Large cardinals and resurrection axioms

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ἔστι δὲ φῦλον ἐν ἀνθρώποισι ματαιότατον,
ὅστις αἰσχύνων ἐπιχώρια παπταίνει τὰ πόρσω,
μεταμώνια θηρεύων ἀκράντοις ἐλπίσιν.

ΠΙΝΔΑΡΟΣ (Γ΄ Πυθιόνικος, 5^{ος} αι. π.Χ.)

Huge hills and mountains of casks on casks were piled upon her wharves, and side by side the world – wandering whale – ships lay silent and safely moored at last; while from others came a sound of carpenters and coopers, with blended noises of fires and forges to melt the pitch, all betokening that new cruises were on the start; that one most perilous and long voyage ended, only begins a second; and a second ended, only begins a third, and so on, for ever and for aye. Such is the endlessness, yea, the intolerableness of all earthly effort.

Herman Melville (Moby – Dick, 1851)

Αντί προλόγου

As indicated by the word *axioms* in the title, the subject matter of the present dissertation is mathematical logic. In particular, it is a dissertation on the field of *set theory*, which studies the abstract concept of mathematical infinity and its formal properties.

The origins of modern set theory can be traced back to the 1870 's in the work of the German mathematician Georg Cantor. Cantor's initial breakthrough, and perhaps his most renowned result, was showing that the cardinality of the natural numbers is strictly smaller than that of the real numbers. In other words, what Cantor proved, amazingly and counter – intuitively, was that infinite collections can have different sizes. Towards the end of the same decade, in 1878, he also established an extremely powerful way to compare infinite cardinalities which, moreover, led him to the formulation of the outstanding *Continuum Hypothesis*.

A few years later, in late 1882, Cantor made yet another breakthrough: he realized that one may orderly count any infinite collection by means of a simple, but profound methodological extension of the familiar counting process which had been used for centuries in the realm of finite numbers. To that end, he concurrently introduced the construction of the sequence of *transfinite ordinal numbers* in such a way that, in his own words, *one might break through any barrier* when forming new numbers along the sequence. Moreover, he observed that, according to his method for comparing cardinalities, the *absolutely infinite* ordinal sequence gives rise to an absolutely infinite hierarchy of strictly increasing sizes of infinities. All these unprecedented results of Cantor can be considered as jointly giving birth to modern set theory.

The mathematical part of the current text is mainly addressed to set – theorists, although on some days I am – unjustifiably – optimistic and expect that other logicians, mathematicians, or people for that matter, will find some relevance of the presented material to their own interests.

Fortunately, and as opposed to general norms regarding journal publications, there are no serious limitations (so far) on the length of a dissertation, a fact of which I have certainly taken advantage. Given this luxury, the majority of the text, including the various arguments in the proofs, was written under the mindset that, when explaining mathematical ideas, it is better being slightly pleonastic than being sketchy. The approach of giving an enhanced picture, one which helps to elucidate the mathematical intuition underlying the occasional technical formalities, is, as I maintain, of vital importance. My hope is that this approach has been successfully reflected throughout the text and that the reader values and relishes this viewpoint as much as I do.

This voyage has come to an end. On the occasion of its conclusion, and in retrospect, it is amusing to recollect all the coincidences which, as if ironically, brought about the actual course of events that led to this finale. There was a time, not too long ago, when the idea of me writing such a dissertation was, as they used to say, *un château en Espagne*. Yet, life is full of surprises. The opening words of the famous English writer echo: *one never knows when the blow may fall*.

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Opening Credits

Like any other manifestation of human expression and intellect, the present dissertation largely reflects, and is a direct offspring of the author's interaction with various people.

Joel Hamkins was exceptionally generous and helpful during the summer of 2011, when, after several enjoyable conversations with his unique expositional style, he brought to my attention the general method of elementary chain constructions and suggested it as a useful tool in the context of $C^{(n)}$ - cardinals; this initial idea eventually led to the development of most of Chapter 2. Moreover, Joel's work and perspective have inspired me to the extent that anyone familiar with them will undoubtedly realize their influence and presence throughout this text.

The importance of David Asperó's intervention cannot be overstated. Via a series of coincidences, we finally had a highly productive discussion, during which David asked the correct questions that quickly gave rise to the formalization of the axioms introduced in Chapter 5. This opened up new directions for me to work on, culminating in the material appearing in the final sections. David is also the one who proposed the name "*Unbounded Resurrection*" for those axioms.

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While being a graduate student in Chicago, Matthias Aschenbrenner was the first to spark my interest in formal logic with his excellent graduate course on undecidability. Mihai Ganea then looked after my training on the fundamentals of first – order logic and has always expressed his confidence in me. I am indeed very grateful to both of them.

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Joan Bagaria, my dear advisor, is the person without whom this dissertation would not have existed (notorious cliché, but true). Joan accepted me as a student at a turning point in (my) life and motivated me anew, giving me a clear goal together with the necessary space and time to work towards it. He has enormously helped me on many different levels, devoting countless hours to my case. He is the one who had the answers but, most importantly, the one who had the questions, the objections, the counterarguments, the suggestions, and the thoughtful and friendly guidance through all of which this work evolved. It certainly was a pleasure. Joan is just great.

Finally, my beloved and cherished friends and family carry such an overwhelming weight that renders meagre any plausible attempt to describe their significance to my life in a few lines. My parents though, Χρήστος και Έλενα, truly deserve a special mention. This dissertation is wholeheartedly dedicated to them both.

Introduction

Set theory is the particular field of mathematics which studies abstract infinite collections – called sets – and their properties. In this sense, it is the theory of mathematical infinity. It has had a plethora of deep and celebrated results ever since its birth, in the late 19^{th} century, through the founding and radical work of Georg Cantor.

The systematic labour of many distinguished mathematicians of the early 20^{th} century led to the formalization of set theory in the framework of first – order logic, thus equipping it with an axiomatic basis which soon became standard. This axiomatization is customarily denoted by the acronym ZFC, standing for Zermelo – Fraenkel set theory with the Axiom of Choice.

ZFC is nowadays widely accepted as the foundation of mathematics; that is to say, the vast majority of mathematical theories can be *interpreted* in set theory. This roughly means that the various mathematical structures from different areas of mathematics can be viewed as sets and, then, all the mathematical theorems about them can be proved from the axioms of ZFC via the usual logical rules of inference. Consequently, set theory, apart from being an ample and rapidly growing mathematical theory in its own right, it also plays the important rôle of encompassing the edifice of mathematical practice, with all the philosophical implications which might arise from the latter feature. It is not clear whether Cantor had such sanguine expectations for the theory which he was just starting to investigate. What started with him, however, continues until this very day to have direct reference to his original insights, formal results, and agitating conjectures.

The *Continuum Hypothesis* (denoted by CH), is the assertion that every infinite subset of the real numbers is either *countable*, that is, equinumerous with the set of natural numbers, or equinumerous with the whole set of reals. Cantor formulated this statement in 1878 and conjectured its truth; nonetheless, after many attempts made by him and other mathematicians during the following years, no proof of it was found. In fact, the solution of the Continuum Hypothesis was the first problem appearing in David Hilbert's famous list of 23 mathematical problems, presented in Paris in the year 1900. For almost four decades after the official challenge posed by Hilbert, no major progress on the matter was made.

In the year 1938 though, sixty years after the original formulation of CH, a partial answer to the problem was given by Kurt Gödel[!]. Gödel showed that, given any model of set theory, one can define a very special substructure of it (called the *constructible universe* L) which, along with the ZFC axioms, also satisfies CH. By that means, he established the relative consistency of CH with ZFC, which moreover meant that the Continuum Hypothesis cannot be refuted from the axioms of set theory. Initially, this might have given heuristic evidence and hope that Cantor's hypothesis is indeed true. Still, after more failed attempts, and in the light of Gödel's eminent *incompleteness theorem*(s), people were eventually led to consider the possibility of the problem being unsolvable in set theory. In order for this alternative to materialize, one would have to demonstrate the relative consistency of the negation of CH with ZFC. For quite some time, this seemed like a daunting task.

Twenty – five years after Gödel's work, in 1963, the mathematician Paul Cohen finally accomplished to exhibit a relative consistency proof for $\neg\,\mathsf{CH}$, thus completing the puzzle: the Continuum Hypothesis was indeed undecidable in ZFC set theory. Perhaps even more strikingly, it was Cohen's method and not the actual result which constituted a groundbreaking advancement. The mehod of *forcing* that he introduced has become an indispensable tool of modern set theory, producing an extraordinary amount of proofs regarding undecidable statements.

Gener(ic)ally speaking, forcing is an extremely flexible and powerful technique which, given an arbitrary model of ZFC (typically referred to as the *ground model*), enables one to construct a strictly larger model of ZFC (usually called a *generic extension*) in a way which allows sufficient control on the statements which hold in the extension. Every forcing construction is guided by a partially ordered set (aka *poset*) $\mathbb{P}$ of the ground model, so that for any given statement φ, its truth or falsity in the generic extension depends on (φ and) the particular internal combinatorial structure of $\mathbb{P}$. The essential flexibility of the method lies in the freedom to select the poset according to our intentions; if this selection is made carefully, we may ensure that certain statements (e.g., the negation of CH) necessarily hold in the

[!] This is intended to be a general introduction to the subject of set theory, aiming at attracting the interest of the non – expert, at refreshing the memory of the expert, as well as at setting out the underlying plot basis for the mathematical content which is about to follow. Those readers interested in the explicit definitions of the various notions and the exact citations of the results mentioned will be excellently guided to them by textbook references such as [28], [30] and [33]. Alternatively, and assuming that the reader has sufficient patience, specific citations will also be provided along the formal development of the pertinent mathematical themes.

extension and, therefore, obtain their relative consistency with the ZFC axioms.

Shortly after its introduction, the forcing machinery attained its full generality through the work of prominent set theorists, triggering the genesis of an ever–expanding list of statements from diverse areas of mathematics, which, one after the other, turned out to be independent from ZFC by forcing arguments. As a result, people quickly became aware of the fact that the axioms of set theory are too weak to settle many important mathematical problems. The natural question, then, was whether there are other axioms that, if added to ZFC, would result in a (reinforced) theory, which would be able to solve some (if not most) of these problems, among which the CH was thought of as the paramount paradigm. In this sense, it can be said that forcing gave rein to thoughts on reinforcing.

The search for new axioms has been a long–standing issue, tantalizing mathematicians and philosophers of set theory for several decades. Two prevailing ideas along these lines, both tracing back to Gödel and –indeed– Cantor himself, are *reflection* and *maximality*.

Reflection is an underlying concept which pervades the body of set theory and is closely tied to the idea of *unknowability of the absolute infinity* of the set–theoretic universe. Intuitively, reflection can be described by saying that, if some property holds in the universe, then it must already hold (that is, "reflect") in some initial segment of it.

In parallel, the concept of maximality refers to the (vague) idea of dispensing with any unnecessary restrictions on the universe of set theory, consistently embracing as many sets as possible. Alternatively, this can be stated as a principle of saturation, i.e., a demand that our universe of discourse is *closed* under a variety of existential requirements and definable operations.

One category of candidates for new axioms consists of the various *large cardinal axioms* which assert the existence of certain strong forms of infinity that are not deducible from ZFC. Some of the early examples were that of an *inaccessible*, that of a *Mahlo* and that of a *measurable* cardinal. Inaccessible cardinals were already considered by Felix Hausdorff in the 1900's, although their current name appeared several years later in the work of Wacław Sierpiński and Alfred Tarski. Mahlo cardinals were studied by Paul Mahlo in the 1910's, whereas measurable ones by Stanisław Ulam in the 1930's.

The list of large cardinals has grown considerably over the years, and it has been enriched with notions coming from a wide spectrum of mathematical interests; yet, it is an impressive fact that all these postulates are found to be linearly ordered in consistency strength, forming an increasing hierarchy of stronger and stronger axioms of infinity. Among others, notable large cardinal notions are *weakly compact*, *Ramsey*, *strong*, *Woodin*, *supercompact*, *extendible* and *huge* cardinals. A very important use of this hierarchy of axioms is to "measure" the consistency strength of any set – theoretic (and thus, mathematical) statement, hence providing an exceedingly dense picture of implications between some of the most potent mathematical assertions.

(Un)fortunately, by the work of Azriel Lévy and Robert Solovay, it was soon realized that large cardinals are unable to resolve our most illustrious problem: the Continuum Hypothesis. Despite this deficit, in 1969, Solovay proved that the existence of a measurable cardinal implies that all sets of reals of a particular complexity (called Σ^1_2 sets) are *Lebesgue measurable*. This was a surprising connection. In fact, although the relation between large cardinals and the structure of the continuum is not yet fully understood, a series of great advances and deep results – especially in the areas of *descriptive set theory*, *inner model theory* and *determinacy* during the last decades – has highlighted the momentous influence that the former have on the latter.

One dominant demonstration of the aforementioned effect, the climax of the work done by Donald Martin, John Steel and W. Hugh Woodin in the 1980 's, is that the axiom of *Projective Determinacy* (denoted by PD) follows from, and is – roughly – equiconsistent with the existence of infinitely many Woodin cardinals. The axiom PD has very strong consequences for the structure of the continuum: it implies, for instance, that every *projective* set of reals is Lebesgue measurable, it has the *Baire property* and the *perfect set property*; in particular, under PD there is no projective counterexample to CH.

In a somewhat different spirit, an important by – product of the development of forcing, during the 1960 's, was the birth of *forcing axioms*. This was initiated by Martin, who isolated a principle that generalizes the *Baire category theorem* and is now known as *Martin's Axiom* (denoted by MA). The original formulation of MA was given in terms of forcing posets and is the assertion that, for any cardinal $\kappa < 2^{\aleph_0}$, any *c.c.c.* poset $\mathbb{P}$ and any family $\mathscr{F} = \{A_\alpha : \alpha < \kappa\}$ of *maximal antichains* of $\mathbb{P}$, there exists a *filter* $G \subseteq \mathbb{P}$ which is $\mathscr{F}$*-generic*, that is, $G \cap A_\alpha \neq \varnothing$, for every $\alpha < \kappa$. Of particular interest is the special case in which $\kappa = \aleph_1$, i.e., the assertion that an $\mathscr{F}$-generic filter may be found for any such family $\mathscr{F}$ of size $\aleph_1$; this instance is denoted by $\mathsf{MA}_{\aleph_1}$ (and it implies $\neg\,\mathsf{CH}$). Martin's Axiom can be seen as a principle of existential closure, asserting, in effect, that the universe is already closed under the existence of such generic filters for c.c.c. partial orders.

Subsequently, other forcing axioms were considered, mainly by expanding the class of posets to which similar closure principles apply, while keeping an attentive eye on possible inconsistencies which might appear; e.g., one may not generalize to the class of *all* posets, since this is easily seen to lead to a contradiction. Two famous strengthenings of Martin's Axiom (in fact, of $\mathsf{MA}_{\aleph_1}$) are the *Proper Forcing Axiom* (denoted by PFA) and *Martin's Maximum* (denoted by MM), both appearing in the 1980's. The PFA was introduced by James Baumgartner and applies to *proper* posets, a notion studied by Saharon Shelah; MM was introduced in a joint work by Matthew Foreman, Menachem Magidor and Saharon Shelah, and refers to *stationary preserving* posets, i.e., posets which preserve the *stationary* subsets of ω_1. Along with their introduction, it was shown that the consistency of both of these forcing axioms follows from that of the existence of a supercompact cardinal.

Strong forms of forcing axioms have dramatic implications for the set – theoretic universe and, in particular, for the continuum and its structure; e.g., PFA (and thus, MM as well) entails PD, the *Singular Cardinal Hypothesis* and, moreover, it implies that the cardinality of the continuum is $\aleph_2$ (hence, it refutes CH). Although it is far from clear whether such assumptions can be considered as natural axioms for set theory, they do conform with the idea of maximality. This aspect is further clarified when one regards forcing axioms as principles of *generic absoluteness*.

In the search for new axioms which would decide problems like the Continuum Hypothesis, one is tempted to think that the effect of forcing should somehow be annihilated. Forcing axioms partly succeed in doing so, as they are intuitively claiming that, for appropriate classes of posets, "sufficient forcing has already been done", i.e., generic filters may already be found in the (current) universe. In other words, such axioms claim that, if the existence of objects satisfying certain properties can be forced, then such objects already exist. In model – theoretic terms, this means that various existential statements are *absolute* between the universe and the generic extensions arising from the relevant posets.

Local examples of this phenomenon are provided by *bounded* forms of forcing axioms, such as the *Bounded Proper Forcing Axiom* (denoted by BPFA) and *Bounded Martin's Maximum* (denoted by BMM). These are weakenings of PFA and MM respectively, having the extra restriction that each of the maximal antichains for which a generic filter may be found should have size at most $\aleph_1$. It then turns out that these bounded versions are equivalent to generic absoluteness statements for bounded segments of the universe. More accurately, BPFA is equivalent to the assertion that, for any proper poset $\mathbb{P}$, the $H_{\aleph_2}$ of the ground model is a Σ_1 - elementary substructure of the $H_{\aleph_2}$ of any generic extension, i.e., $H_{\aleph_2} \prec_1 (H_{\aleph_2})^{V^{\mathbb{P}}}$. A related characterization holds for BMM as well, if one replaces proper posets by stationary preserving ones in the previous equivalence. It has been shown that bounded forcing axioms also have strong consequences for the continuum and its structure; in particular, both BPFA and BMM imply that $2^{\aleph_0} = \aleph_2$. But even in this direction, the road is not paved with rose petals.

Such exemplifications of generic absoluteness, in terms of the H_κ's, quickly run into inconsistency, if we allow too much liberty on the choice regarding the class of posets, the cardinal κ, and the complexity of the formulas considered, or combinations thereof. Be that as it may, and even at the cost of abandoning the general setting, one would like to have adequate control over structures which have some bearing on the continuum, it being one of the central objects of our study.

Focusing then on $L(\mathbf{R})$, the minimal ZF model which contains all the reals, Woodin has shown that the existence of a proper class of Woodin cardinals implies that the (first – order) theory of $L(\mathbf{R})$, with real parameters, is generically absolute for *all* forcing posets of the universe. This kind of absoluteness implies that all sets of reals in $L(\mathbf{R})$ and, therefore, all projective sets of reals are Lebesgue measurable, have the Baire property, etc.

Along his investigations regarding set – theoretic statements of analogous complexity to that of CH (recall that this is Σ_2), Woodin has come along a truly remarkable fact, called *resurrection*. Namely, again in the presence of a proper class of Woodin cardinals and after introducing a forcing notion called *stationary tower*, he showed that if φ is Σ_2 and it can be forced true by any poset, then the fact that it can be forced true is generically absolute. Otherwise stated, we may always resurrect the truth of Σ_2 - statements by *further* forcing, even if we happened to falsify them in our current (messy) forcing constructions.

Woodin has also introduced a strong logical system called Ω–*Logic*, with the feature that its valid statements are generically absolute. It was intended to provide with a framework in which, at least for statements of complexity similar to that of CH over the structure $H_{\aleph_2}$, the effect of forcing is nullified. For this, he also introduced an axiom called $(*)$ which is a strong form of BMM and which, together with ZFC and under suitable assumptions, decides in Ω–Logic the theory of $H_{\aleph_2}$; moreover, Woodin showed[℧] that in such a situation the Continuum Hypothesis necessarily fails.

The interplay and emerging connections between large cardinals, forcing axioms, and the structure of the continuum, have indeed gone a long way. Notwithstanding, there are many obscure territories yet to be explored before we collectively arrive at a sharper, and intuitively more profound understanding of the subtle complexities of the "set – theoretic universe".

The search for new axioms continues, along with our curiosity to reach ever – higher levels of intellectual clarity. As we are ascending, we are indeed going deeper. Or we should, anyway.

[℧] Granted the assumption of the Ω–*Conjecture* which is, as its name indicates, still open.

Annotation of content and overview of results

The current dissertation touches both on large cardinals and on issues related to forcing axioms and generic absoluteness. On the one hand, we take up the hierarchies of $C^{(n)}$ - cardinals, which were introduced in [5] and were shown to have intimate connections with principles of *structural reflection* for the set – theoretic universe.

On the other hand, we also deal with certain classes of resurrection axioms as they are introduced in [29]. We eventually obtain stronger forms of such principles from which we are able to deduce known forcing axioms. Let us now give a brief overview of the particular contents.

The necessary preliminaries may be found, as it should be anticipated, in the first chapter.

In **Chapter 2**, we study several hierarchies of $C^{(n)}$ - cardinals as they are introduced by J. Bagaria (cf. [5]). In the context of an elementary embedding $j : V \longrightarrow M$ associated with some fixed $C^{(n)}$ - cardinal, and under adequate assumptions, we construct appropriate chains of elementary substructures of the model M in order to derive consistency (upper) bounds for the large cardinal notion at hand; in particular, we deal with the $C^{(n)}$ - versions of tallness, superstrongness, strongness, supercompactness, and extendibility. As far as the two latter notions are concerned, we further study their connection, giving an equivalent formulation of extendibility as well.

We also consider the cases of $C^{(n)}$ - Woodin and of $C^{(n)}$ - strongly compact cardinals which were not studied in [5]. Although these notions do not fit in the methodological picture described in the previous paragraph, we nevertheless get characterizations for them in terms of their ordinary counterparts.

In **Chapter 3**, we briefly discuss the interaction of $C^{(n)}$ - cardinals with the forcing machinery, presenting some (quite) basic applications of ordinary techniques.

In **Chapter 4**, we turn our attention to extendible cardinals; by a combination of methods and results from Chapter 2, we establish the existence of apt Laver functions for them. Although the latter was already known (cf. [11]), it is proved from a fresh viewpoint, one which nicely ties with the material of Chapter 5.

On the negative side, we argue that in the case of extendible cardinals one cannot use such Laver functions in order to attain indestructibility results. Along the way, we give an additional characterization of extendibility, and we, moreover, show that the global GCH can be forced while preserving such cardinals.

In **Chapter 5**, we focus on the resurrection axioms as they are introduced by J.D. Hamkins and T. Johnstone (cf. [29]). Initially, we consider the class of stationary preserving posets and, assuming the (consistency of the) existence of an extendible cardinal, we obtain a model in which the resurrection axiom for this class holds.

By analysing the proof of the previous result, we are led to much stronger forms of resurrection for which we introduce a family of axioms under the general name "Unbounded Resurrection". We then prove that the consistency of these axioms follows from that of (the existence of) an extendible cardinal and that, for the appropriate classes of posets, they are strengthenings of the forcing axioms PFA and MM.

We furthermore consider several implications of the unbounded resurrection axioms (e.g., their effect on the continuum, for the classes of c.c.c. and of σ - closed posets) together with their connection with the corresponding ones of [29]. Finally, we also establish some consistency lower bounds for such axioms, mainly by deriving failures of (weak versions of) squares.

We conclude our current mathematical quest with a few final remarks and a small list of open questions, followed by an Appendix on extenders and (some of) their applications.

CHAPTER 1

Prelude

We commence by outlining in this chapter several notational conventions, standard definitions, and well – known results, all of which will be assumed, quoted and – either directly or indirectly – used in the course of the present dissertation. For more details regarding undefined set – theoretic notions and basic background material, the reader is encouraged to consult various classical text references such as [28], [30] and [33].

1.1 Fixing the language

ZFC stands for the familiar first – order axiomatization of Zermelo – Fraenkel set theory, together with the Axiom of Choice. In the few cases where particular fragments of this theory are (locally) relevant, we will use ZF to indicate the absence of choice and ZFC^- to indicate the absence of the Powerset Axiom. The interested reader is referred to [21] for some nuances related to the latter theory. Moreover, KP will stand for Kripke – Platek set theory, a fragment of ZFC which is frequently employed in the development of the forcing apparatus.

For any set X, $trcl(X)$ is its transitive closure, $|X|$ is its cardinality, $\mathcal{P}(X)$ is its powerset and, if κ is any cardinal (even finite), $\mathcal{P}_\kappa X$ and $[X]^{<\kappa}$ both stand for the collection of subsets of X which have cardinality less than κ; moreover, $[X]^\kappa = \{y \subseteq X : |y| = \kappa\}$.

Given any function f, $dom(f)$ is its domain while $range(f)$ is its range, i.e., $range(f) = \{y : \exists x \in dom(f)\,(y = f(x))\}$; additionally, for any $S \subseteq dom(f)$ we write $f \restriction S$ for the restriction of the function to S and, also, we write $f''S$ for the corresponding pointwise image, that is, the collection $\{f(x) : x \in S\}$. We use the three dots in order to indicate partial functions, i.e., $f : X \longrightarrow Y$ means that $dom(f) \subseteq X$, with the inclusion possibly being proper. For any X and Y, ${}^XY$ is the collection of all functions f with $dom(f) = X$ and $range(f) \subseteq Y$; if $|X| = \lambda$ and $|Y| = \kappa$, then $|{}^XY| = \kappa^\lambda$. When the domain X is clear from the context, we write id for the identity function, i.e., $id(x) = x$, for every $x \in X$.

The class of ordinal numbers will be denoted by **ON**. We reserve lower case Greek letters for ordinals, with the letters κ, λ and μ typically used in the case of infinite cardinals. Ordinal intervals are readily comprehensible; e.g., given $\alpha < \beta$, $(\alpha\,,\,\beta)$ is the set of ordinals which lie strictly between α and β. Likewise for half–open or closed intervals. For any well–ordered set A, $\mathrm{ot}(A)$ is its order–type. If $\alpha \in \mathbf{ON}$, the α^{th} aleph number is denoted either by $\aleph_\alpha$ or by ω_α; $\omega = \aleph_0$ is (the cardinality of) the set of natural numbers and $\mathfrak{c}$ is the cardinality of the set of real numbers $\mathbf{R}$, i.e., $\mathfrak{c} = |\mathbf{R}| = |\mathcal{P}(\omega)| = 2^{\aleph_0}$. Furthermore, $\beth_\alpha$ will be the α^{th} beth number. For any infinite ordinal α, $cf(\alpha)$ stands for its cofinality.

If $A \subseteq \mathbf{ON}$, $\sup A$ is the supremum of A (in case A is a set) and $Lim(A)$ is the collection of its limit points, i.e., $\{\xi \,:\, \sup(A \cap \xi) = \xi\}$. Given a limit ordinal α with $cf(\alpha) > \omega$ and some $C \subseteq \alpha$, C is called club in α if $\sup C = \alpha$ and $\alpha \cap Lim(C) \subseteq C$; C is called β-club in α, for some regular $\beta < cf(\alpha)$, if $\sup C = \alpha$ and $\{\xi \in \alpha \cap Lim(C) : cf(\xi) = \beta\} \subseteq C$; a subset $S \subseteq \alpha$ is called stationary in α if $S \cap C \neq \varnothing$ for every club $C \subseteq \alpha$.

We write V for the universe of the well–founded sets and L for Gödel's constructible universe; both are stratified via the usual hierarchy of V_α's (resp. L_α's). For any $x \in V$, $rank(x)$ is the least ordinal α for which $x \in V_{\alpha+1}$. If κ is an infinite cardinal, we let H_κ be the collection of all sets whose transitive closure has size less than κ.

Cantor's *Continuum Hypothesis* will be denoted by CH and is the assertion that $\mathfrak{c} = \aleph_1$; its global version is the *Generalized Continuum Hypothesis*, denoted by GCH, which asserts that for every ordinal α, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$. Finally, SCH will stand for the *Singular Cardinal Hypothesis*, i.e., the assertion that for every singular cardinal λ, if $2^{cf(\lambda)} < \lambda$ then $\lambda^{cf(\lambda)} = \lambda^+$.

1.2 Large cardinals (you would expect)

We will be mainly interested in large cardinals which appear as the critical point of some (non–trivial) elementary embedding; the latter will typically be of the sort $j : V \longrightarrow M$, where M is a transitive class model of ZFC. The critical point in question will be denoted by $cp(j)$ and is the least ordinal moved by the embedding[♮].

Occassionally, we shall also look at "smaller" large cardinals. At any rate, most of the notions which we use in this text are standard; we refer the reader to [28]

[♮] In this sense, $cp(j)$ is the first ordinal to exhibit emotional sensitivity.

and [30] for the corresponding definitions, characterizations and properties of large cardinals such as: inaccessible, Mahlo, measurable, strong, superstrong, Woodin, strongly compact, supercompact, extendible and almost huge. We now give two less popular definitions, which are still interesting in their own right.

Definition 1.1. *A cardinal κ is called $\mathcal{P}^n(\kappa)$ -* **hyper–measurable***, for some $n \geqslant 1$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, ${}^{\kappa}M \subseteq M$ and $V_{\kappa+n} \subseteq M$.*

Clearly, for $n = 1$ this is equivalent to measurability. However, for $n > 1$ this notion transcends measurability in consistency strength, although it is still weaker than strongness.

The next notion is a "miniature" supercompactness one, since we only require closure under κ - sequences.

Definition 1.2 ([26]). *A cardinal κ is called λ -* **tall***, for some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, ${}^{\kappa}M \subseteq M$ and $j(\kappa) > \lambda$; moreover, κ is called* **tall** *if it is λ - tall for every $\lambda > \kappa$.*

Tall cardinals were introduced by J.D. Hamkins. As shown in [26], tallness embeddings can be described by extenders and, moreover, every strong cardinal is tall. On the other hand, it is also shown that, although the existence of a strong cardinal is equiconsistent (modulo ZFC) with the existence of a tall cardinal, a model may be obtained in which the unique tall cardinal is also the unique measurable.

Turning momentarily to stronger notions, we also give an important feature of supercompact cardinals, one which deserves some special attention in light of the material of Chapters 4 and 5.

Theorem 1.3 (Laver). *If κ is supercompact then there is $\ell : \kappa \longrightarrow V_\kappa$ such that, for any cardinal $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$, there is a λ - supercompact embedding $j : V \longrightarrow M$ for κ, with $j(\ell)(\kappa) = x$.*

Such a (partial) function is called *Laver function*, in honour of Richard Laver who introduced the concept and proved the previous theorem (cf. [35]). Without loss of generality, one may assume that such a function has the following additional properties: (i) every $\alpha \in dom(\ell)$ is measurable but not supercompact; (ii) for every $\alpha \in dom(\ell)$, $\ell''\alpha \subseteq V_\alpha$; and (iii) given λ and x as above, the λ - supercompactness embedding can be chosen so that, apart from $j(\ell)(\kappa) = x$, it also satisfies the condition $dom(j(\ell)) \cap (\kappa, \lambda] = \varnothing$.

Finally, diverging from the realm of large cardinals which are incompatible with $V = L$, we present the notion of an *uplifting* cardinal. This, together with its subsequent properties, are all due to J.D. Hamkins and T. Johnstone who introduced them in unpublished work (cf. [29]); they will be relevant for (some of) the results obtained in Chapter 5.

Definition 1.4. *A regular cardinal* κ *is called* **uplifting** *if it satisfies any of the following equivalent conditions:*

(*i*) *There are arbitrarily large regular cardinals* γ *such that* $H_\kappa \prec H_\gamma$.

(*ii*) κ *is inaccessible and there are arbitrarily large inaccessible cardinals* γ *such that* $V_\kappa \prec V_\gamma$.

Notice that if κ is uplifting then this is downwards absolute to L; moreover, V_κ is a Σ_2-elementary substructure of V, denoted by "$V_\kappa \prec_2 V$". In fact, there are unboundedly many $\alpha < \kappa$ with $V_\alpha \prec_2 V$. For an upper bound, if λ is Mahlo then the set $\{\alpha < \lambda : V_\lambda \models$ "α is uplifting"$\}$ is stationary in λ.

Definition 1.5. *Suppose that* κ *is inaccessible. A function* $f : \kappa \longrightarrow V_\kappa$ *is called a* **miniature Laver function** *for* κ, *if* f *is a definable class in the structure* $\langle V_\kappa , \in \rangle$ *and, for every* $x \in V$, *there exists an inaccessible cardinal* γ *such that* $\langle V_\kappa , \in , f \rangle \prec \langle V_\gamma , \in , f^* \rangle$ *and* $f^*(\kappa) = x$, *where* $f^* : \gamma \longrightarrow V_\gamma$ *is the corresponding definable class of* V_γ, *via the same definition.*

Theorem 1.6. *Assume* $V = L$. *Then, every uplifting cardinal* κ *carries a (definable) miniature Laver function.*

Proof. Fix an uplifting cardinal κ. Since $V = L$, we have a canonical, definable, global well-ordering "$\lhd$" of the universe. We now define, in V_κ, the function $f : \kappa \longrightarrow V_\kappa$ as follows: for every $\alpha < \kappa$, if α is an inaccessible but not an uplifting cardinal (in V_κ), we let

$$\xi = \mathrm{ot}\left(\{\alpha < \gamma < \kappa : \gamma \text{ is inaccessible and } V_\alpha \prec V_\gamma\}\right)$$

and then, noting that $\xi < \kappa$, we define $f(\alpha)$ as the ξ^{th} element of the universe, under the well-ordering "$\lhd$"; otherwise, we leave f undefined. It is clear that this is a well-defined class function over the structure $\langle V_\kappa , \in \rangle$. We now check that it has the desired property.

