SOME SPECTRAL PROPERTIES OF THE CANONICAL SOLUTION OPERATOR TO $\bar{\partial}$ ON WEIGHTED FOCK SPACES

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Abstract. We characterize the Schatten class membership of the canonical solution operator to $\bar{\partial}$ acting on $L^2(e^{-2\phi})$, where ϕ is a subharmonic function with $\Delta\phi$ a doubling measure. The obtained characterizations are in terms of $\Delta\phi$. As part of our approach, we study Hankel operators with anti-analytic symbols acting on the corresponding Fock space of entire functions in $L^2(e^{-2\phi})$.

Keywords: Schatten classes, canonical solution operator to $\bar{\partial}$

1. Introduction

For a (nonharmonic) subharmonic function ϕ on $\mathbb{C}$ having the property that $\Delta\phi$ is a doubling measure, the generalized Fock space $\mathcal{F}^2_\phi$ is defined by

$$\mathcal{F}^2_\phi = \{f \in \mathcal{H}(\mathbb{C}) : \|f\|^2_{\mathcal{F}^2_\phi} = \int_{\mathbb{C}} |f(z)|^2 e^{-2\phi(z)}\, dm(z) < \infty\},$$

where $dm(z)$ denotes the Lebesgue measure on $\mathbb{C}$. We let $\mu = \Delta\phi$ and denote by $\rho(z)$ the positive radius for which we have $\mu(D(z,\rho(z))) = 1$, $z \in \mathbb{C}$. The function ρ^{-2} can be regarded as a regularized version of $\Delta\phi$ (see [5, 17]). We consider the canonical solution operator N to $\bar{\partial}$ given by

$$\bar{\partial}Nf = f \quad \text{and} \quad Nf \text{ is of minimal norm in } L^2(e^{-2\phi}),$$

or, equivalently

$$\bar{\partial}Nf = f \quad \text{and} \quad Nf \perp \mathcal{F}^2_\phi.$$

The boundedness and the compactness of N acting on various weighted L^2-spaces have been extensively studied in one or several variables (see [6, 8, 9, 10]). Concerning the Schatten class membership of this operator, it was first shown in [8] that for the particular choice $\phi(z) = |z|^m$, N fails to be Hilbert-Schmidt, and a more involved study was pursued in [10] in the context of several complex variables, where the authors obtain necessary and sufficient conditions for the canonical solution operator to $\bar{\partial}$ to belong to the Schatten class $\mathcal{S}^p$, $p > 0$, when restricted to $(0,1)$-forms with holomorphic coefficients in $L^2(\mu)$, for measures μ with the property that the monomials form an orthogonal family in $L^2(\mu)$. Some particular cases of these results were previously obtained in [16].

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In the present paper we are interested in the setting of subharmonic functions ϕ with $\Delta\phi$ a doubling measure. For this type of weights, it was proven in [18] that N is compact from $L^2(e^{-2\phi})$ to itself if and only if $\rho(z) \to 0$ as $|z| \to \infty$. We continue the investigation in [18] by characterizing the Schatten class membership of N. We find that N fails to be Hilbert-Schmidt and that N belongs to the Schatten class $\mathcal{S}^p$ with $p > 2$ if and only if the following holds

$$\int_{\mathbb{C}} \rho^{p-2}(z)\, dA(z) < \infty. \tag{1}$$

We start our approach by noticing that the restriction of N to $\mathcal{F}^2_\phi$ is actually a (big) Hankel operator with symbol $\bar{z}$. This observation leads us to a study of these properties for Hankel operators on $\mathcal{F}^2_\phi$ with anti-analytic symbols. We would like to point out that Lin and Rochberg [14, 15] considered these problems for Hankel operators with symbols in $L^2(\mathbb{C})$ for a certain class of subharmonic functions ϕ. The case of anti-analytic symbols was investigated in [13, 4, 20] for $\phi(z) = |z|^m$, $m > 0$, and it was shown that a Hankel operator $H_{\bar{g}}$ belongs to $\mathcal{S}^p$ if and only if the symbol g is a polynomial of degree smaller than $m(p-2)/(2p)$. For subharmonic functions ϕ with $\Delta\phi$ a doubling measure, we find that $H_{\bar{g}}$ fails to be Hilbert-Schmidt unless g is constant, and $H_{\bar{g}} \in \mathcal{S}^p$ for $p > 2$, if and only if its symbol satisfies

$$\int_{\mathbb{C}} |g'(z)|^p \rho^{p-2}(z)\, dA(z) < \infty,$$

that is, g is a polynomial whose degree depends on the order of decay of ρ.

Finally, using a result by Russo [19] together with the pointwise estimates obtained in [18] for the kernel of the canonical solution operator N, we show that the condition (1) is actually sufficient for N to belong to $\mathcal{S}^p$ with $p > 2$, even when defined on the whole of $L^2(e^{-2\phi})$.

2. Preliminaries

In this section we gather a few definitions and some known estimates that will be used in our further considerations. We start with some facts about doubling measures. A nonnegative Borel measure μ is called *doubling* if there exists $C > 0$ such that

$$\mu(D(z, 2r)) \le C\mu(D(z, r)),$$

for all $z \in \mathbb{C}$ and $r > 0$. The smallest constant in the previous inequality is called the doubling constant for μ.

