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Aggregate monotonic stable single-valued solutions for cooperative games

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Abstract: We characterize single-valued solutions of transferable utility cooperative games satisfying core selection and aggregate monotonicity. Furthermore, we show that these two properties are compatible with individual rationality, the dummy player property and the symmetry property. We finish characterizing single-valued solutions satisfying these five properties.

Resum: En aquest treball caracteritzem les solucions puntuals de jocs cooperatius d'utilitat transferible que compleixen selecció del core i monotonia agregada. També mostrem que aquestes dues propietats són compatibles amb la individualitat racional, la propietat del jugador fals i la propietat de simetria. Finalment, caracteritzem les solucions puntuals que compleixen les cinc propietats a l'hora.

JEL classification: C71

Keywords: cooperative games, core, aggregate monotonicity, individual rationality, dummy player, symmetry.

1 Introduction

The *core* (Gillies 1959) of a *transferable utility cooperative game* (a game) is the set of feasible outcomes that can not be improved upon by any coalition of players. Since the core of a game may be empty, generalizations and modifications have been considered from the beginning (for details see Kannai 1992). A single-valued solution satisfies *core selection* if it selects a core element for any game with a non-empty core.

In the study of games, several monotonicity properties of single-valued solutions have been introduced. Megiddo (1974) studies *aggregate monotonicity*, which states that when the worth of the grand coalition increases whereas the worths of all other coalitions remain the same, then everyone's payoff should weakly increase. Young (1985) considers a stronger property, called *coalitional monotonicity*: if the worth of a given coalition increases whereas the worths of all other coalitions remain the same, then the payoff of every member of that coalition should weakly increase. He provides a five-agent example showing that this property and core selection are incompatible. Later, Housman and Clark (1998) show the same incompatibility for a four-agent game.

On the other hand, core selection and aggregate monotonicity are compatible on the domain of all games. The per-capita prenucleolus (Grotte 1970), a variant of the classical prenucleolus (Schmeidler 1969), defined by means of the per-capita excesses instead of the classical excesses, satisfies both properties (see for example Young et al. 1982 or Moulin 1988). Calleja et al. (2009) study the set of allocations attainable by single-valued solutions that satisfy core selection and aggregate monotonicity: the aggregate-monotonic core.

However, as quoted in Young et al. (1982): "The per-capita prenucleolus[1] may not be individually rational when the core is empty. While individual rationality may simply be imposed as a constraint (as proposed in Grotte (1970, 1976)), another serious difficulty remains. The per-capita prenucleolus may imply payments to dummies ... (a fact first noted by Reinhard Selten)".

This paper is devoted to analyze the compatibility for single-valued solutions of core selection and aggregate monotonicity, with other desirable and not too demanding properties like *individual rationality*, the *dummy player property* and *symmetry*. Peleg and Sudhölter (2003) contains an excellent summary on properties and known single-valued solutions. In Section 2 we introduce definitions and preliminaries. In Section 3 we review already known single-valued solutions and we see that none of them satisfies core selection, aggregate monotonicity, individual rationality, the dummy player property and symmetry together. Nevertheless, we show that these five properties are compatible. In Section 4 we characterize single-valued solutions satisfying core selection and aggregate monotonicity. Finally, in Section 5 we characterize those single-valued solutions satisfying core selection, aggregate monotonicity, individual rationality, the dummy player property and also the symmetry property.

[1] Weak nucleolus in the terminology used in Young et al. (1982)

2 Preliminaries

We denote by $N = \{1, ..., n\}$ a finite set of players. A coalition is a subset of N and we denote by 2^N the set of all coalitions in N. By $|S|$ we denote the cardinality of the coalition $S \subseteq N$. We use the symbol $\subset$ for strict set inclusions and $\subseteq$ for weak set inclusions. By e_S we denote the characteristic vector of $\mathbb{R}^N$ associated to coalition $S \subseteq N$, $S \neq \varnothing$, i.e. $e_{S,i} = 1$ if $i \in S$ and $e_{S,i} = 0$ if $i \notin S$.

A *transferable utility cooperative game* (a game) is a pair (N, v) (v, for short) where N is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function with $v(\varnothing) = 0$. The number $v(S)$ is the worth of coalition S, that is, what S can achieve on its own. The set of all games with player set N is denoted by G^N.

A vector $x \in \mathbb{R}^N$ is usually called a payoff vector or allocation, and each component x_i is interpreted as the allotment to player $i \in N$. Given a payoff vector $x \in \mathbb{R}^N$ and a coalition $S \subseteq N$, $S \neq \varnothing$, we write $x(S) = \sum_{i \in S} x_i$ for the payoff to coalition S, with $x(\varnothing) = 0$. We denote by $\leq$ in $\mathbb{R}^N$ the standard partial order, i.e. $x \leq y$ if $x_i \leq y_i$ for all $i \in N$.

The *preimputation set* of the game (N, v) consists of those payoff vectors that allocate the worth of the grand coalition: $I^*(v) = \left\{x \in \mathbb{R}^N : x(N) = v(N)\right\}$. The *imputation set* of the game (N, v) consists of those preimputations satisfying individual rationality: $I(v) = \{x \in I^*(v) : x_i \geq v(\{i\}) \text{ for all } i \in N\}$. A game (N, v) is called essential if $v(N) \geq \sum_{i \in N} v(\{i\})$, that is if $I(v) \neq \varnothing$. By E^N we denote the set of all essential games with player set N.

The *core* of the game (N, v) (Gillies, 1959) consists of those preimputations for which every coalition $S \subset N$ receives at least its worth: $C(v) = \{x \in I^*(v) : x(S) \geq v(S) \text{ for all } S \subset N\}$. By B^N we denote the set of all *balanced* games with player set N, that is, those with a non-empty core.

A *single-valued solution* on G^N is a function $F : G^N \to \mathbb{R}^N$ such that $F(v) \in I^*(v)$ for all $v \in G^N$.

In this paper, we deal with a number of properties of single-valued solutions on G^N, two of which are specially relevant in the paper: *core selection* and *aggregate monotonicity*.

Definition 1 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies core selection (CS) if $F(v) \in C(v)$ for all $v \in B^N$.*

In the following we will denote $v <_N v'$ with $v, v' \in G^N$, if $v(S) = v'(S)$ for all $S \subset N$ and $v(N) < v'(N)$.

Definition 2 *(Megiddo, 1974) A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies aggregate monotonicity (AM) if for all $v, v' \in G^N$ with $v <_N v'$, it holds that $F(v) \leq F(v')$.*

Other well known properties are *individual rationality*, the *dummy player property* and *symmetry*.

