DOCUMENTS DE TREBALL
DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E12/277

Consumption, investment and life insurance strategies with heterogeneous discounting

Albert de-Paz

Jesús Marín-Solano

Jorge Navas

Oriol Roch

Adreça correspondencia:
Departament de Matemàtica económica, financera i actuarial
Facultat d'Economia i Empresa
Universitat de Barcelona
Avinguda Diagonal 690
08034 Barcelona (Spain)
Phone: 0034934020101
Email:adepaz@ub.edu

Acknowledgements: Acknowledgments: A preliminary version of this work was presented in the 15th International Congress on Insurance: Mathematics and Economics (Trieste, June 2011). We would like to thank participants for their comments. This work has been partially supported by MEC (Spain) Grant ECO2010-18015.

Abstract

In this paper we analyze how the optimal consumption, investment and life insurance rules are modified by the introduction of a class of time-inconsistent preferences. In particular, we account for the fact that an agent's preferences evolve along the planning horizon according to her increasing concern about the bequest left to her descendants and about her welfare at retirement. To this end, we consider a stochastic continuous time model with random terminal time for an agent with a known distribution of lifetime under heterogeneous discounting. In order to obtain the time-consistent solution, we solve a non-standard dynamic programming equation. For the case of CRRA and CARA utility functions we compare the explicit solutions for the time-inconsistent and the time-consistent agent. The results are illustrated numerically.

Abstract

En aquest treball s'analitza l'efecte que comporta l'introducció de preferències inconsistents temporalment sobre les decisions òptimes de consum, inversió i compra d'assegurança de vida. En concret, es pretén recollir la creixent importància que un individu dóna a la herència que deixa i a la riquesa disponible per a la seva jubilació al llarg de la seva vida laboral. Amb aquesta finalitat, es parteix d'un model estocàstic en temps continu amb temps final aleatori, i s'introdueix el descompte heterogeni, considerant un agent amb una distribució de vida residual coneguda. Per tal d'obtenir solucions consistents temporalment es resol una equació de programació dinàmica no estàndard. Per al cas de funcions d'utilitat del tipus CRRA i CARA es troben solucions explícites. Finalment, els resultats obtinguts s'il·lustren numèricament.

JEL classification: C61; G11; D91

Keywords: heterogeneous discounting; consumption and portfolio rules; life insurance; time-consistency

1 Introduction

The introduction of an uncertain lifetime in portfolio optimization models has proved to be useful in the study of the demand for life insurance, which has usually been derived from a bequest function. The starting point for modern research on the subject dates back to Yaari (1965) who studied the problem of life insurance in a deterministic financial environment with the stochastic time of death as the only source of uncertainty. Later on, Richard (1975) combined the portfolio optimization model in Merton (1969, 1971) with the model in Yaari (1965) to deal with a life-cycle consumption/investment problem in the presence of life insurance and random terminal time. However, the model introduced by Richard (1975) had several unsatisfactory aspects. First, the value function was not well-defined at the final time because the random variable used to model the lifetime was assumed to be bounded. This is a very important point in view of the fact that the problem was analyzed using a dynamic programming approach, which proceeds backward in time. Second, as Leung (1994) pointed out, there is a problem with the existence of interior solutions. In order to overcome these difficulties, Pliska and Ye (2007) incorporated the randomness of the planning horizon by means of the uncertain life model found in reliability theory. In contrast to Richard (1975), in which the random lifetime took values on a bounded interval, in that paper the authors considered an intertemporal model and allowed the random lifetime to take values on $[0, \infty)$. In addition, the authors refined the theory in the following ways. First, the planning horizon was considered to be some fixed point in the future T (the retirement time for the decision maker) in contrast with the model in Richard (1975) in which the planning horizon was interpreted as the finite upper bound on the lifetime. Second, at T a utility was introduced accounting for the agent wealth at the final time. After setting up the HJB equation and deriving the optimal feedback control law, Pliska and Ye (2007) obtained explicit solutions for the family of discounted CRRA utilities. As it is customary in the analysis of intertemporal decision problems, the decision maker considered was characterized by a constant discount rate of time preference, i.e., she discounted the stream of utilities of any category using an exponential discount function with a constant discount rate of time preference according to the Discounted Utility (DU) model introduced in Samuelson (1937). Within this framework, the marginal rate of substitution between payments at different times depends only on the length of the time interval contemplated, being this fact probably the main limitation of the DU model with regard to its capacity to describe the actual time preference patterns.

In fact, the empirical findings on individual behavior seem to challenge some of the predictions of the standard discounting model (see Frederick et al. (2002) for a review of the literature until then). For this reason, variable rates of time preference have received an increasing attention in recent years, in attempts to capture the reported anomalies. In this sense, for instance, several papers focused on the greater impatience of decision makers about the choices in the short run compared with those in the long term using the hyperbolic discount function introduced by Phelps and Pollak (1968). Along the same

lines, Karp (2007) and Marín-Solano and Navas (2010) dealt with the problem with non-constant discounting. Also, in a recent paper by Ekeland et al. (2012), the model of Pliska and Ye (2007) was extended with the introduction of non-constant discount rates.

The choice of the discount function will depend, in general, on the problem under consideration. For instance, in intertemporal problems with a bequest motive, like those studying the demand for life insurance, it is useful to account for the fact that the agent concern about the bequest left to her descendants is not the same when she is young than when she is an adult. A similar effect could be considered in retirement and pension models, in which the willingness to save for a better retirement is likely to be greater at the end of the working life than at the beginning. In addition, for such a long planning horizon the greater impatience in the short run may still play a role, although this bias should evolve according to the different valuations over time of the bequest and the pension plan. In order to capture this asymmetric valuation Marín-Solano and Patxot (2012) introduced the heterogeneous discounting model. According to these authors, the individual preferences at time t take the form

$$\int_t^T e^{-\delta(s-t)} L(x(s), u(s), s)\, ds + e^{-\rho(T-t)} F(x(T), T) \ , \tag{1}$$

i.e., the agent uses a constant discount rate of time preference, but this rate is different for the instantaneous utilities $L(x(s), u(s), s)$ and for the final function $F(x(T), T)$ which, in the previous examples, would account for the bequest or the agent wealth at retirement. The most relevant effect of using any non-constant discount function is that preferences change with time. Impatient agents over-valuing instantaneous utilities in comparison with the final function are characterized by $\rho > \delta$ in equation (1). However, as we approach the end of the planning horizon T the relative value of the final function increases compared with the instantaneous utilities and consequently, the bias to the present decreases with time (see Marín-Solano and Patxot (2012) and de-Paz et al. (2011) for a detailed discussion of this effect).

The aim of this paper is to derive the optimal consumption, investment and life insurance rules for an agent whose concern about both the bequest left to her descendants and her wealth at retirement increases with time. To this end we depart from the model in Pliska and Ye (2007) generalizing the individual time preferences by incorporating heterogeneous discount functions. In contrast to the extension of Pliska and Ye's (2007) model in Ekeland et al. (2012), where an intergenerational problem is introduced by assigning different discount functions to different generations, our setting of heterogeneous discounting focuses on the time preference dynamics of the decision maker, i.e., our setting faces an intragenerational problem. In addition, following Kraft (2003), we derive the wealth process in terms of the portfolio elasticity with respect to the traded assets. This approach allows us to introduce options in the investment opportunity set as well as to enlarge it by any number of contingent claims while maintaining the analytical tractability of the model. Finally, we analyze how the standard solutions are modified depending on the

attitude of the agent towards her changing preferences, showing the differences with some numerical illustrations.

In effect, the individual facing the problem of maximizing (1) can act in two different ways. On the one hand, she could solve the problem by ignoring the fact that her preferences are going to change in the near future, and applying the classical HJB equation. In this case, the strategies obtained will be only optimal from the point of view of her preferences at time t and, in general, will be only obeyed at that time; therefore they are time-inconsistent. On the other hand, she could take into account her changing preferences and obtain the time-consistent strategies by calculating Markov Perfect Equilibria (MPE). These different solutions are usually referred to as naive (in general time-inconsistent) and sophisticated (time-consistent) in the non-constant discounting literature. In order to obtain the MPE, Marín-Solano and Patxot (2012) derived the Dynamic Programming Equation (DPE) in a deterministic framework following a variational approach. The extension to the stochastic case, in which the state dynamics is described by a set of diffusion equations of the form $dx(t) = f(x(t), u(t), t)\, dt + \sigma(x(t), u(t), t) dz(t)$, where $z(t)$ is a standard Wiener process, was studied in de-Paz et al. (2011). In that paper the DPE providing time-consistent solutions was derived following two different approaches. The first one consisted in obtaining the DPE for the heterogeneous discounting problem in discrete time and then taking the formal continuous time limit, following Karp (2007) for the non-constant discounting problem in a deterministic setting (see Marín-Solano and Navas (2010) for the stochastic case). The second one was the variational approach, as in Marín-Solano and Patxot (2012) (which is based on Ekeland and Lazrak (2010)). It is important to remark that, despite the fact that the two approaches are different in nature, the equilibrium conditions coincide.

