$ if $\mathbf{b}[i/b]$ belongs to $Q_{m'}$, together with the fact that $N^+(b_i) = B$ since $b_i = \star$, and that $\mathbf{b}[i/\star]$ does not land in $Q_{m'}$ since no tuple in $Q_{m'}$ has $\star$ in the ith component. The second equality follows from the identity

$$|Q_m| d_i^+(\mathbf{b},Q_{m'}) = d_i^+(Q_m,Q_{m'}) = d_i^-(Q_{m'},Q_m).$$

For the third equality, let $b \in B$ be such that $\mathbf{b}[i/b]$ lands in $Q_{m'}$. Such a b must exist since $d_i^+(\mathbf{b},Q_{m'}) = d_i^+(\mathbf{a},P_{m'})$ and $d_i^+(\mathbf{a},P_{m'}) \geq 1$ as $\mathbf{a}[i/a]$ lands in $P_{m'}$ and $a \in N^+(a_i) = N^+(\star) = A$. Again we are using the fact that no tuple in $Q_{m'}$ has $\star$ in the ith component to make sure that the count $d_i^+(\mathbf{b},Q_{m'})$ does not include $\star$. Now we have $d_i^-(\mathbf{b}[i/b],Q_m) = 1$ since $d_i^-(\mathbf{b}[i/b],Q_m)$ is precisely the number of b' in $N^-(b)\cup\{\star\}$ such that $\mathbf{b}[i/b']$ belongs to Q_m, but the only such b' is $\star$. Indeed, every tuple $\mathbf{b}'$ in Q_m has $\star$ in the ith component since $(\mathbf{b},\mathbf{b}')$ must define a partial k-isomorphism, and $\mathbf{b}$ has it. This together with the identity

$$d^-(Q_{m'},Q_m) = |Q_{m'}| d_i^-(\mathbf{b}[i/b],Q_m)$$

proves the third equality and the claim. ∎

CLAIM 13. *If $|q| < k$ and $b \in B$, then $X_q = \sum_{a\in A} X_{q\cup ab}$.*

Proof. As above, exchange the roles of a and b, and $\mathbf{A}$ and $\mathbf{B}$. ∎

CLAIM 14. *If $|q| < k$, $a \in A$, and $b \in B$, then*

$$\sum_{a' \in A} A_{a,a'} X_{q \cup a'b} = \sum_{b' \in B} X_{q \cup ab'} B_{b',b}.$$

Proof. If q is not a partial mapping, then $X_q = 0$ and $X_{q \cup a'b} = 0$ for every $b \in B$, and the identity is obvious. Assume then that q is a partial mapping and that $|q| < k$. Let $a_1, \ldots, a_r$ be an enumeration without repetitions of $\mathrm{Dom}(q)$. In particular, $r < k$ since we are assuming $|q| < k$. Let $\mathbf{a} = (a_1, \ldots, a_r, a, \star, \ldots, \star)$ be the k-tuple that starts with $a_1, \ldots, a_r$, follows with a, and is padded to length k by adding stars. Similarly, let $\mathbf{b} = (q(a_1), \ldots, q(a_r), b, \star, \ldots, \star)$ be the k-tuple that starts with $q(a_1), \ldots, q(a_r)$, follows with b, and is padded to length k by adding stars.

Set $i = r + 1$ for the rest of the proof, and let $m = c(\mathbf{a})$ and $n = c(\mathbf{b})$. By the same argument as in Claim 12, note that none of the tuples in Q_m or P_n has $\star$ in the ith component since neither $\mathbf{a}$ nor $\mathbf{b}$ has it. We claim that

$$\sum_{a' \in A} A_{a,a'} X_{q \cup a'b} \; = \; \sum_{a' \in N^+(a)} X_{q \cup a'b} \; = \; \frac{d_i^+(\mathbf{a}, P_n)}{|P_n|}. \tag{13}$$

The first equality is obvious. The second equality follows from the definition of $X_{q \cup a'b}$ that sets $X_{q \cup a'b} = 0$ if $\mathbf{a}[i/a']$ does not belong to P_n, and $X_{q \cup a'b} = 1/|P_n|$ if $\mathbf{a}[i/a']$ belongs to P_n, together with the fact that $\mathbf{a}[i/\star]$ does not land in P_n since none of the tuples in P_n has $\star$ in the ith component.

At the same time we claim that

$$\sum_{b' \in B} X_{q \cup ab'} B_{b',b} \; = \; \sum_{b' \in N^-(b)} X_{q \cup ab'} \; = \; \frac{d_i^-(\mathbf{b}, Q_m)}{|Q_m|}. \tag{14}$$

Again the first equality is obvious, and the second equality follows from the definition of $X_{q \cup ab'}$, together with the fact that the tuples in Q_m do not have $\star$ in the ith component.