Definition 3 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies individual rationality (IR) if $F(v) \in I(v)$ for all $v \in E^N$.*

Note that a single-valued solution may satisfy core selection but not individual rationality since it may select a non-individually rational allocation in a game with an empty core.

Let (N, v) be a game. A player $i \in N$ is called a dummy player in (N, v) if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \backslash \{i\}$. A dummy player contributes only his individual worth to all coalitions.

Definition 4 *A single-valued solution, $F : G^N \rightarrow \mathbb{R}^N$, satisfies the dummy player property (DP) if $F_i(v) = v(\{i\})$ for all $v \in G^N$ and all dummy players $i \in N$ in (N, v).*

Let (N, v) be a game. Two players $i, j \in N$ are called symmetric players in (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$. Symmetric players contribute the same to all coalitions they do not belong to.

Definition 5 *A single-valued solution, $F : G^N \rightarrow \mathbb{R}^N$, satisfies symmetry (SYM) if $F_i(v) = F_j(v)$ for all $v \in G^N$ and any pair of symmetric players $i, j \in N$ in (N, v).*

3 Single-valued solutions and properties

Among several single-valued solutions for TU games defined on G^N, the Shapley value and the prenucleolus are maybe the most accepted. Let (N, v) be a game, the Shapley value (Shapley 1953), $\phi(v) \in \mathbb{R}^N$, is defined by

$$\phi_i(v) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|! \, (|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S)\right) \text{ for all } i \in N.$$

The Shapley value satisfies AM, although it does not satisfy CS. Additionally, the Shapley value satisfies DP and SYM, but it does not satisfy IR.

The prenucleolus (Schmeidler 1969), $\nu_*(v) \in \mathbb{R}^N$, is the preimputation $x \in I^*(v)$ that lexicographically minimizes the vector of excesses $e(S, x) = v(S) - x(S)$, $\varnothing \neq S \subset N$, when these are arranged in order of descending magnitude. In comparison with the Shapley value, the prenucleolus satisfies CS, but it does not satisfy AM. Additionally, the prenucleolus satisfies DP and SYM, but it does not satisfy IR.

We are interested in combining core selection with aggregate monotonicity. Although the Shapley value and the prenucleolus does not satisfy CS and AM, these two properties are compatible on the domain of all games, since the per-capita prenucleolus (Grotte 1970) satisfies both properties (see for example Young et al. 1982 or Moulin 1988). The per-capita prenucleolus, $\bar{\nu}_*(v) \in \mathbb{R}^N$, is the preimputation $x \in I^*(v)$ that lexicographically minimizes the vector of per-capita excesses $\bar{e}(S, x) = \frac{v(S) - x(S)}{|S|}$, $\varnothing \neq S \subset N$, when these are arranged in order of descending magnitude.

However, as suggested by Selten and quoted in Young et al. (1982), the per-capita prenucleolus satisfies neither individual rationality nor the dummy player property. In Example 6 the per-capita prenucleolus satisfies neither IR nor the DP property, even for superadditive games[2]. On the other hand, the per-capita prenucleolus satisfies SYM.

Example 6 *Let (N, v) be the four-player game: $v(\{1\}) = 0$, $v(\{2\}) = v(\{3\}) = v(\{4\}) = v(\{1,2\}) = v(\{1,3\}) = v(\{1,4\}) = 1$, $v(\{2,3\}) = v(\{2,4\}) = v(\{3,4\}) = v(\{1,2,3\}) = v(\{1,2,4\}) = v(\{1,3,4\}) = 10$ and $v(\{2,3,4\}) = v(N) = 12$. Note that the game is superadditive and player 1 is a dummy player in (N, v). Now, let (N, v') be such that $v'(S) = v(S)$ for all $S \subset N$ and $v'(N) = 15$. Some computations[3] yield to $C(v') = \{\bar{\nu}_*(v')\} = \{(0, 5, 5, 5)\}$ and $\bar{\nu}_*(v) = (-0'75, 4'25, 4'25, 4'25)$. Then, $\bar{\nu}_*(v) \notin I(v)$ and $\bar{\nu}_{*,1}(v) \neq 0 = v(\{1\})$.*

Table 1 summarizes whether or not the Shapley value, the prenucleolus and the per-capita prenucleolus satisfy these five properties.

	Shapley value	Prenucleolus	Per-capita prenucleolus
CS	$\times$	✓	✓
AM	✓	$\times$	✓
IR	$\times$	$\times$	$\times$ (example 6)
DP	✓	✓	$\times$ (example 6)
S	✓	✓	✓

Table 1. Properties of solutions defined on G^N

A natural and open question is whether or not there exists a single-valued solution defined on G^N satisfying the five properties together.

Another possible approach is to restrict ourselves to essential games and to solutions defined on the set of essential games. The nucleolus and the per-capita nucleolus are solutions defined on E^N.

The nucleolus, $\nu(v) \in \mathbb{R}^N$, is the imputation $x \in I(v)$ that lexicographically minimizes the vector of excesses $e(S, x) = v(S) - x(S)$, $\varnothing \neq S \subset N$, when these are arranged in order of descending magnitude. In the following, we say that the nucleolus satisfies a particular property (CS, AM, IR,...) if it satisfies such a property for essential games. Note that in fact, the nucleolus satisfies IR by definition and, like the prenucleolus, it satisfies CS but not AM (Hokari 2000). Additionally, the nucleolus satisfies DP and SYM.

The per-capita nucleolus, $\bar{\nu}(v) \in \mathbb{R}^N$, is the imputation $x \in I(v)$ that lexicographically minimizes the vector of per-capita excesses $\bar{e}(S, x) = \frac{v(S)-x(S)}{|S|}$, $\varnothing \neq S \subset N$, when these are arranged in order of descending magnitude. As in the case of the nucleolus, we say that the per-capita nucleolus satisfies a particular property (CS, AM, IR,...) if it satisfies such a property for essential games. The per-capita nucleolus satisfies IR by definition, moreover, imposing individual rationality is enough to achieve the dummy player property too

[2] A game v is superadditive if and only if $v(S)+v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$, $S \cap T = \varnothing$.