According to de-Paz et al. (2011), if $V(x,t)$ is the value function of the time-consistent (sophisticated) agent for the problem of maximizing (1) subject to the corresponding state equation, and assuming that it is of class $C^{2,1}$, then $V(x,t)$ satisfies the following DPE

$$\rho V(x,t) - V_t(x,t) - K(x,t) = \tag{2}$$

$$= \sup_{\{u\}} \left\{ L(x,u,t) + V_x(x,t) f(x,u,t) + \frac{1}{2} tr\left(\sigma(x,u,t) \cdot \sigma'(x,u,t) \cdot V_{xx}(x,t)\right) \right\} ,$$

where

$$K(x,t) = (\rho - \delta) E\left[\int_t^T e^{-\delta(s-t)} L(x_s, \phi(x_s, s), s)\, ds \right] \tag{3}$$

with $V(x,T) = F(x,T)$, and being $\phi(x_s, s)$ the equilibrium rule . The subscripts denote the partial derivative. If, for each pair (x,t), there exists a decision rule $u^* = \phi(x,t)$, with corresponding state trajectory $x^*(t)$, such that u^* maximizes the right hand side term of (2), then $u^* = \phi(x,t)$ is the Markov equilibrium rule for the problem with heterogeneous discounting.

It is worth mentioning that, unlike the standard DPE, a new term $K(x,t)$ appears in (2). Checking equation (3) it is obvious that $K(x,t) = 0$ in the standard discounting

case ($\delta = \rho$). By differentiating (3) with respect to t we obtain an "auxiliary dynamic programming equation"

$$\delta K(x,t) - K_t(x,t) = (\delta - \rho)L(x, \phi(x,t), t) + K_x(x,t) \cdot f(x, \phi(x,t), t) + \\ + \frac{1}{2} tr \left(\sigma(x, \phi(x,t), t) \cdot \sigma'(x, \phi(x,t), t) \cdot K_{xx}(x,t) \right) , \tag{4}$$

so that instead of solving (2) and (3), the solution can be characterized by solving the system of partial differential equations (2) and (4) with the corresponding boundary conditions $V(x,T) = F(x,T)$, and $K(x,T) = 0$.

The rest of the paper is organized as follows. In Section 2, we present the model we want to address describing the underlying financial and insurance market as well as the optimal control problem to be solved. In Section 3, we consider the case of CRRA and CARA utility functions and we discuss the time-consistency of the solutions obtained. In Section 4, we provide some numerical illustrations of the main results, comparing our solutions with the standard ones. Finally, Section 5 concludes.

2 The model

Consider a decision maker with a working life that extends from t_0 to T years who is subject to a mortality risk. Let $\tau \in [0, \infty)$ be a random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ representing the agent time of death. We assume that τ has a known distribution function $F(\tau)$ and density function $F'(\tau) = f(\tau)$. At each time $t \in [t_0, \min\{T, \tau\}]$, the agent has to decide how to allocate her personal wealth $W(t)$ between consumption, investment, and life insurance purchase.

The consumption process rate is denoted by $c(t)$. Obviously, the agent enjoys consumption as long as she is alive, i.e., for all $t \leq \min\{T, \tau\}$. The life insurance contract can be purchased by the agent by paying premiums per euro of coverage for age t at a rate denoted by $p(t)$. We assume that contracts of this kind are offered continuously in the insurance market. If $Q(t)$ denotes the total amount of life insurance purchased, the total premium paid at time t is $p(t)Q(t)$. In addition to consumption and purchase of a life insurance policy, we assume that the agent invests the full amount of her savings in a financial market. Let us briefly derive the wealth process when the market comprises two securities, one risk-free and the other risky. The risk-free asset price $M(t)$ is assumed to evolve according to $dM(t) = M(t)rdt$, where $r > 0$ and $M(t_0) = m > 0$, while the risky asset follows a geometric Brownian motion described by

$$dP(t) = P(t)\mu dt + P(t)\sigma dz(t) ,$$

where $P(t_0) = p > 0$ and $z(t)$ is a standard Brownian motion process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space and that its filtration $\{\mathcal{F}_t, t \in [t_0, T]\}$ is the $\mathbb{P}$-augmentation of the filtration generated by $z(t)$. Besides the return on her investment, the agent receives her income at a rate $i(t)$ until

her retirement time or until her death time, whichever happens first. Denoting by w the proportion of savings invested in the risky asset, the wealth process is described by the stochastic differential equation

$$dW = [(r + w(\mu - r))\, W + i(t) - c(t) - p(t)Q(t)]\, dt + w\sigma W dz(t)\,, \tag{5}$$

defined on $[t_0, \min\{T, \tau\}]$, with $W(t_0) = W_0$.

Assume now that the opportunity set for investments is not only composed by the two securities described above but that an option $C(t, P(t))$ on the stock is also available in the market. The introduction of options and other derivatives is a natural generalization of the standard portfolio problem due to their wide use as investment opportunities. However, the extension of the stochastic optimal control approach leads to a much more complicated form of the HJB equation, since the option price $C(t, P(t))$ is a non-linear function of the underlying stock price. Kraft (2003) proposed a kind of two step procedure that greatly simplifies the problem. By introducing the elasticity of the portfolio with respect to the stock price, it is shown that this elasticity can be used as the control variable instead of the share of wealth invested in each asset. Thus, in the first step, investment problems with contingent claims of the form $C(t, P(t))$ can be solved as if the portfolio only contained a risky security and a risk-free security (the reduced portfolio problem). Once the optimal elasticity is obtained, the second step consists in calculating a portfolio tracking this elasticity.

Therefore, according to the elasticity approach, the optimal wealth process can be determined by the optimal elasticity of the portfolio with respect to the stock price. We first define the elasticity of the option price with respect to the price of the underlying, $\epsilon_C = \dfrac{dC/C}{dP/P}$, where dC is obtained using Ito's lemma

$$dC = \left(C_t + C_P P\mu + \frac{1}{2}C_{PP}P^2\sigma^2\right) dt + C_P P\sigma dz(t)\,. \tag{6}$$

An application of the Black-Scholes partial differential equation, $C_t + C_P Pr + \frac{1}{2}C_{PP}P^2\sigma^2 - Cr = 0$, leads to

$$dC = (rC + (\mu - r)C_P P)\, dt + C_P P\sigma dz(t)\,,$$

so that $\epsilon_C = \frac{dC/C}{dP/P} = \frac{C_P P}{C}$ and equation (6) becomes

$$dC = C\,[(r + \epsilon_C(\mu - r))dt + \epsilon_C\sigma dz(t)]\,.$$

Let w_P and w_C denote the proportion of the wealth invested in the risky asset and in the call option, respectively. The remainder $1 - w_P - w_C$ is the proportion invested in the risk-free security. In this case, the wealth process is described by

$$dW = [(r + (w_P + w_C\epsilon_C)(\mu - r))\, W + i(t) - c(t) - p(t)Q(t)]\, dt +$$

$$+ (w_P + w_C\epsilon_C)\sigma W dz(t)\,. \tag{7}$$

In addition, the portfolio's elasticity with respect to the stock price is defined as the weighted sum of the elasticities of the portfolio components $\epsilon = (1-w_P-w_C)\epsilon_M + w_P\epsilon_P + w_C\epsilon_C$. Since $\epsilon_P = 1$ and $\epsilon_M = 0$ respectively, we have

$$\epsilon = w_P + w_C\epsilon_C \,,$$

and the stochastic differential equation describing the wealth process can be written in terms of the elasticity of the investor's portfolio provided that $w = (w_P, w_C)$ is hold constant (static elasticity), i.e.,

$$dW = \left[(r + \epsilon(\mu - r))\, W + i(t) - c(t) - p(t)Q(t)\right] dt + \epsilon\sigma W dz(t)\,, \tag{8}$$

for $t \in [t_0, \min\{T, \tau\}]$, with $W(t_0) = W_0$.

Note that the only difference between equations (8) and (5) is that the control variable w in (5) is replaced by the static elasticity ϵ in (8). In addition, since ϵ is independent of a particular asset, the opportunity set for investment can be enlarged by any number of contingent claims.

The problem for the wage earner is then to choose the consumption, portfolio elasticity and life insurance rules so as to maximize

$$\max_{\{c,\epsilon,Q\}} E\left[\int_t^{T\wedge\tau} e^{-\delta(s-t)}U(c_s)ds + e^{-\rho(\tau-t)}B(Z(\tau),\tau)\mathbb{1}_{\tau\leq T} + \right.$$
$$\left. + e^{-\rho(T-t)}L(W(T))\mathbb{1}_{\tau>T} \mid \tau > t, \mathcal{F}_t\right]\,, \tag{9}$$

where $T \wedge \tau \equiv \min\{T, \tau\}$; $\mathbb{1}_A$ is the indicator function of event A; $U(c)$ is the utility derived from consumption; $L(W(T))$ denotes the utility derived from the wealth available for retirement in case of being alive at T; and $B(Z(\tau), \tau)$ is the utility from the legacy left to her descendants in case of dying before retirement, with $Z(\tau) = W(\tau) + Q(\tau)$. Functions $U(\cdot)$, $B(\cdot)$ and $L(\cdot)$ are assumed to be strictly concave functions on their arguments.

Note that the discount function is the same for $B(Z(\tau), \tau)$ and $L(W(T))$, which are the final functions depending on dying before retirement or not, and it is different from the discount function for the utility derived form consumption, with $\rho > \delta$. In contrast to intergenerational models, in which different generations can be modeled by introducing different discount functions (as in the case of hyperbolic discounting), we are interested in modeling the individual's increasing concern about his/her bequest and his/her wealth available for retirement, i.e., from an intragenerational point of view. As discussed in de-Paz et al. (2011), this asymmetric valuation can not be described by standard exponential discounting or hyperbolic discount functions.