Fix $\mathbf{a}' \in P_n$ and $\mathbf{b}' \in Q_m$. From the definition of common equitable partition we have $d_i^-(\mathbf{a}', P_m) = d_i^-(\mathbf{b}, Q_m)$ and $d_i^+(\mathbf{b}', Q_n) = d_i^+(\mathbf{a}, P_n)$. Moreover, $|P_m| = |Q_m|$ and $|P_n| = |Q_n|$. These, together with either of the two identities

$$|P_n| d_i^-(\mathbf{a}', P_m) = d_i^-(P_n, P_m) = d_i^+(P_m, P_n) = |P_m| d_i^+(\mathbf{a}, P_n),$$
$$|Q_m| d_i^+(\mathbf{b}', Q_n) = d_i^+(Q_m, Q_n) = d_i^-(Q_n, Q_m) = |Q_n| d_i^-(\mathbf{b}, Q_m),$$

give the identity

$$\frac{d_i^-(\mathbf{b}, Q_m)}{|Q_m|} = \frac{d_i^+(\mathbf{a}, P_n)}{|P_n|}.$$

This shows the equality between (13) and (14). ∎

CLAIM 15. *If $|q| < k$, $a \in A$, and $c \in [r]$, then $X_q C_{a,c} = \sum_{b \in B} X_{q \cup ab} D_{b,c}$.*

Proof. First assume that $C_{a,c} = 0$, so that the left-hand side is 0. Then for every $b \in B$ we have either $D_{b,c} = 0$ or $D_{b,c} = 1$ and then $X_{q \cup ab} = 0$ since $q \cup ab$ cannot be a partial isomorphism in this case. Thus, each term in the right-hand side is 0.

Next assume that $C_{a,c} = 1$, so that the left-hand side is X_q. Then $X_{q\cup ab} = 0$ whenever $D_{b,c} = 0$ since $q \cup ab$ cannot be a partial isomorphism in this case. Thus, the right-hand side can be written as

$$\sum_{b\in B} X_{q\cup ab},$$

which equals X_q by (6). ∎

CLAIM 16. *If* $|q| < k$, $b \in B$, *and* $c \in [r]$, *then* $X_q D_{b,c} = \sum_{a\in A} X_{q\cup ab} C_{a,c}$.

Proof. This proof is the same as in the previous claim, exchanging the roles of a and b, and C and D. ∎

These claims show that the proposed assignment satisfies all the equations of $F_k(\mathbf{A}, \mathbf{B})$. Since the components are nonnegative, the lemma follows. ∎

5. Preservation of local linear programs.

Many of the linear programs (LPs) that appear in the combinatorial optimization literature are composed of linear inequalities that are in some sense *local*: the variables involved in the inequality talk about some small *neighborhood* of the graph or hypergraph, or whatever combinatorial structure the LP refers to. In this section we isolate one such definition of *local LP* and show that its polytope of feasible solutions is preserved by the SA levels of fractional isomorphism. This will be of use in the applications of sections 6.

5.1. Local LPs.

Let $\mathbf{A} = (A, E^{\mathbf{A}}, C_1^{\mathbf{A}}, \dots, C_r^{\mathbf{A}})$ be a colored directed graph. Let the size of a tuple $\mathbf{a} \in A^k$, denoted by $|\mathbf{a}|$, be the number of distinct elements in the tuple. For a tuple $\mathbf{a} = (a_1, \dots, a_k) \in A^k$, let us temporarily define $\gamma : \{a_1, \dots, a_k\} \to \{1, \dots, |\mathbf{a}|\}$ to be the unique bijective map such that $\gamma(a_i) \le |(a_1, \dots, a_i)|$ for every $i \in [k]$. We will denote by $[\mathbf{A}, \mathbf{a}]$ the generic colored directed graph isomorphic to the subgraph of $\mathbf{A}$ induced by $\{a_1, \dots, a_k\}$ together with the tuple corresponding to $\mathbf{a}$, which we refer to as its *order-tuple*. Thus, in $[\mathbf{A}, \mathbf{a}]$,

1. the vertices are $\{1, \dots, |\mathbf{a}|\}$,
2. the edges are $\{(\gamma(a), \gamma(a')) : (a, a') \in E^{\mathbf{A}}\}$,
3. the ith color is $\{\gamma(a) : a \in C_i^{\mathbf{A}}\}$, and
4. the order-tuple is $(\gamma(a_1), \gamma(a_2), \dots, \gamma(a_k))$.

For two tuples $\mathbf{a} = (a_1, \dots, a_m)$ and $\mathbf{b} = (b_1, \dots, b_n)$, we write $\mathbf{ab}$ for the concatenation tuple $(a_1, \dots, a_m, b_1, \dots, b_n)$. If $m = n$, we write $\mathbf{a} \mapsto \mathbf{b}$ for the tuple of pairs $((a_1, b_1), \dots, (a_m, b_m))$, which we think of as a partial mapping.

A *basic k-local LP* is specified by rational numbers $d^{[\mathbf{C},\mathbf{c}]}$ and $M_r^{[\mathbf{C},\mathbf{c}]}$ for every generic colored digraph $\mathbf{C}$ of size at most k with order-tuple $\mathbf{c}$ of length at most $2k$ and for every $r \le k$. The instantiation of the system on $\mathbf{A}$ is the system of inequalities that has one variable $x_{\mathbf{a}}$ for every tuple $\mathbf{a} \in A^{\le k}$ and one inequality of the form

$$\sum_{r=1}^{k} \sum_{\substack{\mathbf{a}\in A^r \\ |\mathbf{aa}'|\le k}} M_r^{[\mathbf{A},\mathbf{aa}']}\, x_{\mathbf{a}} \le d^{[\mathbf{A},\mathbf{a}']}$$

for every $\mathbf{a}' \in A^{\le k}$. In words, the coefficient of the variable indexed by $\mathbf{a}$ in the inequality indexed by $\mathbf{a}'$ depends only on $[\mathbf{A}, \mathbf{aa}']$ and the length of $\mathbf{a}$. A *k-local LP* is a union of basic k-local LPs. If L is a k-local LP, its instantiation on $\mathbf{A}$, denoted by $L(\mathbf{A})$, is the union of the instantiations of the basic k-local systems that compose L.