[3] We use that for any two games $v, v' \in G^N$ with $v <_N v'$ it holds that $\bar{\nu}_*(v') = \bar{\nu}_*(v) + \frac{v'(N)-v(N)}{|N|} \cdot e_N$ (see Moulin 1988)

(Derks and Haller 1999). Like the per-capita prenucleolus, it satisfies CS, but unfortunately it does not satisfy AM. Example 7 shows that the per-capita nucleolus does not satisfy aggregate monotonicity[4]. Additionally, the per-capita nucleolus also satisfies SYM.

Table 2 summarizes whether or not the nucleolus and the per-capita nucleolus satisfy these five properties.

	Nucleolus	Per-capita nucleolus
CS	✓	✓
AM	×	×(example 7)
IR	✓	✓
DP	✓	✓
S	✓	✓

Table 2. Properties of solutions defined on E^N

Example 7 *Let (N, v) be the five-player game: $v(\{2,3\}) = v(\{1,4\}) = 5$, $v(\{1,2\}) = v(\{1,2,3\}) = v(N) = 8$ and $v(S) = 0$ otherwise. Now, let (N, v') be such that $v'(S) = v(S)$ for all $S \subset N$ and $v'(N) = 10$. Some computations[5] yield to $\bar{\nu}(v') = (5, 3'8, 1'2, 0, 0)$ and $\bar{\nu}(v) = (4, 4, 0, 0, 0)$. Clearly, the per-capita nucleolus does not satisfy aggregate monotonicity since $\bar{\nu}_2(v') = 3'8 < 4 = \bar{\nu}_2(v)$.*

From the observation of Table 2, once again, it seems natural to ask if there exists a single-valued solution defined on E^N satisfying the five properties together. The answer to this question will appear as a consequence of the study of the compatibility of these five properties for solutions defined on G^N.

The second part of this section is devoted to show that CS, AM, DP, IR and SYM are compatible. With this aim, let us first introduce two useful notions. Let (N, v) be an arbitrary game. We denote by B_v^N the set of balanced games that can be obtained from v by increasing or decreasing only the worth of the grand coalition: $B_v^N = \{v' \in B^N : v'(S) = v(S) \text{ for all } S \subset N\}$. Notice that for all $v \in G^N$, B_v^N is a non-empty subset of B^N.

Definition 8 *The root game (N, v_r) associated to the game (N, v) is the smallest game in B_v^N, i.e. $v_r \in B_v^N$ and $v_r(N) \leq w(N)$ for all $w \in B_v^N$. Additionally, a game (N, v) is said to be rooted if $v = v_r$.*

We denote by G_{root}^N the set of all rooted games. Notice that by definition, $C(w) \neq \varnothing$ for all $w \in G_{root}^N$. As a consequence, any single-valued solution on G^N satisfying core selection must pick an allocation in the core of any rooted game. Note also that any game v can be written in terms of its root game. If,

[4] We thank Javier Arin for fruitful conversations. In particular, for providing us Example 7. Recently, in Kleppe (2010), a PhD thesis defended on January 22, an example showing that the per-capita nucleolus is not aggregate monotonic is also provided.

[5] We use the Kohlberg (1971) criterion for the nucleolus, which can be applied to the per-capita nucleolus (see for example Derks and Haller 1999)

by u_N we denote the unanimity game associated to the grand coalition, where $u_N(N) = 1$ and $u_N(S) = 0$ otherwise, then,

$$v = v_r + (v(N) - v_r(N)) \cdot u_N, \tag{1}$$

where $v(N) - v_r(N)$ does not need to be positive. Indeed, if $v(N) \geq v_r(N)$ then $C(v) \neq \varnothing$, while if $v(N) < v_r(N)$ then $C(v) = \varnothing$.

The next notion is that of *potential dummies*.

Definition 9 *Let (N, v) be a game. A player $i \in N$ is called a potential dummy player in (N, v) if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subset N \setminus \{i\}$. By $PD(v)$ we denote the set of potential dummies in (N, v).*

Note that if $i \in PD(v)$, player i may not be a dummy player in (N, v); however, it will be a dummy player in a game (N, v') with $v'(S) = v(S)$ for all $S \subset N \setminus \{i\}$ and $v'(N) = v(\{i\}) + v(N \setminus \{i\})$. Notice also that $PD(v)$ could be the empty set.

Theorem 10 *Core selection, aggregate monotonicity, individual rationality, the dummy player property and symmetry are compatible for single-valued solutions defined on G^N.*

Proof. We first show the following two claims:

Claim 1: Let w be a rooted game, if $i, j \in N$ are symmetric players in (N, w) and $i \in PD(w)$ then $j \in PD(w)$.

Proof of Claim 1: Let i, j be symmetric players in (N, w) and $i \in PD(w)$, we have to show that $w(S \cup \{j\}) - w(S) = w(\{j\})$ for all $S \subset N \setminus \{j\}$.

We distinguish two cases: first, let it be $S \subset N \setminus \{j\}$ with $i \notin S$. Then, it follows that $w(S \cup \{j\}) - w(S) = w(S \cup \{i\}) - w(S) = w(\{i\}) = w(\{j\})$, where the first and third equalities hold since i, j are symmetric players in (N, w), and the second equality holds since $i \in PD(w)$.

Second, let it be $S \subset N \setminus \{j\}$ with $i \in S$. Then, it follows that $w(S \cup \{j\}) - w(S) = w(((S \setminus \{i\}) \cup \{i\}) \cup \{j\}) - w((S \setminus \{i\}) \cup \{i\}) = w((S \setminus \{i\}) \cup \{j\}) + w(\{i\}) - (w(S \setminus \{i\}) + w(\{i\})) = w((S \setminus \{i\}) \cup \{j\}) - w(S \setminus \{i\}) = w((S \setminus \{i\}) \cup \{i\}) - w(S \setminus \{i\}) = w(\{i\}) = w(\{j\})$, where the fourth and sixth equalities hold since i, j are symmetric players in (N, w), and the second and fifth equalities hold since $i \in PD(w)$.

Here finishes the proof of Claim 1.

Claim 2: Let w be a rooted game, then there exists $x \in C(w)$ such that $x_i = w(\{i\})$ for all $i \in PD(w)$ and $x_i = x_j$ for any pair $i, j \in N$ of symmetric players in (N, w).

Proof of Claim 2: Let w be a rooted game, we first show that there exists $x \in C(w)$ such that $x_i = w(\{i\})$ for all $i \in PD(w)$. In case $PD(w) = \varnothing$ it is trivial. If $PD(w) = N$ then $w(S) = \sum_{i \in S} w(\{i\})$ for all $S \subset N$. Now, take $x = (w(\{1\}), w(\{2\}), ..., w(\{n\}))$, clearly $x(S) = w(S)$ for all $S \subset N$, and being (N, w) rooted we have $w(N) = \sum_{i \in N} w(\{i\})$ and $C(w) = \{(w(\{1\}), w(\{2\}), ...,$

$w(\{n\}))\}$.