Lemma 1. *([5, Lemma 2.1]) Let μ be a doubling measure on $\mathbb{C}$. There exists a constant $\gamma > 0$ such that for any discs D, D' with respective radius $r > r'$ and with $D \cap D' \neq \emptyset$ the following holds*

$$\Big(\frac{\mu(D)}{\mu(D')}\Big)^\gamma \lesssim \frac{r}{r'} \lesssim \Big(\frac{\mu(D)}{\mu(D')}\Big)^{1/\gamma}.$$

From now on we shall assume that ϕ is a subharmonic function on $\mathbb{C}$ such that $\Delta\phi$ is a doubling measure.We denote $D^r(z) = D(z, r\rho(z))$ and for $r = 1$ we simply write $D(z)$ instead of $D^1(z)$. The function ρ has at most polynomial growth/decay (see [17, Remark 1]): there exist constants $C, \beta, \gamma > 0$ such that

$$C^{-1}\frac{1}{|z|^{\gamma}} \le \rho(z) \le C|z|^{\beta}, \quad \text{for} \quad |z| > 1. \tag{2}$$

As an immediate consequence of Lemma 1 one obtains

Lemma 2. [18] *For any $r > 0$ there exists $c > 0$ depending only on r and the doubling constant for $\Delta\phi$ such that*

$$c^{-1}\rho(\zeta) \le \rho(z) \le c\,\rho(\zeta) \quad \textit{for} \quad \zeta \in D^r(z).$$

We also have

Lemma 3. [5, p. 205] *If $\zeta \notin D(z)$, then*

$$\frac{\rho(z)}{\rho(\zeta)} \lesssim \Big(\frac{|z-\zeta|}{\rho(\zeta)}\Big)^{1-\delta}$$

for some $\delta \in (0, 1)$ depending only on the doubling constant for $\Delta\phi$.

For $z, \zeta \in \mathbb{C}$, the distance d_ϕ induced by the metric $\rho^{-2}(z)dz \otimes d\bar z$ is given by

$$d_\phi(z,\zeta) = \inf_\gamma \int_0^1 |\gamma'(t)|\frac{dt}{\rho(\gamma(t))},$$

where γ runs over the piecewise $\mathcal{C}^1$ curves $\gamma : [0, 1] \to \mathbb{C}$ with $\gamma(0) = z$ and $\gamma(1) = \zeta$. We observe now that the metric $\rho^{-2}(z)dz \otimes d\bar z$ is comparable to the Bergman metric: it is well known, see [2] that the Bergman metric $B(\frac{\partial}{\partial z}, z)$ at the point z is given by the solution to the extremal problem

$$B\Big(\frac{\partial}{\partial z}, z\Big) = \frac{\sup\{|f'(z)| :\ f \in \mathcal{F}^2_\phi,\ f(z) = 0; \|f\|_{\mathcal{F}^2_\phi} = 1\}}{\sqrt{K(z,z)}}.$$

where $K(z, \zeta)$ is the Bergman kernel for $\mathcal{F}^2_\phi$. In [17, Lemma 20] it is proved that for all $f \in \mathcal{F}^2_\phi$ with $f(z) = 0$ we have $|f'(z)| \lesssim \frac{e^{\phi(z)}}{\rho^2(z)}\|f\|_{\mathcal{F}^2_\phi}$, thus $B(\frac{\partial}{\partial z}, z) \lesssim 1/\rho(z)$. The other inequality follows taking as $f(\zeta) = C_z(\zeta - z)K(\zeta, z)$ where C_z is taken in such a way that $\|f\|_{\mathcal{F}^2_\phi} = 1$. In view of the estimates of the Bergman kernel stated below it follows that $B(\frac{\partial}{\partial z}, z) \sim 1/\rho(z)$

The following estimates for the Bergman distance d_ϕ hold:

Lemma 4. [17, Lemma 4] *There exists $\delta \in (0, 1)$ such that for every $r > 0$ there exists $C_r > 0$ such that*

$$C_r^{-1}\frac{|z-\zeta|}{\rho(z)} \le d_\phi(z,\zeta) \le C_r\frac{|z-\zeta|}{\rho(z)}, \quad \textit{for } \zeta \in D^r(z),$$

and

$$C_r^{-1}\Big(\frac{|z-\zeta|}{\rho(z)}\Big)^{\delta} \le d_\phi(z,\zeta) \le C_r\Big(\frac{|z-\zeta|}{\rho(z)}\Big)^{2-\delta}, \quad \textit{for } \zeta \in D^r(z)^c.$$

The next result shows that we can replace the weight ϕ by a regular weight $\tilde\phi$ equivalent to it.