Fix $x \in V$ and let $\beta \in \mathbf{ON}$ be such that x is the β^{th} element of the universe, under the well-ordering "$\lhd$". Let δ be the $(\beta + 1)^{\text{st}}$ inaccessible above κ with $V_\kappa \prec V_\delta$ and let $f^* : \delta \longrightarrow V_\delta$ be the corresponding class function over $\langle V_\delta , \in \rangle$, defined as above. By its definition and the choice of δ, it is clear that $f^*(\kappa) = x$. Note also that, since $V_\kappa \prec V_\delta$, by the definition of the function we have that $f^* \restriction \kappa = f$ and $\langle V_\kappa , \in , f \rangle \prec \langle V_\delta , \in , f^* \rangle$ as desired. $\square$

1.3 $C^{(n)}$ - cardinals

Given a natural number n and an ordinal α, we say that α is Σ_n - *correct in the universe* if V_α is a Σ_n - elementary substructure of V (denoted by "$V_\alpha \prec_n V$"); i.e., for every Σ_n - formula φ with parameters in V_α,

$$\varphi \Longleftrightarrow V_\alpha \models \varphi .$$

For every n, let $C^{(n)}$ denote the closed and unbounded proper class of ordinals which are Σ_n - correct in the universe V, that is,

$$C^{(n)} = \{\alpha \in \mathbf{ON} : V_\alpha \prec_n V\}.$$

Obviously, α is Σ_n - correct if and only if it is Π_n - correct. Moreover, the usual arguments show that if $\alpha \in C^{(n)}$, then Σ_{n+1} - formulas (with parameters in V_α) are upwards absolute from V_α to V, whereas Π_{n+1} - formulas (with parameters in V_α) are downwards absolute from V to V_α.

In particular, $C^{(0)} = \mathbf{ON}$. On the other hand, if $\alpha \in C^{(1)}$, then α is already an uncountable strong limit cardinal since, for any $\beta < \alpha$, the statement

$$\exists \gamma \, \exists f \, (\gamma \in \mathbf{ON} \wedge \text{“} f : \gamma \longrightarrow V_\beta \text{ is a surjection”})$$

is Σ_1 in the parameter V_β and so it must hold in V_α. Moreover, if $\alpha \in C^{(1)}$, we similarly have that $\alpha = \beth_\alpha$, and then $H_\alpha = V_\alpha$; hence, $C^{(1)}$ is precisely the class of uncountable cardinals α for which $H_\alpha = V_\alpha$. Unfortunately, such a local characterization is not available for the classes $C^{(n)}$, when $n > 1$.

One can easily show that, for every $n \geqslant 1$, membership in $C^{(n)}$ is expressible by a Π_n (but *not* by any Σ_n) formula. It follows that, for every n, $C^{(n+1)} \subset C^{(n)}$, i.e., the inclusion is proper. Further, the classes $C^{(n)}$ form a *basis* for the Σ_n - definable club classes of ordinals, in the sense that, for any fixed $n \geqslant 1$, if $D \subseteq \mathbf{ON}$ is Σ_n - definable club class then $C^{(n)} \subseteq D$.

The notion of a $C^{(n)}$ - (large) cardinal was introduced by J. Bagaria in [5] (where more details on related material may be found). Unless otherwise stated, the following definitions and results until the end of the current section are all due to J. Bagaria. Throughout, n stands for any fixed natural number.

Definition 1.7. *We say that a cardinal κ is λ - $C^{(n)}$ -* **strong**, *for some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $j(\kappa) \in C^{(n)}$; moreover, we say that κ is $C^{(n)}$ -* **strong** *if it is λ - $C^{(n)}$ - strong for every $\lambda > \kappa$.*

Proposition 1.8. *Every λ - strong cardinal is λ - $C^{(n)}$ - strong. Thus, every strong cardinal is $C^{(n)}$ - strong.*

Definition 1.9. *We say that a cardinal* κ *is* $C^{(n)}$ - **superstrong** *if there exists an elementary embedding* $j : V \longrightarrow M$, *with* M *transitive,* $cp(j) = \kappa$, $V_{j(\kappa)} \subseteq M$ *and* $j(\kappa) \in C^{(n)}$.

Proposition 1.10. *Suppose that* κ *is superstrong, witnessed by the elementary embedding* $j : V \longrightarrow M$. *Then* $j(\kappa) \in C^{(1)}$, *i.e., every superstrong cardinal is* $C^{(1)}$ *- superstrong, witnessed by the same embedding.*

For $n \geqslant 1$, the statements "κ is λ - $C^{(n)}$ - strong" and "κ is $C^{(n)}$ - superstrong" are both Σ_{n+1} - expressible. Consequently, the statement "κ is $C^{(n)}$ - strong" is Π_{n+2} - expressible.

Definition 1.11. *We say that a cardinal* κ *is* λ - $C^{(n)}$ - **supercompact**, *for some* $\lambda > \kappa$, *if there exists an elementary embedding* $j : V \longrightarrow M$, *with* M *transitive,* $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ *and* $j(\kappa) \in C^{(n)}$; *moreover, we say that* κ *is* $C^{(n)}$ - **supercompact** *if it is* λ - $C^{(n)}$ *- supercompact for all* $\lambda > \kappa$.

For every $n \geqslant 1$, the statement "κ is λ - $C^{(n)}$ - supercompact" can be seen to be Σ_{n+1} - expressible, e.g., using the machinery of Martin – Steel extenders; see the Appendix for a more detailed account on such extenders. Therefore, "κ is $C^{(n)}$ - supercompact" is Π_{n+2} - expressible.

Definition 1.12. *We say that a cardinal* κ *is* λ - $C^{(n)}$ - **extendible**, *some* $\lambda > \kappa$, *if there is a* $\theta > \lambda$ *and an elementary embedding* $j : V_\lambda \longrightarrow V_\theta$, *with* $cp(j) = \kappa$, $j(\kappa) > \lambda$ *and* $j(\kappa) \in C^{(n)}$; *moreover, we say that* κ *is* $C^{(n)}$ - **extendible** *if it is* λ - $C^{(n)}$ *- extendible for all* $\lambda > \kappa$.

Proposition 1.13. *For* $n \geqslant 1$, *if* κ *is* $C^{(n)}$ *- extendible then* $\kappa \in C^{(n+2)}$.

The previous proposition is of course true for $n = 0$ as well, in which case it is known that if κ is extendible, then actually $\kappa \in C^{(3)}$.

A slight variation of $C^{(n)}$ - extendibility is given in the following.

Definition 1.14. *We say that a cardinal* κ *is* λ - $C^{(n)+}$ - **extendible**, *some* $\lambda > \kappa$ *with* $\lambda \in C^{(n)}$, *if there exists an ordinal* $\theta \in C^{(n)}$ *and an elementary embedding* $j : V_\lambda \longrightarrow V_\theta$, *with* $cp(j) = \kappa$, $j(\kappa) > \lambda$ *and* $j(\kappa) \in C^{(n)}$; *moreover, we say that* κ *is* $C^{(n)+}$ - **extendible** *if it is* λ - $C^{(n)+}$ *- extendible for all* $\lambda > \kappa$ *with* $\lambda \in C^{(n)}$.

For every $n \geqslant 1$, the statements "κ is λ - $C^{(n)}$ - extendible" and "κ is λ - $C^{(n)+}$ - extendible" are both Σ_{n+1} - expressible. Consequently, "κ is $C^{(n)}$ - extendible" and "κ is $C^{(n)+}$ - extendible" are both Π_{n+2} - expressible.

We now introduce the corresponding $C^{(n)}$ - versions for tall, Woodin and strongly compact cardinals; these notions were not considered in [5], something which we will do in Chapter 2. Evidently, the following definitions are in accordance with the general spirit of [5].

Definition 1.15. *We say that a cardinal κ is* λ - $C^{(n)}$ - **tall**, *for some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\kappa}M \subseteq M$ and $j(\kappa) \in C^{(n)}$; moreover, we say that κ is* $C^{(n)}$ - **tall** *if it is λ - $C^{(n)}$ - tall for all $\lambda > \kappa$.*

Either by using ordinary extenders or, as in the case of supercompactness, by Martin – Steel extenders, we have that for every $n \geqslant 1$, the statement "κ is λ - $C^{(n)}$ - tall" is Σ_{n+1} - expressible; thus, "κ is $C^{(n)}$ - tall" is Π_{n+2} - expressible.

Definition 1.16. *We say that a cardinal δ is* $C^{(n)}$ - **Woodin** *if for every $f \in {}^{\delta}\delta$, there exists a $\kappa < \delta$ with $f''\kappa \subseteq \kappa$, and there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, $V_{j(f)(\kappa)} \subseteq M$, $j(\delta) = \delta$ and $j(\kappa) \in C^{(n)}$.*

Observe that the above definition is in accordance with the local character of Woodin cardinals, i.e., we demand that $j(\delta) = \delta$ so that the various embeddings may be witnessed by extenders which belong to V_δ. Finally, we have the following.

Definition 1.17. *We say that a cardinal κ is* λ - $C^{(n)}$ - **compact**, *some $\lambda > \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, $j(\kappa) \in C^{(n)}$ and so that, for every $X \subseteq M$ with $|X| \leqslant \lambda$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$; moreover, we say that κ is* $C^{(n)}$ - **strongly compact** *if it is λ - $C^{(n)}$ - compact for all $\lambda > \kappa$.*

1.4 Forcing machinery

1.4.1 Notation and all that

Partial orders (aka posets) which are employed in forcing constructions will be denoted by blackboard capital letters such as $\mathbb{P}$, $\mathbb{Q}$ and $\mathbb{R}$. We write $p < q$ to mean than p is stronger than q or, equivalently, p properly extends q. We denote the greatest element of a poset by $\mathbb{1}$ (in particular, we always assume that a poset is non – empty). All forcing posets are assumed to be separative; recall that this does not disturb generality as one may replace any given poset by its (forcing equivalent) separative quotient. Given a poset $\mathbb{P}$, the $\mathbb{P}$ - names are indicated by "dots" and "checks" as usual; we sometimes supress these in order to ease readability, with the intended meaning being clear from the context. The universe of $\mathbb{P}$ - names will be denoted by $V^{\mathbb{P}}$. If $\dot{x}$ is a $\mathbb{P}$ - name and G is a $\mathbb{P}$ - generic filter (over the relevant model), then $\dot{x}_G$ denotes the interpretation of the name by the filter. Moreover, for any $\alpha \in \mathbf{ON}$, $V[G]_\alpha = (V_\alpha)^{V[G]}$ while $V_\alpha[G] = \{\tau_G : \tau \in V^{\mathbb{P}} \cap V_\alpha\}$.

It will sometimes be convenient to work with names which are (recursively) constructed with the aid of a *flat pairing function*. Such a function f plays the rôle of the Kuratowski pairing, with the extra property that for infinite α, if x, $y \in V_\alpha$ then the pair computed by f belongs to V_α; that is, unlike the Kuratowski pair, this function does not increase the rank, except for the finite case. Flat pairing functions are not difficult to construct; e.g., [27] gives one such example, which is apparently due to W.V.O. Quine. We now give a folklore result which will be relevant in later chapters. The present proof follows [27].

Lemma 1.18. *Let $\mathbb{P}$ be a poset with rank$(\mathbb{P}) = \gamma$, for some infinite γ, and suppose that a flat pairing function is used in the construction of the $\mathbb{P}$-names. Then, for any $G \subseteq \mathbb{P}$-generic over V and for any $\alpha \geqslant \gamma \cdot \omega$, we have that $V[G]_\alpha = V_\alpha[G]$.*

Proof. Since $V_\alpha[G] \subseteq V[G]_\alpha$ holds for any α, we only deal with the other inclusion. Let us first explicitly state what we mean by saying that a flat pairing function is used in the construction of the $\mathbb{P}$-names. Having fixed such a function f, we build the universe $V^{\mathbb{P}}$ of $\mathbb{P}$-names recursively, as follows.

We initially let $V_0^{\mathbb{P}} = \varnothing$. For the successor step, given $V_\alpha^{\mathbb{P}}$ for some $\alpha \in \mathbf{ON}$, we let

$$V_{\alpha+1}^{\mathbb{P}} = V_\alpha^{\mathbb{P}} \cup \mathcal{P}(V_\alpha^{\mathbb{P}} \times \mathbb{P}),$$

where we use the flat pairing function f in order to compute (the pairs in) the set $V_\alpha^{\mathbb{P}} \times \mathbb{P}$. For limit ordinals λ, we let $V_\lambda^{\mathbb{P}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{P}}$. Finally, the universe of the constructed $\mathbb{P}$-names is

$$V^{\mathbb{P}} = \bigcup_{\alpha \in \mathbf{ON}} V_\alpha^{\mathbb{P}}.$$

It is clear that the defined hierarchy is cumulative and that, for all $\alpha \in \mathbf{ON}$, the set $V_\alpha^{\mathbb{P}}$ consists of $\mathbb{P}$-names. We now check inductively that, for every ordinal α, $V_\alpha^{\mathbb{P}} \subseteq V_{\gamma+\alpha}$.

The base and limit cases are immediate. For the successor step, suppose that $V_\alpha^{\mathbb{P}} \subseteq V_{\gamma+\alpha}$ for some ordinal α. Then, by the recursive construction, for any $X \in V_{\alpha+1}^{\mathbb{P}}$, either $X \in V_\alpha^{\mathbb{P}}$ or $X \subseteq V_\alpha^{\mathbb{P}} \times \mathbb{P}$. In the former case we are done by the inductive hypothesis. In the latter case, using the fact that $\mathbb{P} \subseteq V_\gamma$, the inductive hypothesis, and that the pairing function does not increase the ranks, we easily get that $X \in V_{\gamma+\alpha+1}$. This concludes the inductive verification of the inclusion $V_\alpha^{\mathbb{P}} \subseteq V_{\gamma+\alpha}$.

Now fix any $G \subseteq \mathbb{P}$-generic over V. We check, again inductively, that for every $\alpha \in \mathbf{ON}$, $V_\alpha^{\mathbb{P}}[G] = V[G]_\alpha$.

The base and limit cases are again immediate. For the successor step, suppose that $V_\alpha^{\mathbb{P}}[G] = V[G]_\alpha$, for some ordinal α. By the inductive hypothesis and the recursive definition of the universe $V^{\mathbb{P}}$,

$$V_{\alpha+1}^{\mathbb{P}}[G] = V[G]_\alpha \cup \{X_G : X \subseteq V_\alpha^{\mathbb{P}} \times \mathbb{P}\}.$$

From this and the inductive hypothesis, it follows that $V_{\alpha+1}^{\mathbb{P}}[G] \subseteq V[G]_{\alpha+1}$. For the converse inclusion, let $S \in V[G]_{\alpha+1} \setminus V[G]_\alpha$ be given. We want to find a name $X \in V_{\alpha+1}^{\mathbb{P}}$ with $X_G = S$. But since $S \subseteq V[G]_\alpha$, the inductive hypothesis implies that for every $z \in S$ there is a name $\tau_z \in V_\alpha^{\mathbb{P}}$ such that $(\tau_z)_G = z$. Now pick any maximal antichain A of $\mathbb{P}$ and, via the use of the flat pairing function, define the following $\mathbb{P}$-name:

$$X = \bigcup_{z \in S} \{\tau_z\} \times A,$$

where notice that $X \subseteq V_\alpha^{\mathbb{P}} \times \mathbb{P}$ and thus, $X \in V_{\alpha+1}^{\mathbb{P}}$. It is straightforward to see that the name X works, i.e., $X_G = S$. Hence, we can conclude that $V_\alpha^{\mathbb{P}}[G] = V[G]_\alpha$, for every $\alpha \in \mathbf{ON}$.

But then, combining the conclusions from the two inductive arguments which we performed, we obtain that, for every ordinal α, $V[G]_\alpha \subseteq V_{\gamma+\alpha}[G]$. Therefore, since $\gamma + \alpha = \alpha$ for every $\alpha \geqslant \gamma \cdot \omega$, we have that for ordinals of the latter form, $V[G]_\alpha = V_\alpha[G]$ as desired. □

If $\mathbb{P}$ is a partial ordering on sequences indexed by ordinals and $s \in \mathbb{P}$, we write *supp*(s) for the support of s, that is, the collection $\{\alpha \in \mathit{dom}(s) : s(\alpha) \neq \mathbb{1}\}$; moreover, in the same context and for any conditions $s, t \in \mathbb{P}$, we say that s is an initial segment of t, which is denoted by $s \sqsubseteq t$, if $t \restriction \mathit{dom}(s) = s$ and $\mathit{dom}(s) = \mathit{dom}(t) \cap \sup\{\xi + 1 : \xi \in \mathit{dom}(s)\}$.

For $n \geqslant 1$, given any poset $\mathbb{P}$, any condition $p \in \mathbb{P}$ and any Σ_n-formula φ, the statement "$p \Vdash \varphi$" is Σ_n-expressible using $\mathbb{P}$ as a parameter. This fact requires that our model satisfies (KP and) Σ_n-collection along with Σ_n-separation; we will remind the reader of this issue in later chapters, wherever it is relevant.

Our terminology on chain conditions and closures of posets is mostly standard. We are explicit regarding the extent of closure of a given $\mathbb{P}$ by writing, for example, "$\leqslant\kappa$-directed closed" in order to mean that we may find lower bounds of directed subsets whose cardinality is at most κ. Correspondingly, the "$<\kappa$" prefix is self–explanatory. When $\kappa = \aleph_1$, we follow the common practice of writing "σ-closed" instead of "$<\aleph_1$-closed". Putting this terminology into action, we now give another folklore result, this time regarding preservation of closure under sequences in forcing extensions (see also Proposition 8.4 in [13]).

Lemma 1.19. *Let $M \subseteq V$ be an inner model of* ZFC *and suppose that for some (cardinal) λ, $V \models {}^{\lambda}M \subseteq M$. Let $\mathbb{P} \in M$ be a forcing notion such that, in V, $\mathbb{P}$ is either λ^+-c.c. or $\leqslant\lambda$-distributive. Then, for any filter G which is $\mathbb{P}$-generic over V, $V[G] \models {}^{\lambda}M[G] \subseteq M[G]$.*

Proof. It is enough to prove the conclusion for λ-sequences of ordinals. First consider the case where the forcing $\mathbb{P}$ is λ^+-c.c. and let G be a $\mathbb{P}$-generic over V. Fix some ordinal sequence $\vec{x} = \langle \beta_\xi : \xi < \lambda \rangle \in {}^{\lambda}M[G] \cap V[G]$ and some $\mathbb{P}$-name $\dot{x}$ such that $\dot{x}_G = \vec{x}$. Also, fix some condition $p \in G$ which forces that $\dot{x}$ is a λ-sequence of ordinals.

For each $\xi < \lambda$, the set $B_\xi = \{q \leqslant p : \exists\gamma\,(q \Vdash \dot{x}(\xi) = \check{\gamma})\}$, i.e., the set of conditions below p which decide the ξ^{th} element of the sequence, is open dense below p and thus, it contains some maximal antichain A_ξ. By the λ^+-c.c. assumption, $|A_\xi| \leqslant \lambda$ and so, by the closure of M in V, $A_\xi \in M$.

Moreover, since for every condition $q \in A_\xi$ there exists some ordinal γ_q with $q \Vdash \dot{x}(\xi) = \check{\gamma}_q$, again by the closure of M, we get that $\langle \gamma_q : q \in A_\xi \rangle \in M$. Thus, in M, we may apply the "mixing lemma" (cf. Chapter VII, Lemma 8.1 in [33]) in order to get a single name $\tau_\xi \in M$ so that, for every $q \in A_\xi$, $q \Vdash \dot{x}(\xi) = \tau_\xi$.

Now, once more by the closure of M, we have that $\langle \tau_\xi : \xi < \lambda \rangle \in M$ and hence $\dot{x}_G = \langle (\tau_\xi)_G : \xi < \lambda \rangle \in M[G]$, by interpreting pointwise the names τ_ξ using the generic. This concludes the λ^+-c.c. case.

Alternatively, suppose that $\mathbb{P}$ is $\leqslant\lambda$-distributive and fix again a filter G which is $\mathbb{P}$-generic over V. Then, by the usual defining equivalent of $\leqslant\lambda$-distributivity, no new λ-sequences of ordinals are added by the forcing. Therefore, any λ-sequence of ordinals which belongs to $V[G]$, already belongs to the ground model V. The conclusion now follows easily, from the assumption $V \models {}^{\lambda}M \subseteq M$ and the fact that $M \subseteq M[G]$. $\square$

The previous lemma is very useful in situations where one starts with a ground model embedding $j : V \longrightarrow M$, with M enjoying some closure under sequences in V, and then one "lifts" the embedding through some forcing $\mathbb{P}$, obtaining a corresponding embedding in some generic extension $V[G]$.

In order for such a lift to be possible in the first place, we have the following criterion.

Lemma 1.20 (Silver). *Suppose that $j : V \longrightarrow M$ is an elementary embedding, with M transitive. Let $\mathbb{P} \in V$ be a poset, let G be $\mathbb{P}$-generic over V and let H be $j(\mathbb{P})$-generic over M. Then, j lifts (uniquely) to $j^* : V[G] \longrightarrow M[H]$ (that is, j^* is an elementary embedding with $j^* \upharpoonright V = j$ and $j^*(G) = H$) if and only if $j''G \subseteq H$.*

As it is customary, we always use the same letter j for the lifted version of the embedding. In practice, we often ensure that the requirement "$j''G \subseteq H$" is satisfied, by exhibiting a particular condition $q \in H$ with the property that q is a lower bound for $j''G$; i.e., such that for every $p \in G$, $q \leqslant j(p)$. Such a condition q is called a *master condition*.

A word of caution should be added. When working in generic extensions, we will be frequently interested in relativized notions such as, for example, the "$H_{\aleph_2}$" of the model $V[G]$, or the "$H_{\mathfrak{c}}$" of $V^{\mathbb{P}}$. These should be written as $(H_{\aleph_2})^{V[G]}$ and $(H_{\mathfrak{c}})^{V^{\mathbb{P}}}$ respectively, stressing the fact that the cardinals "$\aleph_2$" and "$\mathfrak{c}$" are also computed in the corresponding models. In spite of that, we almost always drop the parentheses and write things like $H_{\aleph_2}^{V[G]}$ and $H_{\mathfrak{c}}^{V^{\mathbb{P}}}$. Throughout, in order to avoid ambiguities, we understand such notation by assuming that every defined notion is computed in the sense of the superscript model; if the superscript is missing, then it is understood that the computations take place in V, the fixed initial (ground) model of the argument at hand, whatever that is.

Finally, we give a small list of some abbreviations for popular posets among set theorists. Let κ be a regular cardinal and let λ be any ordinal; we then have the following:

- $Add(\kappa, \lambda)$ is the poset consisting of partial functions $p : \lambda \times \kappa \longrightarrow 2$ where $|p| < \kappa$; the ordering is given by reversed inclusion. This poset is $<\kappa$-directed closed and $(2^{<\kappa})^+$-c.c. Intuitively and as suggested by its name, in the typical case in which $\lambda > \kappa$ is a cardinal, this poset adds λ many new subsets to κ.

  One important special case is the poset $Add(\kappa, 1)$ which adds a Cohen subset to κ by partial functions $p : \kappa \longrightarrow 2$ of size less than κ. If $\kappa = \lambda^+$, then $Add(\kappa, 1)$ forces the GCH at λ, i.e., $2^\lambda = \lambda^+$ holds in any generic extension.

- $Col(\kappa, \lambda)$ is the poset consisting of partial functions $p : \kappa \longrightarrow \lambda$ where $|p| < \kappa$; the ordering is given by reversed inclusion. This poset is $<\kappa$-directed closed and $(|\lambda|^{<\kappa})^+$-c.c. Intuitively, in the typical case in which $\lambda > \kappa$ is a cardinal, this poset collapses λ to have size κ, in any generic extension.

- $Col(\kappa, <\lambda)$ is called the *Lévy collapse* and is the poset consisting of partial functions $p : \lambda \times \kappa \longrightarrow \lambda$ where $|p| < \kappa$ and, for every $\langle \alpha, \beta \rangle \in dom(p)$, $p(\alpha, \beta) \in \alpha$; the ordering is given by reversed inclusion. This poset is $<\kappa$-closed and, in the typical case in which λ is inaccessible, it is also λ-c.c. and makes λ equal to κ^+, in any generic extension.

1.4.2 Forcing axioms

Forcing axioms are generalizations of *Martin's Axiom* for families of $\aleph_1$ - many maximal antichains ($\mathsf{MA}_{\aleph_1}$). Although in this section we do not aim at a thorough treatment or full generality, we do fix a uniform notation for such axioms.

Definition 1.21. *For any (definable) class* Γ *of posets, the* **Forcing Axiom** *for* Γ*, denoted by* $\mathsf{FA}(\Gamma)$*, is the assertion that for every* $\mathbb{Q} \in \Gamma$ *and every collection* $\{A_\alpha : \alpha < \omega_1\}$ *of maximal antichains of* $\mathbb{Q}$*, there exists a filter* $G \subseteq \mathbb{Q}$ *such that* $G \cap A_\alpha \neq \varnothing$*, for all* $\alpha < \omega_1$*.*

Consequently, FA(c.c.c.) is just $\mathsf{MA}_{\aleph_1}$. Note that the axiom FA(σ - closed) is provable in ZFC; in this sense it is not very interesting (but see the stronger version below). In the current dissertation and, in particular, in Chapter 5, we will be mainly interested in the classes of c.c.c., σ - closed, proper, and, also, in the class of stationary preserving posets, that is, posets which preserve the stationary subsets of ω_1. In addition, we shall occasionally consider the case of $\aleph_1$ - semi proper posets as well. For the class of stationary preserving posets, we will frequently use the abbreviation "stat. pres.". The reader is referred to [28] for the relevant definitions and the properties of all the aforementioned notions.

According to the standard set – theoretic terminology, FA(proper) is called the *Proper Forcing Axiom* and is denoted by PFA; also, FA(stat. pres.) is traditionally called *Martin's Maximum* and is denoted by MM. An important consequence of these axioms is that $\mathfrak{c} = \aleph_2$ (see Theorem 31.23 in [28]). Hence, since every c.c.c. poset is also proper (and thus, stationary preserving as well), it follows that

$$\mathsf{MM} \Longrightarrow \mathsf{PFA} \Longrightarrow \mathsf{MA}.$$

A natural weakening of a given forcing axiom is its so – called *bounded* version.

Definition 1.22. *For any (definable) class* Γ *of posets, the* **Bounded Forcing Axiom** *for* Γ*, denoted by* $\mathsf{BFA}(\Gamma)$*, is the assertion that for every* $\mathbb{Q} \in \Gamma$ *and every collection* $\{A_\alpha : \alpha < \omega_1\}$ *of maximal antichains of the algebra* $\mathbb{B} = r.o.(\mathbb{Q}) \setminus \{\mathbf{0}\}$ *with* $|A_\alpha| \leqslant \aleph_1$ *for all* $\alpha < \omega_1$*, there exists a filter* $G \subseteq \mathbb{B}$ *such that* $G \cap A_\alpha \neq \varnothing$*, for all* $\alpha < \omega_1$*.*

Note that the requirement that each A_α is an antichain of the *regular open* Boolean algebra $\mathbb{B}$ (instead of the poset $\mathbb{Q}$ itself) is included in order to avoid trivialities: if the poset $\mathbb{Q}$ does not have any maximal antichains of size $\leqslant \aleph_1$, then the axiom becomes (vacuously) true. Clearly, $\mathsf{FA}(\Gamma)$ implies $\mathsf{BFA}(\Gamma)$. Not surprisingly, BFA(proper) is called the *Bounded Proper Forcing Axiom* and is denoted by BPFA, whereas BFA(stat. pres.) is called *Bounded Martin's Maximum* and is denoted by BMM. It is known that even BPFA implies $\mathfrak{c} = \aleph_2$ (cf. [37]).

In the other direction, two strengthenings of a given forcing axioms are the " + " and the " ++ " versions.

Definition 1.23. *For any (definable) class Γ of posets, the* **Forcing Axiom**$^+$ *for Γ, denoted by $\mathsf{FA}^+(\Gamma)$, is the assertion that for every $\mathbb{Q} \in \Gamma$, for every collection $\{A_\alpha : \alpha < \omega_1\}$ of maximal antichains of $\mathbb{Q}$ and given any $\mathbb{Q}$-name τ for a stationary subset of ω_1 (i.e., $\mathbb{Q} \Vdash$ "$\tau \subseteq \omega_1$ is stationary"), there exists a filter $G \subseteq \mathbb{Q}$ such that $G \cap A_\alpha \neq \varnothing$, for all $\alpha < \omega_1$, and, moreover, so that the set $\tau^G = \{\alpha < \omega_1 : \exists p \in G\,(p \Vdash \alpha \in \tau)\}$ is stationary.*

$\mathsf{FA}^{++}(\Gamma)$ is a similar axiom, where instead of a single $\mathbb{Q}$-name we are given ω_1-many names $\{\tau_\xi : \xi < \omega_1\}$ for stationary subsets of ω_1; we then require that G, apart from intersecting all the A_α's, is such that τ_ξ^G is stationary, for all $\xi < \omega_1$.

The consistency of PFA^{++} as well as that of MM^{++} follows from the consistency of the existence of a supercompact cardinal. The former was shown by J. Baumgartner (see Theorem 31.21 in [28]) while the latter by M. Foreman, M. Magidor and S. Shelah (cf. [20]). Shelah has also shown that MM implies $\mathsf{FA}^+(\sigma\text{-closed})$ (the latter is frequently denoted by $\mathsf{MA}^+(\sigma\text{-closed})$; see Theorem 37.26 in [28]).

We conclude this section with an important characterization, due to J. Bagaria, of bounded forcing axioms in terms of generic absoluteness. Let Γ be any (definable) class of posets; then, we have the following.

Theorem 1.24 ([4]). $\mathsf{BFA}(\Gamma)$ *holds if and only if, for every $\mathbb{Q} \in \Gamma$, $H_{\aleph_2} \prec_1 H_{\aleph_2}^{V^{\mathbb{Q}}}$.*

1.5 Squares and scales

In the final section of this prelude, we present the basic definitions and properties of square sequences and scales, which we will use in Chapter 5. Our account here is far from complete; the interested reader is referred to the excellent exposition given by J. Cummings, M. Foreman and M. Magidor (cf. [14]) for more details.

Definition 1.25. *Let κ be an uncountable cardinal. We say that a sequence of the form $\langle C_\alpha : \alpha \in Lim(\kappa^+) \rangle$ is a $\square_\kappa$-* **sequence** *if the following conditions are satisfied for all $\alpha \in Lim(\kappa^+)$:*

(i) $C_\alpha \subseteq \alpha$ *is a club in* α.

(ii) *If* $cf(\alpha) < \kappa$ *then* $\mathrm{ot}(C_\alpha) < \kappa$.

(iii) *For all* $\beta \in Lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$.

For any uncountable κ, we say that $\square_\kappa$ holds if there exists a $\square_\kappa$-sequence.

The square principle was introduced by Ronald Jensen who also showed that, if $V = L$, then $\square_\kappa$ holds at every uncountable cardinal κ. We remark that one may replace condition (ii) by the (weaker) requirement that, for all $\alpha \in Lim(\kappa^+)$, $\mathrm{ot}(C_\alpha) \leqslant \kappa$; then, the existence of a sequence with this weaker property implies the existence of a $\square_\kappa$ - sequence.

Also, note that $\square_\omega$ is not very interesting: any sequence $\langle\, C_\alpha : \alpha \in Lim(\omega_1)\,\rangle$ with the property that, for each $\alpha \in Lim(\omega_1)$, C_α is unbounded in α and such that $\mathrm{ot}(C_\alpha) = \omega$ satisfies Definition 1.25 (such a sequence is called a *ladder system*).

Square sequences are closely related to the phenomenon of stationary reflection. Given a regular uncountable cardinal κ and $S \subseteq \kappa$ which is stationary in κ, we say that S *reflects at* $\alpha < \kappa$ if $cf(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α. We say that *stationary reflection holds* at κ if every stationary $S \subseteq \kappa$ reflects at some $\alpha < \kappa$. Finally, we say that a stationary $S \subseteq \kappa$ is *non – reflecting* if it does not reflect at any $\alpha < \kappa$.

One basic application of squares is to show that stationary reflection fails: if $\square_\kappa$ holds and $S \subseteq \kappa^+$ is any given stationary subset of κ^+, then there exists some stationary $T \subseteq S$ such that T is non – reflecting (see Theorem 2.1 in [14]).

Jensen also introduced the following weakening of $\square$, called *weak square* (and denoted by "$\square^*$").

Definition 1.26. *Let κ be an uncountable cardinal. We say that a sequence of the form $\langle\, \mathcal{C}_\alpha : \alpha \in Lim(\kappa^+)\,\rangle$ is a $\square^*_\kappa$ -* **sequence** *if the following conditions are satisfied for all $\alpha \in Lim(\kappa^+)$:*

(i) *$\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha)$, $1 \leqslant |\mathcal{C}_\alpha| \leqslant \kappa$ and every $C \in \mathcal{C}_\alpha$ is a club in α.*

(ii) *If $cf(\alpha) < \kappa$ then for all $C \in \mathcal{C}_\alpha$, $\mathrm{ot}(C) < \kappa$.*

(iii) *For all $C \in \mathcal{C}_\alpha$ and every $\beta \in Lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.*

For any uncountable κ, we say that $\square^*_\kappa$ holds if there exists a $\square^*_\kappa$ - sequence.

Like for $\square_\kappa$ - sequences, one may replace condition (ii) in the previous list by the requirement that, for all $\alpha \in Lim(\kappa^+)$ and all $C \in \mathcal{C}_\alpha$, $\mathrm{ot}(C) \leqslant \kappa$. Moreover, as explained in § 1 of [19], one may always assume that the $\square^*_\kappa$ - sequence has the additional property that, for every $\alpha \in Lim(\kappa^+)$, there exists a $C \in \mathcal{C}_\alpha$ with $\mathrm{ot}(C) = cf(\alpha)$.

In general, one may construct a $\square^*_\kappa$ - sequence assuming $\kappa^{<\kappa} = \kappa$. Hence, weak squares are more interesting in the case in which κ is singular. It is known that weak squares are not sufficient in order to get failures of stationary reflection, as in the case of $\square_\kappa$.