In any other case, $\varnothing \neq PD(w) \subset N$, take first an arbitrary element y in $C(w)$. From y, define a new allocation $x \in \mathbb{R}^N$ as follows:

$$x_i = \begin{cases} w(\{i\}) & \text{if } i \in PD(w), \\ y_i + \frac{\sum_{i \in PD(w)} (y_i - w(\{i\}))}{|N \setminus PD(w)|} & \text{if } i \notin PD(w). \end{cases}$$

Then clearly, $x(N) = y(N) = w(N)$. Now, let $S \subset N$ be an arbitrary coalition. To show that $x(S) \geq w(S)$, note first that if $PD(w) \cap S = \{i_1, i_2, ..., i_s\}$, it follows that $w(S) = w(S \setminus \{i_1\}) + w(\{i_1\}) = w(S \setminus \{i_1, i_2\}) + w(\{i_1\}) + w(\{i_2\}) = \cdots = w(S \setminus PD(w)) + \sum\limits_{i \in PD(w) \cap S} w(\{i\})$. Consequently,

$$\begin{aligned} x(S) &= \sum_{i \in PD(w) \cap S} w(\{i\}) + \sum_{i \in S \setminus PD(w)} \left(y_i + \frac{\sum_{i \in PD(w)} (y_i - w(\{i\}))}{|N \setminus PD(w)|} \right) \\ &= \sum_{i \in PD(w) \cap S} w(\{i\}) + \sum_{i \in S \setminus PD(w)} y_i + |S \setminus PD(w)| \frac{\sum_{i \in PD(w)} (y_i - w(\{i\}))}{|N \setminus PD(w)|} \\ &\geq \sum_{i \in PD(w) \cap S} w(\{i\}) + \sum_{i \in S \setminus PD(w)} y_i \geq \sum_{i \in PD(w) \cap S} w(\{i\}) + w(S \setminus PD(w)) \\ &= w(S) \end{aligned}$$

where the last two inequalities follow from $y \in C(w)$. Hence, we have shown that for any rooted game w, there exists $x \in C(w)$ such that $x_i = w(\{i\})$ for all $i \in PD(w)$.

Now, let $y \in C(w)$ be such that $y_i = w(\{i\})$ for all $i \in PD(w)$ and suppose $y_i \neq y_j$ for a pair of symmetric players $i, j \in N$ in (N, w). Define $y' \in \mathbb{R}^N$ as follows:

$$y'_k = \begin{cases} y_k & \text{if } k \neq i, j, \\ y_i & \text{if } k = j, \\ y_j & \text{if } k = i, \end{cases}$$

which clearly belongs to $C(w)$ too. From y and y', we obtain $z = \frac{1}{2}(y + y')$ which satisfies $z \in C(w)$, $z_i = z_j$ and by Claim 1, $z_i = w(\{i\})$ for all $i \in PD(w)$. From z, and in a finite number of steps, we can obtain an allocation $x \in C(w)$ such that $x_i = w(\{i\})$ for all $i \in PD(w)$ and $x_i = x_j$ for any pair of symmetric players $i, j \in N$ in (N, w).

Here finishes the proof of Claim 2.

To continue with the proof of the Theorem, we, next, define a single-valued solution on G^N, $F : G^N \to \mathbb{R}^N$, satisfying CS, AM, IR, DP and SYM. First, for each $w \in G^N_{root}$ we select an arbitrary $x^w \in C(w)$ such that $x^w_i = w(\{i\})$ for all $i \in PD(w)$ and $x^w_i = x^w_j$ for any pair of symmetric players $i, j \in N$ in (N, w). Now, consider an arbitrary game $v \in G^N$. We introduce, associated to v, two allocations: let $x^v_I \in \mathbb{R}^N$ be the allocation defined by $x^v_I = (v(\{1\}), v(\{2\}), ..., v(\{n\}))$, and let $\bar{x}^v_I \in \mathbb{R}^N$ be the allocation defined by $\bar{x}^v_I = x^v_I + e_{N \setminus PD(v)}$. Then, to define the solution, we distinguish three cases:

Case I) If $v_r(N) \neq \sum_{i \in N} v(\{i\})$, then define

$$F(v) = x^v_I + \frac{v(N) - \sum_{i \in N} v(\{i\})}{v_r(N) - \sum_{i \in N} v(\{i\})} \left(x^{v_r} - x^v_I\right)$$

.

Case II) If $v_r(N) = \sum_{i \in N} v(\{i\})$ and $PD(v) \neq N$, then define

$$.F(v) = x_I^v + \frac{v(N) - \sum_{i \in N} v(\{i\})}{|N \backslash PD(v)|} (\bar{x}_I^v - x_I^v)$$

.

Case III) If $v_r(N) = \sum_{i \in N} v(\{i\})$ and $PD(v) = N$, then define

$$F(v) = x_I^v + \frac{v(N) - \sum_{i \in N} v(\{i\})}{|N|} e_N$$

Note that in fact, if $v_r(N) \neq \sum_{i \in N} v(\{i\})$ we are taking the straight line joining the allocations x^{v_r} and x_I^v. In this case, clearly $v_r(N) > \sum_{i \in N} v(\{i\})$ and consequently $x^{v_r} \neq x_I^v$, moreover, since $x^{v_r} \in C(v_r)$ then $x^{v_r} \geq x_I^v$. On the other hand, if $v_r(N) = \sum_{i \in N} v(\{i\})$ note that since $v_r \in B^N$, it holds that $C(v_r) = \{(v(\{1\}), v(\{2\}), ..., v(\{n\}))\}$, or equivalently $x^{v_r} = x_I^v$. Then, we distinguish two cases. In case $PD(v) \neq N$, we take the straight line joining the allocations x_I^v and $\bar{x}_I^v$, which are clearly different and, moreover, $\bar{x}_I^v \geq x_I^v$. While in case $PD(v) = N$, we divide equally among all players the amount $v(N) - v_r(N)$ from the allocation x_I^v.