Proposition 1. [17, Theorem 14] *Let ϕ be a subharmonic function with $\mu = \Delta\phi$ doubling. There exists $\tilde\phi \in C^\infty(\mathbb{C})$ subharmonic such that $|\phi - \tilde\phi| \le c$ with $\Delta\tilde\phi$ doubling and*

$$\Delta\tilde\phi \sim \frac{1}{\rho^2_{\tilde\phi}} \sim \frac{1}{\rho^2_\phi}.$$

We also need the estimates

Lemma 5. [18] *Let ϕ be a subharmonic function with $\mu = \Delta\phi$ doubling. Then for any $\varepsilon > 0$ and $k \ge 0$*

$$\int_{\mathbb{C}} \frac{|z-\zeta|^k}{\exp d_\phi(z,\zeta)^\varepsilon}\, d\mu(z) \le c\, \rho^k(\zeta),$$

where $c > 0$ is a constant depending only on k, ε and on the doubling constant for μ.

Theorem 1. [18] *Let $K(z,\zeta)$ be the Bergman kernel for $\mathcal{F}^2_\phi$. There exist positive constants c and ε (depending only on the doubling constant for $\Delta\phi$) such that for any $z,\zeta \in \mathbb{C}$*

$$|K(z,\zeta)| \le c\frac{1}{\rho(z)\rho(\zeta)}\frac{e^{\phi(z)+\phi(\zeta)}}{\exp d^\varepsilon_\phi(z,\zeta)}.$$

Lemma 6. [18] *There exists $\alpha > 0$ such that*

$$|K(z,\zeta)| \sim K(z,z)^{1/2}K(\zeta,\zeta)^{1/2} \sim \frac{e^{\phi(z)+\phi(\zeta)}}{\rho(z)\rho(\zeta)}, \qquad \textit{if } \ |z-\zeta| < \alpha\rho(z).$$

On the diagonal we have

$$K(z,z) \sim \frac{e^{2\phi(z)}}{\rho^2(z)}, \quad z \in \mathbb{C}. \tag{3}$$

For $\lambda \in \mathbb{D}$, we denote by k_λ the normalized reproducing kernel of $\mathcal{F}^2_\phi$, i.e.

$$k_\lambda(z) = \frac{K(z,\lambda)}{K(\lambda,\lambda)^{1/2}}, \quad z,\lambda \in \mathbb{C}.$$

Finally, let us recall that a compact operator T acting on a Hilbert space belongs to the Schatten class $\mathcal{S}^p$ if the sequence of eigenvalues of $(T^*T)^{1/2}$ belongs to l^p.

3. Hankel operators on $\mathcal{F}^2_\phi$

As already mentioned in the introduction, the canonical solution operator N to $\bar{\partial}$ is defined on $L^2(e^{-2\phi})$ by

$$\bar{\partial}Nf = f \quad \text{and} \quad Nf \perp \mathcal{F}^2_\phi.$$

Let us now consider the restriction of N to $\mathcal{F}^2_\phi$. Notice that if $f \in \mathcal{F}^2_\phi$ and $\bar{z}f \in L^2(e^{-2\phi})$, then

$$Nf = (I - P)(\bar{z}f), \tag{4}$$

where P is the orthogonal projection of $L^2(e^{-2\phi})$ onto $\mathcal{F}^2_\phi$. In general, $\bar{z}f \in L^2(e^{-2\phi})$ does not hold for all $f \in \mathcal{F}^2_\phi$ (see e.g.[13]), but it follows from Theorem 1 that $\bar{z}k_\lambda \in L^2(e^{-2\phi})$ for all $\lambda \in \mathbb{C}$. Since the subset $\mathrm{Span}\{k_\lambda : \lambda \in \mathbb{C}\}$ is dense in $\mathcal{F}^2_\phi$, we deduce from (4) that N coincides with the big Hankel operator acting on $\mathcal{F}^2_\phi$ with symbol $\bar{z}$. Motivated by this last fact, we now aim to study Hankel operators with anti-analytic symbols on $\mathcal{F}^2_\phi$. Given an entire function g so that there exists a dense subset A of $\mathcal{F}^2_\phi$ with $\bar{g}f \in L^2(e^{-2\phi})$ for $f \in A$, the big Hankel operator with symbol $\bar{g}$ is densely defined by

$$H_{\bar{g}}f = \bar{g}f - P(\bar{g}f) = (I - P)(\bar{g}f), \quad f \in A,$$

where P is the orthogonal projection of $L^2(e^{-2\phi})$ onto $\mathcal{F}^2_\phi$. We consider symbols g such that

$$\bar{g}k_\lambda \in L^2(e^{-2\phi}) \text{ for all } \lambda \in \mathbb{C}.$$

It follows from Theorem 1 that, for example, polynomial symbols satisfy this assumption. By the reproducing formula in $\mathcal{F}^2_\phi$ we get

$$H_{\bar{g}}k_\lambda(z) = (\overline{g(z)} - \overline{g(\lambda)})k_\lambda(z), \quad z, \lambda \in \mathbb{C}. \tag{5}$$

For the sake of completeness we shall first characterize the boundedness and compactness of $H_{\bar{g}}$. Let us state the following theorem due to Hörmander which is essential to our approach.