Finally, if the mortality risk is independent of the financial risk, equation (9) with random terminal time transforms into

$$\max_{\{c,\epsilon,Q\}} E\left[\int_t^T \Big(S(s,t)e^{-\delta(s-t)}U(c_s) + f(s,t)e^{-\rho(s-t)}B(Z(s),s)\Big)\, ds + \right.$$
$$\left. + S(T,t)e^{-\rho(T-t)}L(W(T)) \mid \mathcal{F}_t\right] =$$

$$= \max_{\{c,\epsilon,Q\}} \frac{1}{S(t)} E\left[\int_t^T \left(S(s)e^{-\delta(s-t)}U(c_s) + f(s)e^{-\rho(s-t)}B(Z(s),s)\right) ds + \right.$$

$$\left. + S(T)e^{-\rho(T-t)}L(W(T)) \mid \mathcal{F}_t\right], \tag{10}$$

where $S(t) = 1 - F(t)$ denotes the survivor function, $f(s,t) = \frac{f(s)}{S(t)}$ is the conditional density function, and $S(s,t) = \frac{S(s)}{S(t)}$ denotes the conditional survivor function.

3 The case of CRRA and CARA utility functions

In this section we derive explicit solutions for the problem (10) and (8) considering first, utility functions with a constant relative risk aversion, and second, utility functions with a constant absolute risk aversion. We then compare the standard solutions with the time-inconsistent and with the time-consistent solutions for the problem with heterogeneous discounting.

Let c^*,ϵ^* and Q^* denote the optimal consumption, portfolio elasticity and life insurance purchase. Then the current value function at time t is

$$\bar{V}(W,t) = \frac{1}{S(t)} E\left[\int_t^T \left(S(s)e^{-\delta(s-t)}U(c_s^*) + f(s)e^{-\rho(s-t)}B(Z^*(s),s)\right) ds + \right.$$

$$\left. + S(T)e^{-\rho(T-t)}L(W^*(T)) \mid \mathcal{F}_t\right].$$

Throughout the paper we will work with the value function multiplied by the survivor probability function $V(W,t) = S(t)\bar{V}(W,t)$.

Note that, when transforming the functional (9) in (10), the utility function $B(Z(s),s)$ enters in the integral term and it can therefore be viewed as an instantaneous utility. In order to have a constant discount rate for the instantaneous utilities different to that for the final function, we rewrite the objective function (10) as follows. For each $\omega \in \Omega$, we define a new state variable $y_\omega^t(u)$ as

$$y_\omega^t(u) = \int_t^u f(s)e^{-\rho(s-u)}B(Z_\omega(s),s)ds\,.$$

For simplicity, in the following we will omit the subindex ω. Then, maximizing (10) subject to (8) is equivalent to

$$\max_{\{c,\epsilon,Q\}} \frac{1}{S(t)} E\left[\int_t^T S(s)e^{-\delta(s-t)}U(c_s)ds + e^{-\rho(T-t)}\left[y^t(T) + S(T)L(W(T))\right] \mid \mathcal{F}_t\right] \tag{11}$$

subject to (8) and to

$$\dot{y}^t(s) = \rho y^t(s) + f(s)B(Z(s),s)\,. \tag{12}$$

As mentioned in the introduction, the wage earner solving problem (11) subject to (8) and (12) can act in two different ways. The time-consistent agent must solve the DPE (2). Otherwise, the naive agent making decisions at time t without taking into account that

her preferences change with time will maximize (11) subject to (8) and (12) by solving the standard HJB equation

$$\delta V(y^t, W, s) - V_s(y^t, W, s) = \max_{\{c,\epsilon,Q\}} \left\{ S(s)U(c) + \left[\rho y^t(s) + f(s)B(Z(s), s)\right] V_{y^t}(y^t, W, s) + \right.$$

$$\left. + \left[(r + \epsilon(\mu - r))\, W + i(t) - c(t) - p(t)Q(t)\right] V_W(y^t, W, s) + \frac{1}{2}\epsilon^2\sigma^2 W^2 V_{WW}(y^t, W, s) \right\} . \tag{13}$$

The difference between this solution and the solution in the standard case ($\rho = \delta$) comes from the boundary condition used in each problem. While in the standard case the boundary condition is $V(y^t, W, T) = y^t(T) + S(T)L(W(T))$, the value function at T for the time-inconsistent agent, in its current value form, is $V(y^t, W, T) = e^{-(\rho-\delta)(T-t)}(y^t(T) + S(T)L(W(T))$. This boundary condition changes depending on the moment t at which the solution is calculated. In fact, an agent acting in this way constructs her solution by solving the HJB equation (13) together with the family of boundary conditions $V(y^t, W, T) = e^{-(\rho-\delta)(T-t)}(y^t(T) + S(T)L(W(T))$ for $t \in [t_0, T]$ and patching together the solutions obtained. In order to highlight the moment at which the problem is solved, in the following we will denote the value function for the time-inconsistent (naive) agent by $V^t(y^t, W, s)$. In addition, we will omit the superscript t in $y^t(s)$.[1]

3.1 Logarithmic utility function

Consider first the case of logarithmic utility functions

$$U(c_s) = \ln c_s\,, \qquad B(Z(s), s) = a \ln Z(s)\,, \qquad \text{and} \qquad L(W(T)) = b \ln W(T), \tag{14}$$

where a and b are positive real parameters. Let us briefly derive the time-inconsistent strategy solving equation (13) at some particular time $t \in [t_0, T]$, i.e., with the boundary condition

$$V^t(y, W, T) = e^{-(\rho-\delta)(T-t)}(y(T) + bS(T) \ln W(T))\ . \tag{15}$$

From the maximization problem in (13) one easily obtains

$$c^t(s) = \frac{S(s)}{V_W^t}\,, \qquad \epsilon^t(s) = \frac{-(\mu - r)}{\sigma^2 W}\frac{V_W^t}{V_{WW}^t}\,, \qquad Q^t(s) = a\frac{f(s)}{p(s)}\frac{V_y^t}{V_W^t} - W\,, \tag{16}$$

and by guessing $V^t(y, W, s) = \alpha^t(s) \ln(W + \beta^t(s)) + \varphi^t(s)y$ these rules become $c^t(s) = \frac{S(s)}{\alpha^t(s)}(W + \beta^t(s))$, $\epsilon^t(s) = \frac{(\mu_1 - r)}{\sigma^2 W}(W + \beta^t(s))$ and $Q(s)^t = a\frac{f(s)\varphi^t(s)}{p(s)\alpha^t(s)}(W + \beta^t(s)) - W$ where

[1] In the optimal solution from the viewpoint of the t_0-agent, $t = t_0$ in equation (11) (the so called precommitment solution in the literature of hyperbolic discounting), one should add the initial condition $y^{t_0}(t_0) = 0$ in (12). The same initial condition is considered in the time-consistent solution. On the contrary, in the naive solution, the initial condition in the problem for each t-agent is $y^t(t) = 0$, for every t.

$$\alpha^t(s) = bS(T)e^{-\rho(T-t)+\delta(s-t)} + \int_s^T \left(e^{-\delta(\tau-s)}S(\tau) + e^{-\rho(\tau-t)+\delta(s-t)}f(\tau)\right) d\tau$$

and

$$\beta^t(s) = \int_s^T e^{-\int_s^u (r+p(v))dv} i(u)du\,, \qquad \varphi^t(s) = e^{-(\rho-\delta)(s-t)}\,.$$

From the above expressions for $\alpha^t(s)$, $\beta^t(s)$, and $\varphi^t(s)$, it becomes clear that our guessing for the value function is consistent.

Therefore, either the agent is able to commit herself to following the decisions initially taken at t_0 or the rules above will be only obeyed at the time at which they have been calculated, i.e., $s = t$. Thus, either

$$\alpha^{t_0}(s) = bS(T)e^{-\rho(T-t_0)+\delta(s-t_0)} + \int_s^T \left(e^{-\delta(\tau-s)}S(\tau) + e^{-\rho(\tau-t_0)+\delta(s-t_0)}f(\tau)\right) d\tau$$

and $\varphi^{t_0}(s) = e^{-(\rho-\delta)(s-t_0)}$ or

$$\alpha^t(t) = bS(T)e^{-\rho(T-t)} + \int_t^T \left(e^{-\delta(\tau-t)}S(\tau) + e^{-\rho(\tau-t)}f(\tau)\right) d\tau\,, \qquad \varphi^t(t) = 1\,.$$

respectively. With respect to function $\beta^t(s)$, since it does not depend on the moment t, it coincides whether or not the agent can commit herself ($\beta^t(s) = \beta^{t_0}(s) = \beta^s(s)$).

Now we turn the attention to the time-consistent strategy.