There are lots of other weakenings of the basic $\Box$ principle; see [14] and [19] for more information. For our purposes (cf. Chapter 5), it is the *failures* of these principles which are of interest, since they provide us with consistency lower bounds. Such failures follow either from suitable large cardinal hypotheses or from strong forcing axioms. As an example of the former case, Solovay showed that if κ is supercompact, then $\Box_\lambda$ fails for every $\lambda \geqslant \kappa$; as an example of the latter case, Stevo Todorčević has shown that under PFA, $\Box_\lambda$ fails for every uncountable cardinal λ (see Exercise 27.3 and Theorem 31.28 in [28], respectively).

One last notion which we will use (actually, we will use the implications from its non-existence) is that of a *good scale* for a singular cardinal; this was studied by Shelah in the context of his celebrated PCF theory.

Definition 1.27. *Let κ be a singular cardinal and fix some increasing sequence of regular cardinals $\vec{\kappa} = \langle \kappa_i : i < cf(\kappa) \rangle$ with $\sup_i \kappa_i = \kappa$. We say that a sequence of functions of the form $\langle f_\alpha : \alpha < \kappa^+ \rangle$ is a κ^+-* **scale** *(with respect to $\vec{\kappa}$) if the following conditions are satisfied:*

(*i*) *For all $\alpha < \kappa^+$, $f_\alpha \in \prod_{i<cf(\kappa)} \kappa_i$.*

(*ii*) *For all $\alpha < \beta < \kappa^+$, $\{i < cf(\kappa) : f_\alpha(i) \geqslant f_\beta(i)\}$ is bounded in $cf(\kappa)$.*

(*iii*) *For every $g \in \prod_{i<cf(\kappa)} \kappa_i$, there exists some $\alpha < \kappa^+$ such that*

$$\{i < cf(\kappa) : g(i) \geqslant f_\alpha(i)\}$$

is bounded in $cf(\kappa)$.

Equivalently, conditions (*ii*) and (*iii*) can be stated with reference to the ideal I of bounded subsets of $cf(\kappa)$. Condition (*iii*) is sometimes irrelevant to particular applications and, so, some authors tend to drop it from the definition of a scale. Shelah has shown that for every singular κ there exists a κ^+-scale (see, e.g., Theorem 3.53 in [16]).

Definition 1.28. *Let κ be a singular cardinal and fix some increasing sequence of regular cardinals $\vec{\kappa} = \langle \kappa_i : i < cf(\kappa) \rangle$ with $\sup_i \kappa_i = \kappa$. We say that a κ^+-scale $\langle f_\alpha : \alpha < \kappa^+ \rangle$ for κ is* **good** *(with respect to $\vec{\kappa}$) if for all $\alpha < \kappa^+$ with $cf(\alpha) > cf(\kappa)$, there exists an $i < cf(\kappa)$ and some $D \subseteq \alpha$ which is unbounded in α such that*

$$\forall \beta, \gamma \in D \ \forall j > i \, (\beta < \gamma \Longrightarrow f_\beta(j) < f_\gamma(j)).$$

For the proof of the following proposition, the *approachability property* (which is denoted by AP and is a weakening of weak square) is employed; see, for example, [14], [19] and, also, Proposition 4.52 in [16].

Proposition 1.29. *For every singular κ, if $\square^*_\kappa$ holds then every κ^+-scale is good.*

The following is due to Shelah (see [42], or Section 4.7 in [16] for more details).

Theorem 1.30. *For every singular κ, if the SCH fails at κ then there exists a good κ^+-scale.*

Finally, let us briefly mention that, in general, failures of square principles imply the existence of inner models with large cardinals. Some early examples of this situation were, on the one hand, Solovay's proof that if $\square_{\omega_1}$ fails then $\aleph_2$ is Mahlo in L and, on the other, Jensen's proof that, for singular κ, if $\square_\kappa$ fails then there is an inner model with a strong cardinal.

As an example of more recent lower bounds, it has been shown (cf. [45]) that if $\square_\lambda$ fails for some singular strong limit λ, then the *Axiom of Determinacy* (AD) holds in $L(\mathbf{R})$; in particular, this implies the existence of an inner model with infinitely many Woodin cardinals. The interested reader may consult [38] for more results and lower bounds in this direction.

CHAPTER 2

Elementary Chains and $C^{(n)}$ - cardinals

In this chapter we focus on the various $C^{(n)}$ - hierarchies, obtaining several consistency (upper) bounds for the cases of tallness, superstrongness, strongness, supercompactness, and extendibility. Our general method can be roughly described as follows.

Suppose that we are given such a $C^{(n)}$ - cardinal κ and an elementary embedding $j : V \longrightarrow M$ witnessing this fact appropriately. Under the additional assumption that the image $j(\kappa)$ is a regular (or even inaccessible) cardinal in V, we shall construct various elementary chains of substructures of the model M, giving rise to factor elementary embeddings which have analogous strength to that of the initial j.

The aim of these constructions is to ensure that the ordinals below $j(\kappa)$ which arise as images of the large cardinal κ under embeddings of the sort in question, is a sufficiently "rich" subset of $j(\kappa)$; e.g., stationary, α - club for some $\alpha < j(\kappa)$, etc. If $j(\kappa)$ is indeed an inaccessible cardinal, we then check that all the aforementioned factor embeddings can be verified inside the model $V_{j(\kappa)}$ via derived extenders and we, consequently, obtain corresponding consistency upper bounds for each individual $C^{(n)}$ - hierarchy.

We also deal with the case of $C^{(n)}$ - Woodin and that of $C^{(n)}$ - strongly compact cardinals, characterizing each one of them in terms of its ordinary counterpart (i.e., in terms of the usual Woodin and strongly compact cardinals, respectively). Finally, we give connections between the notions of $C^{(n)}$ - supercompactness and of $C^{(n)}$ - extendibility, addressing some issues which were left open in [5].

We begin our study of elementary chains and $C^{(n)}$ - cardinals by first considering the case of tallness, where we shall describe our method in detail.

2.1 Tallness

Suppose that κ is λ-tall for some $\lambda > \kappa$, as witnessed by the elementary embedding $j : V \longrightarrow M$, i.e., M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\kappa}M \subseteq M$. Suppose that, in addition, $j(\kappa)$ is a regular cardinal.

We pick some limit ordinal $\beta \in (\lambda\,,\, j(\kappa))$ and consider the following elementary substructure:

$$X = \{j(f)(x) : \; f \in V,\; f : V_\kappa \longrightarrow V,\; x \in V_\beta^M\} \prec M.$$

To check that $X \prec M$, we use the fact that β is a limit ordinal and we verify the Tarski – Vaught criterion. Let $\phi(y, \vec{v})$ be a formula and suppose that

$$M \models \exists\, y\, \phi(y, j(f_1)(x_1), \ldots, j(f_n)(x_n))\,,$$

for some functions $f_i : V_\kappa \longrightarrow V$ with $f_i \in V$ and some $x_i \in V_\beta^M$, for every $1 \leqslant i \leqslant n$. Since β is limit, $x = \langle\, x_1, \ldots, x_n\rangle \in V_\beta^M$. We define, in V, the (Skolem) function $g = g_{\phi, f_1, \ldots, f_n} : V_\kappa \longrightarrow V$ in the following manner. For every $w \in V_\kappa$,

$$g(w) = \begin{cases} \text{“some” } z \text{ s.t. } \phi(z, f_1(w_1), \ldots, f_n(w_n)) & , \text{ if } w = \langle\, w_1, \ldots, w_n\rangle \\ & \text{ and such a } z \text{ exists} \\ \varnothing & , \text{ otherwise.} \end{cases}$$

We remark that the word “some” in the above definition means that we pick the existential witness in some naturally minimal manner, e.g., having fixed beforehand, for every α, a well – ordering $\lhd_\alpha$ of V_α, we first consider the minimal rank of such a witness z, say γ, and then we choose the least among the various witnesses of rank γ using $\lhd_\gamma$.

We now apply elementarity and consider the function $j(g) : V_{j(\kappa)}^M \longrightarrow M$ evaluated at $x = \langle\, x_1, \ldots, x_n\rangle \in V_\beta^M \subseteq V_{j(\kappa)}^M$. It follows that $j(g)(x) \in X$ and $M \models \phi(j(g)(x), j(f_1)(x_1), \ldots, j(f_n)(x_n))$, which finishes the verification of the Tarski – Vaught criterion. Notice that X is, in fact, the Skolem hull of the $range(j)$ together with V_β^M inside M, with respect to the functions of the form $f : V_\kappa \longrightarrow V$.

Starting with $X_0 = X$ and $\beta_0 = \beta$, we will recursively build, for any $\xi < j(\kappa)$, an increasing (under $\subseteq$) sequence of elementary substructures $X_\xi \prec M$, together with a strictly increasing sequence of corresponding limit ordinals $\beta_\xi < j(\kappa)$, such that each X_ξ is of the form

$$X_\xi = \{j(f)(x) : \; f \in V,\; f : V_\kappa \longrightarrow V,\; x \in V_{\beta_\xi}^M\}.$$

Our aim will be to show that, at appropriate ordinals $\gamma < j(\kappa)$, using the "current" substructure X_γ of this chain, we can define an elementary embedding j_γ which can be nicely represented and which, at the same time, witnesses λ-tallness for the cardinal κ.

So let $\beta_0 = \beta$ and X_0 as defined above. For any $\xi + 1 < j(\kappa)$, given β_ξ and X_ξ, we define

$$\beta_{\xi+1} = \sup(X_\xi \cap j(\kappa)) + \omega$$

and

$$X_{\xi+1} = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{\beta_{\xi+1}}\}.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup\limits_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{\beta_\xi}\},$$

which concludes the recursive definition of the elementary chain.

Remark. As it will soon become clear, the particular ordinal β_0 serves the mere purpose of initializing the construction and is not important for our arguments towards extracting the desired embedding from the elementary chain. In fact, any other limit ordinal greater than λ would also be sufficient. In this sense, although – formally – our construction depends on this initial choice, we supress any further mention to β_0 in order to ease readability. In the few cases where it is needed, we will refer to it as the "initial limit ordinal".

Moreover, for any elementary substructure which is of this particular form, i.e., the Skolem hull of the range of the embedding together with some set in M, we will frequently call the latter set (in our cases, the various V^M_β's) as the set of *seeds* (see [23] for a general theory of seeds). This remark applies to the subsequent sections as well. $\perp$

It is clear that $\langle \beta_\xi : \xi < \gamma \rangle$ is a continuous and strictly increasing sequence of limit ordinals. We check inductively that, in addition, the ordinals β_ξ are bounded below $j(\kappa)$.

Claim. *For every $\xi < j(\kappa)$, $\beta_\xi < j(\kappa)$.*

Proof of claim. The base and limit cases are immediate by choice of $\beta = \beta_0$ and the regularity of $j(\kappa)$. For the successor step, suppose that $\beta_\xi < j(\kappa)$ for some $\xi < j(\kappa)$. It is enough to see that $|X_\xi \cap j(\kappa)| < j(\kappa)$, since then, as $j(\kappa)$ is regular, we will have that $\sup(X_\xi \cap j(\kappa)) < j(\kappa)$ and thus, $\beta_{\xi+1} < j(\kappa)$.

For this, note first that for any given ordinal $\theta \in X_\xi \cap j(\kappa)$ we have $\theta = j(f)(x)$, for some seed $x \in V^M_{\beta_\xi}$ and some function $f \in V$ which can be assumed to be $f : V_\kappa \longrightarrow \kappa$, i.e., taking values in κ. There are at most 2^κ many such functions and at most $|V^M_{\beta_\xi}|$ many such seeds. But now, since $j(\kappa)$ is inaccessible in M and ${}^\kappa M \subseteq M$, we have that $2^\kappa \leqslant (2^\kappa)^M < j(\kappa)$ and also $|V^M_{\beta_\xi}| \leqslant |V_{\beta_\xi}|^M < j(\kappa)$. Thus, $|X_\xi \cap j(\kappa)| < j(\kappa)$ as desired. □

As before, since each β_ξ is limit, $X_\xi \prec M$. Evidently, for any $\xi < j(\kappa)$, $V^M_{\beta_\xi} \subseteq X_\xi$. Also, for any $\xi < \xi' < j(\kappa)$, it is clear that $X_\xi \subseteq X_{\xi'}$. Therefore, an elementary chain of substructures is formed:

$$X_0 \prec X_1 \prec \ldots \prec X_\xi \prec \ldots \quad \prec M.$$

For any $\gamma < j(\kappa)$ with $cf(\gamma) > \kappa$, we may consider the current substructure X_γ, along with the corresponding ordinal β_γ, where $\beta_\gamma = \sup\limits_{\xi<\gamma} \beta_\xi$ and

$$X_\gamma = \bigcup_{\xi<\gamma} X_\xi = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{\beta_\gamma}\}.$$

Clearly, $\beta_\gamma < j(\kappa)$, $cf(\beta_\gamma) = cf(\gamma) > \kappa$, $V^M_{\beta_\gamma} \subseteq X_\gamma$ and $X_\gamma \prec M$.

We then let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse and define the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, with $cp(j_\gamma) = \kappa$. This produces the following commutative diagram of elementary embeddings:

$$\begin{tikzcd}
V \arrow[r, "j"] \arrow[d, "j_\gamma"'] & M \\
M_\gamma \arrow[ur, "k_\gamma = \pi_\gamma^{-1}"'] &
\end{tikzcd}$$

We will now show that the embedding j_γ witnesses the λ-tallness of κ.

The key observation is that, in this situation, $X_\gamma \cap j(\kappa)$ is in fact an ordinal. To check this, let $\alpha \in X_\gamma \cap j(\kappa)$ and let $\alpha' < \alpha$. By construction, there is some $\xi < \gamma$ such that $\alpha \in X_\xi$. Thus, again by construction, $\alpha < \beta_{\xi+1}$ and so, $\alpha' \in V^M_{\beta_{\xi+1}} \subseteq X_{\xi+1} \subseteq X_\gamma$.

The reason for the fact that $X_\gamma \cap j(\kappa)$ is an ordinal, is essentialy the manner in which we have "filled in all the ordinal holes below $j(\kappa)$" along our recursive construction of the substructures X_ξ. It then easily follows that

$$cp(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \beta_\gamma\,.$$

In fact, in such constructions, $X_\gamma \cap j(\kappa)$ is an ordinal if and only if $cp(k_\gamma) = j_\gamma(\kappa)$ in which case, we call the embedding j_γ an *initial factor* of j. Let us be slightly more general and consider the following situation.

Let $j : V \longrightarrow M$ be an elementary embedding with $cp(j) = \kappa$, M transitive and suppose that $X \prec M$ with $range(j) \subseteq X$, $X \cap [\kappa, j(\kappa)) \neq \varnothing$ and $X \cap j(\kappa)$ is bounded in $j(\kappa)$. Let $\pi : X \cong M_0$ be the Mostowski collapse and consider the following diagram of commuting elementary embeddings:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_0"'] & M \\ M_0 \arrow[ur, "k = \pi^{-1}"'] & \end{tikzcd}$$

$j_0 = \pi \circ j : V \longrightarrow M_0$

with $cp(j_0) = \kappa$.

Observe that the imposed requirements on the substructure X ensure that j_0 is well–defined, $M_0 \neq V$ (i.e., $j_0 \neq id$), $cp(j_0) = \kappa$ and $j_0(\kappa) < j(\kappa)$. We then introduce the following notion.

Definition 2.1. *We say that such a* j_0 *is an* **initial factor** *of* j *if* $cp(k) = j_0(\kappa)$.

The following two lemmas are easily verified.

Lemma 2.2. *In the situation described above,* j_0 *is an initial factor of* j *if and only if* $X \cap j(\kappa)$ *is an ordinal. In this case,* $j_0(\kappa) = \sup(X \cap j(\kappa))$. □

Lemma 2.3. *If* j_0 *is an initial factor of* j *(via the collapse* $\pi : X \cong M_0$ *with* $k = \pi^{-1}$*), then* $V^{M_0}_{j_0(\kappa)} = V^M_{j_0(\kappa)} \subseteq range(k)$ *and, therefore,*

$$\{j(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V^M_{j_0(\kappa)}\} \subseteq range(k).$$

□

Returning to our argument, we have that $j_\gamma(\kappa) = \beta_\gamma > \lambda$ and so, in order to conclude that this embedding witnesses λ-tallness, we only need to check that ${}^\kappa M_\gamma \subseteq M_\gamma$. This will essentially come from the fact that the set of seeds that generate M_γ is closed under κ-sequences, a fact which, in turn, follows from ${}^\kappa M \subseteq M$ and $cf(\gamma) > \kappa$. Initially, notice that

$$M_\gamma = \pi_\gamma{}'' X_\gamma = \{j_\gamma(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V^M_{\beta_\gamma}\}.$$

Now suppose that $\{j_\gamma(f_\alpha)(x_\alpha) : \alpha < \kappa\} \subseteq M_\gamma$, where for each $\alpha < \kappa$, $x_\alpha \in V^M_{\beta_\gamma}$ and $f_\alpha \in V$. Let us check that $\langle j_\gamma(f_\alpha)(x_\alpha) : \alpha < \kappa \rangle \in M_\gamma$.

For every $\alpha < \kappa$, there exists some $\xi_\alpha < \gamma$ such that $x_\alpha \in V^M_{\beta_{\xi_\alpha}}$ and so, as $cf(\gamma) > \kappa$, there is a $\xi < \gamma$ with $\{x_\alpha : \alpha < \kappa\} \subseteq V^M_{\beta_\xi}$. Therefore, we obtain that $\langle x_\alpha : \alpha < \kappa \rangle \in V^M_{\beta_\gamma} \subseteq M_\gamma$.

It is also clear that $\langle j_\gamma(f_\alpha) : \alpha < \kappa \rangle = j_\gamma(\langle f_\alpha : \alpha < \kappa \rangle) \restriction \kappa \in M_\gamma$. Hence, in M_γ, we can compute $\langle j_\gamma(f_\alpha)(x_\alpha) : \alpha < \kappa \rangle$ by evaluating pointwise the functions $j_\gamma(f_\alpha)$'s at the corresponding x_α's. We have thus proved the following.

Proposition 2.4. *Suppose that $j : V \longrightarrow M$ is a λ-tall embedding for κ, with $j(\kappa)$ regular. Then, for any given (initial limit ordinal) $\beta_0 \in (\lambda, j(\kappa))$ and for any $\gamma < j(\kappa)$ with $cf(\gamma) > \kappa$, the embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the elementary chain construction after γ steps as above, is an initial factor of j witnessing the λ-tallness of κ.* □

One easily sees that in order to establish the closure under κ-sequences for the last proposition, the only relevant information was the fact that our initial j was a tallness embedding (i.e., ${}^\kappa M \subseteq M$) and that for the set of seeds $V^M_{\beta_\gamma}$ generating the X_γ, we had that $cf(\gamma) = cf(\beta_\gamma) > \kappa$. Thus, we may be slightly more general and state the following.

Corollary 2.5. *Suppose that $j : V \longrightarrow M$ is a λ-tall embedding for κ, with $j(\kappa)$ regular. Suppose that $j_0 : V \longrightarrow M_0 \cong X$ is an initial factor of j via the Mostowski collapse of*

$$X = \{j(f)(x) : \ f \in V, \ f : V_\kappa \longrightarrow V, \ x \in V^M_\theta\} \prec M,$$

where $\theta \in (\lambda, j(\kappa))$ is such that $cf(\theta) > \kappa$. Then, j_0 is λ-tall for κ. □

Our next aim is to consider the class of all the possible images $j_0(\kappa)$ below $j(\kappa)$, where j_0 is any initial factor embedding arising from a Mostowski collapse of some elementary substructure $X \prec M$ of the form

$$X = \{j(f)(x) : \ f \in V, \ f : V_\kappa \longrightarrow V, \ x \in V^M_\theta\},$$

for some $\theta \in (\lambda, j(\kappa))$ with $cf(\theta) > \kappa$. We shall now show that this class is in fact a $[\kappa^+, j(\kappa))$-club in $j(\kappa)$, i.e., it contains suprema of sequences whose length is a (regular) $\delta \in [\kappa^+, j(\kappa))$. Clearly, this class already contains all the images $j_\gamma(\kappa)$ arising from initial factor embeddings coming from our elementary chain construction, for various initial limit ordinals $\beta_0 \in (\lambda, j(\kappa))$ and various lengths $\gamma < j(\kappa)$ with $cf(\gamma) > \kappa$.

So suppose that $\delta \in [\kappa^+, j(\kappa))$ is regular and that we have a strictly increasing sequence $\langle j_i(\kappa) : \ i < \delta \rangle$ of ordinals below $j(\kappa)$ where, for all $i < \delta$, there is some ordinal $\theta_i \in (\lambda, j(\kappa))$ with $cf(\theta_i) > \kappa$, so that the embedding $j_i : V \longrightarrow M_i$ is an initial factor of j arising via the collapse π_i of the substructure

$$X_i = \{j(f)(x) : \ f \in V, \ f : V_\kappa \longrightarrow V, \ x \in V^M_{\theta_i}\} \prec M.$$

Recall that, in such a case, $X_i \cap j(\kappa)$ is an ordinal and, therefore, we have that $j_i(\kappa) = \sup(X_i \cap j(\kappa))$. It is important to point out that, just from the knowledge that j_i comes from the collapse of X_i, we may only conclude that $j_i(\kappa) \geqslant \theta_i$. On the other hand, it follows from Lemma 2.3 that, in fact,

$$\{j(f)(x) : f \in V, x \in V^M_{j_i(\kappa)}\} = \{j(f)(x) : f \in V, x \in V^M_{\theta_i}\}$$

and thus, we may as well assume that $j_i(\kappa) = \theta_i$, for every $i < \delta$. Hence, for any $i < \ell < \delta$, $X_i \subseteq X_\ell$ which, in turn, gives that $X_i \prec X_\ell$ and so, an elementary chain is formed. We may now let $\theta_\delta = \sup\limits_{i<\delta} \theta_i$ and

$$X_\delta = \bigcup_{i<\delta} X_i = \{j(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V^M_{\theta_\delta}\} \prec M.$$

Obviously, $\theta_\delta < j(\kappa)$ and $cf(\theta_\delta) = \delta > \kappa$. Let $\pi_\delta : X_\delta \cong M_\delta$ be the Mostowski collapse and let $j_\delta = \pi_\delta \circ j : V \longrightarrow M_\delta$ with $cp(j_\delta) = \kappa$ be the composed embedding, which forms a commutative diagram as usual. We now have that $X_\delta \cap j(\kappa)$ is an ordinal and then

$$j_\delta(\kappa) = \sup(X_\delta \cap j(\kappa)) = \sup_{i<\delta} \theta_i = \sup_{i<\delta} j_i(\kappa),$$

which shows the desired closure. Clearly, $j_\delta(\kappa) < j(\kappa)$ and, consequently, by Corollary 2.5, j_δ is λ-tall for κ.

Finally, it is also obvious from our construction that the various images $j_0(\kappa)$ of initial factor λ - tall embeddings are unbounded in $j(\kappa)$ (by choosing a sufficiently large initial limit ordinal β_0).

By our discussion so far and by the trivial fact that any λ - tall embedding is actually $< j(\kappa)$ - tall (i.e., is α-tall for every $\alpha < j(\kappa)$), we can conclude the following.

Proposition 2.6. *Suppose that $j : V \longrightarrow M$ witnesses the λ-tallness of κ, with $j(\kappa)$ regular. Then, the collection*

$$D = \{h(\kappa) < j(\kappa) : h \text{ is } \alpha\text{ - tall for } \kappa, \text{ for some } \alpha < j(\kappa)\}$$

contains a $[\kappa^+, j(\kappa))$ - club. □

Next, we deal with the definability of this collection D of images, inside $V_{j(\kappa)}$. Given any α - tall initial factor embedding $j_0 : V \longrightarrow M_0$, where

$$M_0 \cong X = \{j(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V^M_\theta\} \prec M,$$

for some $\alpha \in (\lambda, j(\kappa))$ and some ordinal $\theta < j(\kappa)$ with $cf(\theta) > \kappa$, as we have already remarked, we may assume that $j_0(\kappa) = \theta$. Naturally, we may extract from it the $(\kappa, j_0(\kappa))$ - extender E and construct the corresponding embedding $j_E : V \longrightarrow M_E$ with $cp(j_E) = \kappa$ and $j_E(\kappa) = j_0(\kappa)$. We will then have that

$$M_E = \{j_E(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^{M_0}_{j_0(\kappa)}\}$$

and, also, $V^{M_0}_{j_0(\kappa)} = V^{M_E}_{j_E(\kappa)}$. Furthermore $M_0 \models cf(j_0(\kappa)) > \kappa$, a fact which is computed correctly since ${}^\kappa M_0 \subseteq M_0$. This means that the set of seeds $V^{M_0}_{j_0(\kappa)}$ (which generate M_E) is closed under κ - sequences and so, by arguments which we have already described, it follows that ${}^\kappa M_E \subseteq M_E$, i.e., j_E witnesses α - tallness as well.

Actually more is true as, in such a case, $M_0 = M_E$. This follows from the general fact that, when we construct the model M_E from the $(\kappa, j(\kappa))$ - extender E which is derived from an ambient embedding $j : V \longrightarrow M$, then, considering the corresponding commutative diagram of j, j_E and k_E, we have that

$$range(k_E) = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{j(\kappa)}\}.$$

Hence, in our case, we get that $k_E = id$ and so, it indeed follows that $M_0 = M_E$.

The above observations show that any α - tall initial factor embedding j_0 can be taken to be an extender embedding, still witnessing the α - tallness of κ. Notice that all these extenders belong to $V_{j(\kappa)}$.

Now, if we strengthen the regularity assumption by requiring that $j(\kappa)$ is an *inaccessible*, then, in the ZFC model $V_{j(\kappa)}$, we have that for any such extender E coming from a j_0, $V_{j(\kappa)} \models$ "j_E is α - tall for κ". This follows from the inaccessibility of $j(\kappa)$ and the usual representation of the extender model M_E; these facts enable $V_{j(\kappa)}$ to compute "enough" of j_E and M_E in order to verify that $j_E(\kappa) = j_0(\kappa)$ and ${}^\kappa M_E \subseteq M_E$.

We may thus conclude that, apart from the fact that $V_{j(\kappa)} \models$ "κ is tall", we also have that the collection

$$C_{\text{tall}} = \{h_E(\kappa) < j(\kappa) : h_E \text{ is } \alpha\text{-tall extender embedding, for some } \alpha < j(\kappa)\},$$

which is a class in $j(\kappa)$, is a subclass of D, is definable in $V_{j(\kappa)}$ and, moreover, it contains a $[\kappa^+, j(\kappa))$ - club (in particular, C_{tall} is stationary in $j(\kappa)$). Also, again by inaccessibility, for each $n \in \omega$, we have a club $C^{(n)}_{j(\kappa)} \subseteq j(\kappa)$, consisting of all ordinals below $j(\kappa)$ which are Σ_n - correct in the sense of $V_{j(\kappa)}$. Hence, $C^{(n)}_{j(\kappa)} \cap C_{\text{tall}} \neq \varnothing$, for every $n \in \omega$.

Putting everything together, we have shown the following.

Theorem 2.7. *Suppose that, for some $\lambda > \kappa$, the embedding $j : V \longrightarrow M$ witnesses the λ-tallness of κ, with $j(\kappa)$ inaccessible. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$ - tall".* $\square$

Remark. Observe that we could have assumed that $j(\kappa)$ is inaccessible, right at the beginning of the section, without any change in the construction of the chains. The reason we chose not to is two – fold; on the one hand, we wanted to present the construction under the minimal assumptions and this is accomplished by just requiring the regularity of $j(\kappa)$. On the other hand, the inaccessibility becomes important when we want to have consistency results as the one in the previous theorem, since it guarrantees that $V_{j(\kappa)}$ is a ZFC model (with its own version of Σ_n - correct ordinals).

Since the following sections will have a similar structure when dealing with other $C^{(n)}$ - cardinals, we shall follow the same presentation style and just assume regularity of $j(\kappa)$ for the construction of the corresponding elementary chains. We will then employ the inaccessibility assumption, only when we come to conclude some consistency result, as the one above.

We also address a subtle issue regarding the conclusion of the Theorem 2.7. By Tarski's result regarding the undefinability of truth, there is *no* formula $\phi(n,\alpha)$ with "$\phi(n,\alpha) \longleftrightarrow \alpha \in C^{(n)}$". Hence, as far as satisfaction of formulas in class models is concerned, we are only able to express such a conclusion in terms of an appropriate schema of countably – many formulas: we augment our (standard set – theoretic) language by introducing a constant symbol $\mathbf{k}$, i.e., we consider the language $\mathcal{L}^* = \{\in, \mathbf{k}\}$, and we then let TALL be the countable schema of $\mathcal{L}^*$ - formulas ϕ_n, where for every (meta) $n \in \mathbb{N}$, the formula ϕ_n asserts that "$\mathbf{k}$ is $C^{(n)}$ - tall". In such a case, and for any (definable) non – empty class X and any $z \in X$, a satisfaction statement of the sort "$\langle X , \in , z \rangle \models \mathsf{TALL}$" just means that for any particular (meta) n, we have that $\langle X , \in , z \rangle \models \phi_n$, where the constant k is interpreted as z.

On the other hand, in our situation, we are interested in satisfaction for set models; given the λ - tallness embedding j with $j(\kappa)$ inaccessible, we may actually prove in ZFC that for every $n \in \omega$,

$$V_{j(\kappa)} \models \text{"}\kappa \text{ is } C^{(n)}\text{ - tall"}$$

(which is essentially what we just showed in Theorem 2.7). Even so, the countable schema TALL should not be discarded altogether, as it is relevant when one enquires issues related to consistency strength. This remark also applies to the subsequent sections of the current chapter, where corresponding countable schemata will also be considered. $\perp$

After these clarifications, we point out that Theorem 2.7 certainly gives an upper bound on the consistency strength of the theory $\mathsf{ZFC} + \mathsf{TALL}$, as the latter holds in $\langle V_{j(\kappa)} , \in , \kappa \rangle$, with $\mathbf{k}$ interpreted as κ. In turn, it gives rise to the following natural question(s).

Question 2.8. *What is the consistency strength of the statement "there is a κ such that, for some $\lambda > \kappa$, κ is λ-tall with $j(\kappa)$ regular/inaccessible"?*

An upper bound is the existence of an almost huge cardinal κ (since the image $j(\kappa)$ of such an embedding is inaccessible). Still, one can do better and require that κ is 1-extendible, which is a strictly weaker notion than that of almost hugeness: it is well–known that if κ is 2^κ-supercompact, then there exists a normal $\mathcal{U}$ on κ so that $\{\alpha < \kappa : \alpha \text{ is 1-extendible}\} \in \mathcal{U}$ (see, e.g., Exercise 23.5 in [30]).

As we shall see in the sections to follow, 1-extendibility is an adequate upper bound for results concerning both the case of superstrongness and that of strongness. On the other hand, for supercompactness (and extendibility), it seems as if one cannot avoid requiring the existence of an almost huge cardinal, although the exact bounds are not known.

Having dealt with the case of tallness, all the essential features of our methodology have (hopefully) become apparent. We now proceed with the cases of superstrong and of strong cardinals and, also, later on in this chapter, with those of supercompactness and of extendibility. Since many of the constructions will be analogous in spirit, we shall skip several details and refer to previously established facts when needed.

2.2 Superstrongness

Suppose that κ is a superstrong cardinal and let $j : V \longrightarrow M$ be a witnessing embedding, i.e., M transitive, $cp(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. In addition, suppose that $j(\kappa)$ is regular. Bear in mind that, in such a case, this is actually equivalent to requiring that $j(\kappa)$ is inaccessible, since superstrongness already implies that $j(\kappa) \in C^{(1)}$ (cf. Proposition 1.10). We can thus forget about this distinction in the current section and assume, throughout, that $j(\kappa)$ is inaccessible.

We fix an initial limit ordinal $\beta_0 \in (\kappa, j(\kappa))$ and we recursively construct an elementary chain of substructures of M, starting with seeds in V_{β_0} (note that since $V_{j(\kappa)} \subseteq M$, the superscript "$M$" may be dropped from the sets of seeds). Our aim is to extract an appropriate superstrong initial factor embedding j_γ from the constructed chain. One important difference is that in this case, as opposed to the case of tallness, the ordinal length $\gamma < j(\kappa)$ at which we take the collapse of the current substructure can be any limit ordinal below $j(\kappa)$ (i.e., it can even be ω). The reason is that we are not interested in closure under sequences for the initial factor embedding j_γ.

We start with the chosen limit ordinal β_0 and we let

$$X_0 = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_0}\} \prec M.$$

For any $\xi + 1 < j(\kappa)$, given β_ξ and X_ξ, we let

$$\beta_{\xi+1} = \sup(X_\xi \cap j(\kappa)) + \omega$$

and

$$X_{\xi+1} = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_{\xi+1}}\}.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let again $\beta_\xi = \sup_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_\xi}\},$$

which concludes the recursive definition of the elementary chain.

The regularity of $j(\kappa)$ implies that $\beta_\xi < j(\kappa)$, for all $\xi < j(\kappa)$. At any limit ordinal $\gamma < j(\kappa)$, we consider the substructure X_γ and we let $\pi_\gamma : X_\gamma \cong M_\gamma$ be the Mostowski collapse. We let $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$ be the composed map with $cp(j_\gamma) = \kappa$, producing a commutative diagram of embeddings (where $k_\gamma = \pi_\gamma^{-1} : M_\gamma \longrightarrow M$).

Once again, one easily checks that $X_\gamma \cap j(\kappa)$ is an ordinal which implies that j_γ is indeed an initial factor of j and that

$$cp(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \sup_{\xi<\gamma} \beta_\xi = \beta_\gamma.$$

Therefore, $V_{j_\gamma(\kappa)} \subseteq M_\gamma$ and we have just shown the following.