Theorem 2. [11] *Let $\Omega \subseteq \mathbb{C}$ be a domain and $\phi \in C^2(\Omega)$ be such that $\Delta\phi \geq 0$. For any $f \in L^2_{loc}(\Omega)$ there exists a solution u to $\bar{\partial}u = f$ such that*

$$\int |u|^2 e^{-2\phi} dm \leq \int \frac{|f|^2}{\Delta\phi} e^{-2\phi} dm.$$

Theorem 3. *$H_{\bar{g}}$ extends to a bounded linear operator on $\mathcal{F}^2_\phi$ if and only if $|g'|\rho$ is bounded.*

Proof. Assume first that $|g'|\rho$ is bounded. Then notice that for $f \in \mathrm{Span}\{k_\lambda : \lambda \in \mathbb{C}\}$, $H_{\bar{g}}f$ is the solution to $\bar{\partial}u = \bar{g}'f$ of minimal $L^2(e^{-2\phi})$-norm. By Theorem 2 and Proposition 1 we have

$$\int_{\mathbb{C}} |H_{\bar{g}}f|^2 e^{-2\phi} dm \lesssim \int_{\mathbb{C}} |f|^2 |g'|^2 \rho^2 dm \leq (\sup |g'|\rho)^2 \|f\|^2, \tag{6}$$

which shows that $H_{\bar{g}}$ can be extended to a bounded linear operator on $\mathcal{F}^2_\phi$.

Conversely, assume that $H_{\bar g}$ is bounded. Then we have $\|H_{\bar g}k_\lambda\| < M$ for $\lambda \in \mathbb{C}$, and using relation (5) together with Lemmas 6 and 2 we obtain

$$\begin{aligned} M > \|H_{\bar g}k_\lambda\|^2 &= \int_{\mathbb{C}} |g(z)-g(\lambda)|^2|k_\lambda(z)|^2 e^{-2\phi(z)}dm(z) \\ &\geq \int_{|z-\lambda|<\alpha\rho(\lambda)} |g(z)-g(\lambda)|^2|k_\lambda(z)|^2 e^{-2\phi(z)}dm(z) \\ &\gtrsim \frac{1}{\rho^2(\lambda)}\int_{|z-\lambda|<\alpha\rho(\lambda)} |g(z)-g(\lambda)|^2 dm(z), \end{aligned}$$

for α small enough. By the subharmonicity of $|g|$ and the Cauchy formula applied to $g_\lambda(z) = g(z)-g(\lambda)$ we can now conclude

$$|g'(\lambda)\rho(\lambda)| \lesssim \frac{1}{\rho^2(\lambda)}\int_{|z-\lambda|<\alpha\rho(\lambda)} |g(z)-g(\lambda)|^2 dm(z) < M, \quad \lambda \in \mathbb{C}.$$

□

Remark. The fact that ρ can have at most polynomial decay (see relation (2)) implies that $H_{\bar g}$ is bounded only for polynomial symbols of degree smaller than the order of decay of ρ. Notice also that if $H_{\bar g}$ is bounded, then ρ has to be bounded, since g is a polynomial.

Theorem 4. *$H_{\bar g}$ is compact if and only if $|g'(\lambda)|\rho(\lambda) \to 0$ as $|\lambda| \to \infty$.*

Proof. Assume first that $|g'(\lambda)|\rho(\lambda) \to 0$ as $|\lambda| \to \infty$. As in relation (6) we have

$$\|H_{\bar g}f\|^2 \leq \int_{\mathbb{C}} |g'|^2\rho^2|f|^2e^{-2\phi}dm = \|M_{g'\rho}f\|^2,$$

where $M_{g'\rho} : \mathcal{F}^2_\phi \to L^2(e^{-2\phi})$ is given by $M_{g'\rho}f = g'\rho f$. Hence, if $M_{g'\rho}$ is compact, then $H_{\bar g}$ is compact. We first show that, for $R > 0$, the truncation of $M_{g'\rho}$ given by

$$M^R_{g'\rho}f = \chi_{\{|z|<R\}}\, g'\rho f$$

is compact. To this end, let $\{f_n\}$ be a bounded sequence in $\mathcal{F}^2_\phi$, i.e. $\|f_n\| < M$. Since pointwise evaluation is bounded, we deduce that $\{f_n\}$ is a normal family and it therefore contains a subsequence $\{f_{n_k}\}$ uniformly convergent on compacts to an entire function f. By Fatou's lemma we obtain $f \in \mathcal{F}^2_\phi$. Then $f_{n_k} - f \to 0$ uniformly on compacts and $\|f_n - f\| < 2M$. Hence in order to show that $M^R_{g'\rho}$ is compact, it is enough to show that for any sequence f_n (by abuse of notation) that is bounded in the norm and converges uniformly to zero on compact sets, we have $\|M^R_{g'\rho}f_n\| \to 0$ as $n \to \infty$.

But this is quite easy to see, as

$$\|M^R_{g'\rho}f_n\|^2 \leq \sup_{|z|<R}|f_n|^2 \int_{|z|<R} |g'|^2\rho^2e^{-2\phi}dm \to 0,$$

as $n \to \infty$. Now

$$\|(M_{g'\rho} - M^R_{g'\rho})f\|^2 = \int_{|z|\geq R} |g'|^2\rho^2|f|^2 e^{-2\phi} dm \leq \sup_{|z|>R} |g'|^2\rho^2 \int_{\mathbb{C}} |f|^2 e^{-2\phi} dm, \quad f \in \mathcal{F}^2_\phi,$$

which shows that $\|M_{g'\rho} - M^R_{g'\rho}\| \to 0$ as $R \to \infty$, and therefore $M_{g'\rho}$ is compact, and consequently $H_{\bar g}$ is compact.