Proposition 1 *Assume that $U(c_s)$, $B(Z(s), s)$, and $L(W(T))$ are given by (14). Then $V(y, W, t) = \alpha(t)\ln(W + \beta(t)) + \varphi(t)y$, and the optimal controls are given by*

$$c^*(t) = \frac{S(t)}{\alpha(t)}(W + \beta(t))\,, \qquad \epsilon^*(t) = \frac{(\mu - r)}{\sigma^2 W}(W + \beta(t))\,,$$

$$Q^*(t) = a\frac{f(t)}{p(t)}\frac{\varphi(t)}{\alpha(t)}(W + \beta(t)) - W\,, \tag{17}$$

where

$$\alpha(t) = bS(T)e^{-\rho(T-t)} + \int_t^T \left(e^{-\delta(s-t)}S(s) + e^{-\rho(s-t)}f(s)\right) ds\,, \qquad \varphi(t) = 1\,,$$

$$\beta(t) = \int_t^T e^{-\int_t^s (r+p(v))dv} i(s)ds\,. \tag{18}$$

Proof: According to Proposition 1 in de-Paz et al. (2011), a time-consistent solution can be obtained by solving the DPE (2), which in this specific case becomes

$$\rho V(y, W, t) - K(W, t) - V_t(y, W, t) = \max_{\{c,\epsilon,Q\}} \{S(t)\ln c + [\rho y + af(t)\ln Z(t)]\, V_y(y, W, t) +$$

$$+ [(r + \epsilon(t)(\mu - r))\, W + i(t) - c(t) - p(t)Q(t)]\, V_W(y, W, t) + \frac{1}{2}\epsilon(t)^2\sigma^2 W^2 V_{WW}(y, W, t)\}\,, \tag{19}$$

with $K(W,t)$ given by

$$K(W,t) = (\rho - \delta) E \left[\int_t^T e^{-\delta(s-t)} S(s) \ln c^*(s) ds \right] , \tag{20}$$

where $c^*(s)$ is the equilibrium consumption rule obtained by solving the right hand side in (19). In particular, by applying Corollary 1 in de-Paz et al. (2011), we obtain the system of two coupled partial differential equations

$$\rho V(y,W,t) - K(W,t) - V_t(y,W,t) = \max_{\{c,\epsilon,Q\}} \{ S(t) \ln c + [\rho y + a f(t) \ln Z(t)] V_y(y,W,t) +$$

$$+ [(r + \epsilon(\mu - r)) W + i(t) - c(t) - p(t)Q(t)] V_W(y,W,t) + \frac{1}{2} \epsilon^2 \sigma^2 W^2 V_{WW}(y,W,t) \} , \tag{21}$$

and

$$\delta K(W,t) - K_t(W,t) = (\rho - \delta) S(t) \ln c + [(r + \epsilon(\mu - r)) W +$$

$$+ i(t) - c(t) - p(t)Q(t)] K_W(W,t) + \frac{1}{2} \epsilon^2 \sigma^2 W^2 K_{WW}(W,t) . \tag{22}$$

We guess a solution of the form $V(y,W,t) = \alpha(t) \ln(W + \beta(t)) + \varphi(t) y$, with $V(y,W,T) = y(T) + bS(T) \ln(W(T))$, for the value function and of the form $K(W,t) = A(t) \ln(W + \beta(t))$, with $K(W,T) = 0$, for the function $K(W,t)$. If theses choices prove to be consistent, from the maximization problem in (21) we have that the guessed optimal policies are given by (17). Substituting into (21) and (22), we obtain that the functions $\alpha(t)$, $\beta(t)$, $\varphi(t)$ must satisfy

$$\dot{\alpha}(t) - \rho \alpha(t) = A(t) - S(t) - a f(t) \varphi(t) , \qquad \dot{\beta}(t) - (\mu + p(t)) \beta(t) = i(t) , \qquad \dot{\varphi}(t) = 0 ,$$

together with the boundary conditions $\alpha(T) = bS(T)$, $\beta(T) = 0$, $\varphi(T) = 1$. Solving these equations we get (18). □

With respect to the function $A(t)$, we find that must satisfy $\dot{A}(t) - \delta A(t) = -(\rho - \delta) S(t)$, with $A(T) = 0$. Thus,

$$A(t) = \int_t^T e^{-\delta(s-t)} (\rho - \delta) S(s) ds .$$

Note that this solution coincides with the *a priori* time-inconsistent solution. This feature, also arising in non-constant discounting problems (see Marín-Solano and Navas (2010)), is a property of the logarithmic utilities and it is not preserved for more general utility functions such as the power utilities, as we analyze in the next subsection.

3.2 Power utility function

Next, let us study the problem for the case of power utilities

$$U(c_s) = \frac{c_s^\gamma}{\gamma} , \qquad B(Z(s), s) = a \frac{Z(s)^\gamma}{\gamma} , \qquad \text{and} \qquad L(W(T)) = b \frac{W(T)^\gamma}{\gamma}, \tag{23}$$

with $\gamma < 1$, $\gamma \neq 0$. As in the previous subsection, we first solve equation (13) to obtain the "optimal" solution from the point of view of the agent making decisions at time t

and then we distinguish between the case of acting under commitment and acting without commitment.

We guess a value function of the form $V^t(y,W,s) = \alpha^t(s)\frac{(W+\beta^t(s))^\gamma}{\gamma} + \varphi^t(s)y$, with $V^t(y,W,T) = e^{-(\rho-\delta)(T-t)}\left(bS(T)\frac{W(T)^\gamma}{\gamma} + y(T)\right)$. Then, by maximizing the right hand side of equation (13) we obtain that the "optimal" control rules satisfy $c^t(s) = \left(\frac{\alpha^t(s)}{S(s)}\right)^{\frac{1}{\gamma-1}}(W+\beta^t(s))$, $\epsilon^t(s) = \frac{(\mu-r)}{\sigma^2 W(1-\gamma)}(W+\beta^t(s))$ and $Q^t(s) = \left(\frac{p(s)}{af(s)}\frac{\alpha^t(s)}{\varphi^t(s)}\right)^{\frac{1}{\gamma-1}}(W+\beta^t(s)) - W$, where

$$\beta^t(s) = \int_s^T e^{-\int_s^u (r+p(v))dv} i(u)du\,, \qquad \varphi^t(s) = e^{-(\rho-\delta)(s-t)}\,,$$

$$\alpha^t(s) = \upsilon(s)^{\gamma-1}\left[\int_s^T \upsilon(u)\left(S(u)^{\frac{1}{1-\gamma}} + \left(\frac{ae^{-(\rho-\delta)(u-t)}f(u)}{p(u)^\gamma}\right)^{\frac{1}{1-\gamma}}\right)du + \right.$$
$$\left. + \left(e^{-(\rho-\delta)(T-t)}bS(T)\right)^{\frac{1}{1-\gamma}}\upsilon(T)\right]^{1-\gamma},$$

with

$$\upsilon(s) = exp\left\{-\frac{1}{1-\gamma}\int_0^s\left(\delta - \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}\gamma - \gamma r - \gamma p(u)\right)du\right\},$$

so that our guessing for the value function is consistent.

Once again, the function $\beta^t(s)$ does not depend on t (the moment at which the decision is made) and therefore there is no difference between the committed and the time-inconsistent agent. However, both $\alpha^t(s)$ and $\varphi^t(s)$ show the deviation between these two different behaviors. While the agent who is able to commit herself will compute her decision rule according to

$$\alpha^{t_0}(s) = \upsilon(s)^{\gamma-1}\left[\int_s^T \upsilon(u)\left(S(u)^{\frac{1}{1-\gamma}} + \left(\frac{ae^{-(\rho-\delta)(u-t_0)}f(u)}{p(u)^\gamma}\right)^{\frac{1}{1-\gamma}}\right)du + \right.$$
$$\left. + \left(e^{-(\rho-\delta)(T-t_0)}bS(T)\right)^{\frac{1}{1-\gamma}}\upsilon(T)\right]^{1-\gamma}$$

and $\varphi^{t_0}(s) = e^{-(\rho-\delta)(s-t_0)}$, the time-inconsistent agent will follow the decisions taken only when they are calculated; so at $s = t$

$$\alpha^t(t) = \upsilon(t)^{\gamma-1}\left[\int_t^T \upsilon(u)\left(S(u)^{\frac{1}{1-\gamma}} + \left(\frac{ae^{-(\rho-\delta)(u-t)}f(u)}{p(u)^\gamma}\right)^{\frac{1}{1-\gamma}}\right)du + \right.$$
$$\left. + \left(e^{-(\rho-\delta)(T-t)}bS(T)\right)^{\frac{1}{1-\gamma}}\upsilon(T)\right]^{1-\gamma},$$

and $\varphi(t) = 1$.

With respect to the time-consistent solution, we have:

Proposition 2 *Assume that $U(c_s)$, $B(Z(s),s)$, and $L(W(T))$ are given by (23). Then $V(y,W,t)=\alpha(t)\frac{(W+\beta(t))^\gamma}{\gamma}+\varphi(t)y$, $K(W,t)=A(t)\frac{(W+\beta(t))^\gamma}{\gamma}$, and the optimal controls are given by*

$$c^*(t)=\left(\frac{\alpha(t)}{S(t)}\right)^{\frac{1}{\gamma-1}}(W+\beta(t))\,,\qquad \epsilon^*(t)=\frac{(\mu-r)}{\sigma^2W(1-\gamma)}(W+\beta(t))\,,$$

$$Q^*(t)=\left(\frac{p(t)}{af(t)}\frac{\alpha(t)}{\varphi(t)}\right)^{\frac{1}{\gamma-1}}(W+\beta(t))-W\,, \tag{24}$$

where

$$\beta(t)=\int_t^T e^{-\int_t^s(r+p(v))dv}i(s)ds\,,\qquad \varphi(t)=1\,, \tag{25}$$

while functions $\alpha(t)$ and $A(t)$ are the solution to the following system of differential equations

$$\rho\alpha(t)-A(t)-\dot\alpha(t)=$$

$$\alpha(t)^{\frac{\gamma}{\gamma-1}}\left[(1-\gamma)\left(S(t)^{\frac{1}{1-\gamma}}+\left(\frac{af(t)}{p(t)^\gamma}\right)^{\frac{1}{1-\gamma}}\right)\right]+\gamma\alpha(t)\left[\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}+r+p(t)\right]\,, \tag{26}$$

$$\delta A(t)-\dot A(t)=(S(t))^{\frac{1}{1-\gamma}}\left[(\rho-\delta)\alpha(t)^{\frac{\gamma}{\gamma-1}}-\gamma\alpha(t)^{\frac{1}{\gamma-1}}A(t)\right]+$$

$$+\gamma A(t)\left[\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}+r+p(t)-p(t)\left(\frac{p(t)}{af(t)}\alpha(t)\right)^{\frac{1}{\gamma-1}}\right]\,, \tag{27}$$

with $\alpha(T)=bS(T)$, and $A(T)=0$.