5.2. Examples. Before we go on to show that the feasible solutions to local LPs are preserved by the SA levels of fractional isomorphism, let us give a few examples of local LPs. These examples will actually play a role later in the paper.

Typical constraints. All four examples discussed contain two types of constraints for which it is easy to check the condition of k-locality. One special case that satisfies the condition is an LP consisting of a single inequality with the same coefficient for all $x_{\mathbf{a}}$ in which $\mathbf{a}$ induces a particular colored directed subgraph or one in a set of colored directed subgraphs on the structure. We call such a basic local LP *homogeneous*. The objective functions of many natural LPs are homogeneous local LPs, as we will see.

Another special case is when the coefficient in front of variable $x_{\mathbf{a}}$ in the inequality indexed by $\mathbf{a}'$ is nonzero only if the elements in $\mathbf{a}$ are contained within $\mathbf{a}'$. In this case we have $M_r^{[\mathbf{C},\mathbf{c}]} \neq 0$ only if the first r elements of $\mathbf{c}$ are included in the last $s-r$, where s is the length of $\mathbf{c}$. The nonzero coefficients are allowed to all be different since in the case that $\mathbf{a}_1$ and $\mathbf{a}_2$ are contained in $\mathbf{a}'$, we have $[\mathbf{A}, \mathbf{a}_1\mathbf{a}'] \neq [\mathbf{A}, \mathbf{a}_2\mathbf{a}']$ whenever $\mathbf{a}_1 \neq \mathbf{a}_2$, because they have different order-tuples. We call such a basic k-local LP *bounded*. In particular, any inequality in an LP that mentions only the variables indexed by tuples over up to k points of A is a bounded k-local LP. We see examples below.

Matchings in bipartite graphs. We write the fractional matching polytope for general graphs which, for bipartite graphs, is known to coincide with its integer hull.

Let $\mathbf{G} = (V, E)$ be an undirected graph. The classical way of writing the fractional matching polytope has one variable x_e for each edge $e \in E$ and two types of constraints:

$$\begin{array}{ll} \sum_{e\in\delta(u)} x_e \leq 1 & \text{for } u \in V, \\ 0 \leq x_e \leq 1 & \text{for } e \in E, \end{array}$$

where $\delta(u)$ denotes the set of edges of $\mathbf{G}$ that are incident on u. The classical objective function is

$$\text{maximize } \sum_{e\in E} x_e.$$

In order to write this LP as a local LP, we introduce one variable x_{uv} for every pair of vertices $u, v \in V$, and we add constraints that force these variables to 0 if $\{u, v\}$ is not an edge of the graph and force $x_{uv} = x_{vu}$ for every $u, v \in V$. We also incorporate the objective function as one additional constraint:

$$\tfrac{1}{2}\sum_{u\neq v} x_{uv} \geq W, \tag{15}$$

$$\sum_{v\neq u} x_{uv} \leq 1 \qquad \text{for } u \in V, \tag{16}$$

$$0 \leq x_{uv} \leq 1 \qquad \text{for } u, v \in V, \tag{17}$$

$$x_{uv} = 0 \qquad \text{for } u, v \in V \text{ such that } \{u, v\} \notin E, \tag{18}$$

$$x_{uv} = x_{vu} \qquad \text{for } u, v \in V \text{ with } u \neq v. \tag{19}$$

We check that this is a 2-local LP. First, inequality (15) is a basic homogeneous 2-local system: set $d^{[\mathbf{C},\mathbf{c}]} = -W$ if $\mathbf{C}$ is the empty graph and $\mathbf{c}$ is the empty tuple, and $d^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise; set $M_r^{[\mathbf{C},\mathbf{c}]} = -1/2$ if $r = 2$ and $\mathbf{C}$ is a graph on $\{1, 2\}$ and $\mathbf{c} = (1, 2)$, and $M_r^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise. Second, inequality (16) is a basic 2-local LP: set

$d^{[\mathbf{C},\mathbf{c}]} = 1$ if $\mathbf{C}$ is a graph on $\{1\}$ and $\mathbf{c} = (1)$, and $d^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise; set $M_r^{[\mathbf{C},\mathbf{c}]} = 1$ if $r = 2$ and $\mathbf{C}$ is a graph on $\{1, 2\}$ and $\mathbf{c} = (1, 2, 1)$, and $M_r^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise. The remaining inequalities are basic bounded 2-local systems. Thus, the result is a union of basic 2-local LPs and hence a 2-local LP.

Maximum flows. A network is a directed graph without self-loops, and with two distinguished vertices s and t. We code these as colored directed graphs $\mathbf{G} = (V, E, S, T)$, with color S set to $\{s\}$ and color T set to $\{t\}$. Our networks have unit capacities at every edge.