Suppose now $H_{\bar g}$ is compact. The set $\{k_\lambda\}_{\lambda\in\mathbb{C}}$ is bounded in $\mathcal{F}^2_\phi$. By compactness it follows that the set $\{H_{\bar g}k_\lambda\}_{\lambda\in\mathbb{C}}$ is relatively compact in $L^2(e^{-2\phi})$. Then by the Riesz-Tamarkin compactness theorem (see [3]) we have

$$\lim_{R\to\infty} \int_{|z|>R} |H_{\bar g}k_\lambda|^2 e^{-2\phi} dm = 0, \tag{7}$$

uniformly in λ. Since $H_{\bar g}$ is bounded, we have $B := \sup_\zeta \rho(\zeta) < \infty$. For $|\lambda| > R + B$, the inclusion $\{|z-\lambda| \leq \rho(\lambda)\} \subset \{|z| > R\}$ holds, and then for $\alpha > 0$ sufficiently small we have by Lemma 6

$$\begin{aligned}
\int_{|z|>R} |H_{\bar g}k_\lambda|^2 e^{-2\phi} dm &= \int_{|z|>R} |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z) \\
&\gtrsim \int_{|z-\lambda|<\alpha\rho(\lambda)} |g(z) - g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)} dm(z) \\
&\gtrsim \frac{1}{\rho^2(\lambda)} \int_{|z-\lambda|<\alpha\rho(\lambda)} |g(z) - g(\lambda)|^2 dm(z) \\
&\gtrsim \rho^2(\lambda)|g'(\lambda)|^2,
\end{aligned}$$

where the last step above follows again by the Cauchy formula and the subharmonicity of $|g|$. This shows that

$$\lim_{|\lambda|\to\infty} |g'(\lambda)|\rho(\lambda) = 0.$$

□

In the study of the Schatten class membership of $H_{\bar g}$ we use the following well-known inequality: If T is a compact operator from $\mathcal{F}^2_\phi$ to a Hilbert space $\mathcal{H}$, we have

$$\int_{\mathbb{C}} \|Tk_\lambda\|^p \frac{dm(\lambda)}{\rho^2(\lambda)} \lesssim \|T\|^p_{\mathcal{S}^p}, \tag{8}$$

for $p \geq 2$. To see this, let

$$T = \sum_n \lambda_n \langle \cdot, e_n\rangle f_n,$$

be the canonical form of T, where (e_n) is an orthonormal basis in $\mathcal{F}^2_\phi$, (f_n) is an orthonormal set in H, and the λ_n's are the singular numbers of T. Then

$$TK(\cdot,\lambda) = \sum_n \lambda_n \overline{e_n(\lambda)} f_n, \quad \lambda \in \mathbb{C}.$$

From this we deduce

$$\int_{\mathbb{C}} \|TK(\cdot,\lambda)\|^2 e^{-2\phi(\lambda)} dm(\lambda) = \int_{\mathbb{C}} \sum_n \lambda_n^2 |e_n(\lambda)|^2 e^{-2\phi(\lambda)} dm(\lambda) = \sum_n \lambda_n^2.$$

Hence

$$\int_{\mathbb{C}} \|Tk_\lambda\|^2 \frac{dm(\lambda)}{\rho^2(\lambda)} \sim \int_{\mathbb{C}} \|TK(\cdot,\lambda)\|^2 e^{-2\phi(\lambda)} dm(\lambda) = \|T\|^2_{\mathcal{S}^2}.$$

For $p=\infty$, we have

$$\sup_\lambda \|Tk_\lambda\| \le \|T\|_{\mathcal{S}^\infty}.$$

Then (8) follows by interpolation.

Theorem 5. *Suppose $H_{\bar g}$ is bounded. Then $H_{\bar g} \in \mathcal{S}^p$ with $p>2$ if and only if $g'\rho \in L^p(1/\rho^2)$. Moreover, $H_{\bar g}$ fails to be Hilbert-Schmidt, unless g is constant.*

Proof. Suppose $H_{\bar g} \in \mathcal{S}^p$ with $p \ge 2$. Then by (8) and using arguments similar to those above we have

$$\begin{aligned} \infty > \int_{\mathbb{C}} \|H_{\bar g} k_\lambda\|^p \frac{dm(\lambda)}{\rho^2(\lambda)} &= \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} |g(z)-g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)}\, dm(z) \Big)^{p/2} \frac{dm(\lambda)}{\rho^2(\lambda)} \\ &\gtrsim \int_{\mathbb{C}} \Big(\int_{|z-\lambda|<\alpha\rho(\lambda)} |g(z)-g(\lambda)|^2 |k_\lambda(z)|^2 e^{-2\phi(z)}\, dm(z) \Big)^{p/2} \frac{dm(\lambda)}{\rho^2(\lambda)} \\ &\gtrsim \int_{\mathbb{C}} |g'(\lambda)\rho(\lambda)|^p \frac{dm(\lambda)}{\rho^2(\lambda)}, \end{aligned}$$

for α small enough. With this the necessity is proven. In particular, the above relation shows that $H_{\bar g}$ cannot be Hilbert-Schmidt for nonconstant anti-analytic symbols.