To see that this single-valued solution, F, satisfies core selection, let $v \in B^N$ and then $v(N) \geq v_r(N)$. Suppose first that we are in case I), then clearly $\frac{v(N) - \sum_{i \in N} v(\{i\})}{v_r(N) - \sum_{i \in N} v(\{i\})} \geq 1$ and consequently $F(v) \geq x^{v_r}$, which follows from $x^{v_r} \geq x_I^v$. Now, since $x^{v_r} \in C(v_r)$ we have that $F(v) \in C(v)$. Now, suppose we are in case II), then $\frac{v(N) - \sum_{i \in N} v(\{i\})}{|N \backslash PD(v)|} \geq 0$ since $v(N) \geq v_r(N) = \sum_{i \in N} v(\{i\})$, and consequently $F(v) \geq x_I^v$ which follows from $\bar{x}_I^v \geq x_I^v$. Now, since $x_I^v \in C(v_r)$ we have that $F(v) \in C(v)$. Finally, in case III) we have $\frac{v(N) - \sum_{i \in N} v(\{i\})}{|N|} \geq 0$ since $v(N) \geq v_r(N) = \sum_{i \in N} v(\{i\})$, and consequently $F(v) \geq x_I^v$ and $F(v) \in C(v)$ since $x_I^v \in C(v_r)$.

To see that this single-valued solution, F, satisfies aggregate monotonicity, let $v, v' \in G^N$ with $v <_N v'$. Then, clearly $v(N) < v'(N)$, and it is easy to see by a direct observation of the solution that in any of the three different cases we have $F(v) \leq F(v')$.

To see that this single-valued solution, F, satisfies individual rationality, let $v \in E^N$, then clearly $v(N) \geq \sum_{i \in N} v(\{i\})$. Now suppose that $v_r(N) \neq \sum_{i \in N} v(\{i\})$, in this case, $\frac{v(N) - \sum_{i \in N} v(\{i\})}{v_r(N) - \sum_{i \in N} v(\{i\})} \geq 0$ and together with the fact that $x^{v_r} \geq x_I^v$, it follows that $F(v) \geq x_I^v$ and consequently $F(v) \in I(v)$. On the other hand, if $v_r(N) = \sum_{i \in N} v(\{i\})$ then $\frac{v(N) - \sum_{i \in N} v(\{i\})}{|N \backslash PD(v)|} \geq 0$ if we are in case II) and $\frac{v(N) - \sum_{i \in N} v(\{i\})}{|N|} \geq 0$ in case III), and consequently $F(v) \geq x_I^v$ and $F(v) \in I(v)$.

Symmetry is satisfied since if $v \in G^N$ have a pair of symmetric players $i, j \in N$, then i, j are clearly symmetric players in v_r too. Consequently, since x^{v_r} satisfies $x_i^{v_r} = x_j^{v_r}$, x_I^v satisfies $x_{I,i}^v = v(\{i\}) = v(\{j\}) = x_{I,j}^v$ and $\bar{x}_I^v$, by Claim 1, satisfies $\bar{x}_{I,i}^v = \bar{x}_{I,j}^v$ for any pair of symmetric players $i, j \in N$ in v_r, it follows that $F_i(v) = F_j(v)$.

Finally, to check the dummy player property, let $i \in N$ be a dummy player in a game (N, v). Note first that in fact only a potential dummy can become a dummy player and that $PD(v_r) = PD(v)$, and consequently $i \in PD(v_r)$. Suppose now we are in case I), then we have $x_i^{v_r} = v(\{i\})$ since $i \in PD(v_r)$ and $x_{I,i}^v = v(\{i\})$; and hence, $F_i(v) = v(\{i\})$. Similarly, in case II), we have $x_{I,i}^v = \bar{x}_{I,i}^v = v(\{i\})$ since $i \in PD(v_r)$, and consequently, $F_i(v) = v(\{i\})$. Case III) deserves more attention. Let $i \in N$ be a dummy player in a game (N, v) and $v_r(N) = \sum_{i \in N} v(\{i\})$ with $PD(v_r) = N$. Then, since $v(N) = v(N \setminus \{i\}) + v(\{i\})$ and, as shown in Claim 2, $v(S) = v_r(S) = \sum_{i \in S} v_r(\{i\})$ for all $S \subset N$, we have $v(N) = \sum_{i \in N} v(\{i\})$. Hence $v = v_r$ and $F_i(v) = F_i(v_r) = v(\{i\})$. ■

Corollary 11 *Core selection, aggregate monotonicity, individual rationality, the dummy player property and symmetry are compatible for single-valued solutions defined on E^N.*

Proof. It follows easily from the proof of Theorem 10 defining the same single-valued solution restricted to essential games E^N. ■

4 Core selection and aggregate monotonicity

The main issue of the second part of the paper is to characterize single-valued solution satisfying CS, AM, IR, DP and SYM, and for this aim it is crucial the characterization of the single-valued solutions satisfying core selection and aggregate monotonicity. In order to characterize the single-valued solutions on G^N that satisfy CS and AM, we first introduce *monotonic stable systems*.

A map $\sigma : G_{root}^N \to \mathbb{R}^N$ is a stable root-selection if $\sigma(w) \in C(w)$ for all $w \in G_{root}^N$. A stable root-selection selects a core element for any rooted game. The family of stable root-selections on G_{root}^N is denoted by Σ^N.

A monotonic curve is a function $\gamma : \mathbb{R} \to \mathbb{R}^N$ with $\sum_{i \in N} \gamma_i(s) = s$ for all $s \in \mathbb{R}$, $\gamma(0) = (0, \ldots, 0)$ and $\gamma(s) \leq \gamma(t)$ for each pair $s, t \in \mathbb{R}$ with $s \leq t$. Note, that a monotonic curve will assign non-negative (non-positive) vectors to positive (negative) real numbers. By Λ^N we denote the family of monotonic curves in $\mathbb{R}^N$.

A monotonic curve system is a map $\Gamma : G_{root}^N \to \Lambda^N$. A monotonic curve system selects a monotonic curve, not necessarily the same, for all rooted games. For simplicity we will denote $\Gamma(w)$ by γ^w. By MCS^N we denote the family of monotonic curve systems on G_{root}^N.

Definition 12 *A monotonic stable system is a pair (σ, Γ) with $\sigma \in \Sigma^N$ and $\Gamma \in MCS^N$.*

A monotonic stable system selects, for any rooted game $w \in G_{root}^N$, an element in the core of the game w, and a monotonic curve associated to the game w. Note that these selections can vary from one rooted game to another.