Proposition 2.9. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, for each (initial limit ordinal) $\beta_0 \in (\kappa, j(\kappa))$ and any limit $\gamma < j(\kappa)$, the embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the elementary chain construction as above, is an initial factor of j and is superstrong for κ.* □

Note again that, for any initial β_0, this procedure gives a strictly increasing and continuous sequence of ordinals $\langle \beta_\xi : \xi < j(\kappa) \rangle$ below $j(\kappa)$, all of which are images of κ under initial factor superstrongness embeddings. As one might already expect, this collection of images is a (full) club in $j(\kappa)$.

Corollary 2.10. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, the collection $D = \{h(\kappa) < j(\kappa) : h \text{ is superstrong for } \kappa\}$ contains a club.*

Proof. We show that the collection of images $j_0(\kappa)$ under initial factor superstrong embeddings arising as above, is a club contained in D. Unboundedness is immediate from our construction. To check closure, let $\delta < j(\kappa)$ be regular and suppose that we have a sequence $\langle j_i(\kappa) : i < \delta \rangle$ where, for each $i < \delta$, $j_i : V \longrightarrow M_i$ is a superstrong initial factor of j arising via the Mostowski collapse π_i of

$$X_i = \{j(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V_{\theta_i}\} \prec M,$$

for some $\theta_i < j(\kappa)$ with $j_i(\kappa) = \theta_i$ (this can be assumed without loss of generality, using Lemma 2.3). We then let $\theta_\delta = \sup_{i<\delta} \theta_i < j(\kappa)$ and

$$X_\delta = \bigcup_{i<\delta} X_i = \{j(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V_{\theta_\delta}\} \prec M.$$

If we now consider the Mostowski collapse $\pi_\delta : X_\delta \cong M_\delta$ and define $j_\delta = \pi_\delta \circ j$, one easily verifies that the embedding j_δ is a superstrong initial factor of j with $j_\delta(\kappa) = \theta_\delta = \sup_{i<\delta} j_i(\kappa)$. □

Exactly as in the case of tallness, for each superstrong initial factor embedding $j_0 : V \longrightarrow M_0$, the derived $(\kappa, j_0(\kappa))$-extender E belongs to $V_{j(\kappa)}$ and then, the corresponding extender embedding $j_E : V \longrightarrow M_E$ is superstrong for κ. Moreover, one can again check that, in fact, $M_0 = M_E$. Finally, by inaccessibility, $V_{j(\kappa)} \models$ "j_E is superstrong for κ" for any such extender and so, in particular, $V_{j(\kappa)} \models$ "κ is superstrong". This means that the collection

$$C_{\text{superstrong}} = \{h_E(\kappa) < j(\kappa) : h_E \text{ is a superstrong extender embedding for } \kappa\},$$

is contained in D, is a definable class of $V_{j(\kappa)}$ and contains a club. Then, by considering the clubs $C^{(n)}_{j(\kappa)} \subseteq j(\kappa)$ consisting of all ordinals below $j(\kappa)$ which are Σ_n-correct in the sense of $V_{j(\kappa)}$, we get the following theorem.

Theorem 2.11. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-superstrong".* □

This theorem gives an upper bound on the consistency strength of the schema SUPERSTRONG, which is a schema of countably-many $\{\in, \mathbf{k}\}$-formulas ϕ_n, with each ϕ_n asserting that "$\mathbf{k}$ is $C^{(n)}$-superstrong". On the other hand, for any fixed n, by Proposition 2.4 in [5], if we assume that there is a κ which is 2^κ-supercompact and with $\kappa \in C^{(n)}$, we then get the existence of many $C^{(n)}$-superstrong cardinals below κ.

Recall that if κ is 1-extendible, then it is superstrong (see, for example, Proposition 26.11 in [30]). In fact, the same proof actually shows that κ is superstrong with inaccessible target and, moreover, there is a normal ultrafilter $\mathcal{U}$ on κ so that

$$\{\alpha < \kappa : \alpha \text{ is superstrong with inaccessible target}\} \in \mathcal{U}.$$

Since κ being 2^κ - supercompact implies the existence of many cardinals below κ which are 1 - extendible, the following corollary is an improvement of the aforementioned consistency bound established in [5] (although it is not a direct implication as in [5]).

Corollary 2.12. *If κ is 1 - extendible then there exists a normal ultrafilter $\mathcal{U}$ on κ, such that $\{\alpha < \kappa : \forall n \in \omega\, (V_\kappa \models \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong”})\} \in \mathcal{U}$. In particular, if the theory* ZFC + “$\exists\, \kappa\, (\kappa$ *is* 1 - *extendible*)” *is consistent, then so is the theory* ZFC + SUPERSTRONG.

Proof. Suppose that κ is 1 - extendible as witnessed by $j : V_{\kappa+1} \longrightarrow V_{j(\kappa)+1}$. If we derive the $(\kappa, j(\kappa))$ - extender E from j and consider the corresponding extender embedding $j_E : V \longrightarrow M_E$, then standard arguments show that j_E is superstrong for κ with $j_E(\kappa) = j(\kappa)$ and, furthermore,

$$V_{j(\kappa)+1} \models \text{“}j_E \text{ is superstrong for } \kappa \text{ with inaccessible target } j_E(\kappa) = j(\kappa)\text{”}$$

(where note that $E \in V_{j(\kappa)+1}$).

Also, $C_{\text{superstrong}}$, the definable stationary subclass of $j(\kappa)$ which was mentioned above, belongs to $V_{j(\kappa)+1}$. Additionally, for every $n \in \omega$, we may consider the club class $C^{(n)}_{j(\kappa)} \subseteq j(\kappa)$ consisting of the ordinals that are Σ_n - correct in $V_{j(\kappa)}$; clearly, $C^{(n)}_{j(\kappa)} \in V_{j(\kappa)+1}$ as well. Of course, since $\mathcal{P}(j(\kappa)) \subseteq V_{j(\kappa)+1}$, the latter verifies the fact that $C_{\text{superstrong}}$ is stationary in $j(\kappa)$. Thus, for every $n \in \omega$,

$$\begin{gathered} V_{j(\kappa)+1} \models \exists\, (\kappa, \theta)\text{-extender } E \in V_{j(\kappa)}, \text{ for some } \theta < j(\kappa), \text{ such that} \\ V_{j(\kappa)} \models \text{“}j_E \text{ is superstrong for } \kappa \text{ and } j_E(\kappa) \in C^{(n)}\text{”}. \end{gathered}$$

That is, for every $n \in \omega$,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is } C^{(n)}\text{-superstrong”}.$$

Now, if we define the normal ultrafilter $\mathcal{U}$ on κ derived from the initial embedding j, it then follows that, for every $n \in \omega$,

$$\{\alpha < \kappa : V_\kappa \models \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong”}\} \in \mathcal{U}.$$

Finally, intersecting all these countably – many sets, by the completeness of $\mathcal{U}$, we get that

$$\{\alpha < \kappa : \forall n \in \omega\, (V_\kappa \models \text{“}\alpha \text{ is } C^{(n)}\text{-superstrong”})\} \in \mathcal{U}.$$

□

We observe that in the previous proof, since κ is itself inaccessible, we may "cut off" the universe at V_κ to get a model of $\mathsf{ZFC}+\mathsf{SUPERSTRONG}$ in which there is actually a proper class of α's that satisfy the schema $\mathsf{SUPERSTRONG}$.

Let us also make a remark on the connection between the case of superstrongness and that of tallness. Notice that if $j : V \longrightarrow M$ is superstrong for κ, we may as well assume that $j = j_E$, i.e., it is an extender superstrong embedding with $M = \{j(f)(x) : f \in V, f : V_\kappa \longrightarrow V, x \in V_{j(\kappa)}\}$. So, if $j(\kappa)$ is also regular (and thus, inaccessible), then the same embedding witnesses $< j(\kappa)$-tallness for κ.

Thus, everything we did for tallness can be entirely done under the context of a superstrong embedding with regular (inaccessible) target. We can therefore state the following, which immediately follows from Theorem 2.7.

Corollary 2.13. *Suppose that $j : V \longrightarrow M$ is superstrong for κ, with $j(\kappa)$ inaccessible. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-tall".* □

Consequently, 1-extendibility is an adequate (consistency) upper bound both for the schema TALL and for the schema $\mathsf{SUPERSTRONG}$. Finally, let us conclude this section by stating the following curiosity.

Proposition 2.14. *If κ is 1-extendible, then there is a normal measure $\mathcal{U}$ on κ so that $\{\alpha < \kappa : \alpha \text{ is superstrong with measurable target}\} \in \mathcal{U}$.*

Proof. Let $j : V_{\kappa+1} \longrightarrow V_{j(\kappa)+1}$ witness the 1-extendibility of κ and let $\mathcal{U}$ be the usual normal ultrafilter derived from j. As in the proof of Corollary 2.12, we may derive the $(\kappa, j(\kappa))$-extender E from the extendibility embedding and then consider the map j_E which is superstrong for κ, with inaccessible target $j_E(\kappa) = j(\kappa)$.

We now restrict our attention to $V_{j_E(\kappa)} = V_{j(\kappa)}$, inside which we do our construction in order to produce the club sequence $\langle \beta_\xi : \xi < j(\kappa) \rangle$ of images of initial factor superstrong embeddings. Recall that all these are witnessed by extenders in $V_{j(\kappa)}$. Moreover, recall that $\{\alpha < \kappa : \alpha \text{ is measurable}\} \in \mathcal{U}$ and, therefore, $\{\alpha < j(\kappa) : \alpha \text{ is measurable}\}$ is a subset of $j(\kappa)$ which is stationary in the sense of $V_{j(\kappa)+1}$. But since $\mathcal{P}(j(\kappa)) \subseteq V_{j(\kappa)+1}$, this is computed correctly and thus there is some superstrong initial factor embedding j_β with $j_\beta(\kappa)$ being measurable. Then, as witnessed by this particular (extender of) j_β,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is superstrong with measurable target”}$$

and it now follows that

$$\{\alpha < \kappa : \alpha \text{ is superstrong with measurable target}\} \in \mathcal{U}.$$

□

As a final comment, let us point out that in Proposition 2.14 one may replace measurability by other (local) notions which follow from 1 - extendibility. For example, we may require that the image of the superstrong embeddings is Woodin: if κ is 1 - extendible, then we know that it is superstrong and, moreover, the set $\{\alpha < \kappa : \alpha \text{ is Woodin}\}$ belongs to the normal measure $\mathcal{U}$. In fact, we may even require that such images are measurable Woodin cardinals, measurable Woodins limits of Woodins, etc.

2.3 Strongness

Suppose that κ is λ - strong, for some limit $\lambda > \kappa$, as witnessed by the embedding $j : V \longrightarrow M$, i.e., M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$. In addition, suppose that $j(\kappa)$ is regular.

We shall apply the same ideas in order to build an elementary chain of substructures of M and then produce a λ - strong factor embedding from it. Two remarks are in order here. First, since – as in the case of superstrongness – we are not interested in closure under sequences for the factor embedding, the length at which we take collapses can be any limit ordinal $\gamma < j(\kappa)$.

Additionally, since the crucial requirement for λ - strongness is "$V_\lambda \subseteq M$", we will start our chain by just "throwing in" all the seeds from V_λ, i.e., we let $\beta_0 = \lambda$ and define our first elementary substructure as

$$X_0 = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V_{\beta_0}\} \prec M.$$

From that point on, there two ways to proceed. One can either consider only ordinal seeds, i.e., for *appropriate limit ordinals* (see below) $\beta_\xi \in (\lambda\,,\, j(\kappa))$, we take substructures of the form

$$X_\xi = \{j(f)(\alpha) : f \in V,\, f : \kappa \longrightarrow V,\, \alpha < \beta_\xi\} \prec M$$

(note the modification in the domain of the functions).

Alternatively, we might as well proceed with the rest of the chain by just continuing in the usual manner, using seeds x from $V^M_{\beta_\xi}$ for $\xi > 0$ (where the β_ξ's are defined as in the case of superstrongness). That would do as well, resulting in the desired λ - strong initial factor embedding after γ steps, for any limit ordinal $\gamma < j(\kappa)$. In this case, there is really not much more to say, as the construction and the subsequent results are totally parallel to the case of superstrongness (in particular, we will get a full club contained in the collection of images $h(\kappa) < j(\kappa)$, where h is a λ - strong embedding for κ).

Here, we take the first approach and use ordinal seeds; we do this for completeness of all the ideas developed so far. Although, intuitively, the idea of just using ordinal seeds once we have thrown in the whole V_λ seems quite clear, there are some fine points related to the fact that the ordinals used have to be closed under Gödel pairing in order for the substructures to be elementary in M. For this reason, we referred to them earlier as "appropriate limit ordinals".

Recall that the Gödel pairing function $\mathbf{\Gamma} : \mathbf{ON} \times \mathbf{ON} \longrightarrow \mathbf{ON}$ is Δ_1 and has, thus, absolute evaluation between transitive models. In particular, transitive models agree on whether a given ordinal is closed under $\mathbf{\Gamma}$. Also, recall that given any ordinal α, we may find an ordinal $\beta > \alpha$ closed under $\mathbf{\Gamma}$, by constructing an appropriate ω-sequence of ordinals greater than α, the limit of which is going to be the desired β. Finally, recall that this procedure does not increase the ordinals "too much", in the sense that for any α, $\sup(\mathbf{\Gamma}''(\alpha \times \alpha)) \leqslant \omega^\alpha$. This will ensure that all ordinals closed under Gödel pairing which we will consider, are bounded below the regular cardinal $j(\kappa)$.

Let us start our elementary chain by the already defined $X_0 \prec M$ and with $\beta_0 = \lambda$. Although the sequence of the β_ξ's for $\xi > 1$ will be defined recursively in a uniform manner, β_1 will be special because it will have to make sure that X_0 fits well into the rest of the chain, i.e., X_0 is included in all the subsequent substructures, although these will come just from ordinal seeds.

So, let β_1 be the least limit ordinal closed under $\mathbf{\Gamma}$ above $|V_\lambda|^M$. Noting that $V_\lambda \subseteq M$ and thus, $|V_\lambda| \leqslant |V_\lambda|^M < j(\kappa)$, our remarks on $\mathbf{\Gamma}$ ensure that indeed $\beta_1 < j(\kappa)$. Then, let

$$X_1 = \{j(f)(\alpha) : f \in V,\, f : \kappa \longrightarrow V,\, \alpha < \beta_1\} \prec M.$$

From this point on, we proceed recursively as follows. Given $\beta_\xi < j(\kappa)$ and X_ξ for some $1 \leqslant \xi < j(\kappa)$, we let $\beta_{\xi+1}$ be the least limit ordinal closed under $\mathbf{\Gamma}$ above $\sup(X_\xi \cap j(\kappa))$ and we then let

$$X_{\xi+1} = \{j(f)(\alpha) : f \in V,\, f : \kappa \longrightarrow V,\, \alpha < \beta_{\xi+1}\} \prec M.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let again $\beta_\xi = \sup\limits_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha = \{j(f)(\alpha) : f \in V,\, f : \kappa \longrightarrow V,\, \alpha < \beta_\xi\}.$$

This concludes the recursive definition of the elementary chain. Let us first check that the generated ordinals stay bounded below $j(\kappa)$.

Claim. *For every $\xi \geqslant 1$, $\beta_\xi < j(\kappa)$.*

Proof of claim. We proceed inductively noting that we have already seen that $\beta_1 < j(\kappa)$. Also, the limit case follows from regularity of $j(\kappa)$. So suppose that for some $\xi \geqslant 1$, we have that $\beta_\xi < j(\kappa)$. Let us first check that $|X_\xi \cap j(\kappa)| < j(\kappa)$. But this follows, as before, by a counting argument noting that ${}^{\kappa}2 \subseteq M$ and thus, $2^\kappa \leqslant (2^\kappa)^M < j(\kappa)$, from which we get $|X_\xi \cap j(\kappa)| < j(\kappa)$ again by the regularity of $j(\kappa)$. Hence, $\sup(X_\xi \cap j(\kappa)) < j(\kappa)$ and, then, by our remarks on ordinals closed under $\mathbf{\Gamma}$, we get that $\beta_{\xi+1} < j(\kappa)$. □

Furthermore, we need to check that the first substructure of the chain (generated by seeds in V_λ) fits coherently with the rest of the substructures, which are generated just from ordinal seeds.

Claim. $X_0 \subseteq X_1$.

Proof of claim. Fix a bijection $\pi : \kappa \longrightarrow V_\kappa$ with the property that, for every $\alpha < \kappa$, $\pi \restriction |V_\alpha| : |V_\alpha| \longrightarrow V_\alpha$ is also a bijection (this can be constructed recursively using the inaccessibility of κ). Now suppose that we are given some element $z = j(f)(x) \in X_0$, where $f : V_\kappa \longrightarrow V$ is a function in V and $x \in V_\lambda$.

Define the function $g = f \circ \pi : \kappa \longrightarrow V$ $(g \in V)$ and let $\alpha = (j(\pi))^{-1}(x)$. By elementarity, the bijective function $j(\pi) : j(\kappa) \longrightarrow V^M_{j(\kappa)}$ has the property that $j(\pi) \restriction |V_\lambda|^M : |V_\lambda|^M \longrightarrow V_\lambda$ is also a bijection, since $\lambda < j(\kappa)$. Thus, as $x \in V_\lambda$, it follows that $(j(\pi))^{-1}(x) = \left(j(\pi) \restriction |V_\lambda|^M\right)^{-1}(x) \in |V_\lambda|^M < \beta_1$, i.e., we have just shown that $\alpha < \beta_1$. But then, by elementarity,

$$j(g)(\alpha) = j(f)\,(j(\pi)(\alpha)) = j(f)\left(j(\pi)((j(\pi))^{-1}(x))\right) = j(f)(x) = z\,,$$

which means that $z \in X_1$, completing the argument. □

Having established these facts, it follows that the defined X_ξ's indeed form an increasing (under $\subseteq$) elementary chain in M and so, at any limit $\gamma < j(\kappa)$ we may consider the current substructure X_γ of which we take the Mostowski collapse $\pi_\gamma : X_\gamma \cong M_\gamma$. We then define the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$ with $cp(j_\gamma) = \kappa$, producing a commutative diagram of elementary embeddings (with $k_\gamma = \pi_\gamma^{-1}$), where $\beta_\gamma = \sup\limits_{\xi<\gamma} \beta_\xi$ and

$$X_\gamma = \{j(f)(\alpha) :\ f \in V,\ f : \kappa \longrightarrow V,\ \alpha < \beta_\gamma\} \prec M.$$

By the regularity of $j(\kappa)$, we get that $\beta_\gamma < j(\kappa)$; moreover, one again checks that $X_\gamma \cap j(\kappa)$ is an ordinal, which in turn implies that j_γ is indeed an initial factor of j and then

$$cp(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \sup_{\xi<\gamma} \beta_\xi = \beta_\gamma > \lambda\,.$$

Clearly, $V_\lambda \subseteq M_\gamma$ and we have just shown the following.

Proposition 2.15. *Suppose that $j : V \longrightarrow M$ is λ-strong for κ, for some limit $\lambda > \kappa$, with $j(\kappa)$ regular. Then, for any limit $\gamma < j(\kappa)$, the embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the chain construction as above, is a λ-strong initial factor of j.* □

As in the superstrongness case, this procedure results in a collection of images $j_0(\kappa)$ of λ-strong initial factor embeddings, which contains a club.

Corollary 2.16. *Suppose that $j : V \longrightarrow M$ is λ-strong for κ, for some limit $\lambda > \kappa$, with $j(\kappa)$ regular. Then, $D = \{h(\kappa) < j(\kappa) : h \text{ is } \lambda\text{-strong for } \kappa\}$ contains a club.* □

Moreover, for each λ-strong initial factor $j_0 : V \longrightarrow M_0$, the derived $(\kappa, j_0(\kappa))$-extender E belongs to $V_{j(\kappa)}$ and, also, one easily checks that the corresponding extender embedding $j_E : V \longrightarrow M_E$ is λ-strong for κ. Hence, using the fact that $j(\kappa)$ is regular and above all the relevant information, we get that $V_{j(\kappa)} \models$ "j_E is λ-strong for κ" and so, in particular, we also have that $V_{j(\kappa)} \models$ "κ is λ-strong". This means that

$$C_{\lambda\text{-strong}} = \{h_E(\kappa) < j(\kappa) : h_E \text{ is a } \lambda\text{-strong extender embedding for } \kappa\},$$

is a definable class of $V_{j(\kappa)}$ which contains a club.

If, in addition, we had assumed that $j(\kappa)$ is inaccessible, then we would also get that, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is λ-$C^{(n)}$-strong". Furthermore, as in the cases of tallness and of superstrongness, 1-extendibility is an adequate (consistency) upper bound for the schema STRONG asserting full $C^{(n)}$-strongness, for every n.

Such a conclusion is hardly surprising in the light of Proposition 1.8, which asserts that every λ-strong cardinal is actually λ-$C^{(n)}$-strong; this was obtained in [5] by the method of iterated ultrapowers.

Let us now discuss how the latter method makes the notions of Woodin and of strongly compact cardinals fit into the general context of the $C^{(n)}$-hierarchies.

2.4 Woodinness and Strong Compactness

This section can be thought of as methodologically parenthetical, in the sense that no elementary chain constructions are present here. Nevertheless, we treat the $C^{(n)}$-versions of Woodin and strongly compact cardinals together, for two reasons.

Firstly, they were not considered in [5] where the rest of the $C^{(n)}$-cardinals were introduced; but most importantly, because they both admit similar constructions using the technique of iterated ultrapowers, which completely describe their connection with the notions of Woodin and strongly compact cardinals, respectively.

2.4.1 $C^{(n)}$ - Woodins

We start by recalling our Definition 1.16 of a $C^{(n)}$ - Woodin cardinal.

A cardinal δ is called $C^{(n)}$ - **Woodin** if for every $f \in {}^{\delta}\delta$, there is a $\kappa < \delta$ with $f''\kappa \subseteq \kappa$, and there is an elementary embedding $j : V \longrightarrow M$, with M transitive, $cp(j) = \kappa$, $V_{j(f)(\kappa)} \subseteq M$, $j(\delta) = \delta$ and $j(\kappa) \in C^{(n)}$.

It is not difficult to check that, given a cardinal δ and some n, the statement "δ is $C^{(n)}$ - Woodin" is absolute for any $V_{\delta'}$ with $\delta' > \delta$ and $\delta' \in C^{(n)}$, i.e., for such a cardinal δ', $V_{\delta'} \models$ "δ is $C^{(n)}$ - Woodin" if and only if δ is indeed $C^{(n)}$ - Woodin. Notice also that if the cardinal δ is $C^{(n)}$ - Woodin then (δ is of course Woodin and) $\delta \in Lim(C^{(n)})$. As we shall soon see, this is no coincidence.

With these comments in mind, we now show that $C^{(n)}$ - Woodins form a large cardinal hierarchy of increasing strength.

Lemma 2.17. *For $n \geqslant 1$, if δ is $C^{(n+1)}$ - Woodin then there are unboundedly many $C^{(n)}$ - Woodins below δ. Hence, if δ is the least $C^{(n)}$ - Woodin then it is not $C^{(n+1)}$ - Woodin.*

Proof. Fix some $n \geqslant 1$ and suppose that δ is $C^{(n+1)}$ - Woodin. We further fix an $\alpha < \delta$ and we want to find some $C^{(n)}$ - Woodin cardinal between α and δ.

Let $f \in {}^{\delta}\delta$ be such that $f''\delta \cap \alpha = \varnothing$ and let $\kappa < \delta$ and $j : V \longrightarrow M$ witness $C^{(n+1)}$ - Woodinness with respect to f, i.e., $f''\kappa \subseteq \kappa$, $cp(j) = \kappa$, $V_{j(f)(\kappa)} \subseteq M$, $j(\delta) = \delta$ and $j(\kappa) \in C^{(n+1)}$. Clearly, $\alpha < \kappa < j(\kappa) < \delta$ and, moreover, the following is a true Σ_{n+1} - statement in the parameter α:

$$\exists \delta' \, (\delta' \in C^{(n)} \wedge V_{\delta'} \models \text{“}\exists \text{ some } C^{(n)}\text{ - Woodin above } \alpha\text{”}),$$

since it holds for any $\delta' \in C^{(n)}$ above δ. Therefore, it must hold in $V_{j(\kappa)}$, i.e., there is some $\delta' < j(\kappa)$ so that

$$V_{j(\kappa)} \models (\delta' \in C^{(n)} \wedge V_{\delta'} \models \text{“}\exists \text{ some } C^{(n)}\text{ - Woodin above } \alpha\text{”}).$$

But now, since $j(\kappa) \in C^{(n+1)}$, we indeed have that $\delta' \in C^{(n)}$ and then, clearly, $V_{\delta'} \models$ "$\exists$ some $C^{(n)}$ - Woodin above α". If $\beta \in V_{\delta'}$ is a witness to the latter statement, then by our earlier remarks it follows that β is a $C^{(n)}$ - Woodin cardinal above α. □

The reader might have noticed that the case $n = 0$ is conspicuously missing from Lemma 2.17. The following proposition explains why.

Proposition 2.18. *If δ is Woodin then it is $C^{(1)}$ - Woodin.*

Proof. Suppose that δ is Woodin and fix some function $f \in {}^{\delta}\delta$. We further fix $\kappa < \delta$ with $f''\kappa \subseteq \kappa$ and a (κ, β) - extender $E \in V_\delta$ (for some $\beta < \delta$), such that $j = j_E : V \longrightarrow M_E$ has $cp(j) = \kappa$, $\beta \leqslant j(\kappa) < \delta$, $j(\delta) = \delta$ and $V_{j(f)(\kappa)} \subseteq M_E$.

We shall find an elementary embedding $i : V \longrightarrow M'$ (some transitive M') so that $cp(i) = \kappa$, i still witnesses Woodinnes for δ with respect to the given function and, moreover, $i(\kappa) < \delta$, $i(\delta) = \delta$ and $i(\kappa) \in C^{(1)}$. For this, we use iterated ultrapowers.

We work in M_E. Since $j(\kappa)$ is measurable, let $\mathcal{U} \in M_E$ be a normal, $j(\kappa)$-complete measure on $j(\kappa)$ and let $j_{\mathcal{U}} : M_E \longrightarrow M$ be the ultrapower embedding with critical point $j(\kappa)$. Then, $2^{j(\kappa)} < j_{\mathcal{U}}(j(\kappa)) < \left(2^{j(\kappa)}\right)^+ < \delta$, where the last inequality comes from $j(\kappa) < \delta = j(\delta)$ and the fact that δ is inaccessible.

Now let $\alpha \in (j_{\mathcal{U}}(j(\kappa)), \delta)$ be a true $C^{(1)}$-cardinal above $\left(2^{j(\kappa)}\right)^{M_E}$ (that is, α is $C^{(1)}$ in V). Still working in M_E, we now iterate the map $j_{\mathcal{U}}$ α-many times and let $j_\alpha : M_E \longrightarrow M_\alpha \cong \mathrm{Ult}(M_E, \mathcal{U}_\alpha)$ be the resulting embedding. By standard facts regarding such iterations, we get that $cp(j_\alpha) = j(\kappa)$, $V_{j(\kappa)}^{M_E} \subseteq M_\alpha$, $j_\alpha(j(\kappa)) = \alpha$, $j_\alpha(\delta) = \delta$ (see Corollary 19.7 in [30]) and the iterates are well-founded, so M_α is (taken to be) transitive. We then let $i = j_\alpha \circ j : V \longrightarrow M_\alpha$ be the composed elementary embedding with $cp(i) = \kappa$, $i(\kappa) = j_\alpha(j(\kappa)) = \alpha \in C^{(1)}$ and $i(\delta) = \delta$. It remains to see that i indeed witnesses Woodinness for δ with respect to the given function f.

For this, observe that $i(f)(\kappa) = j_\alpha(j(f))(\kappa) = j_\alpha(j(f))(j_\alpha(\kappa)) = j_\alpha(j(f)(\kappa))$. But since $j(f)(\kappa) < j(\kappa)$ (which follows from the fact that $f''\kappa \subseteq \kappa$) we get that $i(f)(\kappa) = j(f)(\kappa)$ and, therefore, $V_{i(f)(\kappa)} \subseteq M_\alpha$, since $V_{j(\kappa)}^{M_E} \subseteq M_\beta$, for all $\beta \leqslant \alpha$ along the iteration. □

A straightforward modification of the previous proof gives the following.

Corollary 2.19. *If δ is Woodin and $\delta \in Lim(C^{(n)})$, then δ is $C^{(n)}$ - Woodin.* □

Let us point out that this last corollary, together with the discussion before Lemma 2.17, jointly provide us with a characterization of $C^{(n)}$ - Woodin cardinals. In fact, we shall give several equivalent formulations of $C^{(n)}$ - Woodinness; before that, though, one more definition is in order. As usual, n stands for any natural number.

Definition 2.20. *Let κ be a cardinal, let $\lambda \geqslant \kappa$, and let A be any set. We say that κ is λ - $C^{(n)}$ -* **strong for** *A if there is an elementary embedding $j : V \longrightarrow M$ with M transitive, such that $cp(j) = \kappa$, $\lambda < j(\kappa)$, $V_\lambda \subseteq M$, $A \cap V_\lambda = j(A) \cap V_\lambda$ and $j(\kappa) \in C^{(n)}$.*

We are now ready for the next theorem which is based upon, and analogue to Woodin's original result (see Theorem 26.14 in [30]).

Theorem 2.21. *The following are equivalent*:

(i) *δ is a $C^{(n)}$ - Woodin cardinal.*

(ii) *δ is Woodin and $\delta \in Lim(C^{(n)})$.*

(iii) *For every $A \subseteq V_\delta$, the set*

$$S_A^{(n)} = \{\alpha < \delta : \alpha \text{ is } \gamma\text{-}C^{(n)}\text{-strong for } A, \text{ for every } \gamma < \delta\}$$

is stationary in δ.

(iv) *For every $f \in {}^\delta\delta$, there is a $\kappa < \delta$ with $f''\kappa \subseteq \kappa$ and an extender $E \in V_\delta$, so that $cp(j_E) = \kappa$, $j_E(f)(\kappa) = f(\kappa)$, $V_{f(\kappa)} \subseteq M_E$ and $j_E(\kappa) \in C^{(n)}$ (i.e., j_E witnesses $C^{(n)}$ - Woodinness with respect to f).*

Proof. The equivalence of (i) and (ii) is, as already noted, the combination of earlier remarks with Corollary 2.19. Moreover, it is obvious that (iv) implies (i). Let us first deal with the implication $(i) \Longrightarrow (iii)$.

Suppose that δ is a $C^{(n)}$-Woodin cardinal (for some fixed n) and let $A \subseteq V_\delta$ be given. By Woodin's theorem, we know that the set

$$S_A = \{\alpha < \delta : \alpha \text{ is } \gamma\text{-strong for } A, \text{ for every } \gamma < \delta\}$$

is stationary in δ. We will show that $S_A \subseteq S_A^{(n)}$ which is sufficient. So fix some $\alpha \in S_A$ and some $\gamma < \delta$. We want to show that α is, in fact, γ - $C^{(n)}$-strong for A, given that α is γ-strong for A.

Let $j : V \longrightarrow M$ witness the latter, i.e., $cp(j) = \alpha$, $\gamma < j(\alpha)$, $V_\gamma \subseteq M$ and $A \cap V_\gamma = j(A) \cap V_\gamma$. We may assume that $j(\alpha) < \delta = j(\delta)$ because if not, we may derive some $(\alpha, |V_\beta|^+)$-extender E (for some sufficiently large $\beta \in (\gamma, \delta)$) and work with j_E in place of j, noticing that j_E still witnesses γ-strongness for A and has $j_E(\alpha) < \delta = j_E(\delta)$.

Having fixed such an (extender) embedding as in the last paragraph, we use again an iterated ultrapower argument. Since $\delta \in Lim(C^{(n)})$, we may pick some $\lambda > \left(2^{j(\alpha)}\right)^M$ with $\lambda \in C^{(n)}$ and – now working in M – construct the iterated ultrapower embedding $j_\lambda : M \longrightarrow M_\lambda$ with $cp(j_\lambda) = j(\alpha)$, $j_\lambda(j(\alpha)) = \lambda$ and $j_\lambda(\delta) = \delta$, and with all the iterates being well-founded (so M_λ is taken to be transitive).

We then let $i = j_\lambda \circ j : V \longrightarrow M_\lambda$ be the composed elementary embedding with $cp(i) = \alpha$, $\gamma < i(\alpha) = \lambda \in C^{(n)}$ and $i(\delta) = \delta$. Moreover, we have that $V_\gamma \subseteq M_\lambda$

because $V_\gamma \subseteq V^M_{j(\alpha)} \subseteq M_\lambda$. In order to check that this embedding witnesses γ - $C^{(n)}$ - strongness for A, it remains to see that $A \cap V_\gamma = i(A) \cap V_\gamma$. For this, we have the following string of equalities:

$$i(A) \cap V_\gamma = j_\lambda(j(A)) \cap V_\gamma = j_\lambda(j(A) \cap V_\gamma) = j_\lambda(A \cap V_\gamma) = A \cap V_\gamma\,,$$

where $j_\lambda(V_\gamma) = V_\gamma$ because $V_\gamma \in V^M_{j(\alpha)}$ and similarly for the last equality. This concludes the proof of $(i) \Longrightarrow (iii)$.

It is important to note that in the argument described above, given any $A \subseteq V_\delta$, any $\gamma < \delta$ and any $\alpha \in S_A$, the embedding i which witnesses the γ - $C^{(n)}$ - strongness for A of α, can be taken so that $i(\alpha) < \delta = i(\delta)$ and, in fact, to be an extender embedding (simply derive the obvious $(\alpha, i(\alpha))$ - extender).

With these remarks in mind, the final implication $(iii) \Longrightarrow (iv)$ is an immediate consequence of the corresponding one in Woodin's theorem. □

Let us now turn to the case of $C^{(n)}$ - strongly compact cardinals.