Proof: To obtain the time-consistent solution we must solve the DPE (2). Specifically, according to Corollary 1 in de-Paz et al. (2011) we must solve the set of DPE

$$\rho V(y,W,t)-K(W,t)-V_t(y,W,t)=\max_{\{c,\epsilon,Q\}}\left\{S(t)\frac{c^\gamma}{\gamma}+\left[(r+\epsilon(\mu-r))\,W+i(t)-\right.\right.$$

$$\left.\left.-c(t)-p(t)Q(t)\right]V_W(y,W,t)+\left(\rho y+af(t)\frac{c^\gamma}{\gamma}\right)V_y(y,W,t)+\frac{1}{2}\epsilon^2\sigma^2W^2V_{WW}(y,W,t)\right\}\,, \tag{28}$$

$$\delta K(W,t)-K_t(W,t)=(\rho-\delta)S(t)\frac{c^\gamma}{\gamma}$$

$$+\left[(r+\epsilon(\mu-r))\,W+i(t)-c(t)-p(t)Q(t)\right]K_W(W,t)+\frac{1}{2}\epsilon^2\sigma^2W^2K_{WW}(W,t)\,. \tag{29}$$

We try as a candidates for the value function and to the function $K(W,t)$, $V(y,W,t)=\alpha(t)\frac{(W+\beta(t))^\gamma}{\gamma}+\varphi(t)y$ and $K(W,t)=A(t)\frac{(W+\beta(t))^\gamma}{\gamma}$ respectively. Maximizing the right hand side of (28) we obtain (24). Finally, substituting the guessed functions and the corresponding optimal controls in (28-29), together with the terminal conditions $V(y,W,T)=y(T)+bS(T)\frac{W(T)^\gamma}{\gamma}$ and $K(W,T)=0$, we obtain that functions $\beta(t)$ and $\varphi(t)$ are given by (25), while functions $\alpha(t)$ and $A(t)$ are the solution to the system of differential equations (26-27). □

3.3 Exponential utility function

Finally, let us solve the problem for the case of (constant absolute risk aversion) exponential utility functions

$$U(c_s) = \frac{-1}{\gamma} e^{-\gamma c_s}, \qquad B(Z(s), s) = \frac{-a}{\gamma} e^{-\gamma Z(s)}, \quad \text{and} \quad L(W(T)) = -b e^{-\gamma W(T)}, \tag{30}$$

with $\gamma > 0$. Once again, we first derive the "optimal" rules for the point of view of an agent deciding at $t \in [t_0, T]$ by means of equation (13). Then we differentiate between the agent who is able to commit herself and the time-inconsistent agent.

By guessing $V^t(y, W, s) = -a e^{-\gamma(\alpha^t(s) + \beta^t(s) W)} + \varphi^t(s) y$, the maximization problem in (13) gives

$$c^t(s) = \alpha^t(s) + \beta^t(s) W - \frac{1}{\gamma} \ln\left(\frac{a\gamma\beta^t(s)}{S(s)}\right), \quad \epsilon^t(s) = \frac{(\mu - r)}{\sigma^2 \gamma \beta^t(s) W}, \tag{31}$$

$$Q^t(s) = \alpha^t(s) + \beta^t(s) W - \frac{1}{\gamma} \ln\left(\frac{p(s)}{f(s)} \frac{\gamma\beta^t(s)}{\varphi^t(s)}\right) - W. \tag{32}$$

We substitute (31) and (32) in (13), and after several calculations, we obtain that the functions $\alpha^t(s)$, $\beta^t(s)$, and $\varphi^t(s)$ satisfy

$$\begin{aligned} \alpha^t(s) &= \frac{-1}{\gamma} \left[\left(\ln\left(\frac{b}{a} S(T)\right) - (\rho - \delta)(T - t) \right) e^{-\int_s^T (1+p(v))\beta^t(v) dv} + \right. \\ &\qquad \left. + \int_s^T \left(\vartheta^t(u) e^{-\int_s^u (1+p(v))\beta^t(v) dv} \right) du \right], \\ \beta^t(s) &= \frac{1}{e^{-\int_s^T (r+p(v)) dv} + \int_s^T \left((1+p(u)) e^{-\int_s^u (r+p(v)) dv} \right) du}, \end{aligned} \tag{33}$$

and $\varphi^t(s) = e^{-(\rho-\delta)(s-t)}$, where

$$\vartheta^t(u) = \beta^t(u) \left[1 + p(u) + \ln\left(\frac{a\gamma\beta^t(u)}{S(u)}\right) - p(u) \ln\left(\frac{p(u)}{f(u)} \frac{\gamma\beta^t(u)}{\varphi^t(u)}\right) - i(u)\gamma \right] - \delta - \frac{1}{2} \frac{(\mu - r)^2}{\sigma^2}.$$

Although the function $\beta^t(s)$ does not depend on t, and hence $\beta^t(s) = \beta^{t_0}(s) = \beta^s(s)$, the "optimal" policies change depending on whether the agent reconsiders her previous decisions or is committed with the initial ones. In the last case, the decision maker will compute her policies according to

$$\begin{aligned} \alpha^{t_0}(s) &= \frac{-1}{\gamma} \left[\left(\ln\left(\frac{b}{a} S(T)\right) - (\rho - \delta)(T - t_0) \right) e^{-\int_s^T (1+p(v))\beta(v) dv} + \right. \\ &\qquad \left. + \int_s^T \left(\vartheta^{t_0}(u) e^{-\int_s^u (1+p(v))\beta(v) dv} \right) du \right], \end{aligned}$$

and $\varphi^{t_0}(s) = e^{-(\rho-\delta)(s-t_0)}$. If she is not committed with the initial decisions, she will be continuously modifying her calculated choices for the future. Consequently $\alpha^t(s)$ and $\varphi^t(s)$ will be only obeyed at $s = t$, i.e.,

$$\alpha(t) = \frac{-1}{\gamma} \left[\left(\ln\left(\frac{b}{a} S(T)\right) - (\rho - \delta)(T - t) \right) e^{-\int_t^T (1+p(v))\beta(v) dv} + \right.$$

$$+\int_t^T \left(\vartheta(u)e^{-\int_t^u (1+p(v))\beta(v)dv}\right) du\Bigg] ,$$

and $\varphi(t) = 1$.

Next, let us derive the time-consistent solution.

Proposition 3 *Assume that $U(c_s)$, $B(Z(s), s)$, and $L(W(T))$ are given by (30). Then $V(y, W, t) = -ae^{-\gamma(\alpha(t)+\beta(t)W)} + \varphi(t)y$, $K(W, t) = A(t)e^{-\gamma(\alpha(t)+\beta(t)W)}$, and the optimal controls are given by*

$$c^*(t) = \alpha(t) + \beta(t)W - \frac{1}{\gamma}\ln\left(\frac{a\gamma\beta(t)}{S(t)}\right), \quad \epsilon^*(t) = \frac{(\mu - r)}{\sigma^2\gamma\beta(t)W},$$

$$Q^*(t) = \alpha(t) + \beta(t)W - \frac{1}{\gamma}\ln\left(\frac{p(t)}{f(t)}\frac{\gamma\beta(t)}{\varphi(t)}\right) - W, \tag{34}$$

where

$$\beta^t(s) = \frac{1}{e^{-\int_s^T (r+p(v))dv} + \int_s^T \left((1+p(u))e^{-\int_s^u (r+p(v))dv}\right) du}, \qquad \varphi(t) = 1, \tag{35}$$

while functions $\alpha(t)$ and $A(t)$ are the solution to the following system of differential equations

$$a\gamma\dot{\alpha}(t) + a\rho + A(t) = a\beta(t)(1 - \gamma p(t)) - a\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} +$$
$$+\left[\alpha(t)(1 + p(t)) - \frac{1}{\gamma}\left(\ln\left(\frac{a\gamma\beta(t)}{S(t)}\right) + p(t)\ln\left(\frac{p(t)}{f(t)}\gamma\beta(t)\right)\right) + i(t)\right] a\gamma\beta(t), \tag{36}$$

$$\dot{A}(t) - \delta A(t) - \gamma A(t)\dot{\alpha}(t) = a(\rho - \delta)\beta(t) + \frac{1}{2}\frac{(\mu_1 - r)^2}{\sigma^2}A(t) -$$
$$-\left[\alpha(t)(1 + p(t)) - \frac{1}{\gamma}\left(\ln\left(\frac{a\gamma\beta(t)}{S(t)}\right) + p(t)\ln\left(\frac{p(t)}{f(t)}\gamma\beta(t)\right)\right) - i(t)\right]\gamma A(t)\beta(t), \tag{37}$$

with $\alpha(T) = \frac{-1}{\gamma}\ln\left(\frac{b}{a}S(T)\right)$, and $A(T) = 0$.