The classical LP for st-flows has one variable x_e for every $e \in E$ and two types of constraints:

$$\begin{aligned} &\textstyle\sum_{e\in\delta^-(u)} x_e - \sum_{e\in\delta^+(u)} x_e = 0 && \text{for } u \in V \setminus \{s, t\},\\ &0 \le x_e \le 1 && \text{for } e \in E, \end{aligned}$$

where $\delta^-(u)$ denotes the set of edges of $\mathbf{G}$ *entering* u, and $\delta^+(u)$ denotes the set of edges of $\mathbf{G}$ *leaving* u. The objective is to maximize the flow going out of s:

$$\text{maximize } \textstyle\sum_{e\in\delta^+(s)} x_e.$$

In order to write this LP as a local LP, we introduce one variable x_{uv} for every pair of vertices $u, v \in V$ and add constraints that force x_{uv} to be nonzero only on edges $(u, v) \in E$. We also incorporate the objective function as a constraint:

$$\textstyle\sum_{v\neq s} x_{sv} \ge W, \tag{20}$$

$$\textstyle\sum_{v\neq u} x_{vu} - \sum_{v\neq u} x_{uv} = 0 \qquad \text{for } u \in V \setminus \{s, t\}, \tag{21}$$

$$0 \le x_{uv} \le 1 \qquad \text{for } u, v \in V, \tag{22}$$

$$x_{uv} = 0 \qquad \text{for } u, v \in V \text{ such that } (u, v) \notin E. \tag{23}$$

Inequality (20) is a basic homogeneous 2-local LP: set $d^{[\mathbf{C},\mathbf{c}]} = -W$ if $\mathbf{C}$ is the empty graph and $\mathbf{c}$ is the empty tuple, and $d^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise; set $M_r^{[\mathbf{C},\mathbf{c}]} = -1$ if $r = 2$ and $\mathbf{C}$ is a graph on $\{1, 2\}$ with color S on vertex 1 and $\mathbf{c} = (1, 2)$, and $M_r^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise. Equation (21) is a union of two basic 2-local LPs with opposite signs: one for $\le$ and one for $\ge$. In the first, set $d^{[\mathbf{C},\mathbf{c}]} = 0$ for every $\mathbf{C}$ and $\mathbf{c}$, and $M_r^{[\mathbf{C},\mathbf{c}]} = 1$ if $r = 2$ and $\mathbf{C}$ is a graph on $\{1, 2\}$ where 2 is not colored S or T and $\mathbf{c} = (1, 2, 2)$, and set $M_r^{[\mathbf{C},\mathbf{c}]} = -1$ if $r = 2$ and $\mathbf{C}$ is a graph on $\{1, 2\}$ where 1 is not colored S or T and $\mathbf{c} = (1, 2, 1)$, and $M_r^{[\mathbf{C},\mathbf{c}]} = 0$ otherwise. The remaining inequalities are basic bounded 2-local LPs.

SA levels of vertex-cover. Let $\mathbf{G} = (V, E)$ be an undirected graph. The vanilla LP for vertex cover is

$$0 \le x_u \le 1 \qquad \text{for } u \in V, \tag{24}$$

$$x_u + x_v \ge 1 \qquad \text{for } \{u, v\} \in E. \tag{25}$$

The objective function is

$$\text{minimize} \quad \textstyle\sum_{u\in V} x_u.$$

The corresponding t-level SA system is defined on the variables y_I for every $I \subseteq V$ with $|I| \le t$. For $I, J \subseteq V$, let $S(I, J) = \sum_{J'\subseteq J} (-1)^{|J'|} y_{I\cup J'}$, which is the linearization of the extended monomial $\prod_{i\in I} x_i \prod_{j\in J}(1-x_j)$. In particular, $S(I,J)=0$ if I and J are not disjoint. In the following definition I and J range over all disjoint subsets of V such that $|I\cup J|\le t-1$. We also incorporate the objective function as a constraint:

(26) $\sum_{u\in V} y_{\{u\}} \le W,$

(27) $y_\emptyset = 1,$

(28) $0 \le S(I\cup\{u\},J) \le S(I,J)$ for $u\in V$,

(29) $S(I\cup\{u\},J) + S(I\cup\{v\},J) \ge S(I,J)$ for $\{u,v\}\in E$.

Inequalities (28) and (29) come from multiplying (24) and (25) by the extended monomial $\prod_{i\in I} x_i \prod_{j\in J}(1-x_j)$ and linearizing.

To put it in the form of a local LP we need the variables to be indexed by tuples, so we replace y_I for $I=\{v_1,\dots,v_r\}$, $r\le t$, by $y_{\mathbf{a}}$ for $\mathbf{a}=(v_1,\dots,v_r)$, and for every permutation $\pi:[r]\to[r]$ we add the constraint $y_{\mathbf{a}} = y_{\mathbf{a}\circ\pi}$. Constraints should also be indexed by tuples, so (29) is really a pair of (equivalent) constraints: one for (u,v) and one for (v,u).

Inequality (26), which comes from the objective function, is a homogeneous 1-local LP, which implies it is also a homogeneous $(t+1)$-local LP. The remaining constraints are bounded $(t+1)$-local LPs.

SA levels of max-cut. Again, let $\mathbf{G}=(V,E)$ be an undirected graph. The LP relaxation for max-cut known as the metric polytope has one variable x_{uv} for every pair of vertices $u,v\in V$ and the constraints below:

(30) $0\le x_{uv}\le 1$ for $u,v\in V$,

(31) $x_{uv} = x_{vu}$ for $u,v\in V$,

(32) $x_{uw} \le x_{uv} + x_{vw}$ for $u,v,w\in V$,

(33) $x_{uv} + x_{vw} \le 2 - x_{uw}$ for $u,v,w\in V$.