2.4.2 $C^{(n)}$ - strongly compacts

Recall that a cardinal κ is γ - compact, for some $\gamma \geqslant \kappa$, if and only if there is a fine measure on $\mathcal{P}_\kappa\gamma$. The latter is equivalent to the existence of an elementary embedding $j : V \longrightarrow M$ with M transitive, $cp(j) = \kappa$ and with the property that, for any $X \subseteq M$ with $|X| \leqslant \gamma$, there is a $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$. We then say that κ is strongly compact if it is γ - compact for every $\gamma \geqslant \kappa$.

We use the method of iterated ultrapowers in order to show that every strongly compact cardinal is actually $C^{(n)}$ - strongly compact (cf. Definition 1.17).

Theorem 2.22. *Suppose that for some $\gamma \geqslant \kappa$, κ is γ - compact. Then, κ is γ - $C^{(n)}$ - compact.*

Proof. We fix some n and some $\gamma \geqslant \kappa$ and we let $j : V \longrightarrow M$ be an elementary embedding witnessing the γ - compactness of κ. Clearly, $j(\kappa) > \gamma$ and $j(\kappa)$ is measurable in M.

Let $\mathcal{U} \in M$ be an M - normal measure on $j(\kappa)$ and let us fix some ordinal $\alpha \in C^{(n)}$ such that $cf(\alpha) > \gamma$ and $\alpha > \left(2^{j(\kappa)}\right)^M$. We iterate the ultrapower construction inside M, starting with $\mathcal{U}$ and repeating for α - many steps. Let $j_\alpha : M \longrightarrow M_\alpha$ be the resulting embedding with $cp(j_\alpha) = j(\kappa)$, where M_α is (taken to be) transitive by the well – foundedness of all the structures along the iteration. Again, by known facts regarding such iterations, we have that $j_\alpha(j(\kappa)) = \alpha$ and we let $h : V \longrightarrow M_\alpha$ be the composed elementary embedding, i.e., $h = j_\alpha \circ j$, with $cp(h) = \kappa$ and with $h(\kappa) = \alpha \in C^{(n)}$. We now check that h is γ - compact for κ.

Suppose that $X \subseteq M_\alpha$ and, without loss of generality, suppose $|X| = \gamma$ (in particular, γ is a cardinal). By the representation of iterated utrapowers (see Lemma 19.6 in [30]) we may assume that each $z \in X$ is of the form $j_\alpha(f)(\kappa_{\xi_1}, \ldots, \kappa_{\xi_m})$, where $f : [j(\kappa)]^m \longrightarrow M$ is a function that belongs to M and, for each $1 \leqslant i \leqslant m$, κ_{ξ_m} is an element along the critical sequence $\langle \kappa_\xi : \xi < \alpha \rangle$ (where $\kappa_0 = j(\kappa)$). Thus, we have the representation

$$X = \{ j_\alpha(f_i)(\vec{\kappa}_i) : i < \gamma \}$$

where, for each $i < \gamma$, $f_i \in M$ is a function with domain (included in) $[j(\kappa)]^{<\omega}$ and $\vec{\kappa}_i$ is some finite tuple of ordinals along the critical sequence.

Now, since $cf(\alpha) > \gamma$, there exists some ordinal $\delta < \alpha$ such that, for every $i < \gamma$, $max(\vec{\kappa}_i) < \kappa_\delta$. In other words, since there are, in total, γ - many κ_ξ's involved in the representation of the elements of X, there is some stage δ of the iteration so that the current critical point κ_δ is above them all. Evidently, $[\kappa_\delta]^{<\omega} \in M_\alpha$ and $M_\alpha \models |[\kappa_\delta]^{<\omega}| < \alpha$.

Moreover, since $\{f_i : i < \gamma\} \subseteq M$ and j is γ - compact, there is some $Y_0 \in M$ with $\{f_i : i < \gamma\} \subseteq Y_0$ and $M \models |Y_0| < j(\kappa)$. By elementarity, it then follows that $\{j_\alpha(f_i) : i < \gamma\} \subseteq j_\alpha(Y_0) \in M_\alpha$ and $M_\alpha \models |j_\alpha(Y_0)| < \alpha$. Therefore, in M_α, we may use the set $j_\alpha(Y_0)$ and the set $[\kappa_\delta]^{<\omega}$ in order to define the desired Y that covers X. We let

$$Y = \{ g(\vec{s}) : \text{for some } m \in \omega,\ g \in j_\alpha(Y_0) \text{ is a function on } [\alpha]^m \text{ and } \vec{s} \in [\kappa_\delta]^m \}$$

and we then have that $Y \in M_\alpha$, $X \subseteq Y$ and $M_\alpha \models |Y| < \alpha$ which concludes the proof. □

From the previous theorem, we immediately get the following characterization.

Corollary 2.23. *κ is strongly compact $\Longleftrightarrow$ κ is $C^{(n)}$ - strongly compact.* □

Summarizing, we have obtained characterizations of $C^{(n)}$ - Woodin and $C^{(n)}$ - strongly compact cardinals which, in addition, do not seem to leave much space for investigating further these notions in their own right, as they reduce them to their ordinary counterparts.

We now move on to the cases of supercompact and of extendible cardinals where things are much more subtle and interesting. Apropos, this also means that the methodological parenthesis is, effectively, closed and we can now return to our familiar elementary chain arguments.

2.5 Supercompactness

Suppose that κ is λ-supercompact, for some $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$, as witnessed by the embedding $j : V \longrightarrow M$, i.e., M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$. In addition, suppose that $j(\kappa)$ is regular.

In this case, as opposed to the cases of tallness and of (super)strongness, we build our elementary chain using a slightly different collection of seeds. Namely, we also include $j''\lambda$ which (belongs to M and) will serve as the "prototype" λ-sequence; we shall then use it in order to obtain the closure under sequences for the final (initial factor) embedding that we are aiming for.

Let us start by picking some initial limit ordinal $\beta_0 \in (\lambda, j(\kappa))$ and by letting

$$X_0 = \{j(f)(j''\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_0}\} \prec M,$$

where note that the domain of the functions has been modified accordingly. For any $\xi + 1 < j(\kappa)$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\kappa)) + \omega$ and

$$X_{\xi+1} = \{j(f)(j''\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_{\xi+1}}\}.$$

If $\xi < j(\kappa)$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup\limits_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha = \{j(f)(j''\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_\xi}\},$$

which concludes the recursive definition of the elementary chain.

We observe that M is actually closed under $\lambda^{<\kappa}$-sequences and thus, any function of the form $f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow \kappa$ belongs to M. Therefore, by the inaccessibility of $j(\kappa)$ in M, $|2^{\lambda^{<\kappa}}| < j(\kappa)$. Now, using the regularity of $j(\kappa)$, a counting argument shows that for each $\xi < j(\kappa)$, $\beta_\xi < j(\kappa)$.

At any limit ordinal $\gamma < j(\kappa)$ with $cf(\gamma) > \lambda$, we consider the current substructure X_γ of which we take the Mostowski collapse $\pi_\gamma : X_\gamma \cong M_\gamma$. We then define the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, with $cp(j_\gamma) = \kappa$, producing a commutative diagram of elementary embeddings (with $k_\gamma = \pi_\gamma^{-1}$), where $\beta_\gamma = \sup\limits_{\xi<\gamma} \beta_\xi$ and

$$X_\gamma = \{j(f)(j''\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_\gamma}\} \prec M.$$

One again checks that $X_\gamma \cap j(\kappa)$ is an ordinal, which implies that j_γ is indeed an initial factor of j and that

$$cp(k_\gamma) = j_\gamma(\kappa) = \sup(X_\gamma \cap j(\kappa)) = \sup_{\xi<\gamma} \beta_\xi = \beta_\gamma > \lambda,$$

with $cf(\beta_\gamma) = cf(\gamma) > \lambda$. Therefore, in order to conclude that the embedding j_γ is λ-supercompact, we only need to check that ${}^\lambda M_\gamma \subseteq M_\gamma$. For this, note first that since $j''\lambda \cup \{j''\lambda\} \subseteq X_0 \subseteq X_\gamma$, we obtain that

$$M_\gamma = \{j_\gamma(f)(j_\gamma{}''\lambda, x) : f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\ x \in V^M_{\beta_\gamma}\}.$$

Clearly, we also have that $j_\gamma{}''\lambda \in M_\gamma$ and, therefore, if we consider the map $j_\gamma \restriction \lambda : \lambda \longrightarrow j_\gamma{}''\lambda$ (as an order – type function), we get that $j_\gamma \restriction \lambda \in M_\gamma$ as well. We now use this function in order to show that M_γ is closed under λ-sequences.

Let $\{j_\gamma(f_i)(j_\gamma{}''\lambda, x_i) : i < \lambda\} \subseteq M_\gamma$, where for $i < \lambda$, $x_i \in V^M_{\beta_\gamma}$ and $f_i \in V$. Since $cf(\beta_\gamma) = cf(\gamma) > \lambda$ and ${}^\lambda M \subseteq M$, we have that $\langle x_i : i < \lambda \rangle \in V^M_{\beta_\gamma} \subseteq M_\gamma$. It will be enough to show that $\langle j_\gamma(f_i) : i < \lambda \rangle \in M_\gamma$ as well, since in that case, we can compute in M_γ the sequence $\langle j_\gamma(f_i)(j_\gamma{}''\lambda, x_i) : i < \lambda \rangle$ by evaluating pointwise the functions $j_\gamma(f_i)$'s at the corresponding x_i's together with $j_\gamma{}''\lambda$.

Now, $j_\gamma(\langle f_i : i < \lambda \rangle)$ is a function $G : j_\gamma(\lambda) \longrightarrow M_\gamma$ that belongs to M_γ. Using G and $j_\gamma \restriction \lambda$, define in M_γ the function $F : \lambda \longrightarrow M_\gamma$ by letting, for every $\alpha < \lambda$, $F(\alpha) = G(j_\gamma(\alpha))$. But then, for every $\alpha < \lambda$,

$$F(\alpha) = j_\gamma(\langle f_i : i < \lambda \rangle)(j_\gamma(\alpha)) = j_\gamma(\langle f_i : i < \lambda \rangle(\alpha)) = j_\gamma(f_\alpha)\,,$$

i.e., $F = \langle j_\gamma(f_i) : i < \lambda \rangle \in M_\gamma$ and we are done. We have thus shown the following.

Proposition 2.24. *Suppose that $j : V \longrightarrow M$ is λ-supercompact for κ, for some $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$ and with $j(\kappa)$ regular. Then, for each (initial limit ordinal) $\beta_0 \in (\lambda, j(\kappa))$ and each $\gamma < j(\kappa)$ with $cf(\gamma) > \lambda$, the embedding $j_\gamma : V \longrightarrow M_\gamma$ arising from the elementary chain construction as above, is an initial factor of j witnessing λ-supercompactness of κ.* □

By our usual methods, one obtains the following corollary.

Corollary 2.25. *Suppose that $j : V \longrightarrow M$ witnesses the λ-supercompactness of κ, for some $\lambda > \kappa$ with $\lambda^{<\kappa} = \lambda$ and with $j(\kappa)$ regular. Then, the collection*

$$D = \{h(\kappa) < j(\kappa) : h \textit{ is } \lambda\textit{-supercompact for } \kappa\}$$

contains a $[|\lambda|^+, j(\kappa))$-club. □

Unlike the cases of tallness and of (super)strongness where the various initial factor embeddings were witnessed inside $V_{j(\kappa)}$ by (short) derived extenders, in the case of supercompactness such extenders are not sufficient for our purposes and, so, the situation seems, at first sight, problematic in this respect.

Of course, normal measures are also out of the question: recall that for every ultrapower embedding j_U coming from a normal fine measure U on $\mathcal{P}_\kappa\lambda$, we have that $2^{\lambda^{<\kappa}} \leqslant (2^{\lambda^{<\kappa}})^M < j_U(\kappa) < (2^{\lambda^{<\kappa}})^+$ (see Proposition 22.11 in [30]); therefore, for sufficiently large initial limit ordinals β_0 with $(2^{\lambda^{<\kappa}})^+ < \beta_0 < j(\kappa)$, our constructed j_γ will be far from represented by such measures.

To overcome these limitations and capture supercompactness embeddings, one option is to turn to (quite) long extenders; another would be to use an alternative "combinatorial" reformulation of λ-supercompactness in terms of Martin – Steel extenders, which are generalizations of the ordinary extender notion having as their support any prescribed transitive set Y.

For the moment, we choose the second option and develop our ideas according to the generality of Martin – Steel extenders. Thus, in what follows and unless otherwise stated, the term "extender" refers to the Martin – Steel form (this will be consistently indicated by the presence of some "Y" as the support set). Towards the end of the current chapter (cf. Corollary 2.37) we shall also comment on how ordinary (albeit long) extenders may be used in order to describe λ-supercompactness embeddings.

Those readers who are not familiar with the theory of Martin – Steel extenders are encouraged to consult Section A.3 of the Appendix. There, apart from various details on such objects and their properties, a proof of the following theorem may also be found (cf. Theorem A.14).

Theorem 2.26. *A cardinal κ is λ-supercompact if and only if there exists a (κ, Y)-extender E, with Y transitive, such that $\{\kappa\} \cup [Y]^{<\omega} \cup {}^\lambda Y \cup j_E{}''Y \subseteq Y$ and $j_E(\kappa) > \lambda$, where j_E is the extender elementary embedding.* □

In fact, as shown in Section A.3 of the Appendix, given such a (κ, Y)-extender E, the corresponding embedding j_E is λ-supercompact. Conversely, given any λ-supercompact embedding, there is way to extract from it an appropriate transitive set Y so that the derived (κ, Y)-extender meets all the displayed requirements. Moreover, in this case, $j_E(\kappa) = j(\kappa)$.

Note that the above characterization can be formulated as a Σ_2-statement about κ and λ, since one can require the existence of some (thought of as sufficiently large) ordinal $\alpha > \lambda$, so that $E \in V_\alpha$ and all the clauses regarding the extender embedding j_E are faithfully verified inside V_α.

Returning to our chain construction and as far as the representation of initial factor embeddings inside $V_{j(\kappa)}$ is concerned, it would thus be sufficient to require, in addition, that $j(\kappa)$ is an inaccessible member of $C^{(2)}$. In such a case, for any initial factor λ-supercompact embedding j_β arising from our construction, the fact that there is some (κ, Y)-extender witnessing its λ-supercompactness will be reflected inside $V_{j(\kappa)}$ and hence we will be able to conclude that, for every $n \in \omega$, κ is λ-$C^{(n)}$-supercompact inside $V_{j(\kappa)}$.

Going one step further, we would like to witness full $C^{(n)}$-supercompactness, not just locally for a fixed λ. Contemplating momentarily on the way we produce the elementary chain, it is clear that the essential feature which guarrantees closure under λ-sequences is the fact that $j''\lambda \in M$; this enables us to define the appropriate substructures in the first place. In other words, the fact that we pick the length of the chain to have cofinality strictly above λ is a secondary issue since, in the absence of $j''\lambda$, we would not even be able to start the appropriate construction.

Therefore, in order to witness full $C^{(n)}$-supercompactness below the inaccessible $j(\kappa)$ by an elementary chain construction, it would be enough if the initial embedding were such that $j''\alpha \in M$ for every $\alpha < j(\kappa)$. Hence, it suffices to take j an *almost huge* embedding (in which case, $j(\kappa)$ is inaccessible). As we will shortly see, this assumption already implies the existence of the relevant extenders inside $V_{j(\kappa)}$, i.e., requiring that $j(\kappa) \in C^{(2)}$ is unnecessary. Putting all these ideas together, we now show the following.

Theorem 2.27. *Suppose that $j : V \longrightarrow M$ witnesses the almost hugeness of κ. Then, for every $n \in \omega$, $V_{j(\kappa)} \models$ "κ is $C^{(n)}$-supercompact".*

Proof. Suppose that κ is almost huge, as witnessed by the elementary embedding $j : V \longrightarrow M$, i.e., M transitive, $cp(j) = \kappa$ and ${}^{<j(\kappa)}M \subseteq M$. In particular, $V_{j(\kappa)} \subseteq M$ and $j(\kappa)$ is inaccessible. As we are aiming towards full $C^{(n)}$-supercompactness below $j(\kappa)$, we may as well consider only regular $\lambda < j(\kappa)^\flat$. Then, for any such fixed λ, since ${}^{\lambda}M \subseteq M$ and $j''\lambda \in M$, we may pick some initial limit ordinal $\beta_0^{(\lambda)} \in (\lambda, j(\kappa))$ and perform our construction exactly as in the λ-supercompactness case, i.e., construct, for any $\xi < j(\kappa)$, the corresponding limit ordinal $\beta_\xi^{(\lambda)} < j(\kappa)$ and the substructure

$$X_\xi^{(\lambda)} = \{ j(f)(j''\lambda, x) : f \in V,\, f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\, x \in V_{\beta_\xi^{(\lambda)}} \} \prec M,$$

producing the elementary chain $X_0^{(\lambda)} \prec \ldots \prec X_\xi^{(\lambda)} \prec \ldots \prec M$. As before, at any limit $\gamma < j(\kappa)$ with $cf(\gamma) \geqslant \lambda^+$, we may pause and take the transitive collapse of the current substructure, in order to produce a λ-supercompact initial factor embedding $j_\gamma^{(\lambda)}$ with $j_\gamma^{(\lambda)}(\kappa) = \beta_\gamma^{(\lambda)}$, where $\beta_\gamma^{(\lambda)}$ is the current ordinal of the produced sequence $\langle \beta_\xi^{(\lambda)} : \xi < j(\kappa) \rangle$.

Thus, for any fixed regular $\lambda \in (\kappa, j(\kappa))$, we produce a corresponding class of ordinals in $j(\kappa)$, namely, $C_\lambda = \{ \beta_\gamma^{(\lambda)} : cf(\gamma) = \lambda^+ \}$ which consists of the

$^\flat$ And since κ is certainly supercompact in the model $V_{j(\kappa)}$, it follows by a result of Solovay (see, for example, Lemma 20.11 in [28]) that, for any regular $\lambda \in (\kappa, j(\kappa))$, we have $\lambda^{<\kappa} = \lambda$.

images of κ under the initial factor λ-supercompactness embeddings that are taken exactly at limits of cofinality λ^+. Of course, each C_λ is λ^+-club and thus stationary in $j(\kappa)$.

Now, we may let $C_{\text{s.c.}} = \bigcup\{C_\lambda : \lambda \in (\kappa\,,\, j(\kappa)),\ \lambda \text{ regular}\}$ and then $C_{\text{s.c.}}$ is a stationary subset of $j(\kappa)$ which is a disjoint union of stationary subsets. Since $j(\kappa)$ is inaccessible, we may also consider the clubs $C^{(n)}_{j(\kappa)} \subseteq V_{j(\kappa)}$ of ordinals that are Σ_n-correct in the sense of $V_{j(\kappa)}$.

Then, for every $n \in \omega$ and every regular $\lambda \in (\kappa\,,\, j(\kappa))$, there is an initial factor λ-supercompact embedding j_0, with $j_0(\kappa) \in C^{(n)}_{j(\kappa)}$. We now use the characterization of Theorem 2.26 in order to show that all these embeddings are witnessed by extenders inside $V_{j(\kappa)}$.

Fix some $n \in \omega$ and suppose that we have, for some regular $\lambda \in (\kappa\,,\, j(\kappa))$, an initial factor λ-$C^{(n)}$-supercompact embedding $j_0 : V \longrightarrow M_0$ coming from our construction, i.e., once again, the embedding arises via a transitive collapse of an elementary substructure $X \prec M$ of the form above, coming together with a corresponding limit ordinal $\theta = \sup(X \cap j(\kappa))$. Before we describe how to extract the appropriate $Y \subseteq M_0$ which meets the requirements of Theorem 2.26 and which will serve as the support of the relevant extender, let us first look at the behaviour of j_0 on ordinals below the inaccessible $j(\kappa)$.

Claim. *For any $\alpha < j(\kappa)$, $j_0(\alpha) < j(\kappa)$.*

Proof of claim. Recall that by construction of j_0, $j_0(\alpha) = \mathrm{ot}(X \cap j(\alpha))$. We now show that this order-type is below $j(\kappa)$ by showing that $|X \cap j(\alpha)| < j(\kappa)$. The latter is verified by a counting argument, resembling the ones we have repeatedly employed; i.e., we argue that any $\xi \in X \cap j(\alpha)$ is represented as $\xi = j(f)(j''\lambda\,, x)$, for some function $f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow \alpha$ and some seed $x \in V_\theta$, where $\theta < j(\kappa)$. But then, the inaccessibility of $j(\kappa)$ implies that there are, in total, less than $j(\kappa)$-many such functions and seeds and thus, $|X \cap j(\alpha)| < j(\kappa)$. $\square$

Given the claim, we now describe how to produce the appropriate transitive $Y \subseteq M_0$. The idea is simple: we start with $j_0''\lambda$ (which belongs to M_0) and we recursively close under all the relevant properties. We repeat λ^+-many times (taking unions at limit stages) and the resulting set is the desired Y. Formally, we define by transfinite recursion on λ^+:

$$
\begin{aligned}
Y_0 &= \mathit{trcl}(\{j_0''\lambda\}) \\
Y_{\alpha+1} &= \mathit{trcl}(Y_\alpha \cup [Y_\alpha]^{<\omega} \cup {}^\lambda Y_\alpha \cup j_0''Y_\alpha) \\
Y_\alpha &= \bigcup_{\xi<\alpha} Y_\xi\,, \text{ if } \alpha \text{ is limit}
\end{aligned}
$$

and we then let $Y = Y_{\lambda^+}$. It is straightforward to check that $Y \subseteq M_0$, Y is transitive and $\{\kappa\} \cup [Y]^{<\omega} \cup {}^{\lambda}Y \cup j_0{}''Y \subseteq Y$. Moreover, by the fact that $j_0{}''j(\kappa) \subseteq j(\kappa)$ which is what the claim gives, it follows that $Y \in V_{j(\kappa)}$. Now let E be the (κ, Y)-extender derived from the embedding j_0. Recall that any derived extender of this sort comes with a corresponding ordinal $\zeta > \kappa$ which is the least one so that $Y \subseteq j_0(V_\zeta)$ (and then, the various ultrafilters of the extender are on ${}^{a}V_\zeta$, for $a \in [Y]^{<\omega}$). In our case, $\zeta < j(\kappa)$ again due to $j_0{}''j(\kappa) \subseteq j(\kappa)$. We may thus conclude that $E \in V_{j(\kappa)}$.

We now consider the extender ultrapower embedding $j_E : V \longrightarrow M_E$ which forms a commutative diagram with j_0 and the third factor $k_E : M_E \longrightarrow M_0$. Recall that, in general, in such a situation we have that $Y \subseteq M_E$ and $k_E \restriction Y = id$ (see Section A.3 of the Appendix for more details). Since we are moreover given that $j_0{}''Y \subseteq Y$, we thus get that $j_E \restriction Y = j_0 \restriction Y$ and $j_E{}''Y = j_0{}''Y \subseteq Y$; in particular, $j_E(\kappa) = j_0(\kappa) > \lambda$.

Therefore, by Theorem 2.26, j_E is a λ-supercompact embedding such that $j_E(\kappa) = j_0(\kappa)$. Moreover, since everything is bounded below the inaccessible $j(\kappa)$ and $E \in V_{j(\kappa)}$, we can verify inside the latter the λ-$C^{(n)}$-supercompactness of κ using the extender characterization, i.e., we get that

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is } \lambda\text{-}C^{(n)}\text{-supercompact”}$$

and so, for every $n \in \omega$, $V_{j(\kappa)} \models$ “κ is $C^{(n)}$-supercompact”. □

The last theorem gives an upper bound on the consistency strength of the corresponding schema SC, which is a schema of countably-many $\{\in, \mathbf{k}\}$-formulas ϕ_n, with each ϕ_n asserting that “$\mathbf{k}$ is $C^{(n)}$-supercompact”.

Hence, as a corollary, if the theory ZFC + “$\exists\,\kappa\,(\kappa$ is almost huge)” is consistent, then so is the theory ZFC + SC. We moreover show that this bound is sharp in the following sense.

Corollary 2.28. *If the theory* ZFC + “$\exists\,\kappa\,(\kappa$ *is almost huge*)” *is consistent, then so is the theory* ZFC + SC + “$\forall\,\lambda\,(\lambda$ *is not almost huge*)”.

Proof. Suppose that there exists an almost huge cardinal and let κ be the least one. By Theorem 2.27, if $j : V \longrightarrow M$ witnesses the almost hugeness of κ, then $V_{j(\kappa)}$ is a model of ZFC + SC (with $\mathbf{k}$ interpreted as κ). We show that in $V_{j(\kappa)}$, there is no almost huge cardinal. Towards a contradiction, suppose otherwise and let λ be the least almost huge cardinal in the sense of $V_{j(\kappa)}$.

Now recall that, as in the case of hugeness, the least almost huge cardinal is strictly smaller than the least supercompact, provided that they both exist. Thus, since κ is certainly supercompact in $V_{j(\kappa)}$, we get that $\lambda < \kappa$. But this is a contradiction since "λ is almost huge" is a Σ_2-statement and $j(\kappa)$ is inaccessible, i.e., λ would have to be an almost huge cardinal below κ. $\square$

In particular, as yet another corollary we get that the least almost huge cardinal, if it exists, is not $C^{(2)}$-almost huge. This was already shown in [5] by the use of direct reflection of the related Σ_2-statement.

Before continuing any further, we make the following remarks which clarify some details related to our constructions and, in particular, to the ones employed in the proof of Theorem 2.27.

Remarks.

1. Note that, in the proof of Theorem 2.27, any initial factor λ-supercompact embedding j_0 which arises from our chain is, in addition, superstrong. Thus, when we recursively construct the transitive Y which serves as the support of the witnessing extender inside $V_{j(\kappa)}$, we may as well include $V_{j_0(\kappa)}$ at the very first level Y_0, together with $j_0{}''\lambda$. This ensures that the extender embedding j_E will be superstrong as well, since, as we have seen, $j_E(\kappa) = j_0(\kappa)$.

2. Although it may well be the case that there are no supercompacts below the almost huge cardinal κ (e.g., if the latter is the least one), there are many supercompacts below κ in the sense of the model $V_{j(\kappa)}$. This comes from direct reflection using the normal ultrafilter $\mathcal{U}$ on κ which is derived from the initial almost huge embedding j. Here note that for any $\alpha < \kappa$, α is supercompact in V_κ if and only if α is $< \kappa$-supercompact in $V_{j(\kappa)}$, which in turn is equivalent to α being supercompact in $V_{j(\kappa)}$. Additionally, by a direct reflection argument using the normal measure, for every $n \in \omega$, we have that $\{\alpha < \kappa : V_\kappa \models \text{"}\alpha \text{ is } C^{(n)}\text{-supercompact"}\} \in \mathcal{U}$.

3. In the proof of Theorem 2.27, it is important to keep track of where exactly the various constructions are made so that any meta-mathematical doubts are avoided. Our route was the following. We started with an almost huge embedding $j : V \longrightarrow M$ and we then constructed, in V, the various elementary chains of substructures of M. We showed that for every $n \in \omega$ and any regular $\lambda < j(\kappa)$, there exists some initial factor λ-supercompact embedding j_0 arising from the chain, so that $j_0(\kappa) \in C^{(n)}$ in the sense of $V_{j(\kappa)}$. Finally, again in V, we inductively built the appropriate transitive Y which was used in order to extract the corresponding extender from j_0. We showed that this extender belongs to $V_{j(\kappa)}$ and the latter may use it in order to faithfully verify the fact that κ is λ-$C^{(n)}$-supercompact.

Let us also point out that, regarding the second of the above remarks, we have used the fact that if α is $<\kappa$ - supercompact and κ is supercompact, then α is fully supercompact. A natural question is whether this generalizes to the case of $C^{(n)}$ - supercompactness, for $n>0$. We now briefly address this question and show that if, as in our first remark, the witnessing extenders are in fact superstrong, then it does.

For this, we use the following lemma. Before that, let us mention that whenever we write "supercompact and superstrong", we temporarily mean that the two notions are witnessed *simultaneously* by the same embedding, i.e., the supercompactness embedding which we consider at any given argument, happens to be superstrong as well.

In the next section we shall give a more detailed account of simultaneous supercompactness and superstrongness (cf. Definition 2.34) and, furthermore, we will show that, in the "$C^{(n)}$" case, such an assumption already implies $C^{(n)}$ - extendibility (cf. Proposition 2.35).

Consequently, given an analogous result of J. Bagaria for $C^{(n)}$ - extendibles (cf. Proposition 1.13), the following lemma is hardly surprising.

Lemma 2.29. *If κ is $C^{(n)}$ - supercompact and superstrong, then $\kappa \in C^{(n+2)}$.*

Proof. We argue inductively. For $n=0$, it is well – known that if κ is supercompact then $\kappa \in C^{(2)}$. So, assume that the statement holds for some $n \geqslant 0$ and suppose that κ is $C^{(n+1)}$ - supercompact and superstrong. By the inductive hypothesis, we have $\kappa \in C^{(n+2)}$.

Let $\psi = \exists\, x\, \phi(x,y)$ be a Σ_{n+3} - formula, where $\phi(x,y)$ is Π_{n+2} and the parameter(s) y belongs to V_κ. If $V_\kappa \models \psi$ then, since $\kappa \in C^{(n+2)}$, we get that ψ holds. Conversely, suppose that ψ holds and fix some witness, i.e., fix some $\overline{x}$ such that $\phi(\overline{x},y)$ holds.

Now let $j: V \longrightarrow M$ be any λ - $C^{(n+1)}$ - supercompact and superstrong embedding for κ, with $\lambda > rank(\overline{x})$. Then, $\overline{x} \in V_{j(\kappa)} \subseteq M$ and $j(\kappa) \in C^{(n+1)}$. It now follows that $\phi(\overline{x},y)$, being a Π_{n+2} - statement, holds in $V_{j(\kappa)}$ and so, since $V_{j(\kappa)} \subseteq M$, we get that $V^M_{j(\kappa)} \models \exists\, x\, \phi(x,y)$.

By elementarity and the fact that the parameter(s) is fixed by j, we obtain that $V_\kappa \models \exists\, x\, \phi(x,y)$ which concludes the proof. □

Of course, this lemma combined with the fact that the statement "α is $C^{(n)}$ - supercompact" is Π_{n+2} - expressible immediately implies the following.

Corollary 2.30. *Suppose that κ is $C^{(n)}$ - supercompact, witnessed by superstrong extenders and suppose that $\alpha < \kappa$ is $<\kappa$ - $C^{(n)}$ -supercompact. Then, α is (fully) $C^{(n)}$ - supercompact.* □

We remark that this last corollary is valid even when the witnessing extenders are ordinary, instead of being of the Martin – Steel form, i.e., when each E is some (long) (α, β) - extender, which is additionally superstrong.

As we have pointed out, in the particular model $V_{j(\kappa)}$ obtained in the proof of Theorem 2.27 in which κ is $C^{(n)}$ - supercompact for every n, the conclusion of Corollary 2.30 for $\alpha < \kappa$ follows trivially from the elementarity and the critical point of the initial almost huge embedding j. That is, for any $\alpha < \kappa$, α is $C^{(n)}$ - supercompact in V_κ if and only if it is $C^{(n)}$ - supercompact in $V_{j(\kappa)}$.

Moreover, for any such fixed $\alpha < \kappa$ and for any $\gamma < \kappa$, the very same extenders belonging to V_κ witness the "γ - $C^{(n)}$ - supercompactness" of α either in V_κ or in $V_{j(\kappa)}$. In particular, $V_{j(\kappa)}$ thinks that κ is a limit of Σ_n - correct ordinals and thus, for every $n \in \omega$, $V_{j(\kappa)} \models \kappa \in Lim(C^{(n)})$.

In fact, we now give a small list of the properties that κ enjoys inside that particular model $V_{j(\kappa)}$. Regarding part (iii) below, recall Definition 1.14.

Proposition 2.31. *Suppose that κ is almost huge, as witnessed by the embedding $j : V \longrightarrow M$. Let $\mathcal{U}$ be the usual normal measure on κ derived from j. Then, for any $n \in \omega$, the following hold in the* (ZFC) *model $V_{j(\kappa)}$:*

(i) $\kappa \in Lim(C^{(n)})$.

(ii) κ is $C^{(n)}$ - supercompact and

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)} \text{ - supercompact}\} \in \mathcal{U}.$$

(iii) κ is $C^{(n)+}$ - extendible and

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)+} \text{- extendible}\} \in \mathcal{U}.$$

Proof. Parts (i) and (ii) have already been established by the preceding discussion.

For (iii), fix some n and some $\lambda \in (\kappa, j(\kappa))$ so that $V_{j(\kappa)} \models \lambda \in C^{(n)}$ (recall that there are unboundedly many such λ below $j(\kappa)$); now, by the closure of M and the inaccessibility of $j(\kappa)$, $j \restriction V_\lambda \in M$ and then, $j \restriction V_\lambda : V_\lambda \longrightarrow V_{j(\lambda)}$ is elementary, has critical point κ, $(j \restriction V_\lambda)(\kappa) > \lambda$, $j(\lambda) < j(j(\kappa))$ and, moreover, $j(V_{j(\kappa)}) \models j(\lambda) \in C^{(n)}$, where all clauses are computed in the model M (that is why the superscript "M" is missing from the "$V_{j(\lambda)}$").

Let us temporarily fix a formula $\varphi(\lambda, \mu, \kappa)$ asserting that "there exists a λ-extendibility embedding h for κ with $\mu = h(\lambda)$". From the point of view of M, we have just argued that for every $\lambda \in (\kappa, j(\kappa))$ with $V_{j(\kappa)} \models \lambda \in C^{(n)}$, there exists a $\mu < j(j(\kappa))$ such that $\varphi(\lambda, \mu, \kappa)$ holds and, moreover, $j(V_{j(\kappa)}) \models \mu \in C^{(n)}$.