Proof: According to Corollary 1 in de-Paz et al.(2011), Markov Perfect Equilibria can be obtained by solving the set of two coupled PDE

$$\rho V(y, W, t) - K(W, t) - V_t(y, W, t) =$$
$$\max_{\{c,\epsilon,Q\}}\left\{\frac{-1}{\gamma}e^{-\gamma c}S(t) + [(r + \epsilon(\mu - r))W + i(t) - c(t) - p(t)Q(t)]V_W(y, W, t) + \right. \tag{38}$$
$$\left. + \left(\rho y + f(t)\frac{-a}{\gamma}e^{-\gamma(W+Q)}\right)V_y(y, W, t) + \frac{1}{2}\epsilon^2\sigma^2W^2V_{WW}(y, W, t)\right\},$$

$$\delta K(W, t) - K_t(W, t) = (\rho - \delta)\frac{-1}{\gamma}e^{-\gamma c}S(t) +$$
$$+[(r + \epsilon(\mu - r))W + i(t) - c(t) - p(t)Q(t)]K_W(W, t) + \frac{1}{2}\epsilon^2\sigma^2W^2K_{WW}(W, t). \tag{39}$$

We guess as a candidate to the value function $V(y, W, t) = -ae^{-\gamma(\alpha(t)+\beta(t)W)} + \varphi(t)y$, and with respect to $K(W, t)$ we try $K(W, t) = A(t)e^{-\gamma(\alpha(t)+\beta(t)W)}$. If these choices proves to be consistent, then from (38) we get (34). By substituting in (38-39) and collecting terms in W, on the one hand, and collecting terms in x, on the other hand, we get that $\beta(t)$ and $\varphi(t)$ are given by (35). With respect to the functions $\alpha(t)$ and $A(t)$, we obtain that they must be the solution to the system of differential equations (36-37). □

4 Numerical illustrations

In this section we provide some numerical examples to illustrate the results for the case of power utility functions. As a baseline case, we consider a 25 years old agent endowed with an initial wealth of 1000 euros and with an initial wage of 25000 euros which grows at 3% every year until $T = 65$, when the agent retires. The agent exhibits a risk aversion parameter of $\gamma = -3$ and her heterogeneous preferences are characterized by $\delta = 0.03$ and $\rho = 0.1$. We assume that the individual is subject to an instantaneous force of mortality or hazard rate given by the Gompert law of mortality $\lambda(t) = \frac{1}{h}e^{\frac{(t-\eta)}{h}}$, $t \geq 0$. Following Milevsky (2006), we take $\eta = 82.3$ and $h = 11.4$. Due to the well-known relationship between the hazard rate and the density and survivor probability functions we have $f(t) = \lambda(t)e^{-\int_0^t \lambda(s)ds}$ and $S(t) = e^{-\int_0^t \lambda(s)ds}$. Regarding the life insurance market, we assume that the insurance company sets the premium in order to make a profit. In general, the insurance is said to be actuarially fair when the expected profit rate equals 0, which in this case means $p(t) = \lambda(t)$. Consequently, in order to be profitable the insurance company must charge a loading factor θ accounting for the percentage markup from the fair value of insurance, i.e., $p(t) = (1 + \theta)\lambda(t)$. For this particular example we consider $\theta = 10\%$ so that the premium per euro of coverage at age t is $p(t) = (1+0.1)\lambda(t)$. Finally, we assume that the risk-free asset yields a return of $r = 0.03$ while the risky security has an expected return of $\mu = 0.09$ and volatility $\sigma = 0.3$.

Before comparing our solutions with the standard solutions, note that the agent makes all her decisions according to her total available wealth (her current wealth $W(t)$ plus the present value of her future earnings $\beta(t)$). Although the present value of future earnings has a positive effect in all the control variables, Figure 1 shows that in this case the current wealth has a negative effect on the total amount of insurance purchased, i.e., the more wealthy the agent is, the less life insurance she purchases. However, since the wage earner has a a small current wealth relative to her future earnings, she depends on her wages to make her decisions. Figure 2 shows the present value of future earnings, two possible trajectories of the total available wealth together with the corresponding time-consistent life insurance rule, and their expected values.

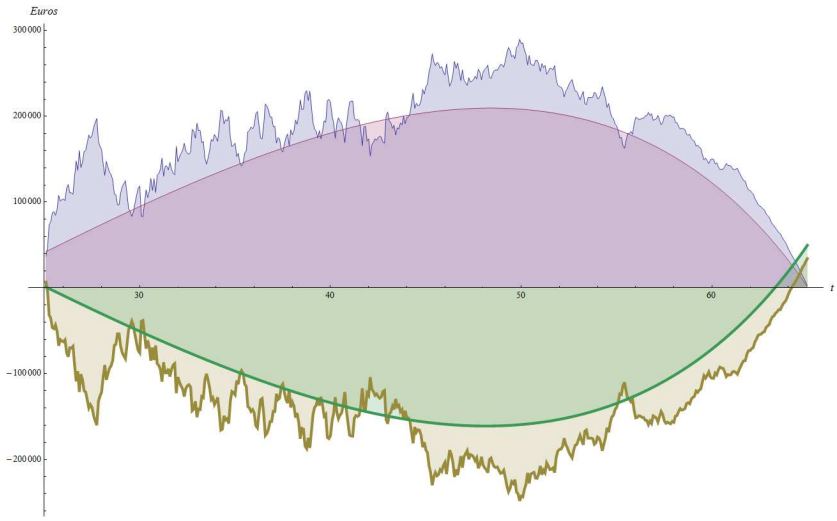

Figure 1: Simulated $W(t)$ (thick), expected $W(t)$ (thick), simulated time-consistent life insurance rule (thin) and expected time-consistent life insurance rule (thin).

Note that in spite of the negative current wealth, the amount of life insurance purchased is enough to leave a positive bequest if premature death occurs.

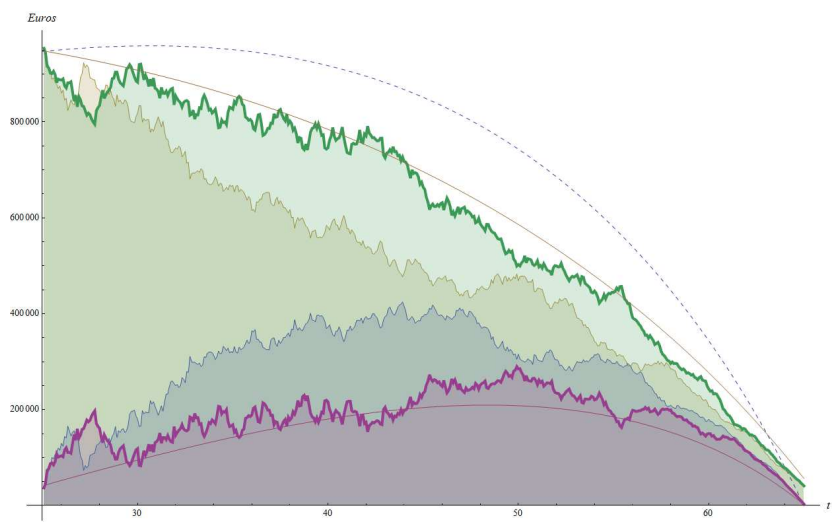

Figure 2: Present value of future income (dashed). Two possible trajectories of the total available wealth and the corresponding time-consistent life insurance rule (the thickness reflects the correspondence). Smooth lines represents the expected values.

For this reason the following comparisons are made for different values of initial wage. Figures 3 to 7 show the differences between standard and heterogeneous behaviors. At the beginning of the planning horizon, the wage earner with heterogeneous discounting is more impatient than the agent with standard discounting, since we have assumed $\rho > \delta$. However, as time goes on the bias to the present decreases as her concern about her bequest and her retirement increases. In order to highlight how the heterogeneous preferences

evolve over time, we focus first on the differences with the standard case from the point of view of a 25 years old agent who is able to commit herself with the decisions initially taken. Then we look at how these differences change if the agent reconsiders her choices at any time (time-inconsistency), and we end by analyzing the differences from the point of view of the time-consistent agent.

On the one hand, if the wage earner does not modify the decisions made at the age of 25, when she underestimates the bequest left to her descendants and her wealth at retirement, one should expect her to purchase less life insurance and to consume more than the standard agent. On the other hand, if the agent does not commit herself, her policies should change according to her preferences at different ages. Therefore, she should purchase more life insurance and consume less than the committed agent. Finally, although the time-consistent agent also overvalues the instantaneous utilities at the beginning of the planing horizon, she knows that her preferences are going to change in the near future. In this case, her policies should reflect the equilibrium between her preferences at different times.