The objective function is

$$\text{maximize } \tfrac{1}{2}\textstyle\sum_{\{u,v\}\in E} x_{uv}.$$

The corresponding t-level SA system is defined on the variables y_I for every $I\subseteq V^2$ with $|I|\le t$. In the following system I and J range over all disjoint subsets of V^2 such that $|I\cup J|\le t-1$. We also incorporate the objective function as a constraint:

(34) $\frac{1}{2}\sum_{\{u,v\}\in E} y_{\{uv\}} \ge W,$

(35) $y_\emptyset = 1,$

(36) $0\le S(I\cup\{uv\},J)\le S(I,J),$

(37) $S(I\cup\{uv\},J) = S(I\cup\{vu\},J),$

(38) $S(I\cup\{uw\},J) \le S(I\cup\{uv\},J) + S(I\cup\{vw\},J),$

(39) $S(I\cup\{uv\},J) + S(I\cup\{vw\},J) \le 2S(I,J) - S(I\cup\{uw\},J).$

To put it in the form required by Theorem 2 we need the variables to be indexed by tuples, so we replace y_I for $I=\{(v_1,v'_1),\dots,(v_r,v'_r)\}$, $r\le t$, by $y_{\mathbf{a}}$ for $\mathbf{a}=(a_1,\dots,a_r)$ with $a_i=(v_i,v'_i)$, and for every permutation $\pi:[r]\to[r]$ we add the constraint $y_{\mathbf{a}} = y_{\mathbf{a}\circ\pi}$.

Similarly to the case of vertex-cover, the first constraint, which comes from the objective function, is a homogeneous 2-local LP, which implies it is also a homogeneous $(2t+1)$-local LP. The remaining constraints are bounded $(2t+1)$-local LPs.

5.3. Preservation of feasible solutions. Next we show that local LPs have the good feature that their polyhedra of feasible solutions are preserved by sufficiently high levels of the SA relaxation of fractional isomorphism. More precisely, to preserve k-local LPs, k levels suffice. The full statement is the following.

THEOREM 2. *Let L be a k-local LP, and let $\mathbf{A}$ and $\mathbf{B}$ be colored digraphs such that $\mathbf{A} \equiv^k_{\mathrm{SA}} \mathbf{B}$. Then $L(\mathbf{A})$ is feasible if and only if $L(\mathbf{B})$ is feasible. Furthermore, if $x_{\mathbf{a}}$ is a solution of $L(\mathbf{A})$, then $x_{\mathbf{b}} = \sum_{\mathbf{a}\in A^r} X_{\mathbf{a}\mapsto\mathbf{b}}\, x_{\mathbf{a}}$ is a solution of $L(\mathbf{B})$, where $\mathbf{X}$ denotes the solution witnessing $\mathbf{A} \equiv^k_{\mathrm{SA}} \mathbf{B}$, and r is the length of $\mathbf{b}$.*

The rest of this section is devoted to the proof of Theorem 2. Let us fix $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A} \equiv^k_{\mathrm{SA}} \mathbf{B}$, and let $\mathbf{X}$ be a solution of $F_k(\mathbf{A},\mathbf{B})$ witnessing this fact. We start with a straightforward lemma about the properties of $\mathbf{X}$ that we will use several times.

LEMMA 7. *Let $0 \le r, s \le k$ be integers, let $\mathbf{a} \in A^r$ and $\mathbf{a}' \in A^s$ be such that $|\mathbf{aa}'| \le k$, and let $\mathbf{b} \in B^r$. Then*

$$X_{\mathbf{a}\mapsto\mathbf{b}} = \sum_{\substack{\mathbf{b}'\in B^s \\ [\mathbf{A},\mathbf{aa}']=[\mathbf{B},\mathbf{bb}']}} X_{\mathbf{aa}'\mapsto\mathbf{bb}'}.$$

Proof. The proof is a simple induction on s. For $s=0$ the statement trivially holds. Next, for $s \ge 1$, suppose $\mathbf{a}' = \mathbf{a}''a$, where $\mathbf{a}'' \in A^{s-1}$ and $a \in A$. Applying Claim 2 from section 4 we have

$$\sum_{\substack{\mathbf{b}''\in B^{s-1}, b\in B \\ [\mathbf{A},\mathbf{aa}''a]=[\mathbf{B},\mathbf{bb}''b]}} X_{\mathbf{aa}''a\,\mapsto\,\mathbf{bb}''b} = \sum_{\substack{\mathbf{b}''\in B^{s-1} \\ [\mathbf{A},\mathbf{aa}'']=[\mathbf{B},\mathbf{bb}'']}} \sum_{b\in B} X_{\mathbf{aa}''a\,\mapsto\,\mathbf{bb}''b}.$$

Equation (6) of the SA system shows that the right-hand side is

$$\sum_{\substack{\mathbf{b}''\in B^{s-1} \\ [\mathbf{A},\mathbf{aa}'']=[\mathbf{B},\mathbf{bb}'']}} X_{\mathbf{aa}''\mapsto\mathbf{bb}''},$$

and the induction hypothesis gives that this is precisely $X_{\mathbf{a}\mapsto\mathbf{b}}$. ◻

We proceed with the proof of the theorem. It is sufficient to prove the statement for a basic k-local LP L given by $M_r^{[\mathbf{C},\mathbf{c}]}$ and $d^{[\mathbf{C},\mathbf{c}]}$. Let $x_{\mathbf{a}}$ be a feasible solution for $L(\mathbf{A})$. Thus for every $\mathbf{a}' \in A^{\le k}$ we have

$$\sum_{r=1}^{k} \sum_{\substack{\mathbf{a}\in A^r \\ |\mathbf{aa}'|\le k}} M_r^{[\mathbf{A},\mathbf{aa}']}\, x_{\mathbf{a}} \le d^{[\mathbf{A},\mathbf{a}']}. \tag{40}$$