But now, by the usual reflection of the normal measure, we get that the set of ordinals $\alpha<\kappa$ so that

$$\forall\,\lambda\in(\alpha\,,\,\kappa)\,(V_\kappa\models\lambda\in C^{(n)}\longrightarrow\exists\,\mu<j(\kappa)\,(\varphi(\lambda\,,\mu\,,\,\alpha)\,\wedge\,V_{j(\kappa)}\models\mu\in C^{(n)})),$$

belongs to $\mathcal{U}$. Let us call this set A. Fix any $\alpha\in A$ and fix any $\lambda\in(\alpha\,,\,\kappa)$ with $V_\kappa\models\lambda\in C^{(n)}$. Furthermore, fix a $\mu<j(\kappa)$ witnessing that $\alpha\in A$, i.e., such that $\varphi(\lambda\,,\mu\,,\,\alpha)$ holds and $V_{j(\kappa)}\models\mu\in C^{(n)}$.

Since $\mu<j(\kappa)$, by the inaccessibility of the latter we have that the witnessing λ-extendibility embedding for α actually belongs to $V_{j(\kappa)}$, i.e.,

$$V_{j(\kappa)}\models\exists\,\mu\,(\varphi(\lambda\,,\mu\,,\,\alpha)\,\wedge\,\mu\in C^{(n)}).$$

Therefore, by elementarity, for any such $\alpha\in A$ and any fixed $\lambda\in(\alpha\,,\,\kappa)$ with $V_\kappa\models\lambda\in C^{(n)}$, there exists a $\mu<\kappa$ so that

$$V_\kappa\models\varphi(\lambda\,,\mu\,,\,\alpha)\,\wedge\,\mu\in C^{(n)},$$

i.e., such extendibility embeddings for α may be witnessed inside V_κ. Note how we have successively bounded the μ's, first below $j(\kappa)$ and now below κ, ensuring that they are also in the relative $C^{(n)}$ of these structures. This discussion shows that the set

$$B=\{\alpha<\kappa:\,V_\kappa\models\forall\,\lambda>\alpha\,(\lambda\in C^{(n)}\longrightarrow\exists\,\mu\,(\varphi(\lambda\,,\mu\,,\,\alpha)\,\wedge\,\mu\in C^{(n)}))\}$$

includes A and hence it also belongs to the normal measure $\mathcal{U}$. Consequently, by "reversing" the reflection argument of the measure, we obtain that

$$V_{j(\kappa)}\models\forall\,\lambda>\kappa\,(\lambda\in C^{(n)}\longrightarrow\exists\,\mu\,(\varphi(\lambda\,,\mu\,,\,\kappa)\,\wedge\,\mu\in C^{(n)})).$$

But then, if we pick any $\lambda\in(\kappa\,,\,j(\kappa))$ with $V_{j(\kappa)}\models\lambda\in C^{(n)}$ and consider any witnessing extendibility embedding $h:V_\lambda\longrightarrow V_\mu$ for κ in $V_{j(\kappa)}$, we get that, in the sense of the latter structure, all three: κ, λ and μ are in $C^{(n)}$. Hence, $h(\kappa)\in C^{(n)}$ as well. This shows that κ is $C^{(n)+}$-extendible in $V_{j(\kappa)}$. In turn, once more by a reflection argument,

$$\{\alpha<\kappa:\,V_\kappa\models\text{“}\alpha\text{ is }C^{(n)+}\text{-extendible”}\}\in\mathcal{U}$$

and then, using the initial embedding j it immediately follows that, in $V_{j(\kappa)}$,

$$\{\alpha<\kappa:\,\alpha\text{ is }C^{(n)+}\text{-extendible}\}\in\mathcal{U},$$

which concludes part (*iii*) and, in effect, the proof. $\square$

As yet another corollary, we get that the assumption of almost hugeness is sufficient for the consistency of all the $C^{(n)}$-cardinals considered so far, in a strong sense: in the model $V_{j(\kappa)}$ and for any $n \in \omega$, the cardinal κ is $C^{(n)}$-supercompact, $C^{(n)}$-superstrong and, clearly, $C^{(n)}$-tall and $C^{(n)}$-strong as well. In addition, we just showed that it is also $C^{(n)+}$-extendible. Evidently, if we consider the *least* almost huge cardinal of the universe, the corresponding versions of Corollary 2.28 are obtained for all these cases.

Having brought into the discussion the case of extendibility this way, we now leave behind the particular model $V_{j(\kappa)}$ and turn to the general setting, looking more closely at $C^{(n)}$-extendible cardinals.

2.6 Extendibility

Given the already established consistency results of the previous section, let us now focus on the further study of the connection between the hierarchies of $C^{(n)}$-extendible and $C^{(n)}$-supercompact cardinals. We begin with the following.

Theorem 2.32. *Suppose that κ is $\lambda+1$-$C^{(n)}$-extendible, for some $n > 0$ and some $\lambda = \beth_\lambda > \kappa$ with $cf(\lambda) > \kappa$. Then, κ is λ-$C^{(n)}$-supercompact and superstrong.*

Proof. Fix some $n > 0$ and some $\lambda = \beth_\lambda > \kappa$ with $cf(\lambda) > \kappa$. Furthermore, let $j : V_{\lambda+1} \longrightarrow V_{j(\lambda)+1}$ be an elementary embedding that witnesses the $\lambda+1$-$C^{(n)}$-extendibility of κ, i.e., $cp(j) = \kappa$, $j(\kappa) > \lambda + 1$ and $j(\kappa) \in C^{(n)}$.

Now let E be the $(\kappa, j(\lambda))$-extender derived from j, that is, E is of the form $\langle E_a : a \in [j(\lambda)]^{<\omega} \rangle$ where, each E_a is a κ-complete ultrafilter on $[\lambda]^{|a|}$ defined as usual: for $X \subseteq [\lambda]^{|a|}$, $X \in E_a$ if and only if $a \in j(X)$.

Despite the fact that the ambient embedding j is between sets and not inner models, this definition makes sense since, for any $m \in \omega$, $\mathcal{P}([\lambda]^m) \subseteq V_{\lambda+1}$ and $[j(\lambda)]^{<\omega} \subseteq V_{j(\lambda)}$. Moreover, all the relevant information regarding this extender, i.e., the ultrafilters, the set $[j(\lambda)]^{<\omega}$, the various projection functions witnessing coherence, etc., are in $V_{j(\lambda)+1}$ where we can faithfully verify that E is indeed a $(\kappa, j(\lambda))$-extender. Note that E is just a long version of an ordinary extender and, in our situation, the rôle of ζ, which is the least ordinal such that $j(\lambda) \leqslant j(\zeta)$, is being played by λ.

Let $j_E : V \longrightarrow M_E$ be the associated extender embedding with $cp(j_E) = \kappa$. Although there is not a "full" third factor embedding k_E commuting with j and j_E, we may nonetheless get a restricted version of the usual commutative diagram, defining $k_E^* : V^{M_E}_{j_E(\lambda)} \longrightarrow V_{j(\lambda)}$, by letting

$$k_E^*([\,a, [f]\,]) = j(f)(a)$$

for all $[\,a, [f]\,] \in V^{M_E}_{j_E(\lambda)}$, where $a \in [j(\lambda)]^{<\omega}$ and $f : [\lambda]^{|a|} \longrightarrow V_\lambda$. We remark that the definition makes sense since any such function, representing an element in $V^{M_E}_{j_E(\lambda)}$, actually belongs to $V_{\lambda+1}$. Moreover, it is easily checked that k_E^* is a well–defined $\{\in\}$-embedding and so, in particular, injective. We then get the commutative diagram

$$\begin{tikzcd}
V_\lambda \arrow[r, "j \restriction V_\lambda"] \arrow[d, "j_E \restriction V_\lambda"'] & V_{j(\lambda)} \\
V^{M_E}_{j_E(\lambda)} \arrow[ur, "k_E^*"'] &
\end{tikzcd}$$

where $j \restriction V_\lambda = k_E^* \circ (j_E \restriction V_\lambda)$. Next, we show that k_E^* is in fact the identity.

Since $\lambda = \beth_\lambda$, fix some bijection $g : [\lambda]^1 \longrightarrow V_\lambda$ and observe that $g \in V_{\lambda+1}$. Then, by elementarity, we have that $j(g) : [j(\lambda)]^1 \longrightarrow V_{j(\lambda)}$ is also a bijection and $j(g) \in V_{j(\lambda)+1}$. Thus, for every $x \in V_{j(\lambda)}$, there is some $\xi < j(\lambda)$ with $x = j(g)(\{\xi\})$. But this means that for every $x \in V_{j(\lambda)}$, $x = k_E^*([\,\{\xi\}, [g]\,])$, where $[\,\{\xi\}, [g]\,]$ is an element of $V^{M_E}_{j_E(\lambda)}$, i.e., k_E^* is also surjective. Therefore, it must be the identity, since its domain and range are transitive sets. It now follows that $V^{M_E}_{j_E(\lambda)} = V_{j(\lambda)}$, i.e., $V_{j(\lambda)} \subseteq M_E$ and so, j_E is superstrong and, for every ordinal $\alpha \leqslant \lambda$, $j_E(\alpha) = j(\alpha)$. In particular, $j_E(\kappa) = j(\kappa)$ and, also, noting that $cf(j(\lambda)) > \lambda$ (computed in V), we have $j_E{}''\lambda = j{}''\lambda \in V_{j(\lambda)}$ and so $j_E{}''\lambda \in M_E$.

Thus, it will be enough to show that ${}^\lambda M_E \subseteq M_E$ in order to conclude the embedding j_E witnesses the λ-$C^{(n)}$-supercompactness of κ. But this is shown exactly as in the description of our elementary chain construction in the supercompactness case: we use the fact that $j_E{}''\lambda \in M_E$ and the nice representation of the latter, in order to get closure under λ-sequences. Let us briefly repeat the argument here for completeness.

Recall that $M_E = \{j_E(f)(a) : a \in [j(\lambda)]^{<\omega},\, f : [\lambda]^{|a|} \longrightarrow V,\, f \in V\}$. Suppose that $\{j_E(f_i)(a_i) : i < \lambda\} \subseteq M_E$, aiming at showing that the sequence $\langle\, j_E(f_i)(a_i) : i < \lambda\,\rangle$ belongs to M_E. First, recall that $cf(j(\lambda)) > \lambda$ from which we get that $\langle\, a_i : i < \lambda\,\rangle \in V_{j(\lambda)} \subseteq M_E$. Thus, it will be enough to show that $\langle\, j_E(f_i) : i < \lambda\,\rangle \in M_E$ since in that case, M_E can compute the desired sequence $\langle\, j_E(f_i)(a_i) : i < \lambda\,\rangle$ by evaluating pointwise the functions $j_E(f_i)$'s at the corresponding a_i's.

For this, since $j_E{}''\lambda \in M_E$, we also have that $j_E \restriction \lambda \in M_E$, where the map $j_E \restriction \lambda : \lambda \longrightarrow j_E{}''\lambda$ is viewed in M_E as an order–type function. Moreover, it is clear that $G = j_E(\langle f_i : i < \lambda \rangle)$ belongs to M_E, where $G : j_E(\lambda) \longrightarrow M_E$. Now define, in M_E, the function $F = G \circ j_E \restriction \lambda : \lambda \longrightarrow M_E$ and let us check that, in fact, $F = \langle j_E(f_i) : i < \lambda \rangle$ which will conclude the proof. But this follows from elementarity and the definition of F since, for every $i < \lambda$, we have

$$F(i) = G(j_E(i)) = j_E(\langle f_i : i < \lambda \rangle)(j_E(i)) = j_E(\langle f_i : i < \lambda \rangle(i)) = j_E(f_i)\,.$$

□

We immediately get the following corollary. This answers affirmatively the relevant question which was posed in [5].

Corollary 2.33. *If κ is $C^{(n)}$-extendible then it is also $C^{(n)}$-supercompact.* □

It is clear that in the statement of Theorem 2.32 we may as well require that the image $j(\kappa)$ of the produced λ-supercompact embedding is inaccessible. Moreover, this can be also strengthened to Mahlo, weakly compact and even measurable and Woodin, if one requires sufficient extendibility for κ.

To see this, first recall that all the aforementioned large cardinal notions are local, i.e., they can be faithfully verified in some initial segment of the universe whose ordinal rank is explicitly related to the large cardinal in question. For instance, the Mahloness, the weak compactness or the Woodinness of α are properties which can all be verified inside $V_{\alpha+1}$. On the other hand, the measurability of α can be verified inside $V_{\alpha+2}$.

Thus, by just assuming that κ is 1-extendible, we do have that $j(\kappa)$ is a true Mahlo, weak compact and even Woodin cardinal. Analogously, the 2-extendibility of κ gives that $j(\kappa)$ is a true measurable. More generally, for any limit ordinal $\lambda > \kappa$, if κ is $\lambda + 1$-extendible then the witnessing embedding is such that $j(\kappa)$ is a Woodin measurable cardinal.

Let us also point out that, in the proof of the Theorem 2.32, we in fact get more than mere superstrongness for the derived extender embedding $j_E : V \longrightarrow M_E$. We actually have that $V_{j(\lambda)} \subseteq M_E$, which follows from the extendibility of the initial embedding j. As we are about to characterize ($C^{(n)}$-) extendibility in terms of such embeddings, it is appropriate to introduce the following notion.

Definition 2.34. *A cardinal κ is λ-* **supercompact and** *θ-* **superstrong***, for some $\lambda, \theta \geqslant \kappa$, if there exists an elementary embedding $j : V \longrightarrow M$ with M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\lambda}M \subseteq M$ and $V_{j(\theta)} \subseteq M$.*

For the global version(s) of this notion, the absence of one of the two parameters indicates universal quantification on the parameter missing; e.g., κ is supercompact and θ-superstrong, for some fixed $\theta \geqslant \kappa$, if and only if it is λ-supercompact and θ - superstrong, for every $\lambda \geqslant \kappa$. On the other hand, the absence of both parameters is intended to mean that the same λ is quantified for both of them, i.e., κ is supercompact and superstrong if and only if it is λ - supercompact and λ - superstrong, for every $\lambda \geqslant \kappa$.

We stress the fact that the above notion transcends supercompactness in the sense that if κ is the least supercompact, then it is not λ - supercompact and κ - superstrong, for any λ. In fact, as we shall soon see, global supercompactness and κ - superstrongness is equivalent to extendibility.

So, in our newly established terminology, Theorem 2.32 actually shows that, for any $n > 0$ and any $\lambda = \beth_\lambda > \kappa$ with $cf(\lambda) > \kappa$, if the cardinal κ is $\lambda + 1$ - $C^{(n)}$ - extendible then it is λ - $C^{(n)}$ - supercompact and λ - superstrong, witnessed by the derived (ordinary but) long $(\kappa, j(\lambda))$ - extender E.

Although we do not know the exact relation between $C^{(n)}$ - extendibles and $C^{(n)}$ - supercompacts, they seem to be different notions, i.e., even some partial converse of Corollary 2.33 seems unlikely without any extra assumptions. However, we now show that, under the extra assumption of κ - superstrongness, the converse of Corollary 2.33 does hold. This will be an application of Lemma 2.29.

Proposition 2.35. *If κ is $C^{(n)}$ - supercompact and κ - superstrong, then it is $C^{(n)}$ - extendible.*

Proof. Fix $n \geqslant 1$ and suppose that κ is $C^{(n)}$ - supercompact and κ - superstrong (i.e., usual superstrongness). Fix some $\lambda > \kappa$ with $\lambda \in C^{(n+2)}$ and recall that $\lambda = |V_\lambda|$. Now let $j : V \longrightarrow M$ be an elementary embedding which witnesses the fact that κ is λ - $C^{(n)}$ - supercompact and κ - superstrong, i.e., M transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^\lambda M \subseteq M$, $j(\kappa) \in C^{(n)}$ and $V_{j(\kappa)} \subseteq M$.

By Lemma 2.29, $\kappa \in C^{(n+2)}$ and then, by elementarity, $M \models j(\kappa) \in C^{(n+2)}$ and $M \models j(\lambda) \in C^{(n+2)}$. Now note that, by the closure of the target model, the restricted embedding $j \restriction V_\lambda : V_\lambda \longrightarrow V^M_{j(\lambda)}$ belongs to M and this witnesses the fact that

$$M \models \text{“}\kappa \text{ is } < \lambda\text{ - }C^{(n)}\text{ - extendible”}.$$

Moreover, as $j(\kappa) \in C^{(n)}$, we have that $V_{j(\kappa)} \models \lambda \in C^{(n+1)}$ and then, using that $V_{j(\kappa)} \subseteq M$ and $M \models j(\kappa) \in C^{(n+2)}$, we also get that $M \models \lambda \in C^{(n+1)}$. It now follows that the “$<\lambda$ - $C^{(n)}$ - extendibility” of κ in M can be verified inside V_λ, that is, $M \models V_\lambda \models$ “κ is $C^{(n)}$ - extendible”. Therefore, since $V_\lambda \subseteq M$, the previous statement is computed correctly, i.e., $V_\lambda \models$ “κ is $C^{(n)}$ - extendible”. Hence, recalling that $\lambda \in C^{(n+2)}$, we conclude that κ is indeed $C^{(n)}$ - extendible.

The case $n = 0$, connecting standard extendibility to supercompactness and κ - superstrongness, is similar. The only difference is that one needs to start with a $\lambda \in C^{(3)}$, since the property of being extendible is Π_3-expressible; one again checks that $M \models \lambda \in C^{(2)}$ and then argues as above. □

As a direct combination of the last proposition and Theorem 2.32, we get the following characterization.

Corollary 2.36. *A cardinal κ is $C^{(n)}$-extendible if and only if it is $C^{(n)}$-supercompact and κ-superstrong.* □

In particular, for any fixed $\lambda \geqslant \kappa$, we also get the interesting equivalence that the cardinal κ is $C^{(n)}$-supercompact and λ-superstrong if and only if it is $C^{(n)}$-supercompact and κ-superstrong. In other words, for the global version, usual superstrongness of the supercompact embeddings is equivalent to the apparently stronger λ-superstrongness, for $\lambda \geqslant \kappa$.

Let us also recall that the above characterization is in accordance with the results in [5], where it was shown that every $C^{(n)}$-extendible cardinal belongs to $C^{(n+2)}$. Also, for $n = 1$, it makes the question regarding the relation between supercompact and $C^{(1)}$-supercompact cardinals even more intriguing.

Before concluding this chapter, we momentarily return to the supercompactness case. As the reader should have already noticed, the proof of Theorem 2.32 gives a way of describing supercompactness embeddings by ordinary extenders (albeit long ones). Since this was advertised earlier on in the chapter, let us make it clear and precise in the following corollary.

Corollary 2.37. *Suppose that $j : V \longrightarrow M$ is λ-supercompact for κ, for some $\lambda > \kappa$. Let $\theta = \beth_\theta \geqslant \lambda$ with $cf(\theta) > \kappa$ and let E be the $(\kappa, j(\theta))$-extender derived from j. Then, the extender embedding $j_E : V \longrightarrow M_E$ is λ-supercompact for κ with $j_E(\kappa) = j(\kappa)$.*

Proof. Fix $\lambda > \kappa$ and $j : V \longrightarrow M$, a λ-supercompact embedding for κ. Further, fix $\theta = \beth_\theta \geqslant \lambda$ with $cf(\theta) > \kappa$ and let E be the $(\kappa, j(\theta))$-extender derived from j. Then, recalling the proof of Theorem 2.32 one easily checks that the same idea goes through.

Namely, we consider the (in this case, full) third factor embedding k_E, arguing that it is surjective (and thus, the identity) for sets in M_E of rank below $j_E(\theta)$, i.e., $k_E \upharpoonright V^{M_E}_{j_E(\theta)} = id$ and $V^{M_E}_{j_E(\theta)} = V^M_{j(\theta)}$. Moreover, for all $\alpha \leqslant \theta$, we have that $j_E(\alpha) = j(\alpha)$ and, also, as $cf(j(\theta)) > \lambda$ (computed in V), we get that $j_E{}''\lambda = j''\lambda \in V^M_{j(\theta)}$ and $j_E{}''\lambda \in M_E$ as well. To conclude, one shows closure under λ-sequences for M_E just as we did in the proof of Theorem 2.32, noting again that the assumption $cf(\theta) > \kappa$ ensures that $cf(j(\theta)) > \lambda$ which is exactly what is needed for the rest of the argument to work. □

Observe once more that if the initial embedding j happens to have some degree of superstrongness, then this is carried over to the extender embedding; e.g., in the above proof as it stands, if j was also θ - superstrong, then the same would be true for j_E.

Moreover, since the arising extender embedding is such that $j_E(\kappa) = j(\kappa)$, the previous proof works for $C^{(n)}$ - supercompactness as well, i.e., if $j : V \longrightarrow M$ witnesses the λ - $C^{(n)}$ - supercompactness of κ, $\theta = \beth_\theta \geqslant \lambda$ with $cf(\theta) > \kappa$ and we let E be the $(\kappa, j(\theta))$ - extender derived from j, then j_E is also λ - $C^{(n)}$ - supercompact for κ.

From our discussion so far, and except for the cases of Woodinness and of strong compactness, the relation between the various large cardinals and their corresponding $C^{(n)}$ - versions has not been fully clarified. In this context, it seems that the notion of supercompactness is of central importance, with that of tallness being essentially a special case.

The main question would then be, whether one can *separate* ordinary supercompactness from $C^{(n)}$ - supercompactness, perhaps via some forcing arguments. For example, when $n = 1$, one could aim at "killing" the $C^{(1)}$ - supercompactness of a cardinal, while preserving its supercompactness.

For another example, in the particular model $V_{j(\kappa)}$ arising from the almost huge embedding (cf. Proposition 2.31), one could try to make the first supercompact strictly smaller that the first $C^{(1)}$ - supercompact. But, even for these cases, it is still unclear how such a forcing argument might look like. For $n > 1$ the situation becomes even more elusive, as we have no good control on the behaviour of the Σ_n - correct ordinals with respect to forcing.

Having said all that, we do not bypass the forcing machinery altogether and we are happy to announce a brief forcing interlude which is about to follow in the next chapter. There, we shall present some (quite) basic properties arising from the interaction between forcing and $C^{(n)}$ - cardinals, along with some hints regarding the obstacles described above. We hope that you follow the relaxed perspective of this interlude and that you enjoy it, dear reader.

CHAPTER 3

Forcing *INTERLUDE*

We softly begin our interlude by recalling that if $\alpha \in C^{(n)}$, for some $n \geqslant 1$, then V_α satisfies Kripke – Platek set theory and Σ_n - separation along with Σ_n - collection. Thus, for any poset $\mathbb{P} \in V_\alpha$, we may freely do forcing over V_α with the extra knowledge that, for any particular Σ_n - formula, the forcing relation is going to be Σ_n - definable over V_α, using the poset $\mathbb{P}$ as a parameter.

Let us now see how some forcing notions interact with the $C^{(n)}$ - cardinals.

Lemma 3.1. *Let $\mathbb{P}$ be a forcing notion and κ a cardinal.*

(i) *Suppose that $|\mathbb{P}| < \kappa$ and $\kappa \in C^{(n)}$, for some $n \geqslant 1$. Then, we have that $\mathbb{P} \Vdash \check{\kappa} \in C^{(n)}$.*

(ii) *Suppose that $\mathbb{P}$ does not change V_κ. Then, $\mathbb{P} \Vdash \check{\kappa} \in C^{(1)}$ if and only if $\kappa \in C^{(1)}$.*

(iii) *Suppose that $\mathbb{P}$ does not change V_κ and $\kappa \in C^{(2)}$. Then, for every $\lambda < \kappa$, if $\mathbb{P} \Vdash \check{\lambda} \in C^{(2)}$ then $\lambda \in C^{(2)}$.*

Proof. For (i), without loss of generality, we may assume that $rank(\mathbb{P}) = \gamma < \kappa$ and that a flat pairing function has been used in the construction of the $\mathbb{P}$ - names. Now let G be $\mathbb{P}$ - generic over V. By Lemma 1.18 and since κ is a cardinal, we get that $V_\kappa[G] = V[G]_\kappa$.

Fix some $n \geqslant 1$, some Π_{n-1} - formula $\phi(y, v_1, \ldots, v_k)$ and some $\mathbb{P}$ - names $\dot{x}_1, \ldots, \dot{x}_k$, such that $V[G] \models \exists y\, \phi(y, (\dot{x}_1)_G, \ldots, (\dot{x}_k)_G)$, where $(\dot{x}_i)_G \in V[G]_\kappa$ for each $1 \leqslant i \leqslant k$. Then, there is some condition $p \in G$ which forces this fact in V, i.e., the Σ_n - statement "$p \Vdash \exists y\, \phi(y, \dot{x}_1, \ldots, \dot{x}_k)$" holds. Now, given that $V_\kappa[G] = V[G]_\kappa$, all the $\dot{x}_i$'s may be assumed to belong to V_κ. Therefore, since $\kappa \in C^{(n)}$, we get that $V_\kappa \models$ "$p \Vdash \exists y\, \phi(y, \dot{x}_1, \ldots, \dot{x}_k)$".

But then, as the poset $\mathbb{P}$ belongs to V_κ and we may use the generic object G freely in order to force over the latter, we get that

$$V_\kappa[G] \models \exists\, y\, \phi(y, (\dot{x}_1)_G, \ldots, (\dot{x}_k)_G).$$

The same argument going backwards, shows that if $V_\kappa[G] = V[G]_\kappa$ satisfies a Σ_n - formula then the same is true for $V[G]$. This implies that $V[G] \models \kappa \in C^{(n)}$ and we are done.

For (ii), we recall that being a $C^{(1)}$ - cardinal is a Π_1 - expressible property and hence the forward implication is true regardless of any assumption on the poset, i.e., it follows from the fact that such a property is downwards absolute, from any generic extension to the ground model.

For the converse, assume that $\kappa \in C^{(1)}$ in the ground model, that is, κ is an uncountable (strong limit) cardinal with $V_\kappa = H_\kappa$. From this and the assumption that $V_\kappa = V[G]_\kappa$, it easily follows that $(V_\kappa)^{V[G]} = (H_\kappa)^{V[G]}$. Moreover, κ remains an uncountable strong limit cardinal in $V[G]$ and thus, $V[G] \models \kappa \in C^{(1)}$ which concludes part (ii).

For (iii), suppose that $\kappa \in C^{(2)}$ and that $\mathbb{P}$ does not change V_κ. Fix some G which is $\mathbb{P}$ - generic over V and some $\lambda < \kappa$ so that $V[G] \models \lambda \in C^{(2)}$.

By part (ii), $\lambda \in C^{(1)}$ and $V[G] \models \kappa \in C^{(1)}$. Now let ψ be a Σ_2 - formula, whose parameter(s), if any, belongs to V_λ, and suppose that ψ holds in the ground model. Then, $V[G]_\kappa = V_\kappa \models \psi$ and so, since κ is $C^{(1)}$ in the extension, we have that $V[G] \models \psi$. Now, going downwards, we first get $V[G]_\lambda \models \psi$ by the correctness of λ in $V[G]$ and finally, $V_\lambda \models \psi$ by the assumption on $\mathbb{P}$. □

For $n > 1$, the lack of a (local) "combinatorial" characterization of membership in the class $C^{(n)}$ is a serious obstacle in showing the converse of (i), or of generalizing (ii) of Lemma 3.1. Note that (iii) is only a partial result in this direction. In fact, we now argue that the converse of (iii) does not necessarily hold since the following situation is possible (i.e., consistent).

It is known that, relative to large cardinal assumptions at the level of hyper–measurability, it is consistent that the GCH holds at successor cardinals but fails at limits. More precisely, given a $\mathcal{P}^3(\lambda)$ - hyper–measurable cardinal λ, there is a model of ZFC in which $2^\alpha = \alpha^{++}$ for every limit cardinal α, whereas the GCH holds everywhere else (cf. [12]). Suppose we are in such a model V and consider any $\kappa \in C^{(2)}$. Now force with the canonical poset $Add(\kappa^+, 1)$ which is $\leqslant \kappa$ - closed (hence it does not change V_κ) and makes the GCH true at κ. Then, the cardinal κ and, in fact, all cardinals below it which were Σ_2 - correct in V, are no longer Σ_2 - correct in the extension. To see this, notice that the statement "there exists a limit cardinal at which the GCH holds" is Σ_2, $\kappa \in C^{(1)}$ remains true by Lemma 3.1 (*ii*) and the continuum function is not altered at cardinals below κ.

This example shows that it is possible to kill the "$C^{(2)}$-ness" of all cardinals up to κ, while preserving V_κ (and hence preserving all $C^{(1)}$'s below it)$^\sharp$. What is more, it highlights the (not very surprising) fact that there is no general, local, combinatorial characterization of $C^{(2)}$-ness, i.e., a characterization in terms of the existence of some object A satisfying local properties, verifiable inside any sufficiently large V_α which contains A.

For suppose that such a characterization were available. Work again inside the model where the GCH fails at limits but holds at successor cardinals. Fix some $\kappa \in C^{(2)}$. Then, the latter fact has to be witnessed by the existence of an object A. Fix some sufficiently large α with $A \in V_\alpha$, so that the purported characterization is faithfully verified in V_α, i.e., $V_\alpha \models$ "A witnesses that $\kappa \in C^{(2)}$". Pick any $\lambda \in C^{(2)}$ above α and force the GCH at λ without changing V_λ. By the previous paragraphs, κ no longer belongs to $C^{(2)}$ in the generic extension. On the other hand, since V_λ is preserved after the forcing, the Δ_1-expressible local fact that $V_\alpha \models$ "A satisfies the characterization" is still true.

Let us now turn to the preservation of (some of) the $C^{(n)}$-large cardinals considered so far, under appropriate forcing. We begin by first considering small forcing.

Lemma 3.2. *Suppose that κ is $C^{(n)}$-tall (superstrong, supercompact, extendible), for some $n \geqslant 1$, and let $\mathbb{P}$ be a forcing notion with $|\mathbb{P}| < \kappa$. Then, κ remains $C^{(n)}$-tall (resp. superstrong, supercompact, extendible) in $V^{\mathbb{P}}$.*

Proof. We may assume that $\mathbb{P} \in V_\kappa$. Let us consider the case of supercompactness. Fix an $n \geqslant 1$ and a (cardinal) $\lambda > \kappa$ and let $j : V \longrightarrow M$ witness the λ-$C^{(n)}$-supercompactness of κ in V. We let G be $\mathbb{P}$-generic over V and we show that the embedding can be lifted through the forcing.

Since $\mathbb{P} \in V_\kappa$, we have that $j(\mathbb{P}) = \mathbb{P}$ and so G is also $j(\mathbb{P})$-generic over M, with $j''G = G$. Thus, the embedding lifts to $j : V[G] \longrightarrow M[G]$ in the generic extension, where $cp(j) = \kappa$ and $j(\kappa) > \lambda$ still hold. Moreover, since $|\mathbb{P}| < \kappa$, $\mathbb{P}$ is certainly λ^+-c.c. and thus, by Lemma 1.19, we get that $V[G] \models {}^\lambda M[G] \subseteq M[G]$.

Finally, by Lemma 3.1 (*i*), $V[G] \models j(\kappa) \in C^{(n)}$ and this finishes the verification that the lifted embedding witnesses the λ-$C^{(n)}$-supercompactness of κ in the generic extension.

$^\sharp$ For another such example, but arising from considerations of significantly larger consistency strength, the reader is referred to Chapter 4 and the concluding remarks after Corollary 4.9.

The case of tallness is now obvious, as the argument is essentially the same. For superstrongness, considering again a flat pairing function as in the proof of Lemma 3.1 (i), we obtain that $V[G]_{j(\kappa)} = V_{j(\kappa)}[G]$ and hence $V[G]_{j(\kappa)} \subseteq M[G]$ by virtue of $V_{j(\kappa)} \subseteq M \subseteq M[G]$.

To conclude, by Corollary 2.36, the case of extendibility follows from a straightforward combination of the cases of supercompactness and of superstrongness. $\square$

We now state the following, which is a more informative version of Lemma 3.2. Again, we focus on the case of supercompactness.

Lemma 3.3. *Suppose that E is a (κ, Y) - extender with $j_E : V \longrightarrow M_E$ being a λ - $C^{(n)}$ - supercompact embedding for κ, for some $\lambda > \kappa$. Let $\mathbb{P}$ be a forcing notion with $|\mathbb{P}| < \kappa$. Then, j_E lifts through $\mathbb{P}$ to a λ - $C^{(n)}$ - supercompact extender embedding.*

Proof. Fix G a $\mathbb{P}$ - generic over V. The fact that the given embedding lifts to $j_E : V[G] \longrightarrow M_E[G]$ in order to witness λ - $C^{(n)}$ - supercompactness of κ in $V[G]$, follows from Lemma 3.2. We now show that the extender representation is also preserved along the way.

Recall that any such extender embedding comes with a nice representation of its ultrapower (see the Appendix) and so, in the ground model,

$$M_E = \{ j_E(f)(j_E^{-1} \restriction j_E(a)) : a \in [Y]^{<\omega}, \ f : {}^a(V_\zeta) \longrightarrow V, \ f \in V \},$$

where ζ is the least ordinal such that $Y \subseteq j_E(V_\zeta)$. We now argue that, for the lifted embedding, $M_E[G]$ has a representation in terms of functions in the generic extension.

Claim. $M_E[G] = \{ j_E(f)(j_E^{-1} \restriction j_E(a)) : a \in [Y]^{<\omega}, \ f \in V[G] \}$.

Proof of claim. Of course, by "$f \in V[G]$" it is meant that the functions are again of the form $f : {}^a(V[G]_\zeta) \longrightarrow V[G]$. The claim is actually an instance of the more general fact that, if an embedding is generated by a set of seeds and it lifts to some generic extension, then the same set of seeds generates the lifted version. A proof of this can be found either in [23] or in [27] and is – essentially – as follows.