Figure 3 shows the difference of the life insurance purchased by the committed 25 years old agent and the standard discounting case. Departing from a similar level of life insurance purchased, the difference is negative from that moment until the ages close to the retirement date, when it becomes positive. This means that the individual using the heterogeneous discount function postpones the purchase of life insurance when she is 25 years old to the later adulthood. Note that the deviation attains the maximum length around the age of 50 and decreases from that point onwards. In addition, for a given age, an increasing initial wage leads the agent to buy more life insurance under the standard preferences than under the heterogeneous ones, except at ages closer to 65 years. In Figure 4, we compare the life insurance purchased by the time-inconsistent agent and the standard solution. In this case, the difference is positive since the agent reconsiders her choice at each time point according to her increasing concern about the bequest. The comparison of the time-consistent and standard behaviors is shown in Figure 5. The difference is also positive although it is larger than the difference in Figure 4, i.e., time-consistent planning leads the agent to buy more life insurance than the time-inconsistent one. Note that, in contrast to the committed agent, the agent with heterogeneous discounting (both the time-inconsistent and the time-consistent) reacts to an increase in her salary by buying more life insurance than the standard agent.

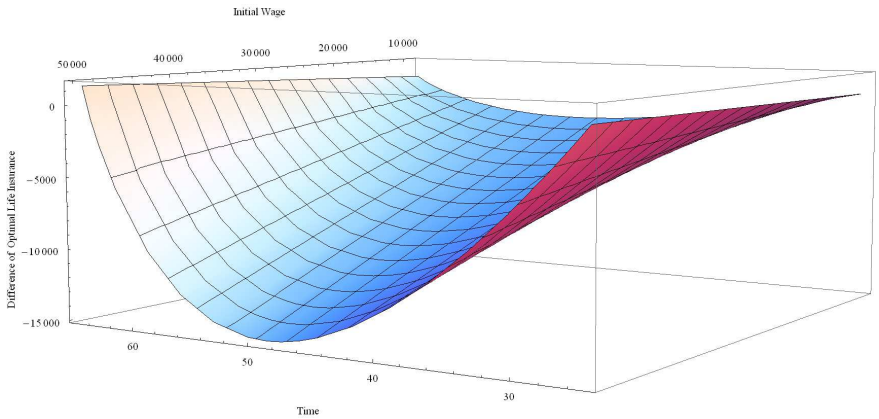

Figure 3: Difference of optimal life insurance between the heterogeneous agent committed with her preferences at the age of 25 and the standard agent for different values of initial wage.

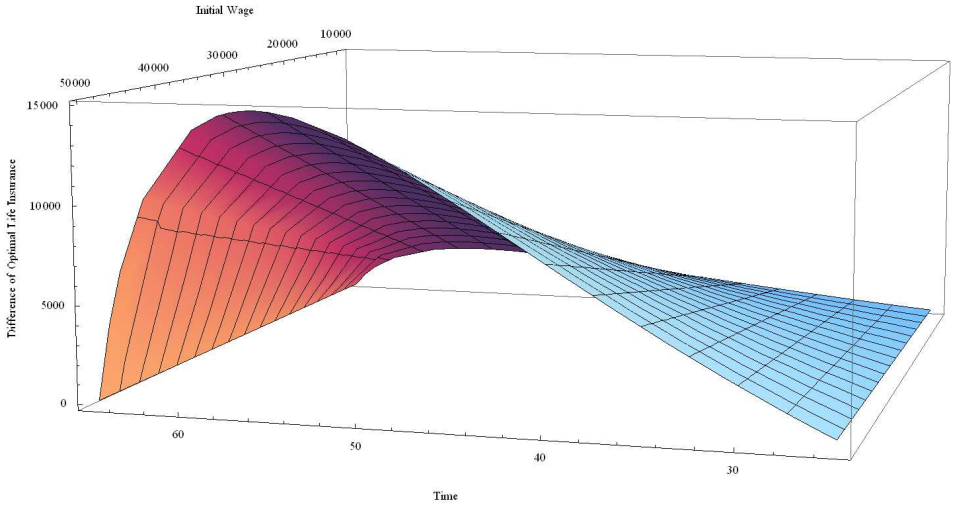

Figure 4: Difference of optimal life insurance between the time-inconsistent heterogeneous agent and the standard agent for different values of initial wage.

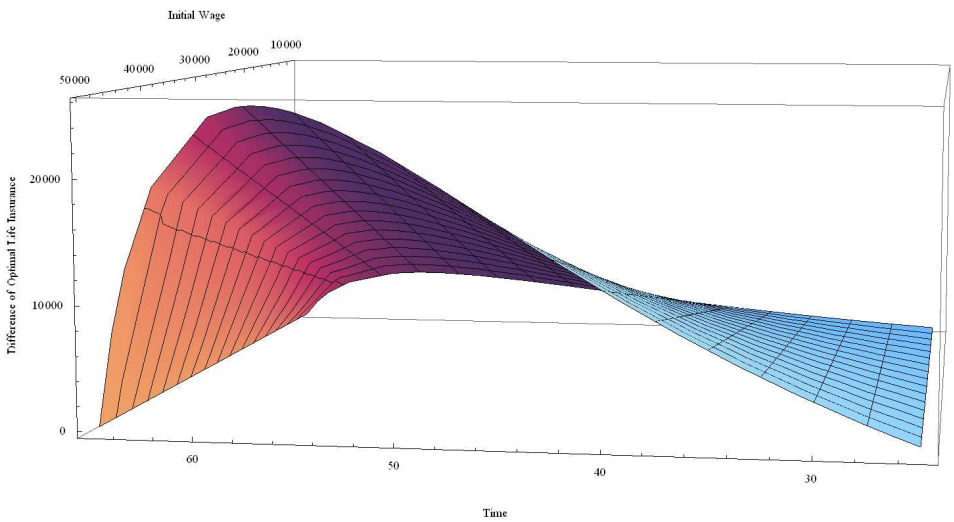

Figure 5: Difference of optimal life insurance between the time-consistent heterogeneous agent and the standard agent for different values of initial wage.

In figure 6 we show the life insurance paths (simulated and expected values) for the committed, the time-inconsistent and the time-consistent agent and for the baseline initial wage (25000 euros).

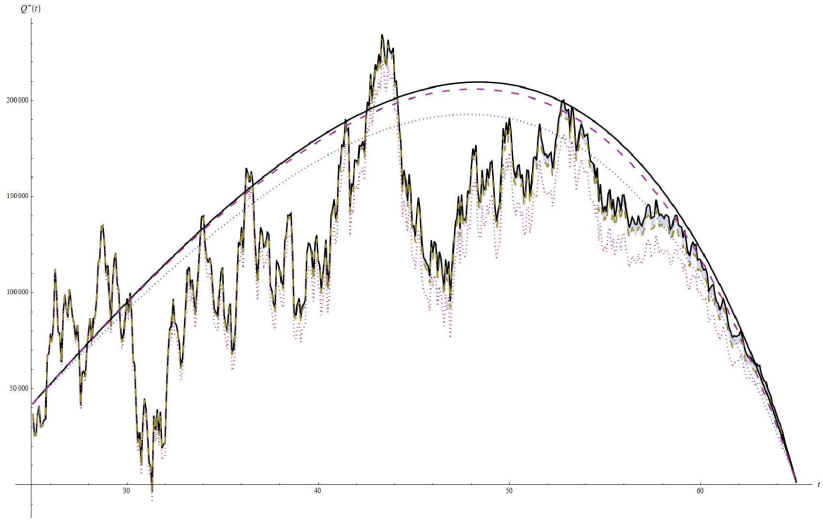

Figure 6: Comparison of the optimal life insurance purchase for time-consistent (solid), time-inconsistent (large dashing) and committed 25-years old (small dashing) agents.

Figure 7 highlights the deviation of consumption patterns for different initial wages. Consumption brings immediate benefit so the heterogeneous agent, who is more impatient, decides to allocate larger amounts to consumption at least in the first periods. The committed agent ends up allocating larger amounts to consumption than the standard

agent at all ages, since her path reflects the preferences from the perspective of the 25 years old. The time-inconsistent wage earner starts consuming more than the standard. However, as time goes on she modifies (reduces) her previous choices according to her decreasing bias to the present. As a result, her consumption path intersects the standard one between the ages of 45 and 50, and ends in a lower level. Finally, the time-consistent trajectory starts above the other three solutions and ends below them. This is so because this agent makes her plan knowing how her preferences are going to evolve and she decides to take advantage of the different levels of impatience at each time point. Thus, her consumption is greater while she more impatient, since she knows that in the future her preferences will lead her to consume less. Observe that an increase in the initial wage shifts the curves upwards though, unlike the life insurance purchase, it hardly modifies the differences between the four consumption paths.