Note also that associated to any monotonic stable system $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, we can define a single-valued solution $F^{\sigma,\Gamma} : G^N \to \mathbb{R}^N$ on G^N by

$F^{\sigma,\Gamma}(v) = \sigma(v_r) + \gamma^{v_r}(v(N) - v_r(N))$. Moreover, any single-valued solution on G^N, $F : G^N \to \mathbb{R}^N$, could be described by an efficient, not necessarily stable, selection for all rooted games, and a curve system, selecting a curve, not necessarily monotonic, for all rooted games.

In next theorem, we characterize the single-valued solutions satisfying core selection and aggregate monotonicity using monotonic stable systems.

Theorem 13 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies core selection and aggregate monotonicity if and only if there exists a monotonic stable system $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, such that $F = F^{\sigma,\Gamma}$.*

Proof. Let $F^{\sigma,\Gamma} : G^N \to \mathbb{R}^N$ be such that $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$. Then it is straightforward that $F^{\sigma,\Gamma}$ satisfies core selection and aggregate monotonicity.

Now, let $F : G^N \to \mathbb{R}^N$ be a core selection and aggregate monotonic single-valued solution. Define $\sigma : G^N_{root} \to \mathbb{R}^N$ by $\sigma(w) = F(w)$ for all $w \in G^N_{root}$, by core selection $F(w) \in C(w)$ for all $w \in G^N_{root}$, hence $\sigma \in \Sigma^N$. Next, define $\Gamma : G^N_{root} \to \Lambda^N$ by $\gamma^w(s) = F(w + s \cdot u_N) - F(w)$ for all $s \in \mathbb{R}$ and all $w \in G^N_{root}$. Clearly, for every $w \in G^N_{root}$ we have $\gamma^w(0) = (0, \dots, 0)$, $\sum_{i \in N} \gamma^w_i(s) = s$ for all $s \in \mathbb{R}$ and, for each pair $s, t \in \mathbb{R}$ with $s \leq t$ it holds that

$$\begin{aligned}\gamma^w(s) &= F(w + s \cdot u_N) - F(w) \\ &\leq F(w + t \cdot u_N) - F(w) \\ &= \gamma^w(t),\end{aligned}$$

where the inequality follows since F satisfies aggregate monotonicity. Consequently, γ^w is a monotonic curve for all $w \in G^N_{root}$ and then $\Gamma \in MCS^N$. Hence, (σ, Γ) is a monotonic stable system.

To finish, let us show that $F^{\sigma,\Gamma} = F$ with $\sigma \in \Sigma^N$ and $\Gamma \in MCS^N$ as defined above. Let v be an arbitrary game, then

$$\begin{aligned}F^{\sigma,\Gamma}(v) &= \sigma(v_r) + \gamma^{v_r}(v(N) - v_r(N)) \\ &= F(v_r) + F(v_r + (v(N) - v_r(N)) \cdot u_N) - F(v_r) = F(v).\end{aligned}$$

where the last equality follows from equation (1). ■

5 Dummy player, individual rationality and symmetry

Next, we characterize all single-valued solutions satisfying any combination of CS and AM with IR, and/or DP and/or SYM. We will start with the characterization of the single-valued solutions on G^N satisfying core selection, aggregate monotonicity and individual rationality. First, we need to define *individually rational adapted* monotonic stable systems.

Definition 14 *We say that a monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, is individually rational adapted if for all $w \in G^N_{root}$ it holds that $x^w_I - \sigma(w) \in Im(\gamma^w)$, where $x^w_I = (w(\{1\}), ...w(\{n\}))$ and $Im(\gamma^w)$ denotes the image of γ^w.*

Note that for a given rooted game w, an individually rational adapted monotonic stable system imposes that the monotonic curve γ^w contains the vector $x_I^w - \sigma(w)$. This will be crucial to ensure that a solution defined by a monotonic stable system selects precisely x_I^w in the game w_I, defined as $w_I(S) = w(S)$ for all $S \subset N$ and $w_I(N) = \sum_{i\in N} w(\{i\})$.

With individually rational adapted monotonic stable systems we characterize single-valued solutions satisfying core selection, aggregate monotonicity and individual rationality.

Theorem 15 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies core selection, aggregate monotonicity and individual rationality if and only if there exists an individually rational adapted monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, such that $F = F^{\sigma,\Gamma}$.*

Proof. This is straightforward following the proof of Theorem 13 and showing the next two points:

1) If $F^{\sigma,\Gamma} : G^N \to \mathbb{R}^N$ is such that $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$ with (σ, Γ) being individually rational adapted then $F^{\sigma,\Gamma}$ satisfies IR. Let $v \in E^N$, then $F^{\sigma,\Gamma}(v) = \sigma(v_r) + \gamma^{v_r}(v(N) - v_r(N)) \geq \sigma(v_r) + \gamma^{v_r}\left(\sum_{i\in N} v(\{i\}) - v_r(N)\right) = \sigma(v_r) + x_I^{v_r} - \sigma(v_r) = x_I^{v_r}$. Here, the inequality follows from γ^{v_r} being a monotonic curve and $v(N) \geq \sum_{i\in N} v(\{i\})$. The second equality holds since $x_I^{v_r} - \sigma(v_r) \in Im(\gamma^{v_r})$ and clearly, the only $s \in \mathbb{R}$ such that $\gamma^{v_r}(s) = x_I^{v_r} - \sigma(v_r)$ is $s = \sum_{i\in N} v(\{i\}) - v_r(N)$. Hence $F^{\sigma,\Gamma}(v) \geq x_I^{v_r}$ and $F^{\sigma,\alpha}$ satisfies IR .

2) If $F : G^N \to \mathbb{R}^N$ is a single-valued solution satisfying CS, AM and IR then $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$ defined as in the proof of Theorem 13, that is, $\sigma(w) = F(w)$ and $\gamma^w(s) = F(w + s \cdot u_N) - F(w)$ for all $s \in \mathbb{R}$ and all $w \in G^N_{root}$, is individually rational adapted. Let $w \in G^N_{root}$, and let $w_I \in G^N$ be such that $w_I(S) = w(S)$ for all $S \subset N$ and $w_I(N) = \sum_{i\in N} w(\{i\})$, then since $I(w_I) = \{x_I^w\}$ and F satisfies IR, it follows that $F(w_I) = x_I^w$. Consequently, $\gamma^w(w_I(N) - w(N)) = F(w + (w_I(N) - w(N)) \cdot u_N) - F(w) = F(w_I) - F(w) = x_I^w - \sigma(w)$, and $x_I^w - \sigma(w) \in Im(\gamma^w)$. ■

Let us now show an incompatibility result. First, we define a property that we call *equal surplus division*, which is a stronger version of AM.

