Multigraded Structures and the Depth of Blow-up Algebras
Gemma Colomé i Nin

A first goal of this thesis is to contribute to the knowledge of cohomological properties of non-standard multigraded modules. In particular we study the Hilbert function of a non-standard multigraded module, the asymptotic depth of the homogeneous components of a multigraded module and the asymptotic depth of the Veronese modules. To reach our purposes, we generalize some cohomological invariants to the non-standard multigraded case and we study properties on the vanishing of local cohomology modules. In particular we study the generalized depth of a multigraded module.

In chapters 2, 3 and 4, we consider multigraded rings $S$, finitely generated over the local ring $S_0$ by elements of degrees $\gamma_1, \ldots, \gamma_r$ with $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{ir}, 0) \in \mathbb{N}^r$ and $\gamma_{ii} \neq 0$ for $i = 1, \ldots, r$. In Chapter 2, we prove that the Hilbert function of a multigraded $S$-module is quasi-polynomial in a cone of $\mathbb{N}^r$. Moreover the Grothendieck-Serre formula is satisfied in our situation as well.

In Chapter 3, using the quasi-polynomial behavior of the Hilbert function of the Koszul homology modules of a multigraded $S$-module $M$ with respect to a system of generators of the maximal ideal of $S_0$, we can prove that the depth of the homogeneous components of $M$ is constant for degrees in a subnet of a cone of $\mathbb{N}^r$ defined by $\gamma_1, \ldots, \gamma_r$. In some cases we can assure constant depth in all the cone. By considering the multigraded blow-up algebras associated to ideals $I_1, \ldots, I_r$ in a Noetherian local ring $(R, \mathfrak{m})$, we can prove that the depth of $R/I_1^{n_1} \cdots I_r^{n_r}$ is constant for $n_1, \ldots, n_r$ large enough.

In Chapter 4, we study the depth of Veronese modules $M(a, b)$ for $a, b$ large enough. In particular we prove that in almost-standard case (i.e. with $\gamma_i = (0, \ldots, 0, \gamma_{i1}, 0, \ldots, 0)$, $\gamma_{ii} > 0$, for $i = 1, \ldots, r$) with $S_0$ a quotient of a regular local ring, this depth is constant for $a, b$ in some regions of $\mathbb{N}^r$. To reach this result we need a previous study about Veronese modules and about the vanishing of local cohomology modules. In particular we prove that, in the more general case, if $S_0$ is a quotient of a regular local ring, the generalized depth is invariant by taking Veronese transforms. Moreover in the almost-standard case the generalized depth coincides with the index of finite graduation of the local cohomology modules with respect to the homogeneous maximal ideal.

A second goal of the thesis is the study of the depth of blow-up algebras associated to an ideal. In Chapter 5 we obtain refined versions of some conjectures on the depth of the associated graded ring of an ideal. By using certain non-standard bigraded structures, the integers that appear in Guerrieri’s Conjecture and in Wang’s Conjecture can be interpreted as a multiplicities of some bigraded modules. In particular we have given an answer to the question formulated by A. Guerrieri and C. Huneke in 1993. We have proved that given an $\mathfrak{m}$-primary ideal $I$ in a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension $d > 0$ with minimal reduction $J$, assuming that the lengths of the homogeneous components of the Valabrega-Valla module of $I$ and $J$ are less than or equal to 1, then the depth of the associated graded ring of $I$ is greater than or equal to $d - 2$.

Finally, in Chapter 6, the study of the Hilbert function of certain submodules of the bigraded modules studied before, allows us to prove some cases in which the Hilbert function of an $\mathfrak{m}$-primary ideal in a one-dimensional Cohen-Macaulay local ring is non-decreasing.