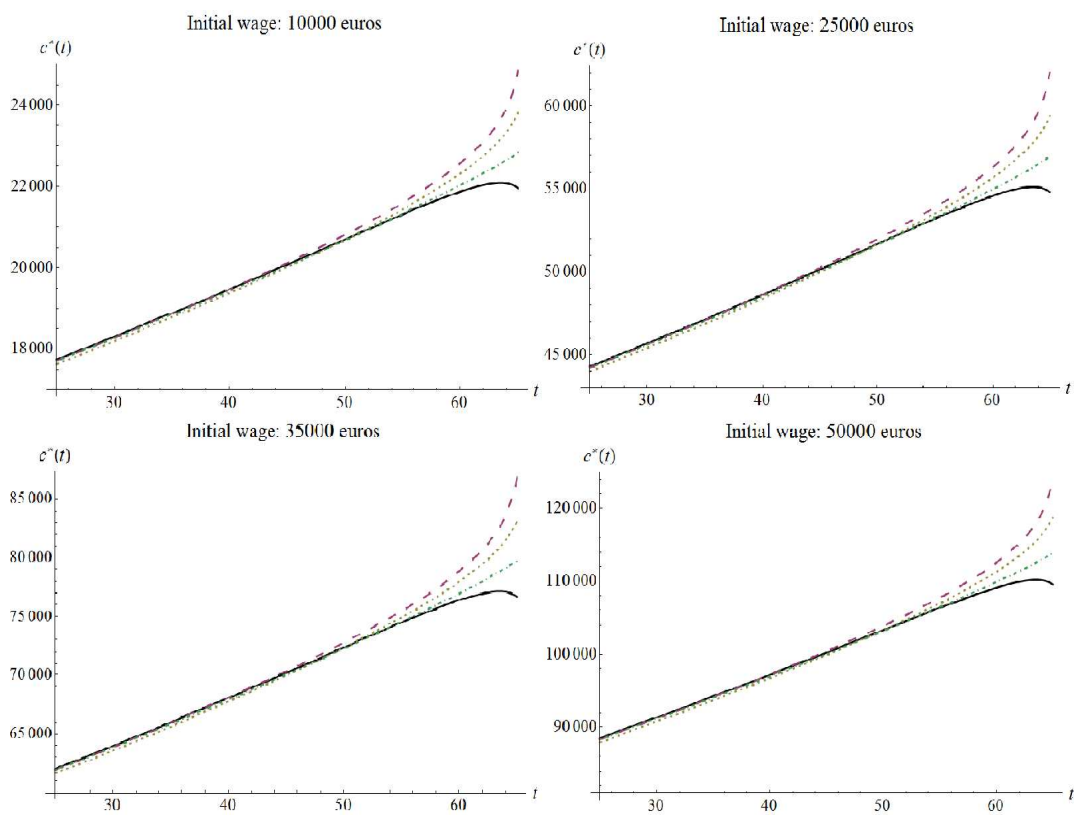

Figure 7: Consumption paths for the standard case (dotted), the committed agent (large dashing), the time-inconsistent agent (dot-dahsed) and the time-consistent agent (solid).

To conclude this section, we analyze how the time-consistent life insurance and consumption rules are modified when we vary the heterogeneous preferences. Figures 8 and 9 show that the previous results are intensified if the discount rate for the final function ρ is increased. For $\delta = 0.03$ we plot the different paths taking $\rho = 0.06$, $\rho = 0.15$ and $\rho = 0.2$. In particular, Figure 8 shows that the life insurance purchase increases with ρ,

while Figure 9 shows that the consumption path rotates as ρ increases, i.e., the agent consumes more when she is young and less when she is older.

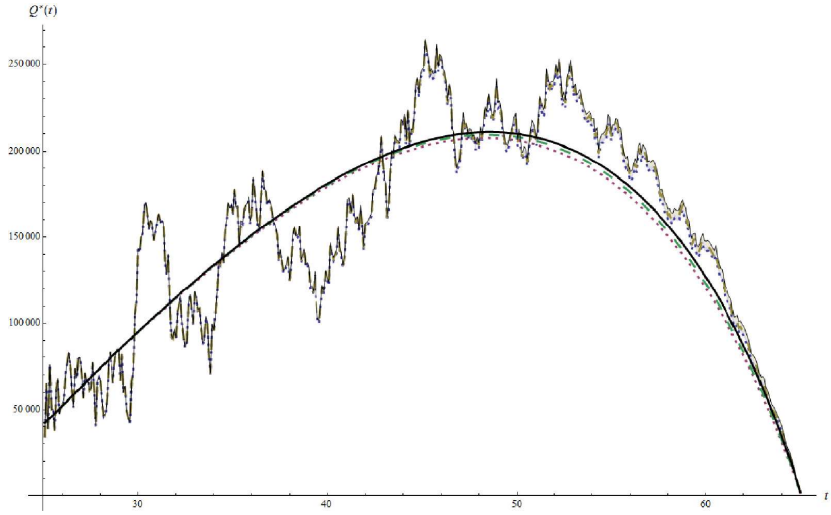

Figure 8: Sensitivity of the time-consistent life insurance for different values of ρ, $\rho = 0.06$ (small dashing), $\rho = 0.15$ (large dashing), $\rho = 0.2$ (solid).

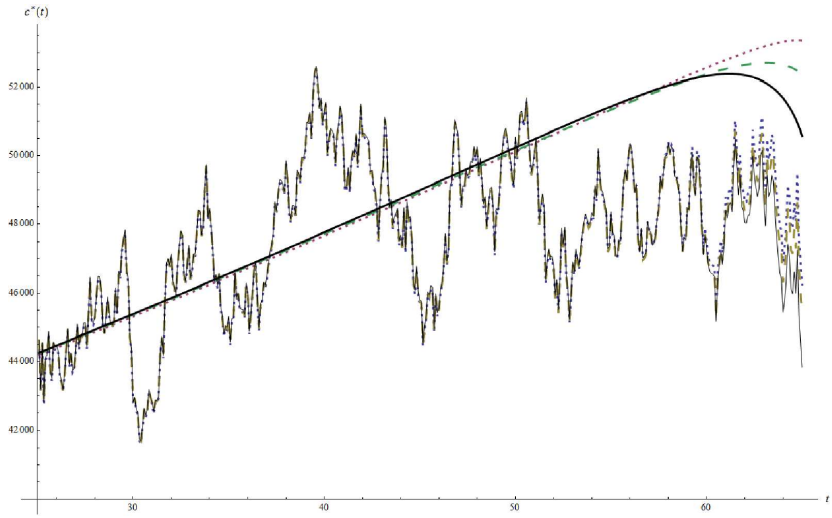

Figure 9: Sensitivity of the time-consistent consumption for different values of ρ, $\rho = 0.06$ (small dashing), $\rho = 0.15$ (large dashing), $\rho = 0.2$ (solid).

5 Conclusions

In this paper, we have studied the effects of introducing heterogeneous discounting into a stochastic continuous time model with random lifetime in which the wage earner decides between three different strategies: consumption, investment and life insurance purchase. In contrast with the standard case, heterogeneous preferences capture the different valu-

ations that the individual gives to the bequest left to her descendants and to her wealth at retirement along the planning horizon. Consequently, the optimal policies for an agent using the heterogeneous discount function differ from those for an agent with standard discounting. In order to illustrate these effects, we have departed from the model in Pliska and Ye (2007) generalizing the individual time preferences with the heterogeneous discount function introduced by Marín-Solano and Patxot (2012). In addition, we have derived the wealth dynamics in terms of the portfolio elasticity (Kraft (2003)). This procedure allows us to generalize the investment problem by introducing contingent claims in the opportunity set while maintaining the analytical tractability of the model. Explicit solutions have been obtained for the case of CRRA and CARA utility functions for both the time-inconsistent and the time-consistent agent. The implications of the use of the heterogeneous discount function have been illustrated, showing the differences between our results and the standard ones.

References

[1] de-Paz, A., Marín-Solano, J. and Navas, J. (2011). Heterogeneous discounting in consumption-investment problems. Time consistent solutions. Available at SSRN: http://ssrn.com/abstract=1953544 or http://dx.doi.org/10.2139/ssrn.1953544

[2] Ekeland, I. and Lazrak, A. (2010). The golden rule when preferences are time inconsistent. Mathematics and Financial Economics 4 (1), 29-55.

[3] Ekeland, I., Mbodji, O. and Pirvu, T. (2012). Time-consistent portfolio management. SIAM Journal of Financial Mathematics 3, 1-32.

[4] Frederick, S., Loewenstein, G. and O'Donoghue, T. (2002). Time discounting and time preference: a critical review. Journal of Economic Literature 40, 351-401.

[5] Karp, L. (2007). Non-constant discounting in continuous time. Journal of Economic Theory 132, 557-568.

[6] Kraft, H. (2003). Elasticity approach to portfolio optimization. Mathematical Methods of Operations Research 58, 159-182.

[7] Leung, S.F. (1994). Uncertain lifetime, the theory of the consumer, and the life cicle hypothesis. Econometrica 62, 1233-1239.

[8] Marín-Solano, J. and Navas, J. (2010). Consumption and portfolio rules for time-inconsistent investors. European Journal of Operational Research 201 (3), 860-872.

[9] Marín-Solano, J. and Patxot, C. (2012). Heterogeneous discounting in economic problems. Optimal Control Applications and Methods 33 (1), 32-50.

[10] Merton, R.C. (1969). Lifetime portfolio selection under uncertainty: the continuous case. Reviews of Economical Statistics 51, 247-257.

[11] Merton, R.C. (1971). Optimum consumption and portfolio rules in a continuous time model. Journal of Economic Theory 3, 373-413.

[12] Milevsky, M.A. (2006). The Calculus of Retirement Income. Financial Models for Pension Annuities and Life Insurance. Cambridge University Press, Cambrifge, UK.

[13] Phelps, E.S. and Pollak, R.A. (1968). On second-best national saving and game-equilibrium growth. Review of Economic Studies 35, 185-199.

[14] Pliska, S.R. and Ye, J. (2007). Optimal life insurance purchase and consumption/investment under uncertain lifetime. Journal of Banking and Finance 31, 1307-1319.

[15] Richard, S.F. (1975). Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a constinuous time model. Journal of Financial Economics 2, 187-203.

[16] Samuelson, P.A. (1937). A note on measurement of utility. Review of Economic Studies 4 (2), 155-161.

[17] Yaari, M.E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. The Review of Economic Studies 32 (2), 137-150.