We need to show that for every $\mathbf{b}' \in B^{\le k}$ it holds that

$$\sum_{r=1}^{k} \sum_{\substack{\mathbf{b}\in B^r \\ |\mathbf{bb}'|\le k}} M_r^{[\mathbf{B},\mathbf{bb}']} \sum_{\mathbf{a}\in A^r} X_{\mathbf{a}\mapsto\mathbf{b}}\, x_{\mathbf{a}} \le d^{[\mathbf{B},\mathbf{b}']}. \tag{41}$$

In the following, let $0 \le s \le k$ be such that $\mathbf{b}' \in B^s$. Using Lemma 7 the left-hand side of (41) becomes

$$\sum_{r=1}^{k} \sum_{\substack{\mathbf{b}\in B^r \\ |\mathbf{bb}'|\le k}} M_r^{[\mathbf{B},\mathbf{bb}']} \sum_{\mathbf{a}\in A^r} \sum_{\substack{\mathbf{a}'\in A^s \\ [\mathbf{A},\mathbf{aa}']=[\mathbf{B},\mathbf{bb}']}} X_{\mathbf{aa}'\mapsto\mathbf{bb}'}\, x_{\mathbf{a}}.$$

Rearranging the sums with care we can rewrite this as

$$\sum_{\substack{\mathbf{a}'\in A^s\\ [\mathbf{A},\mathbf{a}']=[\mathbf{B},\mathbf{b}']}} \sum_{r=1}^{k} \sum_{\substack{\mathbf{b}\in B^r\\ |\mathbf{bb}'|\le k}} \sum_{\substack{\mathbf{a}\in A^r\\ [\mathbf{A},\mathbf{aa}']=[\mathbf{B},\mathbf{bb}']}} M_r^{[\mathbf{B},\mathbf{bb}']}\, X_{\mathbf{aa}'\mapsto\mathbf{bb}'}\, x_{\mathbf{a}}$$

$$= \sum_{\substack{\mathbf{a}'\in A^s\\ [\mathbf{A},\mathbf{a}']=[\mathbf{B},\mathbf{b}']}} \sum_{r=1}^{k} \sum_{\substack{\mathbf{a}\in A^r\\ |\mathbf{aa}'|\le k}} \sum_{\substack{\mathbf{b}\in B^r\\ [\mathbf{A},\mathbf{aa}']=[\mathbf{B},\mathbf{bb}']}} M_r^{[\mathbf{B},\mathbf{bb}']}\, X_{\mathbf{aa}'\mapsto\mathbf{bb}'}\, x_{\mathbf{a}}$$

$$= \sum_{\substack{\mathbf{a}'\in A^s\\ [\mathbf{A},\mathbf{a}']=[\mathbf{B},\mathbf{b}']}} \sum_{r=1}^{k} \sum_{\substack{\mathbf{a}\in A^r\\ |\mathbf{aa}'|\le k}} M_r^{[\mathbf{A},\mathbf{aa}']}\, x_{\mathbf{a}} \sum_{\substack{\mathbf{b}\in B^r\\ [\mathbf{A},\mathbf{aa}']=[\mathbf{B},\mathbf{bb}']}} X_{\mathbf{aa}'\mapsto\mathbf{bb}'}.$$

In the last line we used the fact that the condition $[\mathbf{A},\mathbf{aa}'] = [\mathbf{B},\mathbf{bb}']$ implies $M_r^{[\mathbf{A},\mathbf{aa}']} = M_r^{[\mathbf{B},\mathbf{bb}']}$. Using again Lemma 7 the last expression becomes

$$\sum_{\substack{\mathbf{a}'\in A^s\\ [\mathbf{A},\mathbf{a}']=[\mathbf{B},\mathbf{b}']}} \sum_{r=1}^{k} \sum_{\substack{\mathbf{a}\in A^r\\ |\mathbf{aa}'|\le k}} M_r^{[\mathbf{A},\mathbf{aa}']}\, x_{\mathbf{a}}\, X_{\mathbf{a}'\mapsto\mathbf{b}'}$$

$$= \sum_{\substack{\mathbf{a}'\in A^s\\ [\mathbf{A},\mathbf{a}']=[\mathbf{B},\mathbf{b}']}} X_{\mathbf{a}'\mapsto\mathbf{b}'} \sum_{r=1}^{k} \sum_{\substack{\mathbf{a}\in A^r\\ |\mathbf{aa}'|\le k}} M_r^{[\mathbf{A},\mathbf{aa}']}\, x_{\mathbf{a}}$$

$$\le \sum_{\substack{\mathbf{a}'\in A^s\\ [\mathbf{A},\mathbf{a}']=[\mathbf{B},\mathbf{b}']}} X_{\mathbf{a}'\mapsto\mathbf{b}'}\, d^{[\mathbf{A},\mathbf{a}']} = d^{[\mathbf{B},\mathbf{b}']}.$$

In the last line we used (40), together with the fact that the condition $[\mathbf{A},\mathbf{a}'] = [\mathbf{B},\mathbf{b}']$ implies $d^{[\mathbf{A},\mathbf{a}']} = d^{[\mathbf{B},\mathbf{b}']}$, another application of Lemma 7, and $X_\emptyset = 1$ by (9). This completes the proof of Theorem 2.