To prove the sufficiency, assume $g'\rho \in L^p(1/\rho^2)$. Then a subharmonicity argument shows that $|g'(\lambda)|\rho(\lambda) \to 0$ as $|\lambda| \to \infty$. As in the proof of Theorem 4 we have

$$\|H_{\bar g} f\| \lesssim \|M_{g'\rho} f\|, \quad f \in \mathcal{F}^2_\phi.$$

Therefore $M_{g'\rho} \in \mathcal{S}^p$ for some $p>2$, implies $H_{\bar g} \in \mathcal{S}^p$. Indeed, this follows from the criterion (see [7]): A linear operator $S: H_1 \to H_2$, where H_1, H_2 are separable Hilbert spaces, belongs to $\mathcal{S}^p, p \ge 2$, if and only if $\sum \|Se_n\|^p < \infty$, for any orthonormal basis $\{e_n\}$ of H_1. We notice that for $f, h \in \mathcal{F}^2_\phi$ we have

$$\langle M^*_{g'\rho} M_{g'\rho} f, h\rangle = \langle M_{g'\rho} f, M_{g'\rho} h\rangle = \int_{\mathbb{C}} f\bar h |g'|^2 \rho^2 e^{-2\phi}\, dm = \langle T_{|g'|^2\rho^2} f, h\rangle,$$

where $T_{|g'|^2\rho^2}$ is the Toeplitz operator on $\mathcal{F}^2_\phi$ with symbol $|g'|^2\rho^2$. In order to show that $M_{b'\rho} \in \mathcal{S}^p$, we are going to prove that $T_{|g'|^2\rho^2} = M^*_{g'\rho} M_{g'\rho} \in \mathcal{S}^{p/2}$. Since $|g'(\lambda)|\rho(\lambda) \to 0$ as $|\lambda| \to \infty$, the proof of the sufficiency in Theorem 4 shows that $M_{g'\rho}$ is compact, and hence $T_{|g'|^2\rho^2}$ is compact. Denote $G = |g'|^2\rho^2$ for convenience. The operator T_G is also positive and self-adjoint, and it is then given by

$$T_G = \sum_n \lambda_n \langle \cdot, e_n\rangle e_n,$$

where λ_n are the singular numbers of T_G, and e_n is an orthonormal basis in $\mathcal{F}^2_\phi$. Then

$$\lambda_n = \langle T_G e_n, e_n\rangle = \int_{\mathbb{C}} |e_n|^2 G e^{-2\phi}\, dm,$$

and by Jensen's inequality we get

$$\lambda_n^{p/2} \le \int_{\mathbb{C}} G^{p/2}|e_n|^2 e^{-2\phi}\, dm$$

using the fact that $|e_n|^2 e^{-2\phi}\, dm$ is a probability measure on $\mathbb{C}$. Taking into account the fact that $K(z,\zeta) = \sum e_n(z)\overline{e_n(\zeta)}$, we can sum up over n in the previous relation to deduce

$$\begin{aligned}\sum_n \lambda_n^{p/2} &\le \sum_n \int_{\mathbb{C}} G^{p/2}|e_n|^2 e^{-2\phi}\, dm\\ &= \int_{\mathbb{C}} G(z)^{p/2} K(z,z) e^{-2\phi(z)}\, dm(z)\\ &\lesssim \int_{\mathbb{C}} G(z)^{p/2}\frac{1}{\rho^2(z)}\, dm(z) < \infty,\end{aligned}$$

by our assumption. Thus $T_G \in \mathcal{S}^{p/2}$, and consequently $H_{\bar g} \in \mathcal{S}^p$. □

4. The canonical solution to $\bar\partial$ on $L^2(e^{-2\phi})$

For $g = z$ in Theorem 5 we obtain that the restriction of the canonical solution operator N to $\bar\partial$ to the generalized Fock space $\mathcal{F}^2_\phi$ is never Hilbert-Schmidt and it belongs to $\mathcal{S}^p$ for $p > 2$ if and only if

$$\int_{\mathbb{C}} \rho^{p-2}(z)\, dm(z) < \infty. \tag{9}$$

The aim of this section is to show that the condition above is sufficient for N to belong to $\mathcal{S}^p$, even when defined on the whole of $L^2(e^{-2\phi})$.