Let $\tau_G \in M_E[G]$ be some arbitrary element, where $\tau \in M_E$ is a $j_E(\mathbb{P})$ - name. By the representation of M_E, there is some $a \in [Y]^{<\omega}$ and some ground model function $f : {}^a(V_\zeta) \longrightarrow V$ with $\tau = j_E(f)(j_E^{-1} \restriction j_E(a))$. We may assume that for every $s \in {}^a(V_\zeta)$, $f(s)$ is a $\mathbb{P}$ - name. Now define, in $V[G]$ and using f, the function $g : {}^a(V[G]_\zeta) \longrightarrow V[G]$ so that, for every $s \in {}^a(V[G]_\zeta)$, if $f(s)$ is defined (and is thus a $\mathbb{P}$ - name), then $g(s) = (f(s))_G$, otherwise $g(s) = \varnothing$. An easy computation shows that $j_E(g)(j_E^{-1} \restriction j_E(a)) = \big(j_E(f)(j_E^{-1} \restriction j_E(a))\big)_{j_E(G)} = \tau_G$ and the claim follows. $\square$

Given the claim, if we now extract, in $V[G]$, the (κ, Y)-extender E^* from the lifted embedding, it is easily seen that, in fact, the lifted embedding is exactly the extender ultrapower by E^*, in $V[G]$. $\square$

Observe that the same argument works even if one has ordinary extenders instead of Martin–Steel ones. Also, as noted in the proof of the claim, the key point was the fact that the ground model embedding had a nice representation with regard to some class of functions and some set of seeds. As the argument is quite general, the same proof works for the cases of tall, of superstrong, and of extendible cardinals as well, where for extendibility we again appeal to the equivalent formulation in terms of simultaneous supercompactness and superstrongness.

We now consider preservation of $C^{(n)}$-tall and of $C^{(n)}$-supercompact cardinals under sufficiently distributive forcing notions.

Lemma 3.4. *Suppose that κ is $C^{(n)}$-tall and let $\mathbb{P}$ be a $\leqslant\kappa$-distributive forcing. Then, κ remains $C^{(n)}$-tall in $V^{\mathbb{P}}$.*

Proof. We show that, for sufficiently large $\lambda > \kappa$, every extender embedding which witnesses the λ-$C^{(n)}$-tallness of κ, lifts through the forcing. So, let us fix some n, some ordinal $\lambda > \max\{\kappa, |\mathbb{P}|\}$ and some extender embedding $j : V \longrightarrow M$ which witnesses the λ-$C^{(n)}$-tallness of κ, i.e., M is transitive, $cp(j) = \kappa$, $j(\kappa) > \lambda$, ${}^{\kappa}M \subseteq M$ and $j(\kappa) \in C^{(n)}$. Recall that, in such a case, we may assume that

$$M = \{j(f)(x) : f \in V,\, f : V_\kappa \longrightarrow V,\, x \in V^M_{j(\kappa)}\}.$$

Let G be $\mathbb{P}$-generic over V. In order to show that j lifts through the forcing, we check that $j\,''G$ generates a $j(\mathbb{P})$-generic filter over M. For this, let $D \in M$ be an open dense subset of $j(\mathbb{P})$. By the above representation of M, there is some function $f : V_\kappa \longrightarrow V$ in the ground model and some seed $x \in V^M_{j(\kappa)}$ such that $D = j(f)(x)$. Without loss of generality, we may assume that for every $w \in V_\kappa$, $f(w)$ is an open dense subset of the poset $\mathbb{P}$.

Now consider $D_0 = \bigcap\limits_{w \in V_\kappa} f(w)$ which, by the distributivity of the forcing, is open dense in $\mathbb{P}$. Therefore, $G \cap D_0 \neq \varnothing$ which in turn implies that $j\,''G \cap D \neq \varnothing$, since $j\,''D_0 \subseteq D$. This shows that $j\,''G$ meets every open dense subset of $j(\mathbb{P})$ in M and so it generates, on the M-side, a generic filter $H \subseteq j(\mathbb{P})$; thus, we may indeed lift the embedding in order to obtain $j : V[G] \longrightarrow M[j(G)]$, where we necessarily have that $j(G) = H$.

In the extension $V[G]$, we still have that $cp(j) = \kappa$ and $j(\kappa) > \lambda$ for the lifted version. Moreover, since $j(\kappa) > |\mathbb{P}|$, Lemma 3.1 (i) implies that $j(\kappa)$ belongs to $C^{(n)}$ in $V[G]$. Finally, the $\leqslant\kappa$-distributivity of the forcing ensures that no new κ-sequences of ordinals are introduced. That is, any κ-sequence of ordinals which belongs to $V[G]$, actually lies in V and therefore in M, by the closure of the latter. Hence, $V[G] \models {}^{\kappa}M[j(G)] \subseteq M[j(G)]$ and so the lifted embedding witnesses the λ - $C^{(n)}$ - tallness of κ in the generic extension. □

As we show next, a similar argument works for $C^{(n)}$ - supercompactness. The difference here is that we can no longer expect a uniform distributivity bound on the forcing; in other words, we only have partial preservation.

Lemma 3.5. *Suppose that κ is $C^{(n)}$ - supercompact and suppose that, for some* (*cardinal*) *$\lambda > \kappa$, the poset $\mathbb{P}$ is $\leqslant \lambda^{<\kappa}$ - distributive. Then, κ remains λ - $C^{(n)}$ - supercompact in $V^{\mathbb{P}}$.*

Proof. Fix some n and some $\theta > \max\{\lambda\,, |\mathbb{P}|\}$ and let $j : V \longrightarrow M$ witness the θ - $C^{(n)}$ - supercompactness of κ. Now consider the elementary substructure

$$X_0 = \{j(f)(j\,''\lambda\,,\,x) : f \in V,\, f : \mathcal{P}_\kappa\lambda \times V_\kappa \longrightarrow V,\, x \in V^M_{j(\kappa)}\} \prec M,$$

which gives rise to a (not necessarily initial) factor λ - supercompact embedding $j_0 : V \longrightarrow M_0$, via the Mostowski collapse $\pi_0 : X_0 \cong M_0$ as usual. Since $j_0(\kappa) = j(\kappa)$, j_0 is moreover λ - $C^{(n)}$ - supercompact for κ.

Let G be $\mathbb{P}$ - generic over V. Exactly as in Lemma 3.4, we show that j_0 lifts through the forcing, by verifying that $j_0{}''G$ generates a $j_0(\mathbb{P})$ - generic filter over M_0. Note that the $\leqslant \lambda^{<\kappa}$ - distributivity of the forcing is exactly the modification needed in order for the same argument to work.

Finally, again by the distributivity of the forcing, we obtain the required closure under λ - sequences for the lifted embedding $j_0 : V[G] \longrightarrow M_0[j_0(G)]$ and, therefore, we conclude that κ remains λ - $C^{(n)}$ - supercompact in the generic extension, as desired. □

Remark. Note that in both Lemmas 3.4 and 3.5, the assumption of full $C^{(n)}$ - tallness (resp. $C^{(n)}$ - supercompactness) is what enables us to employ Lemma 3.1(i) in order to conclude that the image $j(\kappa)$ of the ground model embedding *which we choose* to lift remains a Σ_n - correct cardinal in the generic extension. For $n = 0$ this is clearly not an issue. Indeed, for the ordinary notions, we have level – by – level preservations: it is known that λ - tallness (resp. λ - supercompactness) embeddings lift through any $\leqslant \kappa$ - distributive (resp. $\leqslant \lambda^{<\kappa}$ - distributive) forcing, by essentially the same arguments.

Furthermore, again by the relevant result of Solovay, for every *regular* λ above a supercompact κ, we have that $\lambda^{<\kappa} = \lambda$, which in such cases simplifies slightly the assumption of Lemma 3.5. $\perp$

This concludes our interlude. We now move on to the next chapter, where we discuss how some ideas from Chapter 2 can be applied to the case of extendible cardinals. In particular, we will combine our characterization of extendibles – in terms of simultaneous supercompactness and superstrongness – with the method of elementary chain constructions; we shall then see what happens.

CHAPTER 4

Extendible cardinals, Laver functions, and the GCH

In 1978, Richard Laver established a landmark result: he showed that the supercompactness of a cardinal κ can be made indestructible under $<\kappa$ - directed closed forcing (cf. [35]). For this, he introduced a type of partial functions on the cardinal κ which have the property of "anticipating" any arbitrary element of the universe, via appropriate supercompactness embeddings for κ. Such functions, which exist for any supercompact cardinal, are now called *Laver functions*.

Following Laver's innovation, a vast area of applications and extensions has emerged, giving many wonderful results during the years: Baumgartner used Laver functions in order to obtain the consistency of PFA (see Theorem 31.21 in [28]); Foreman, Magidor, and Shelah used analogous techniques for their proof of the consistency of MM (cf. [20]). Hamkins has considered several generalized Laver principles (cf. [25]) and has also introduced the *Lottery preparation* (cf. [24]); Corazza has shown the existence of Laver sequences for extendible, super–almost huge, superhuge cardinals, and more (cf. [11]). At the same time, there have been various results on indestructibility issues (e.g., by Apter et al. on the indestructibility of strongly compacts; cf. [2]). And all these are just to mention a few. Indeed, Laver functions have proven to be an extremely versatile and fruitful tool in the context of large cardinals.

In the first section of this chapter we consider the case of extendible cardinals, showing that such functions can be built recursively, using the framework introduced by Laver. Although this particular result is not a novelty, our proof uses a nice combination of the extendibility characterization from Chapter 2 (cf. Corollary 2.36) and of an elementary chain construction.

After that, and on the negative side, we argue that one cannot use such functions in order to derive indestructibility results for extendible cardinals, contrary to what happens in the case of supercompactness. The essential obstacle in the current setting is apparently the fact that every extendible cardinal is Σ_3-correct in the universe and thus, is able to reflect much information regarding the GCH patterns.

Along the way, we obtain one more characterization of extendibility and we also show that one may force the global GCH in the universe while preserving such cardinals. Let us begin.

4.1 Extendibility Laver functions

In Section 2.6 we characterized extendibility in terms of supercompactness and (λ-) superstongness (for $\lambda \geqslant \kappa$). We therefore define what it means to be a Laver function for an extendible cardinal as follows; recall that the three – dot notation is used to denote partial functions.

Definition 4.1. *Suppose that κ is extendible. A function $\ell \vdots \kappa \longrightarrow V_\kappa$ is an* **extendibility Laver function** *for κ if it satisfies the following requirements:*

(i) Every $\alpha \in dom(\ell)$ is supercompact but not extendible.

(ii) For every $\alpha \in dom(\ell)$, $\ell''\alpha \subseteq V_\alpha$.

(iii) For every (cardinal) $\lambda \geqslant \kappa$ and any $x \in H_{\lambda^+}$ there is an (extender) elementary embedding $j : V \longrightarrow M$ which is λ-supercompact and λ-superstrong for κ, and such that $j(\ell)(\kappa) = x$.

As in the case of ordinary Laver functions, we build an extendibility Laver function by recursively considering the "points of possible failure" of (*iii*), which is the essential requirement. The reader may consult [27] or [28] for a detailed account on such a proof strategy.

Theorem 4.2. *Every extendible cardinal carries an extendibility Laver function.*

Proof. We fix an extendible cardinal κ and some well – ordering $\lhd_\kappa$ of V_κ. For any $\alpha < \kappa$, assuming that $\ell \restriction \alpha$ has already been defined, we define the function ℓ at α only if the following conditions are satisfied:

(1) α is supercompact but not extendible and $\ell''\alpha \subseteq V_\alpha$.

(2) There exists some (cardinal) $\lambda \geqslant \alpha$ and some $x \in H_{\lambda^+}$ such that, for every (extender) elementary embedding $j : V \longrightarrow M$ which is λ-supercompact and λ-superstrong for α, $j(\ell \restriction \alpha)(\alpha) \neq x$.

In such a case, we fix λ_α, the least cardinal for which condition (2) holds (for some $x \in H_{\lambda_\alpha^+}$), and we then define $\ell(\alpha)$ to be the $\lhd_\kappa$-minimal element $x \in H_{\lambda_\alpha^+}$, among all witnesses to condition (2), for this fixed cardinal λ_α. Otherwise, we leave ℓ undefined at α. This concludes the recursive definition of the function.

Before showing that ℓ is indeed an extendibility Laver function, let us first check that, if condition (2) is satisfied for some $\alpha < \kappa$, then the least such λ_α must necessarily be below κ. This will justify the use of $\lhd_\kappa$ to pick the value $\ell(\alpha)$ and will consequently show that the range of ℓ is included in V_κ, i.e., the constructed function is indeed of the form $\ell : \kappa \longrightarrow V_\kappa$.

For this, we use the fact that every extendible cardinal is Σ_3-correct in the universe and show that condition (2) can be expressed as a Σ_3-statement in the parameters α and $\ell \restriction \alpha$. Let $\varphi(\alpha, \lambda, E)$ be a fixed formula asserting that "λ is a cardinal and E is a λ-supercompact and λ-superstrong extender for α". We remark here that the extender E can be either of the Martin–Steel form or an ordinary one (e.g., as in Corollary 2.37). Then, condition (2) is easily seen to be equivalent to the Σ_3-statement $\exists \lambda \exists x\, \psi(\lambda, x, \alpha, \ell \restriction \alpha)$, where the Π_2-formula $\psi(\lambda, x, \alpha, \ell \restriction \alpha)$ is as shown below:

$$\begin{aligned}\forall E, \mu, Y, Z\,(&\text{“}\mu \text{ is strong limit, } cf(\mu) > rank(E) > \lambda\text{”} \wedge \text{“}Z = V_\mu\text{”} \wedge \\ &\lambda, x, E, Y \in Z \wedge Z \models \text{“}Y = H_{\lambda^+}\text{”} \wedge x \in Y \wedge \\ &Z \models \varphi(\alpha, \lambda, E) \longrightarrow Z \models \text{“}j_E(\ell \restriction \alpha)(\alpha) \neq x\text{”}).\end{aligned}$$

Intuitively, μ being a strong limit of cofinality above all the relevant information makes V_μ capable of verifying faithfully both the fact that E is λ-supercompact and λ-superstrong for α, and also that $j_E(\ell \restriction \alpha)(\alpha) \neq x$.

We now show that the constructed $\ell : \kappa \longrightarrow V_\kappa$ is indeed an extendibility Laver function for κ. First of all, it is clear by its construction that conditions (i) and (ii) of Definition 4.1 are satisfied. We now check condition (iii).

Towards a contradiction, suppose that there exists a least cardinal $\lambda \geqslant \kappa$ and some $x \in H_{\lambda^+}$ such that condition (iii) of Definition 4.1 fails. Then, using the formula displayed above and for these fixed λ and x, we have that $\psi(\lambda, x, \kappa, \ell)$ is a true Π_2-statement asserting exactly this failure.

Now fix some $\theta \in C^{(2)}$ with $\theta > \lambda$ and let $\overline{\theta} > \theta$ be inaccessible. We need the following.

Claim. *There exists an elementary embedding $j : V \longrightarrow M$, witnessing the $\overline{\theta}$-supercompactness and $\overline{\theta}$-superstrongness of κ, with $j(\kappa)$ and $j(\overline{\theta})$ inaccessibles and with $M \models$ "κ is not extendible".*

Proof of claim. We refer to the arguments in the proof of Theorem 2.32. Let $h : V_{\overline{\theta}+1} \longrightarrow V_{h(\overline{\theta})+1}$ witness the $\overline{\theta}+1$-extendibility of κ, with image $h(\kappa) > \overline{\theta}+1$

as small as possible (and with $h(\kappa)$ and $h(\overline{\theta})$ inaccessibles).

It follows that $V_{h(\kappa)} \models$ "κ is not $\overline{\theta}+1$-extendible" since, otherwise, any witnessing embedding $i \in V_{h(\kappa)}$ would indeed be a $\overline{\theta}+1$-extendibility map for κ, as $h(\kappa) \in C^{(1)}$. But such an i would necessarily have $i(\kappa) < h(\kappa)$, contradicting the minimality of the latter.

Now derive E, the $(\kappa, h(\overline{\theta}))$-extender from h and consider the extender embedding $j = j_E : V \longrightarrow M_E$. As in Theorem 2.32, it now follows that j is $\overline{\theta}$-supercompact and $\overline{\theta}$-superstrong for κ, with $j(\kappa) = h(\kappa)$ and $j(\overline{\theta}) = h(\overline{\theta})$ inaccessibles. To conclude the proof of the claim, it follows by the previous paragraph that $V_{j(\kappa)} \models$ "κ is not extendible" and then, since $M_E \models j(\kappa) \in C^{(3)}$ by elementarity, we also have that $M_E \models$ "κ is not extendible" as desired. □

Having fixed such a $j : V \longrightarrow M$ given by the claim, we now show that, in M, $j(\ell)$ is defined at κ. For this, recall that $j(\ell) \restriction \kappa = \ell$, which implies that $j(\ell)''\kappa \subseteq V_\kappa$ in M. Moreover, since $V_{j(\kappa)} \subseteq M$, M has all the relevant normal fine measures and so $M \models$ "κ is $< j(\kappa)$-supercompact". Thus, since $M \models$ "$j(\kappa)$ is supercompact", κ is supercompact in M, whereas, by the above claim, κ is not extendible in M.

In addition, $V_{j(\kappa)} \models \theta \in C^{(2)}$ since Π_2-statements are downwards absolute to $C^{(1)}$-ordinals. Thus, again by $V_{j(\kappa)} \subseteq M$ and $M \models j(\kappa) \in C^{(2)}$, we have that $M \models \theta \in C^{(2)}$. Moreover, as $\theta \in C^{(2)}$, $V_\theta \models \psi(\lambda, x, \kappa, \ell)$ and thus, it follows that $M \models \psi(\lambda, x, \kappa, \ell)$. In fact, it is true in M that λ is the least cardinal for which $\psi(\lambda, x, \kappa, \ell)$ holds, for some $x \in H_{\lambda^+}$, i.e., in M, $\lambda = \lambda_\kappa$ in our notation. This is because, for any $\overline{\lambda} < \lambda$ and any $\overline{x} \in H_{\overline{\lambda}^+}$, it is true in V that there exists some appropriate $\overline{\lambda}$-supercompact and $\overline{\lambda}$-superstrong extender $\overline{E}$ for which $j_{\overline{E}}(\ell)(\kappa) = \overline{x}$. Since this is reflected in $V_\theta \subseteq M$, it is also true in M.

All this shows that conditions (1) and (2) of our recursive definition are satisfied in M, for $j(\ell) \restriction \kappa$ and κ. Thus, there is some $y \in H_{\lambda^+}$ such that $j(\ell)(\kappa) = y$ (where this y is chosen using the well-ordering $j(\lhd_\kappa)$ in M). Observe that, precisely by the definition of $j(\ell)(\kappa) = y$,

$$(\star) \qquad M \models \psi(\lambda, y, \kappa, j(\ell) \restriction \kappa)$$

(where $j(\ell) \restriction \kappa = \ell$) and it is this fact that will give us the desired contradiction.

We now use an elementary chain construction, using the fact that j is λ-supercompact and $\overline{\theta}$-superstrong, with $j(\kappa)$ and $j(\overline{\theta})$ inaccessibles. In the present construction, we slightly diverge from our treatment of elementary chains in the case of supercompactness (cf. Section 2.5) since here, we are aiming at a factor embedding which is, in addition, $\overline{\theta}$-superstrong. In particular, the ordinals β_ξ which we will consider along the chain will start above $j(\kappa)$ and will be bounded in (the inaccessible) $j(\overline{\theta})$. At any rate, the underlying ideas should be familiar. Let us start by picking some initial limit ordinal $\beta_0 \in (j(\kappa)\,,\,j(\overline{\theta}))$ and by letting

$$X_0 = \{j(f)(j\,''\lambda\,,x) :\ f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_{\overline{\theta}} \longrightarrow V,\ x \in V_{\beta_0}\} \prec M.$$

We also pick some $\gamma < j(\overline{\theta})$ with $cf(\gamma) > \lambda$, which will serve as the length of our chain. Then, for any $\xi+1 < \gamma$, given β_ξ and X_ξ, we let $\beta_{\xi+1} = \sup(X_\xi \cap j(\overline{\theta}))+\omega$ and

$$X_{\xi+1} = \{j(f)(j\,''\lambda\,,x) :\ f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_{\overline{\theta}} \longrightarrow V,\ x \in V_{\beta_{\xi+1}}\}.$$

If $\xi < \gamma$ is limit and we have already defined β_α and X_α for every $\alpha < \xi$, we let $\beta_\xi = \sup\limits_{\alpha<\xi} \beta_\alpha$ and

$$X_\xi = \bigcup_{\alpha<\xi} X_\alpha\,.$$

Finally, we let $\beta_\gamma = \sup\limits_{\alpha<\gamma} \beta_\alpha$ and

$$X_\gamma = \bigcup_{\alpha<\gamma} X_\alpha = \{j(f)(j\,''\lambda\,,x) :\ f \in V,\ f : \mathcal{P}_\kappa\lambda \times V_{\overline{\theta}} \longrightarrow V,\ x \in V_{\beta_\gamma}\} \prec M.$$

Note that in the definition of the $\beta_{\xi+1}$'s, $j(\overline{\theta})$ has replaced the $j(\kappa)$ which we used for our chain constructions in Chapter 2. Recalling then our arguments from Section 2.5, the inaccessibility of $j(\overline{\theta})$ gives that $\beta_\gamma < j(\overline{\theta})$, where clearly $cf(\beta_\gamma) = cf(\gamma) > \lambda$. As usual, we take the Mostowski collapse $\pi_\gamma : X_\gamma \cong M_\gamma$ and we then define the composed map $j_\gamma = \pi_\gamma \circ j : V \longrightarrow M_\gamma$, with $cp(j_\gamma) = \kappa$, producing a commutative diagram of elementary embeddings (with $k_\gamma = \pi_\gamma^{-1}$).

Along the lines of Section 2.5, one easily checks that j_γ is a λ-supercompact and $\overline{\theta}$-superstrong factor of j, where in fact, $cp(j_\gamma) = \kappa$, $j_\gamma(\kappa) = j(\kappa)$ and

$$cp(k_\gamma) = j_\gamma(\overline{\theta}) = \sup(X_\gamma \cap j(\overline{\theta})) = \beta_\gamma\,.$$

Furthermore, recall that (as in the proof of Theorem 2.27), for every $\alpha < j(\overline{\theta})$, we have that $j_\gamma(\alpha) < j(\overline{\theta})$ and then, the (Martin – Steel) extender E which is derived from j_γ and which witnesses its λ-supercompactness and $\overline{\theta}$-superstrongness, actually belongs to $V_{j(\overline{\theta})}$ and thus to M. Finally, again by the inaccessibility of $j(\overline{\theta})$, the model $V_{j(\overline{\theta})}$ can faithfully verify that E is a λ-supercompact and $\overline{\theta}$-superstrong extender for κ. Hence, M certainly thinks that "E is λ-supercompact and λ-superstrong for κ" and, moreover, it correctly computes the value $j_E(\ell)(\kappa)$. We are about to contradict $(\star)$ by showing that

$$j_\gamma(\ell)(\kappa) = j_E(\ell)(\kappa) = y.$$

For this, first of all and without loss of generality, the transitive set $Y \subseteq M_\gamma$ which serves as the support of the extender E, may be taken to include $j_\gamma(\ell)$, which is an element of $V^{M_\gamma}_{j_\gamma(\kappa)+1} = V_{j(\kappa)+1}$ (recall again the construction of such a Y in the proof of Theorem 2.27, and also the remarks following that proof). Then, as $k_E : M_E \longrightarrow M_\gamma$ is the identity on Y and commutes with $j_\gamma : V \longrightarrow M_\gamma$ and $j_E : V \longrightarrow M_E$, we will have that $j_\gamma(\ell) = j_E(\ell)$.

Consequently, it is enough to show that $j_\gamma(\ell)(\kappa) = y$. But this follows easily from the fact that j_γ is a factor of j with $cp(k_\gamma) = j_\gamma(\overline{\theta})$. Just notice that, by construction of j_γ, all the four: κ, λ, H_{λ^+} and y belong to V_{β_γ} and are, thus, fixed by the collapse π_γ. It now follows that

$$j_\gamma(\ell)(\kappa) = \pi_\gamma(j(\ell)(\kappa)) = \pi_\gamma(y) = y\,,$$

which gives the desired contradiction and concludes the proof. $\square$

Remark. Given the fact that extendible cardinals are involved, it is natural to ask whether we can require that the extendibility Laver function, apart from *anticipating* any prescribed set $x \in H_{\lambda^+}$ of the universe (meaning that $j(\ell)(\kappa) = x$ for some appropriate λ-supercompact and λ-superstrong embedding j), is such that $j(\kappa)$ is, in addition, an inaccessible cardinal (or even Mahlo, weakly compact, measurable, Woodin). Recall here the remark following Definition 2.34.

Indeed, we can "build-in" any of these extra properties, into the recursive construction of the Laver function. That is, for the case of, say, Woodin, we modify condition (2) of the recursion and require the following:

If there exists some (cardinal) λ and some $x \in H_{\lambda^+}$ so that, for every (extender) elementary embedding $j : V \longrightarrow M$ which is λ-supercompact and λ-superstrong for α and has $j(\alpha)$ Woodin, we have $j(\ell \restriction \alpha)(\alpha) \neq x$, *then* we fix the least λ for which this property holds (for some $x \in H_{\lambda^+}$) and we define $\ell(\alpha)$ as the $\lhd_\kappa$-minimal element $x \in H_{\lambda^+}$, among all possible candidates which satisfy this property, for this fixed λ.

It is now straightforward to check that, modulo the obvious changes (e.g., in the formula $\psi(\lambda\,, x\,, \alpha\,, \ell \restriction \alpha)$, we incorporate the extra clause "$j_E(\alpha)$ is Woodin"), the rest of the argument goes through smoothly. In particular, for the embedding j in the claim of the proof, we pick it in such a way that the $j(\kappa)$ is a Woodin cardinal; this is clear. Then, at the final step where we use the elementary chain argument, as $j_\gamma(\kappa) = j(\kappa)$, M indeed verifies that j_γ is, additionally, Woodin, and we thus arrive again at the desired contradiction.

From now on, if it ever becomes relevant, we will assume any of these extra properties (or even combinations of them as, e.g., Woodin measurable) as part of the definition of an extendibility Laver function. $\perp$

Arguably, the most important feature of the usual Laver function is the fact that the λ-supercompact embedding j may be chosen (with some extra care) so that, apart from anticipating the fixed set $x \in H_{\lambda^+}$, is such that the domain of $j(\ell)$ includes large empty spaces, in the sense that $dom(j(\ell)) \cap (\kappa, \lambda] = \varnothing$.

Recall that it is exactly this property which is exploited in an essential way in order to lift various embeddings through the Laver preparation, deriving indestructibility results for the supercompactness of κ.

We now show that, without loss of generality, this extra feature is enjoyed by extendibility Laver functions as well. For this, let us (very) temporarily denote by ℓ^* any extendibility Laver function which is built in the following way: we recursively construct the function as in the proof of Theorem 4.2 but, in addition to conditions (1) and (2) appearing there, we also impose the requirement:

$$(3) \quad \alpha \in dom(\ell^*) \Longrightarrow \alpha \geqslant \sup\{\lambda_\gamma : \gamma \in \alpha \cap dom(\ell^*)\},$$

where, according to our notation, λ_γ denotes the least (cardinal) for which there is a set $x \in H_{\lambda_\gamma^+}$ which is never anticipated by any λ_γ-supercompact and λ_γ-superstrong embedding for γ.

It is straightforward to check that this extra requirement does not interfere with the rest of the proof of Theorem 4.2, i.e., any such (modified) ℓ^* is still an extendibility Laver function for κ. We then have the following.

Lemma 4.3. *Suppose that κ is extendible and let $\ell^* : \kappa \longrightarrow V_\kappa$ be a modified extendibility Laver function as just described. Then, for every (cardinal) $\lambda \geqslant \kappa$ and every $x \in H_{\lambda^+}$, there is an (extender) elementary embedding $j : V \longrightarrow M$ which is λ-supercompact and λ-superstrong for κ, with $j(\ell^*)(\kappa) = x$ and with $dom(j(\ell^*)) \cap (\kappa, \lambda] = \varnothing$.*

Proof. Fix a cardinal $\lambda \geqslant \kappa$ and some $x \in H_{\lambda^+}$. Furthermore, fix some $\theta \in C^{(2)}$ with $\theta > \lambda$. Clearly, $x \in H_{\theta^+}$ and so, since ℓ^* is an extendibility Laver function, let $j : V \longrightarrow M$ be a θ-supercompact and θ-superstrong (extender) embedding for κ, with $j(\ell^*)(\kappa) = x$. Exactly as in our earlier arguments, $M \models \theta \in C^{(2)}$.

Now, $\kappa \in dom(j(\ell^*))$ and so let us call β the next ordinal in $dom(j(\ell^*))$ above κ; we will show that $\beta > \lambda$ which is what we want. For this, we appeal to the extra requirement (3) which we introduced to the recursive definition of the function ℓ^*.

Since $\theta \in C^{(2)}$, for every $\overline{\lambda} \leqslant \lambda$ and any $\overline{x} \in H_{\overline{\lambda}^+}$, the fact that $\overline{x}$ is anticipated by some $\overline{\lambda}$-supercompact and $\overline{\lambda}$-superstrong extender embedding for κ, is correctly reflected inside V_θ; then, in turn, it also holds in M, since $M \models \theta \in C^{(2)}$. Hence, as computed in M, the least cardinal α for which there is some $z \in H_{\alpha^+}$ which is never anticipated by any α-supercompact and α-superstrong extender embedding for κ has to be greater than λ, i.e., $\lambda_\kappa > \lambda$ in M. But then, the extra condition (3) which we imposed automatically implies that $\beta \geqslant \lambda_\kappa > \lambda$. □

Once we have extendibility Laver functions with the extra feature guaranteed by the last lemma, one naturally (or naively) hopes for indestructibility results using the standard framework of the Laver preparation. Nevertheless, as we have already advertised, an extendible cardinal κ can never be made indestructible (e.g., under $<\kappa$-directed closed forcing) by set forcing and we now turn to the proof of this fact.

Actually, we shall show that this "deficit" of extendible cardinals applies to any Σ_3-correct cardinal and to a wide range of forcing notions, although the property of $<\kappa$-directed closure serves as our motivational and prototypical example.

The key observation is the fact that Σ_3-correct cardinals reflect much information regarding the GCH patterns of the universe and thus, as one can do all sorts of modifications to the GCH above such a cardinal κ using a great variety of posets, one cannot expect to preserve at the same time the Σ_3-correctness of κ. For instance, we might already observe the following.

Lemma 4.4. *Suppose that $\kappa \in C^{(3)}$ and the* GCH *holds everywhere above κ. Then, for any $\lambda > \kappa$, the poset $Add(\lambda, \lambda^{++})$ destroys the Σ_2-correctness of κ.*

Proof. Note that the sentence "there exists some α such that, for all $\beta > \alpha$, $2^\beta = \beta^+$" is Σ_3 and clearly true in V, by the assumption on the GCH above κ. Hence, as it must hold in V_κ as well, fix some $\alpha < \kappa$ so that the GCH holds above α in the structure V_κ.

But now it is clear that, for any $\lambda > \kappa$, after forcing with the canonical poset $\mathbb{P} = Add(\lambda, \lambda^{++})$, we make the GCH fail at λ while preserving all of V_κ (and so, in particular, the GCH pattern below κ as well). Hence, in $V^{\mathbb{P}}$, κ cannot be Σ_2-correct anymore, since the Σ_2-statement "there exists some $\lambda > \alpha$ such that the GCH fails at λ" is not reflected correctly in V_κ. $\square$

Our next goal is to show that, in the case of extendibility, the GCH assumption is harmless as one can always force the global GCH while preserving extendible cardinals. This sort of preservation is already known for other large cardinal notions: R.B. Jensen inaugurated the list with the case of measurables; T.K. Menas then did it for supercompacts, and J.D. Hamkins followed with I1 embeddings (see [27] for the first two, and [22] for the third case). More recently, similar results were obtained for versions of superstrongs, due to S.D. Friedman, and for 1-extendibles and Vopěnka's Principle, due to A. Brooke – Taylor (see [10] and [9] respectively).

Towards this goal, we now turn to a different characterization of extendibility, one which better suits our current purposes.

4.2 (yet) Another characterization

Motivated by the analogous result for 1-extendibility given in [8], we give a straightforward generalization of the same idea to cover full extendibility.

Proposition 4.5. *Let* κ *be a cardinal and fix some* $\lambda = \beth_\lambda \geqslant \kappa$. *Then,* κ *is* $\lambda + 1$*-extendible if and only if there exists some* (*cardinal*) μ *and an elementary embedding* $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ *with* $cp(j) = \kappa$ *and* $j(\kappa) > \lambda + 1$.

Proof. Suppose that κ is $\lambda + 1$-extendible, for some fixed $\lambda = \beth_\lambda \geqslant \kappa$, and let $h : V_{\lambda+1} \longrightarrow V_{h(\lambda)+1}$ be a witnessing elementary embedding, with $cp(h) = \kappa$ and $h(\kappa) > \lambda + 1$. Let us put $\mu = h(\lambda)$, which is clearly a cardinal. We shall use h in order to define an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$, such that $cp(j) = \kappa$ and $j \restriction (\lambda + 1) = h \restriction (\lambda + 1)$. For this, we employ ordinary coding techniques in order to describe any set in H_{λ^+} using appropriate codes in $V_{\lambda+1}$.