6. Applications. In this section we highlight some applications of the Transfer Lemma. First, we use known results from polyhedral combinatorics to get new definability results in finite model theory. Second, we use known constructions in finite model theory to get instances of high SA rank.

In the following, let MAX-FLOW denote the LP for *st*-flows, as discussed in section 5. Similarly, let BIPARTITE-MATCHING denote the standard LP for matchings in bipartite graphs. For every integer $t \ge 1$, let VERTEX-COVER^t denote the tth level of SA of the standard linear programming relaxation of vertex-cover, and let MAX-CUT^t denote the tth level of SA of the metric polytope relaxation of max-cut.

6.1. New definability results. For a local LP L, we say that L is *preserved* by an equivalence $\equiv$ if, whenever $\mathbf{A} \equiv \mathbf{B}$ and $L(\mathbf{A})$ has a solution, $L(\mathbf{B})$ also has a solution. More generally, if L is a local LP with an associated objective function max $\mathbf{c}^{\mathrm{T}}\mathbf{x}$ for which the constraint $\mathbf{c}^{\mathrm{T}}\mathbf{x} \ge W$ is also a local LP for every value W, then we say that *the optimum value of L is preserved by* $\equiv$ if the expanded local LP $L \cup \{\mathbf{c}^{\mathrm{T}}\mathbf{x} \ge W\}$ is preserved by $\equiv$ for every W.

The examples mentioned are all k-local LPs, for appropriate k, with the objective function incorporated as a constraint. A corollary to Theorem 2 is that the optimum value of MAX-FLOW is preserved by $\equiv^2_{\mathrm{SA}}$. Similarly, the optimum value of the LP BIPARTITE-MATCHING is preserved by $\equiv^2_{\mathrm{SA}}$, the optimum of VERTEX-COVER^t is preserved by $\equiv^{t+1}_{\mathrm{SA}}$, and the optimum of MAX-CUT^t is preserved by $\equiv^{2t+1}_{\mathrm{SA}}$. By the

Transfer Lemma, the optimum values of these LPs are also preserved by $\equiv^3_{\mathrm{C}}$ in the first two cases and by $\equiv^{t+2}_{\mathrm{C}}$ and $\equiv^{2t+2}_{\mathrm{C}}$ in the last two.

One consequence of this is, for example, that the class of bipartite graphs that have a perfect matching is definable by a sentence of the logic $\mathrm{C}^3_{\infty\omega}$. Indeed, by a standard result of finite model theory, each $\equiv^3_{\mathrm{C}}$-equivalence class is definable by a $\mathrm{C}^3_{\infty\omega}$-sentence (see Lemma 1.39 in [29]), and therefore it suffices to take the (infinitary) disjunction of the sentences that define the equivalence classes of the bipartite graphs that have a perfect matching. The preservation by $\equiv^3_{\mathrm{C}}$ guarantees the correctness. The same sort of argument carries over to MAX-FLOW on *st*-networks. Thus, for example, the class of *saturable* networks is $\mathrm{C}^3_{\infty\omega}$-definable, where a saturable network is one in which enough flow can be pushed through it to fill the capacity of all arcs leaving the source. Obviously, the same would work for networks on which a 1/3-fraction of the capacity, say, can be filled.

A less direct application concerns the max-cut problem on $\mathbf{K}_5$-minor-free graphs. A nontrivial result in polyhedral combinatorics states that for graphs $\mathbf{G}$ that do not have $\mathbf{K}_5$ as a minor, optimizing over the projection of the metric polytope to the edges of $\mathbf{G}$ yields the integral optimal cut of $\mathbf{G}$ [5]. Since this is what the LP MAX-CUT^1 is and the optimum of MAX-CUT^1 is preserved by $\equiv^4_{\mathrm{C}}$, we get that the class of $\mathbf{K}_5$-minor-free graphs that have a partition that cuts at least half the edges is $\mathrm{C}^4_{\infty\omega}$-definable by the same argument as before. Obviously, the choice to cut half the edges is arbitrary. Let us note that from the results in [15] on counting logics being able to express all polynomial-time properties on classes of minor-free graphs, this definability result would follow for $\mathrm{C}^k_{\infty\omega}$ replacing $\mathrm{C}^4_{\infty\omega}$ for *some* k (that is very likely big). This is because optimizing a linear function over the metric polytope can be done in polynomial time by linear programming. Our argument shows that $k = 4$ is enough, and it is interesting that the two proofs are very different.

It should be noted that the method outlined here yields definability results for infinitary logic only, and it is not clear how to apply it to obtain definability results in its uniform fragment IFP+C (inflationary fixed-point logic plus counting).

6.2. SA rank lower bounds. In this subsection we show how to build instances of high rank from the methods for proving inexpressibility results for counting logics.

Suppose we are asked to show that the property of having a vertex-cover of at most a third of the vertices is not definable in the logic $\mathrm{C}^k_{\infty\omega}$. Here k could be a function of the number of vertices n. The way to do so is by exhibiting two n-vertex graphs $\mathbf{G}$ and $\mathbf{H}$, one of which has a vertex-cover of size at most $n/3$ and the other does not, and yet $\mathbf{G}$ and $\mathbf{H}$ are indistinguishable by the logic in the sense that $\mathbf{G} \equiv^k_{\mathrm{C}} \mathbf{H}$. As a matter of fact, this is a complete method in the sense that if the property is really not definable in the logic, then such graphs $\mathbf{G}$ and $\mathbf{H}$ are guaranteed to exist (see [9]). Let us now see what this tells us about the SA levels of the LP-relaxation for vertex-cover.