Definition 16 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies equal surplus division (ESD) if for all $v, v' \in G^N$ with $v <_N v'$, it holds that $F(v') = F(v) + \frac{v'(N) - v(N)}{|N|} e_N$.*

A single-valued solution, $F : G^N \to \mathbb{R}^N$, that satisfies CS and IR cannot satisfy ESD. Consider, for instance, the three-player game $w(\{1, 2\}) = w(\{1, 3\}) = w(\{1, 2, 3\}) = 1$ and $w(S) = 0$ otherwise, which is clearly a rooted game and satisfies $w(N) > \sum_{i\in N} w(\{i\})$. This game has a unique core element which is $(1, 0, 0)$, and any single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfying CS must hold $F(w) = (1, 0, 0)$. On the other hand, note that players 2 and 3 receive exactly their individual worth. Now, suppose that F satisfies

ESD too, and take the game $v \in G^N$ with $v(S) = w(S)$ for all $S \subset N$ and $v(N) = \frac{1}{2}$. Then $F(v) = (\frac{5}{6}, -\frac{1}{6}, -\frac{1}{6}) \notin I(v)$ and consequently F does not satisfy IR. In fact, if F satisfies ESD then $\gamma^w(s) = \frac{s}{|N|}e_N$ and clearly when $s = \sum_{i \in N} w(\{i\}) - w(N) = -\frac{1}{2}$ we have $\gamma^w(s) = (-\frac{1}{6}, -\frac{1}{6}, -\frac{1}{6}) \neq x_I^w - F(w)$ and consequently $x_I^w - F(w) \notin Im(\gamma^w)$. It can be easily checked that the same happens in Example 6. We then state the following corollary.

Corollary 17 *For $|N| \geq 3$ there is no single-valued solution on G^N that satisfies core selection, equal surplus division and individual rationality.*

Let us remark that among the single-valued solutions presented in Section 2, those that satisfy AM, the Shapley value and the per-capita prenucleolus, satisfy ESD too. However, to define a solution satisfying CS, AM and IR we cannot use a solution which satisfies ESD. In Theorem 10 we provide a single-valued solution satisfying these three properties, but to do it we lose the simplicity that ESD solutions allow.

To continue, we will next characterize the single-valued solutions on G^N satisfying core selection, aggregate monotonicity and the dummy player property. To this aim, we first show that potential dummy players satisfy an important property which we call the dummy coincidence property. Suppose an arbitrary game v with a pair of potential dummies $i, j \in PD(v)$. Obviously, player i will become a dummy player in a game v', obtained from v, only varying the worth of the grand coalition. It is also obvious that player j will become a dummy player in a game v'', obtained also from v, only varying the worth of the grand coalition. What is somehow surprising is that at the end the games v' and v'' will be the same game. In the next Lemma we state and prove the dummy coincidence property.

Lemma 18 *Let $i \in N$ be a dummy player in (N, v) and $j \in PD(v)$, then j is also a dummy player in (N, v) and $v(N) = \sum_{i \in PD(v)} v(\{i\}) + v(N \backslash PD(v))$. Moreover, if $v \in B^N$ then $v = v_r$.*

Proof. Let (N, v) be a game, i a dummy player in (N, v) and $j \in PD(v)$, then $v(N) - v(N \backslash \{i\}) = v(\{i\})$. Take $S = N \backslash \{i, j\}$, since $j \in PD(v)$, then $v((N \backslash \{i, j\}) \cup \{j\}) - v(N \backslash \{i, j\}) = v(\{j\})$. Hence, $v(N) = v(\{i\}) + v(N \backslash \{i\}) = v(\{i\}) + v(\{j\}) + v(N \backslash \{i, j\})$. Moreover, since i is a dummy player, we also have $v((N \backslash \{i, j\}) \cup \{i\}) - v(N \backslash \{i, j\}) = v(\{i\})$ and, consequently, $v(N \backslash \{i, j\}) = v(N \backslash \{j\}) - v(\{i\})$.

To show that j is a dummy player in (N, v) it is enough to see that

$$\begin{aligned} v(N) - v(N \backslash \{j\}) &= v(\{i\}) + v(\{j\}) + v(N \backslash \{i, j\}) - v(N \backslash \{j\}) \\ &= v(\{i\}) + v(\{j\}) + (v(N \backslash \{j\}) - v(\{i\})) - v(N \backslash \{j\}) \\ &= v(\{j\}). \end{aligned}$$

Consequently, any potential dummy in the game (N, v) is also a dummy player in the game (N, v) and, then $v(N) = \sum_{i \in PD(v)} v(\{i\}) + v(N \backslash PD(v))$.

Finally, to prove that if $v \in B^N$ then $v = v_r$, note that for any $v \in B^N$ we have $v(N) \geq v_r(N)$. Now, suppose $v(N) = \sum_{i \in PD(v)} v(\{i\}) + v(N \backslash PD(v)) > v_r(N)$; this inequality, clearly, involves a contradiction with v_r being balanced. Hence, $v(N) = v_r(N)$ and $v = v_r$. ■

The above lemma has some important consequences. If one considers an arbitrary game v with a set of potential dummies, $PD(v)$, and increases or decreases only the worth of the grand coalition, at a particular value, the value $\sum_{i \in PD(v)} v(\{i\}) + v(N \backslash PD(v))$, all potential dummies will become dummy players. Note also that $PD(v) = PD(v_r)$ and $v(S) = v_r(S)$ for all $S \subset N$, then, for simplicity, and for a given rooted game $w \in G^N_{root}$, we denote $s_D = \left(\sum_{i \in PD(w)} w(\{i\}) + w(N \backslash PD(w))\right) - w(N)$. Let us now define *dummy adapted* monotonic stable systems.

Definition 19 *We say that a monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, is dummy adapted if for all $w \in G^N_{root}$ it holds that $\gamma_i^w(s_D) = w(\{i\}) - \sigma_i(w)$ for all $i \in PD(w)$.*

With dummy adapted monotonic stable systems we characterize the single-valued solutions satisfying core selection, aggregate monotonicity and the dummy player property.