For the integral kernel $C(z,\zeta)$ of N, i.e.

$$Nf(z) = \int_{\mathbb{C}} e^{\phi(z)-\phi(\zeta)} C(z,\zeta) f(\zeta)\, dm(\zeta), \quad f \in L^2(e^{-2\phi}),$$

the following estimates were obtained in [18]

Theorem 6. [18] *There exists $\varepsilon > 0$ such that*

$$|C(z,\zeta)| \lesssim \begin{cases} |z-\zeta|^{-1}, & |z-\zeta| \le \rho(z),\\ \rho^{-1}(z)\exp(-d_\phi(z,\zeta)^\varepsilon), & |z-\zeta| \ge \rho(z).\end{cases}$$

To prove our main result we use these estimates together with a criterion for an integral operator to belong to Schatten classes for $p \ge 2$ obtained in [19]. Given a measure space (X,μ), let $G(x,y)$ be a complex-valued measurable function on $X\times X$ and denote $G^*(x,y)=\overline{G(y,x)}$. Consider the mixed normed space

$$L^p(L^q)=\Big\{G:\ \int\Big(\int |G(x,y)|^q d\mu(y)\Big)^{p/q} d\mu(x)<\infty\Big\}$$

Theorem 7. [19] *Let $p\geq 2$ and let (X,μ) be as above. If $G,G^*\in L^p(L^{p'})$, where $1/p+1/p'=1$, then the integral operator with kernel $G(x,y)$ given by*

$$Tf(x)=\int G(x,y)f(y)d\mu(y),\quad f\in L^2(d\mu),$$

belongs to $\mathcal{S}^p$.

A first version of the above theorem was proven in [19] (see also [12]) and subsequently improved in [1], where sharper conditions on the kernel G were given.

Theorem 8. *The operator N is never Hilbert-Schmidt. For $p>2$, N belongs to the Schatten class $\mathcal{S}^p$ if and only if (9) holds.*

Proof. The necessity follows from Theorem 5. It remains to prove the sufficiency. Assume ρ satisfies (9) for some $p>2$. In order to prove that $N\in\mathcal{S}^p$, we want apply Theorem 7. To this end consider the unitary operator $U:L^2\to L^2(e^{-2\phi})$ given by

$$Uf=fe^{\phi}.$$

Then $N\in\mathcal{S}^p$ if and only if $U^*NU\in\mathcal{S}^p$. Notice that

$$U^*NUf(z)=\int_{\mathbb{C}} C(z,\zeta)f(\zeta)\,dm(\zeta),\quad f\in L^2.$$

Now it is enough to show that the kernel $C(z,\zeta)$ of U^*NU satisfies the conditions in Theorem 7, and then the conclusion will easily follow. We shall first estimate

$$\|C\|^p_{L^p(L^{p'})}=\int_{\mathbb{C}}\Big(\int_{\mathbb{C}}|C(z,\zeta)|^{p'}\,dm(\zeta)\Big)^{p/p'}dm(z). \tag{10}$$

Theorem 6 implies

$$\begin{aligned}\int_{\mathbb{C}}|C(z,\zeta)|^{p'}\,dm(\zeta) &\lesssim \int_{|z-\zeta|\leq\rho(z)}\frac{dm(\zeta)}{|z-\zeta|^{p'}}+\int_{|z-\zeta|>\rho(z)}\frac{dm(\zeta)}{\rho(z)^{p'}\exp(p'd^{\varepsilon}_{\phi}(z,\zeta))}\\ &\lesssim \rho(z)^{2-p'}+\int_{|z-\zeta|>\rho(z)}\frac{dm(\zeta)}{\rho(z)^{p'}\exp d^{\varepsilon_1}_{\phi}(z,\zeta)}, \end{aligned}\tag{11}$$

for $0<\varepsilon_1<\varepsilon$. Now for $|z-\zeta|\leq\rho(z)$ or $|z-\zeta|\leq\rho(\zeta)$ we have $\rho(z)\sim\rho(\zeta)$ by Lemma 2. On the other hand, for $(z,\zeta)\in\{|z-\zeta|>\rho(z)\}\cap\{|z-\zeta|>\rho(\zeta)\}$, Lemmas 3-4 imply

$$\frac{\rho(\zeta)^2}{\exp d^{\varepsilon_1}_{\phi}(z,\zeta)}\lesssim\frac{\rho(z)^2}{\exp d^{\varepsilon_2}_{\phi}(z,\zeta)},$$

for some $\varepsilon_2 > 0$. Using this in (11) we get

$$\begin{aligned}\int_{\mathbb{C}} |C(z,\zeta)|^{p'}\, dm(\zeta) &\lesssim \rho(z)^{2-p'} + \rho(z)^{2-p'} \int_{|z-\zeta|>\rho(z)} \frac{1}{\exp d_\phi^{\varepsilon_2}(z,\zeta)} \frac{dm(\zeta)}{\rho^2(\zeta)} \\ &\lesssim \rho(z)^{2-p'},\end{aligned}$$

where the last step above follows by Proposition 1 and Lemma 5. Returning to (10) we obtain

$$\|C\|^p_{L^p(L^{p'})} = \int_{\mathbb{C}} \rho(z)^{(2-p')p/p'} dm(z) = \int_{\mathbb{C}} \rho(z)^{p-2} dm(z) < \infty,$$

by our assumption. It remains to show that $\|C^*\|_{L^p(L^{p'})} < \infty$. Although the estimates are analogous in this case, we include them for the sake of completeness. We have

$$\|C^*\|^p_{L^p(L^{p'})} = \int_{\mathbb{C}} \Big(\int_{\mathbb{C}} |C(z,\zeta)|^{p'}\, dm(z) \Big)^{p/p'} dm(\zeta). \tag{12}$$