Since $\mathbf{G} \equiv^k_{\mathrm{C}} \mathbf{H}$, by the Transfer Lemma we get $\mathbf{G} \equiv^{k-1}_{\mathrm{SA}} \mathbf{H}$. On the other hand, VERTEX-COVER^{k-2} is a $(k-1)$-local LP, so its optimum value is preserved by $\equiv^{k-1}_{\mathrm{SA}}$ by Theorem 2. Therefore $\mathbf{G}$ and $\mathbf{H}$ give the same optimum value of VERTEX-COVER^{k-2}, which must be at most $n/3$ since the integral optimum is always an upper bound on the relaxation of a minimization problem. We conclude that $\mathbf{H}$ is a graph on which the optimum of VERTEX-COVER^{k-2} is strictly smaller than the minimum vertex-cover since, by construction, $\mathbf{H}$ does not have a vertex-cover of size at most $n/3$. If we manage to afford $k = \Omega(n)$, where n is the number of vertices in $\mathbf{H}$, we get an optimal rank lower bound, up to constant factors.

The sketched plan can be carried out for many LPs, including VERTEX-COVER and MAX-CUT, to get SA rank lower bounds. In the rest of this section we outline the ingredients that are needed for this.

A well-known construction due to Cai, Fürer, and Immerman [7] gives explicit pairs of nonisomorphic n-vertex graphs $\mathbf{G}$ and $\mathbf{H}$ such that $\mathbf{G} \equiv_{\mathrm{C}}^{\Omega(n)} \mathbf{H}$. It was later observed in [2] that such graphs can be thought of as systems of linear equations over GF(2), call them $\mathbf{S}$ and $\mathbf{T}$, that remain $\equiv_{\mathrm{C}}^{\Omega(n)}$-indistinguishable, yet one is satisfiable and the other is not. This time n refers to the number of variables in the systems. At this point an approach suggests itself: apply the standard reduction from the solvability of linear equations over GF(2) to vertex-cover to get pairs of graphs, call them $\mathbf{G}'$ and $\mathbf{H}'$, and hope that they stay $\equiv_{\mathrm{C}}^{\Omega(n)}$-indistinguishable, where now n is the number of vertices in these graphs. And indeed, if done with care, this actually works. One easy way to guarantee that $\mathbf{G}'$ and $\mathbf{H}'$ are $\equiv_{\mathrm{C}}^{\Omega(n)}$-indistinguishable is by showing that the reduction is definable in the logic $\mathrm{C}_{\infty\omega}^{k}$ for a fixed constant k.

7. Discussion and open problems. Isomorphism is the finest of all binary relations on finite structures. There are other interesting relations such as embeddings and homomorphisms that could be phrased as 0-1 LPs and then relaxed. The SA levels of these would then yield tighter and tighter approximations. On the combinatorial side, embeddings and homomorphisms also admit relaxations through corresponding pebble games. In the case of homomorphisms, this is the existential k-pebble game popularized by Kolaitis and Vardi in the context of constraint satisfaction problems [21]. Does a version of the Transfer Lemma apply in this case too? One of the directions is easy, and a version of this was actually anticipated in [3], but it seems that the lack of counting in the homomorphism game could be a serious obstacle for the other.

On a different line of thought, the most promising outcome of our main result is the connection it sets between polyhedral combinatorics and finite model theory. In section 6 we have shown how rather elementary arguments are able to exploit the knowledge in one field to get results in the other. We hope that more sophisticated arguments could lead to stronger results. Let us point out two interesting possibilities.

In the direction from polyhedral combinatorics to finite model theory, it would be interesting to exploit the sophisticated constructions of integrality gap instances in the world of lift-and-project methods. One of the admitted bottlenecks of the pebble-game technique for proving inexpressibility results is the lack of general methods for building pairs of structures with different properties that stay sufficiently indistinguishable. Perhaps the methods for building integrality gap instances, say as in [8] through metric-embedding arguments from functional analysis, could be of use for building such objects. A concrete example where this could be applied is to the problem of perfect matchings on general graphs. In short, the question reduces to building, for every constant $k \geq 2$, a pair of $\equiv_{\mathrm{C}}^{k}$-equivalent (or $\equiv_{\mathrm{SA}}^{k}$-equivalent) graphs $\mathbf{G}_0$ and $\mathbf{G}_1$ in which $\mathbf{G}_0$ has a perfect matching but $\mathbf{G}_1$ does not. This would show that the class of general graphs having a perfect matching is not definable in the logic $\mathrm{C}_{\infty\omega}^{k}$ for any k, thus solving a problem in [6]. The recent progress in understanding the SA levels of the matching polytope could perhaps be also useful here [28].

In the direction from finite model theory to polyhedral combinatorics, new results could follow if the construction in [7] were strengthened to a pair of indistinguishable instances of the unique-games problem with a large gap in their optimal values. With such a lower bound in hand one would likely be able to exploit the reductions from unique-games to vertex-cover in [19] to get instances where an optimal integrality gap of 2 could resist up to $\Omega(n)$ levels of SA, which is currently not known. At any rate, exploring the gap-creating reductions underlying PCP constructions in the context of finite model theory appears to be an attractive line of research worth pursuing in itself.

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