Theorem 20 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies core selection, aggregate monotonicity and the dummy player property if and only if there exists a dummy adapted monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, such that $F = F^{\sigma,\Gamma}$.*

Proof. This is straightforward following the proof of Theorem 13 and showing the next two points:

1) If $F^{\sigma,\Gamma} : G^N \to \mathbb{R}^N$ is such that $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$ with (σ, Γ) being dummy adapted then $F^{\sigma,\Gamma}$ satisfies DP. Let $i \in N$ be a dummy player in game (N, v). If $v \in B^N$, by Lemma 18, $v = v_r$, and then we have $F^{\sigma,\Gamma}(v) = \sigma(v_r) \in C(v)$. Now, since $F^{\sigma,\Gamma}(v)$ is a core element it follows that $F_i^{\sigma,\Gamma}(v) = v(\{i\})$. If $v \notin B^N$, then $F_i^{\sigma,\Gamma}(v) = \sigma_i(v_r) + \gamma_i^{v_r}(v(N) - v_r(N)) = \sigma_i(v_r) + \gamma_i^{v_r}(s_D) = \sigma_i(v_r) + v(\{i\}) - \sigma_i(v_r) = v(\{i\})$, where the second equality follows from Lemma 18 and the definition of s_D, and the third equality from (σ, Γ) being dummy adapted. Hence, $F^{\sigma,\Gamma}$ satisfies the dummy player property.

2) If $F : G^N \to \mathbb{R}^N$ is a single-valued solution satisfying CS, AM and DP then $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$ defined as in the proof of Theorem 13 is dummy adapted. Let $w \in G^N_{root}$ and $i \in PD(w)$, then $\gamma_i^w(s_D) = F_i(w + s_D \cdot u_N) - F_i(w) = w(\{i\}) - \sigma_i(w)$, which follows since F satisfies the dummy player property, and by Lemma 18 and the definition of s_D, player i becomes a dummy player in the game $(N, w + s_D \cdot u_N)$. ■

Again, an incompatibility result appears. A single-valued solution, $F : G^N \to \mathbb{R}^N$, that satisfies CS and DP cannot satisfy equal surplus division. Consider, for instance the three-player rooted game $w(\{1\}) = 0$, $w(\{1,2,3\}) = 2$ and

$w(S) = 1$ otherwise, that satisfies $w(N) > \sum_{i \in PD(w)} w(\{i\}) + w(N \backslash PD(w))$ and where player 1 is the only potential dummy player. This game has a unique core element $(0, 1, 1)$ and any single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfying CS must hold $F(w) = (0, 1, 1)$. On the other hand, note that player 1 receives exactly his individual worth. Now, suppose that F satisfies ESD too, and take the game $v \in G^N$ with $v(S) = w(S)$ for all $S \subset N$ and $v(N) = 1$. Then $F(v) = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, but the game v satisfies $v(N) = \sum_{i \in PD(w)} w(\{i\}) + w(N \backslash PD(w))$ and consequently, player 1 is a dummy player in (N, v). Hence, F does not satisfy DP. In fact, if F satisfies ESD then $\gamma^w(s) = \frac{s}{|N|} e_N$, and clearly when $s = s_D$ and player 1 becomes a dummy player, he cannot receive exactly $w(\{1\})$. Again, this is also what happens in Example 6. We then state the following corollary.

Corollary 21 *For $|N| \geq 3$ there is no single-valued solution on G^N that satisfies core selection, equal surplus division and the dummy player property.*

We want to remark here again that to define a solution satisfying CS, AM and DP we cannot use a solution which satisfies equal surplus division, and this is a reason why the single-valued solution we provide in Theorem 10 lacks simplicity.

Finally, we characterize the single-valued solutions satisfying core selection, aggregate monotonicity and symmetry. Note that given an arbitrary game (N, v) with a pair of symmetric players $i, j \in N$, then $i, j \in N$ are also symmetric players in a new game (N, v') with $v(S) = v'(S)$ for all $S \subset N$ and $v(N) \neq v'(N)$. First, we need to define *symmetry adapted* monotonic stable systems.

Definition 22 *We say that a monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, is symmetry adapted if for all $w \in G^N_{root}$ and any pair $i, j \in N$ of symmetric players in (N, w) it holds that $\sigma_i(w) = \sigma_j(w)$ and $\gamma^w_i(s) = \gamma^w_j(s)$ for all $s \in \mathbb{R}$.*

With symmetry adapted monotonic stable systems we characterize the single-valued solutions satisfying core selection, aggregate monotonicity and symmetry.

Theorem 23 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies core selection, aggregate monotonicity and symmetry if and only if there exists a symmetry adapted monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, such that $F = F^{\sigma,\Gamma}$.*

Proof. This is straightforward following the proof of Theorem 13, and showing the next two points:

1) It is easy to see that if $F^{\sigma,\Gamma} : G^N \to \mathbb{R}^N$ is such that $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$ with (σ, Γ) being symmetry adapted then $F^{\sigma,\Gamma}$ satisfies SYM.

2) If $F : G^N \to \mathbb{R}^N$ is a single-valued solution satisfying CS, AM and SYM then $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$ defined as in the proof of Theorem 13 is symmetry adapted. Let $w \in G^N_{root}$ and $i, j \in N$ a pair of symmetric players in (N, w), then it follows that $\sigma_i(w) = F_i(w) = F_j(w) = \sigma_j(w)$ from the symmetry of F. Moreover, $\gamma^w_i(s) = F_i(w + s \cdot u_N) - F_i(w) = F_j(w + s \cdot u_N) - F_j(w) = \gamma^w_j(s)$ for all $s \in \mathbb{R}$, since i, j are also symmetric players in the game $(N, w + s \cdot u_N)$ for all $s \in \mathbb{R}$ and F satisfies symmetry. ■

To conclude, we state without a proof (it follows easily from Theorems 13, 15, 20 and 23) the following corollary, in which we characterize the set of all single-valued solutions on G^N that satisfy core selection, aggregate monotonicity, the dummy player property, individual rationality and symmetry together. From Theorem 10, we know the existence of solutions satisfying these five properties together. In fact, we know that the conditions we impose on monotonic stable systems to get IR, DP and SYM are all of them compatible.

Corollary 24 *A single-valued solution, $F : G^N \to \mathbb{R}^N$, satisfies core selection, aggregate monotonicity, the dummy player property, individual rationality and symmetry if and only if there exists an individually rational, dummy and symmetry adapted monotonic stable system, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, such that $F = F^{\sigma,\Gamma}$.*

As a final remark, note that to characterize the set of single-valued solutions satisfying the combination of CS and AM with IR, and/or DP, and/or SYM, it is enough to impose to monotonic stable systems, $(\sigma, \Gamma) \in \Sigma^N \times MCS^N$, the corresponding conditions to be individually rational, and/or dummy, and/or symmetry adapted.

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