As before, by Theorem 6 and Lemma 2 we get

$$\begin{aligned}\int_{\mathbb{C}} |C(z,\zeta)|^{p'}\, dm(z) &\lesssim \int_{|z-\zeta|\le\rho(z)} \frac{dm(z)}{|z-\zeta|^{p'}} + \int_{|z-\zeta|>\rho(z)} \frac{dm(z)}{\rho(z)^{p'} \exp(p' d_\phi^{\varepsilon}(z,\zeta))} \\ &\lesssim \int_{|z-\zeta|\le c\rho(\zeta)} \frac{dm(z)}{|z-\zeta|^{p'}} + \int_{|z-\zeta|>\rho(z)} \frac{dm(z)}{\rho(z)^{p'} \exp(p' d_\phi^{\varepsilon}(z,\zeta))} \\ &\lesssim \rho(\zeta)^{2-p'} \Big(1 + \int_{|z-\zeta|>\rho(z)} \frac{1}{\exp d_\phi^{\varepsilon_1}(z,\zeta)} \frac{dm(z)}{\rho^2(\zeta)} \Big),\end{aligned}$$

where $c > 0$, and the last step above follows by Lemmas 3-4. By Proposition 1 and Lemma 5 we obtain

$$\int_{\mathbb{C}} |C(z,\zeta)|^{p'}\, dm(z) \lesssim \rho(\zeta)^{2-p'},$$

and hence by (12) we get

$$\|C^*\|^p_{L^p(L^{p'})} \lesssim \int_{\mathbb{C}} |\rho(\zeta)|^{p-2}\, dm(\zeta) < \infty.$$

With this the proof is complete. □

References

[1] J. Arazy, S. D. Fisher, S. Janson, J. Peetre, Membership of Hankel operators on the ball in unitary ideals, *J. London Math. Soc.* **43** (1991), 485-508.

[2] S. Bergman, *The Kernel Function and Conformal Mapping*, 2nd ed., A.M.S. Survey V, Providence, 1970.

[3] M. S. Berger, *Nonlinearity and functional analysis*, Academic Press, New York-London, 1977.

[4] H. Bommier-Hato, E. H. Youssfi, Hankel operators on weighted Fock spaces, *Integral Equations Operator Theory* **59** (2007), 1-17.

[5] M. Christ, On the $\overline{\partial}$ equation in weighted L^2 norms in C^1,*J. Geom. Anal.* **1**(1991), 193-230.

[6] S. Fu, E. J. Straube, Compactness in the $\overline{\partial}$-Neumann problem, Complex analysis and geometry (Columbus, OH, 1999), 141-160, Ohio State Univ. Math. Res. Inst. Publ., 9, de Gruyter, Berlin, 2001.

[7] I. C. Gohberg and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators. Translations of Mathematical Monographs, Amer. Math. Soc., Providence, R.I., 1969.

[8] F. Haslinger, The canonical solution operator to $\overline{\partial}$ restricted to spaces of entire functions, *Ann. Fac. Sci. Toulouse Math.* (6) **11** (2002), 57-70.

[9] F. Haslinger and B. Helffer, Compactness of the solution operator to $\overline{\partial}$ in weighted L^2-spaces, *J. Funct. Anal.* **243** (2007), 679-697.

[10] F. Haslinger and B. Lamel, Spectral properties of the canonical solution operator to $\overline{\partial}$, *J. Funct. Anal.* **255** (2008), 13-24.

[11] L. Hörmander, L^2 estimates and existence theorems for the $\bar{\partial}$ operator, *Acta Math.* **113** (1965), 89-152.

[12] S. Janson, T. H. Wolff, Schatten classes and commutators of singular integral operators, *Ark. Mat.* **20** (1982), 301-310.

[13] W. Knirsch, G. Schneider, Continuity and Schatten-von Neumann p-class membership of Hankel operators with anti-holomorphic symbols on (generalized) Fock spaces, *J. Math. Anal. Appl.* **320** (2006), 403-414.

[14] P. Lin, R. Rochberg, Hankel operators on the weighted Bergman spaces with exponential type weights, *Integral Equations Operator Theory* **21** (1995), 460-483.

[15] P. Lin, R. Rochberg, Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights, *Pacific J. Math.* **173** (1996), 127-146.

[16] S. Lovera, E. H. Youssfi, Spectral properties of the $\overline{\partial}$-canonical solution operator, *J. Funct. Anal.* **208** (2004), 360-376.

[17] N. Marco, X. Massaneda and J. Ortega-Cerdà, Interpolating and sampling sequences for entire functions, *Geom. Funct. Anal.* **13** (2003), 862-914.

[18] J. Marzo and J. Ortega-Cerdà, Pointwise estimates for the Bergmam kernel of the weighted Fock space, *J. Geom. Anal.*, **19** (2009), 890-910.

[19] B. Russo , On the Hausdorff-Young theorem for integral operators, *Pacific J. Math.* **68** (1977), 241-253.

[20] G. Schneider, A note on Schatten-class membership of Hankel operators with anti-holomorphic symbols on generalized Fock-spaces, *Math. Nachr.* **282** (2009), 99-103.

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