Let $x \in H_{\lambda^+}$ and fix some (any) bijection $f_x : |trcl(\{x\})| \longrightarrow trcl(\{x\})$. Consider the binary relation E_x on $dom(f_x)$ which is defined so that, for any $\alpha, \beta \in dom(f_x)$,

$$\langle \alpha , \beta \rangle \in E_x \Longleftrightarrow f_x(\alpha) \in f_x(\beta) .$$

Notice that, since $dom(f_x)$ is some cardinal $\leqslant \lambda$, $E_x \in V_{\lambda+1}$ and, moreover, if $\langle T_x , \in \rangle$ is the Mostowski collapse of $\langle dom(f_x) , E_x \rangle$, then $T_x \cong trcl(\{x\})$ and thus, as both are transitive sets, $T_x = trcl(\{x\})$. It is clear that from T_x one may recover easily the set x, it being the unique maximal element of T_x under "$\in$".

This procedure gives a way of coding sets $x \in H_{\lambda^+}$ by corresponding sets $E_x \subseteq \eta \times \eta$, for some cardinal $\eta \leqslant \lambda$, with $E_x \in V_{\lambda+1}$. Then we may define, inside $V_{\lambda+1}$, a class $\mathcal{C}_\lambda \subseteq V_{\lambda+1}$ consisting exactly of such coding sets; that is, $X \in \mathcal{C}_\lambda$ if and only if X is a well-founded, extensional, binary relation on some cardinal $\eta \leqslant \lambda$, (where η is the union of the domain and the range of X) and so that, if $X \neq \varnothing$, then X has a unique maximal element.

Now, for any $X \in \mathcal{C}_\lambda$, let $fld(X)$ denote the field of X, i.e., the cardinal η which equals the union of the domain and the range of X and, moreover, if $X \neq \varnothing$, let $max(X)$ denote the unique ordinal in $fld(X)$ which is maximal with respect to the relation X. Next, for any X and Y in $\mathcal{C}_\lambda$, define the relation "$=^*$" by declaring that

$$X =^* Y \Longleftrightarrow \exists f \, (f : \langle fld(X) , X \rangle \xrightarrow{iso} \langle fld(Y) , Y \rangle),$$

i.e., the two relations are isomorphic. Also, for any $X \in \mathcal{C}_\lambda$ and any $a \in fld(X)$, let $X_a = \bigcup_{n \in \omega} A_n^{(a)}$ where $A_0^{(a)} = \{\langle x , a \rangle \in X : x \in fld(X)\}$ and, recursively for

$n \in \omega$, $A^{(a)}_{n+1} = \{\langle x, z\rangle \in X : z \in dom(A_n)\}$. Now define the relation "$\in^*$" by stipulating that

$$X \in^* Y \iff \exists a \in fld(Y)\,(\langle a, max(Y)\rangle \in Y \wedge X =^* Y_a).$$

Clearly, the relations "$=^*$" and "$\in^*$" are definable in $V_{\lambda+1}$. For any $X \in \mathcal{C}_\lambda$, let us denote its Mostowski collapse by $\pi(X)$. Then, one easily checks that, for any X and Y in $\mathcal{C}_\lambda$,

$$X =^* Y \iff \pi(X) = \pi(Y)$$

and

$$X \in^* Y \iff \pi(max(X)) \in \pi(max(Y)).$$

By the way, observe that given any $x \in H_{\lambda^+}$ and any code $E_x \in \mathcal{C}_\lambda$ arising from some bijection $f_x : |trcl(\{x\})| \longrightarrow trcl(\{x\})$, we have that $x = \pi(max(E_x))$.

All the above make us capable of translating any first-order formula φ whose parameters range over H_{λ^+}, into an equivalent formula φ^* whose parameters range over $V_{\lambda+1}$: we replace any $x \in H_{\lambda^+}$ by a corresponding code $E_x \in \mathcal{C}_\lambda$ for it; we replace the standard set-theoretic relations "$=$" and "$\in$" by the definable relations "$=^*$" and "$\in^*$" respectively; finally, quantification is taken to range over $\mathcal{C}_\lambda$. That is, for any first-order formula $\varphi(v_1, \ldots, v_n)$ and any $x_i \in H_{\lambda^+}$, for $1 \leqslant i \leqslant n$,

$$H_{\lambda^+} \models \varphi(x_1, \ldots, x_n) \iff V_{\lambda+1} \models \varphi^*(E_{x_1}, \ldots, E_{x_n}),$$

for some (any) bijections $f_{x_i} : |trcl(\{x_i\})| \longrightarrow trcl(\{x_i\})$, for $1 \leqslant i \leqslant n$. It is also clear that all the above can be done equally well for μ in place of λ. In fact, we have a class $\mathcal{C}_\mu$ which is definable in $V_{\mu+1}$ by the exact same definition, with the only difference being that one replaces λ by μ, i.e., a coding set X is now an appropriate relation on some cardinal $\eta \leqslant \mu$. Obviously, $\mathcal{C}_\lambda \subseteq \mathcal{C}_\mu$. Of course, the definitions of $fld(X)$, $max(X)$, and the relations $=^*$ and $\in^*$ remain the same. Hence, we similarly have a translation of any first-order formula φ whose parameters range over H_{μ^+}, into an equivalent formula whose parameters range over $V_{\mu+1}$. Abusing the notation slightly, we again call φ^* this translation, keeping in mind that the quantification now ranges over the class $\mathcal{C}_\mu$ as defined in $V_{\mu+1}$.

At this point, by the elementarity of h, for any X and Y in $\mathcal{C}_\lambda$,

$$V_{\lambda+1} \models X =^* Y \iff V_{\mu+1} \models h(X) =^* h(Y)$$

and

$$V_{\lambda+1} \models X \in^* Y \iff V_{\mu+1} \models h(X) \in^* h(Y).$$

Then, inductively, for any formula $\varphi(v_1, \ldots, v_n)$ and $X_i \in \mathcal{C}_\lambda$, for $1 \leqslant i \leqslant n$, $V_{\lambda+1} \models \varphi^*(X_1, \ldots, X_n)$ if and only if $V_{\mu+1} \models \varphi^*(h(X_1), \ldots, h(X_n))$. We now define the map $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ by letting, for every $x \in H_{\lambda^+}$,

$$j(x) = \pi(max(h(E_x))),$$

for some (any) bijection $f_x : |trcl(\{x\})| \longrightarrow trcl(\{x\})$, giving rise to the code E_x. We evidently have that $j \restriction (\lambda+1) = h \restriction (\lambda+1)$. Let us finally check that j is an elementary embedding. For this, fix any formula $\varphi(v_1, \ldots, v_n)$, any $x_i \in H_{\lambda^+}$ and any corresponding codes $E_{x_i} \in V_{\lambda+1}$, for $1 \leqslant i \leqslant n$. We have the following equivalences:

$$\begin{aligned} H_{\lambda^+} \models \varphi(x_1, \ldots, x_n) &\iff V_{\lambda+1} \models \varphi^*(E_{x_1}, \ldots, E_{x_n}) \\ &\iff V_{\mu+1} \models \varphi^*(h(E_{x_1}), \ldots, h(E_{x_n})) \\ &\iff H_{\mu^+} \models \varphi(j(x_1), \ldots, j(x_n)), \end{aligned}$$

which conclude the proof of the forward direction of the proposition.

Conversely, suppose that for some $\lambda = \beth_\lambda \geqslant \kappa$ and some cardinal μ, we have an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $cp(j) = \kappa$ and $j(\kappa) > \lambda + 1$. Clearly, $j(\lambda) = \mu$. Furthermore, as λ is a beth fixed point, $V_{\lambda+1}$ is a definable class in H_{λ^+}, namely,

$$V_{\lambda+1} = \{x \in H_{\lambda^+} : x \subseteq H_\lambda\}.$$

This means that we may relativize any first-order formula to $V_{\lambda+1}$, within H_{λ^+}. Of course, the analogous facts are true for $V_{\mu+1}$ and H_{μ^+} correspondingly. By these observations, one easily verifies that $j \restriction V_{\lambda+1} : V_{\lambda+1} \longrightarrow V_{\mu+1}$ is an elementary embedding witnessing the $\lambda+1$-extendibility of κ. □

As an immediate corollary, we get the following characterization of extendibility in terms of the H_λ's.

Corollary 4.6. *A cardinal κ is extendible if and only if for all $\lambda = \beth_\lambda \geqslant \kappa$, there is some (cardinal) μ and an elementary embedding $j : H_{\lambda^+} \longrightarrow H_{\mu^+}$ with $cp(j) = \kappa$ and $j(\kappa) > \lambda + 1$.* □

4.3 Forcing the global GCH

We now use the characterization just obtained in order to show that the global GCH can be forced while preserving extendible cardinals. For this, we shall use a *class length* forcing iteration $\mathbb{P}$; intuitively, this will just be the Easton iteration of the "obvious" canonical posets $Add(\alpha^+, 1)$, which will successively force the GCH at every infinite cardinal α of the universe.

It is true that we have not accounted for the nuances related to class forcing in the prelude. One excuse is that we shall not be needing the full generality of the theory of class forcing. Another excuse is that this is the only place in the text where such a forcing is used. In this sense, the current section and especially Theorem 4.8 can be thought of as another local methodological digression, just like Section 2.4 was for Chapter 2.

Withal, the reader is reminded that our treatment here follows closely [8], where a very informative presentation of such class iterations is given; in particular, one may find details regarding the following definition of the poset which we will employ, as well as several of its properties which we will invoke. We hope that this reference compensates for the lack of an introductory exposition.

Definition 4.7. *The* **canonical forcing $\mathbb{P}$ for the global** GCH *is the class length reverse Easton iteration of* $\langle \dot{\mathbb{Q}}_\alpha : \alpha \in \mathbf{ON} \rangle$, *where* $\mathbb{P}_0 = \{\mathbb{1}\}$ *and, for each* α, *if* α *is an infinite cardinal in* $V^{\mathbb{P}_\alpha}$, *then* $\dot{\mathbb{Q}}_\alpha$ *is the canonical* $\mathbb{P}_\alpha$ *- name for the poset* $Add(\alpha^+, 1)^{V^{\mathbb{P}_\alpha}}$. *At limit stages we take direct limits at inaccessibles, and inverse limits otherwise. Finally,* $\mathbb{P}$ *is the direct limit of the* $\mathbb{P}_\alpha$ *'s, for* $\alpha \in \mathbf{ON}$.

The iteration $\mathbb{P}$ preserves ZFC, preserves inaccessible cardinals and forces the GCH everywhere. Moreover, at any inaccessible cardinal α, the iteration factors as $\mathbb{P}_\alpha * \mathbb{P}_{\text{tail}}$, where $|\mathbb{P}_\alpha| = \alpha$ and $\mathbb{P}_{\text{tail}}$ is (forced to be) $\leqslant \alpha$ - directed closed. It is also known (see [15] for details) that the weak homogeneity[‡] of the individual GCH forcings carries over to the whole, class – length iteration and any initial segment of it. We are now ready to prove the following.

Theorem 4.8. *Every extendible cardinal* κ *is preserved by the canonical forcing* $\mathbb{P}$ *for the global* GCH.

Proof. Fix an extendible cardinal κ and further fix some inaccessible $\lambda > \kappa$. By the results of the previous section, let $j : H_{\lambda^+} \longrightarrow H_{j(\lambda)^+}$ be an embedding witnessing the $\lambda + 1$ - extendibility of κ in V; that is, $cp(j) = \kappa$, $j(\kappa) > \lambda + 1$ and $j(\lambda)$ inaccessible.

Let G be $\mathbb{P}$ - generic over V; it is our aim to show that this ground model embedding j lifts to an embedding of the form $j : H^{V[G]}_{\lambda^+} \longrightarrow H^{V[G]}_{j(\lambda)^+}$, witnessing the $\lambda + 1$ - extendibility of κ in $V[G]$, which will be enough in order to conclude the theorem. For this, we factor the whole forcing iteration as

$$\mathbb{P}_\kappa * \dot{\mathbb{P}}_{[\kappa\, ,\, \lambda)} * \dot{\mathbb{P}}_{[\lambda\, ,\, \infty)}\ ,$$

[‡] Recall that a poset $\mathbb{Q}$ is called *weakly homogeneous* if for any two $p\, , q \in \mathbb{Q}$ there is an automorphism $\sigma : \mathbb{Q} \to \mathbb{Q}$ such that $\sigma(p)$ and q are compatible.

where the notation should be self–explanatory; e.g., $\dot{\mathbb{P}}_{[\kappa,\lambda)}$ is the ($\mathbb{P}_\kappa$-name for the) partial iteration of forcings which occur at stages between κ and λ. We will lift the ground model embedding in two (and a half) steps, according to the above factorization. Throughout the proof, let us denote by G_κ, $G_{[\kappa,\lambda)}$ and, more generally, $G_{(\alpha,\beta)}$, the corresponding *projections* of the generic filter G, which are generics for the corresponding partial iterations of $\mathbb{P}$. With the expectation that this does not produce any confusion, whenever we drop the "dot" from any factor of the iteration, it means that we are considering the corresponding poset in the current, partial, generic extension of V, arising from the interpetation of the corresponding name by the current, partial, generic filter; e.g., $\mathbb{P}_{[\kappa,\lambda)} = (\dot{\mathbb{P}}_{[\kappa,\lambda)})_{G_\kappa}$, considered in the partial generic extension $V[G_\kappa]$.

As our first step, we lift through the initial forcing $\mathbb{P}_\kappa$, where we observe that $\mathbb{P}_\kappa \in H_{\kappa^+}$ and thus, G_κ is certainly $\mathbb{P}_\kappa$-generic over H_{λ^+}. Accordingly, the partial filter $G_{j(\kappa)}$ is $\mathbb{P}_{j(\kappa)}$-generic over $H_{j(\lambda)^+}$, where $\mathbb{P}_{j(\kappa)} = j(\mathbb{P}_\kappa) \in H_{j(\lambda)^+}$. Since the forcing $\mathbb{P}_\kappa$ is a direct limit and $cp(j) = \kappa$, it is easily checked that $j\,''G_\kappa \subseteq G_{j(\kappa)}$ and hence we may indeed perform the first lift of the embedding:

$$j : H_{\lambda^+}[G_\kappa] \longrightarrow H_{j(\lambda)^+}[G_{j(\kappa)}].$$

For the second step, it is our aim to lift further through the forcing $\mathbb{P}_{[\kappa,\lambda)}$, i.e., through $(\dot{\mathbb{P}}_{[\kappa,\lambda)})_{G_\kappa}$. We remark that this makes sense, since $\dot{\mathbb{P}}_{[\kappa,\lambda)}$ has size λ and $\mathbb{P}_{[\kappa,\lambda)} \in H_{\lambda^+}[G_\kappa]$. Now, it is clear that $G_{[\kappa,\lambda)}$ is $\mathbb{P}_{[\kappa,\lambda)}$-generic over $H_{\lambda^+}[G_\kappa]$. Similarly, $G_{[j(\kappa),j(\lambda))}$ is $\mathbb{P}_{[j(\kappa),j(\lambda))}$-generic over $H_{j(\lambda)^+}[G_{j(\kappa)}]$, where, by elementarity, $\mathbb{P}_{[j(\kappa),j(\lambda))} = j(\mathbb{P}_{[\kappa,\lambda)})$.

Thus, the only problem in performing the second lift, is to ensure that the lifting criterion $j\,''G_{[\kappa,\lambda)} \subseteq G_{[j(\kappa),j(\lambda))}$ is satisfied. For this, although one may indeed find a relevant master condition r as we shall soon see, it is not necessarily the case that our (fixed beforehand) generic G, is such that its segment $G_{[j(\kappa),j(\lambda))}$ contains this condition. So, our plan will be to first find a master condition r and then argue, using the (weak) homogeneity of the GCH forcings, that we may modify appropriately the part $G_{[j(\kappa),j(\lambda))}$ of our generic filter in order to produce, in $V[G]$, a filter G^* which will contain the condition r, which will be $\mathbb{P}_{[j(\kappa),j(\lambda))}$-generic over $H_{j(\lambda)^+}[G_{j(\kappa)}]$ and so that $V[G_{j(\lambda)}] = V[G_{j(\kappa)}][G^*]$, i.e., it will result in the same generic extension for the forcing at hand.

In order to find the master condition, recall that $\mathbb{P}_{[\kappa,\lambda)}$ has size λ in $H_{\lambda^+}[G_\kappa]$ (and so in $H_{j(\lambda)^+}[G_{j(\kappa)}]$ as well) an also, we clearly have that $j\,''\lambda \in H_{j(\lambda)^+}[G_{j(\kappa)}]$. Therefore, as $G_{[\kappa,\lambda)}$ appears explicitly in the partial filter $G_{j(\kappa)}$, we may combine $j\,''\lambda$ with some enumeration of $\mathbb{P}_{[\kappa,\lambda)}$ in order to get that $j \restriction \mathbb{P}_{[\kappa,\lambda)} \in H_{j(\lambda)^+}[G_{j(\kappa)}]$; thus, $j\,''G_{[\kappa,\lambda)} \in H_{j(\lambda)^+}[G_{j(\kappa)}]$ as well (and has size λ there). Now, since $j\,''G_{[\kappa,\lambda)}$ is a directed subset of $\mathbb{P}_{[j(\kappa),j(\lambda))}$ and the latter is $\leqslant j(\kappa)$-directed closed in $H_{j(\lambda)^+}[G_{j(\kappa)}]$, there is indeed a lower bound for $j\,''G_{[\kappa,\lambda)}$, i.e., there exists some

$r \in \mathbb{P}_{[j(\kappa), j(\lambda))}$ with $r \leqslant j''G_{[\kappa, \lambda)}$; this is the desired master condition. As we have pointed out, there is no reason to expect that $r \in G_{[j(\kappa), j(\lambda))}$. We now modify the filter $G_{[j(\kappa), j(\lambda))}$ in order to produce the appropriate G^*, with $r \in G^*$.

Working for the moment in the model $H_{j(\lambda)^+}[G_{j(\kappa)}]$, since $\mathbb{P}_{[j(\kappa), j(\lambda))}$ is weakly homogeneous, the set of conditions t for which there exists some automorphism $e : \mathbb{P}_{[j(\kappa), j(\lambda))} \longrightarrow \mathbb{P}_{[j(\kappa), j(\lambda))}$ with $e(t) \leqslant r$ is dense. Using the fact that $G_{[j(\kappa), j(\lambda))}$ is generic, we may find such a condition $t \in G_{[j(\kappa), j(\lambda))}$. Note that this t cannot be found working in $H_{j(\lambda)^+}[G_{j(\kappa)}]$, since we are appealing to the further generic filter $G_{[j(\kappa), j(\lambda))}$; even so, it indeed exists and it certainly belongs to $H_{j(\lambda)^+}[G_{j(\kappa)}]$, together with the corresponding automorphism e. Then, by standard forcing facts (cf. Chapter VII, Theorem 7.11 in [33]) it follows that, if we let G^* be the filter generated by the pointwise image $e''G_{[j(\kappa), j(\lambda))}$, then G^* is $\mathbb{P}_{[j(\kappa), j(\lambda))}$-generic over $H_{j(\lambda)^+}[G_{j(\kappa)}]$ with $r \in G^*$ and, moreover,

$$H_{j(\lambda)^+}[G_{j(\lambda)}] = H_{j(\lambda)^+}[G_{j(\kappa)}][G^*],$$

i.e., we have succeeded in finding the appropriate filter which contains our fixed master condition. We may thus conclude the second lift of the embedding, obtaining

$$j : H_{\lambda^+}[G_\kappa][G_{[\kappa, \lambda)}] \longrightarrow H_{j(\lambda)^+}[G_{j(\kappa)}][G^*],$$

or, equivalently,

$$j : H_{\lambda^+}[G_\lambda] \longrightarrow H_{j(\lambda)^+}[G_{j(\lambda)}].$$

As the final (half) step of the argument, we show that the currently lifted embedding is sufficient in order to witness the $\lambda+1$-extendibility of κ in the extension $V[G]$. For this, we argue that, in fact,

$$H_{\lambda^+}^{V[G]} = H_{\lambda^+}^{V[G_\lambda]} = H_{\lambda^+}[G_\lambda].$$

First, notice that the rest of the iteration above λ, that is, $\mathbb{P}_{[\lambda, \infty)}$ is (forced to be) $\leqslant \lambda$-closed and so it does not affect H_{λ^+}; i.e., $H_{\lambda^+}^{V[G]} = H_{\lambda^+}^{V[G_\lambda]}$.

Let us now check that $H_{\lambda^+}^{V[G_\lambda]} = H_{\lambda^+}[G_\lambda]$ as well. We remark that the latter structure, being a generic extension of the ZFC^- model H_{λ^+} by the generic filter G_λ, is also a ZFC^- model. As the right – to – left inclusion is clear, we fix any element $X \in H_{\lambda^+}^{V[G_\lambda]}$ and we want to find an appropriate $\mathbb{P}_\lambda$-name witnessing that $X \in H_{\lambda^+}[G_\lambda]$. But then, exactly as in the proof of Proposition 4.5, X can be obtained in $V[G_\lambda]$ by the Mostowski collapse of some appropriate coding subset of $\lambda \times \lambda$ (where recall that the whole process did not use the Powerset Axiom).

Since all subsets of $\lambda \times \lambda$ in $V[G_\lambda]$ have nice names which lie in H_{λ^+} (essentially because, due to the size of $\mathbb{P}_\lambda$, all its antichains belong to H_{λ^+}), we get that all such coding subsets of $\lambda \times \lambda$ belong to $H_{\lambda^+}[G_\lambda]$. Therefore, $X \in H_{\lambda^+}[G_\lambda]$ since it can be retrieved there by the Mostowski collapse of its code.

In a totally analogous manner, we have that $H^{V[G]}_{j(\lambda)^+} = H_{j(\lambda)^+}[G_{j(\lambda)}]$ as well; hence, the lifted embedding is indeed of the form $j : H^{V[G]}_{\lambda^+} \longrightarrow H^{V[G]}_{j(\lambda)^+}$ and the proof is complete. $\square$

After all the preceding discussion, we can finally return to the issue of indestructibility and show that extendible cardinals do not enjoy such niceties. In this setting, we temporarily fix a broad ambient family of posets.

Let us arbitrarily declare that a property R of posets is *cofinally sympathetic to non*–GCH, if for all β there exists some (cardinal) $\alpha > \beta$ so that the canonical poset $Add(\alpha, \alpha^{++})$ (which kills the GCH at α) satisfies R.

Typical examples are intended to be the various closure properties, such as being "κ-directed closed", for some regular cardinal κ; all these are certainly cofinally sympathetic to non–GCH. On the other hand, chain conditions are not of this sort, as we cannot expect them to hold cofinally in the ordinals. To take the extreme example, we may consider the property of being "c.c.c." which, except for the basic Cohen forcings for adding subsets of ω, it is never satisfied by the canonical (killing) GCH posets.

As a direct combination of Theorem 4.8 and of the easy observation stated in Lemma 4.4, we can now show the following.

Corollary 4.9. *If κ is extendible, then no (set) forcing which preserves the Σ_3-correctness of κ can make its Σ_2-correctness indestructible under posets satisfying R, for any property R which is cofinally sympathetic to non*–GCH.

Proof. Fix some property R which is cofinally sympathetic to non–GCH and assume, towards a contradiction, that there is a (set) forcing notion $\mathbb{P}$ which preserves the Σ_3-correctness of κ and which makes its Σ_2-correctness indestructible under R.

By Theorem 4.8, we may force to get a model in which κ is extendible and the global GCH holds. Then, we perform the purported forcing $\mathbb{P}$ and we thus obtain a model V in which κ is Σ_3-correct and –allegedly– an indestructible Σ_2-correct cardinal. Noticing that $\mathbb{P}$ is a set forcing, the GCH pattern for sufficiently large cardinals of the universe is not altered; that is, for every $\alpha > |\mathbb{P}|$, $2^\alpha = \alpha^+$ holds in V. Thus, there is some $\beta < \kappa$ so that, for all $\alpha \in (\beta, \kappa)$, $2^\alpha = \alpha^+$.

Now let $\gamma > max\{\kappa, |\mathbb{P}|\}$ be such that $\mathbb{Q} = Add(\gamma, \gamma^{++})$ satisfies property R. But then, forcing with $\mathbb{Q}$ preserves the whole of V_κ, and hence the GCH pattern below κ, while at the same time it kills the GCH at γ. This means that the Σ_2-statement "there exists some $\alpha > \beta$ so that the GCH fails at α", is not reflected correctly in the V_κ of $V^{\mathbb{Q}}$. This is a contradiction. $\square$

It follows that, not only can we not make an extendible cardinal κ indestructible, but also, forcing the global GCH makes it extremely "destructible": any poset killing the GCH above κ kills many of its large cardinal properties (e.g., κ can no longer be supercompact or strong, as it is not even Σ_2-correct). In addition, the same argument shows that any poset killing the GCH above κ kills the Σ_2-correctness of all $C^{(2)}$-cardinals up to κ as well. Note the analogy between this observation and the example given in Chapter 3, commenting on the non–existence of a local definition for the class $C^{(2)}$.

Recalling Lemma 4.4, similar observations apply to any Σ_3-correct cardinal κ which is compatible with the *eventual* GCH in the universe; for any such cardinal, apart from the lack of indestructibility results, we moreover have that under the assumption of the eventual GCH, say above some cardinal θ, any poset killing the GCH above θ kills the Σ_2-correctness of κ (and, also, the Σ_2-correctness of many $C^{(2)}$-cardinals below κ as well).

As a final comment, we mention a different way of killing the extendibility of a cardinal κ, while preserving its inaccessibility: we start by forcing the global GCH in the universe and we then perform an Easton forcing $\mathbb{Q}$ to kill the GCH at every *regular* cardinal below κ; such a forcing preserves cofinalities and, since κ is Mahlo, it is also κ-c.c. (see Chapter VIII, § 4 in [33]). In the resulting model, the GCH fails at every regular below κ while it continues to hold everywhere above it. Consequently, κ cannot remain Σ_2-correct since the statement "the GCH holds at some regular α" is not reflected correctly. In the present setting, this example hopefully does some (partial) justice to chain conditions, which were neglected by properties that are cofinally sympathetic to non – GCH.

We now abandon indestructibility matters and turn to some applications of extendibility Laver functions, in the next (and final) chapter of this dissertation.

CHAPTER 5

Resurrection Axioms

The *Resurrection Axioms*, introduced in unpublished work by J.D. Hamkins and T. Johnstone (cf. [29]), are motivated by several generic absoluteness results and their limitations[↺]. For instance, it is well – known (cf. [3] and, independently, [44]) that Martin's Axiom (MA) is equivalent to asserting that, for any c.c.c. poset $\mathbb{P}$,

$$H_{\mathfrak{c}} \prec_1 H_{\mathfrak{c}}^{V^{\mathbb{P}}},$$

where recall that, in such a setting and according to our convention, each occurence of the symbol $\mathfrak{c}$ stands for the relativized size of the continuum, as it is independently computed in the models V and $V^{\mathbb{P}}$, respectively. For another example, the Bounded Proper Forcing Axiom (BPFA) is equivalent to asserting that, for any proper poset $\mathbb{P}$,

$$H_{\aleph_2} \prec_1 H_{\aleph_2}^{V^{\mathbb{P}}}.$$

Likewise for Bounded Martin's Maximum (BMM), replacing proper posets by stationary preserving ones in the above statement. For more details on characterizations of this sort, see [4]. Using a uniform notation for such absoluteness results, given any (definable) regular κ, any fixed n and any (definable) class Γ of posets,

$$\mathcal{A}(H_\kappa\,,\,\Sigma_n\,,\,\Gamma)$$

denotes the assertion that, for any poset $\mathbb{P} \in \Gamma$, H_κ is a Σ_n - elementary substructure of $H_\kappa^{V^{\mathbb{P}}}$. Hence, e.g., $\mathcal{A}(H_{\aleph_1}\,,\,\Sigma_1\,,\,\text{all posets})$ is true by the Lévy – Shoenfield absoluteness theorem, while BPFA is equivalent to $\mathcal{A}(H_{\aleph_2}\,,\,\Sigma_1\,,\,\text{proper})$.

Unfortunately, such absoluteness statements quickly run into inconsistency, already for $\kappa = \aleph_2$. For example, if (some) posets in Γ collapse ω_1, then it is clear that $\mathcal{A}(H_{\aleph_2}\,,\,\Sigma_1\,,\,\Gamma)$ is inconsistent. Moreover, considering $n > 1$ leads to

[↺] W.H. Woodin's work related to the *stationary tower* forcing provides further background and early considerations of the phenomenon of resurrection; see also the Introduction.

the same problems, with $\mathcal{A}(H_{\aleph_2}, \Sigma_2, \Gamma)$ being false for most classes of forcing notions; e.g., $\mathcal{A}(H_{\aleph_2}, \Sigma_2, \sigma\text{-closed})$ cannot hold if CH fails.

In the particular case of the forcing axiom BPFA, note that we may not replace Σ_1 by Σ_2 in the above characterization, since it is known that BPFA implies $\mathfrak{c} = \aleph_2$, whereas the canonical forcing $Add(\omega_1, 1)$ which forces CH is certainly proper. In addition, one may not replace $\aleph_2$ by $\mathfrak{c}$ either, since the use of the canonical poset $\mathbb{Q} = Add(\omega_1, 1)$ again leads to a contradiction: just observe that $\aleph_1^V = \aleph_1^{V^{\mathbb{Q}}} = \mathfrak{c}^{V^{\mathbb{Q}}}$ and hence $H_{\mathfrak{c}} \not\subseteq H_{\mathfrak{c}}^{V^{\mathbb{Q}}}$.

All these limitations motivate the idea of resurrection from this perspective; that is, we require the existence of an appropriate (name for a) poset $\dot{\mathbb{R}}$ such that, by *further* forcing by it, we resurrect the full elementarity of the structure $H_{\mathfrak{c}}$ into that of the whole forcing extension, i.e.,

$$H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}.$$

With these ideas in mind, let us proceed to the formal definition, as it is introduced by Hamkins and Johnstone.

Definition 5.1 ([29]). *For any (definable) class Γ of posets, the* **Resurrection Axiom** *for Γ, denoted by* RA(Γ)*, is the assertion that for any $\mathbb{Q} \in \Gamma$, there exists a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$, with $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", such that*

$$H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}.$$

Several axioms of this sort are studied in [29], by either varying the class Γ or by considering *weak resurrection*, where one does not impose any requirement on the further (name for a) poset $\dot{\mathbb{R}}$. Among other results, it is shown, for instance, that "RA(proper) + $\neg$CH" has consistency strength below a Mahlo (precisely, that of an uplifting cardinal; see Definition 1.4).

On the other hand, the resurrection axiom for the class Γ of posets which preserve the stationary subsets of ω_1 is not dealt with in [29]; our first goal is to study this case in the next section.

5.1 The case of stationary preserving posets

In what follows, we consistently denote by RA(stat. pres.) the resurrection axiom for the class of posets which preserve the stationary subsets of ω_1. We shall show that its consistency follows from the consistency of (the existence of) an extendible cardinal. For this, we use the techniques of Foreman, Magidor and Shelah (cf. [20]), who obtained the consistency of MM from that of (the existence of) a supercompact cardinal.

In our situation, we replace the supercompactness assumption by extendibility and we then use an extendibility Laver function in order to define the forcing iteration along the lines of [20]. Let us first recall the definitions of the relevant properties of posets which will be involved in the proof.

Definition 5.2. *A poset $\mathbb{P}$ is called* **stationary preserving** (*for subsets of ω_1*) *if every stationary $S \subseteq \omega_1$ remains stationary in $V^{\mathbb{P}}$.*

In particular, if $\mathbb{P}$ is stationary preserving then $\omega_1^V = \omega_1^{V^{\mathbb{P}}}$. Recall that every c.c.c., σ-closed, or proper poset, is indeed stationary preserving. An important intermediate notion, lying between proper and stationary preserving, is that of $\aleph_1$-*semi properness* and was introduced by Shelah (cf. Chapter X in [43]).

Definition 5.3. *A poset $\mathbb{P}$ is called* $\aleph_1$-**semi proper** *if for all sufficiently large regular λ, there is a club $C \subseteq [H_\lambda]^{\aleph_0}$ (consisting of countable elementary substructures of H_λ) so that, for all $X \in C$ with $\mathbb{P} \in X$ and for all $p \in X \cap \mathbb{P}$, there is a $q \leqslant p$ which is $(X, \mathbb{P})$-semigeneric, i.e., for every $\mathbb{P}$-name $\tau \in X$ for a countable ordinal and for every $G \subseteq \mathbb{P}$-generic over V with $q \in G$, $\tau_G \in X$.*

It can be shown that we may equivalently require that the defining conditions hold for $\lambda = (2^{|\mathbb{P}|})^+$ (provided that $\mathbb{P} \in H_\lambda$). Moreover, notice that the statement "$\mathbb{P}$ is $\aleph_1$-semi proper" is Σ_2-expressible, using $\mathbb{P}$ as a parameter.

Shelah has shown that under *revised countable support* (RCS) iterations, the property of $\aleph_1$-semi properness is preserved. For more details on such iterations and further development of the theory of proper and improper forcing, see [43]. Although, in general, the property of being $\aleph_1$-semi proper is stronger than that of being stationary preserving, there are cases in which the two notions coincide. This coincidence is traditionally denoted by:

$$(\dagger) \qquad \forall\, \mathbb{Q}\, (\mathbb{Q} \text{ is } \aleph_1\text{-semi proper} \iff \mathbb{Q} \text{ is stationary preserving}).$$

The (†) principle follows from $\mathsf{MA}^+(\sigma\text{-closed})$ (cf. [20]) and has itself large cardinal strength (see Theorem 26 in [20] and, then, § 17 in [13]). In our setting, the preservation of semi properness using RCS will be enough, together with the following important result regarding SPFA, i.e., the $\aleph_1$-*Semi Proper Forcing Axiom* (see Theorem 37.10 in [28]).

Theorem 5.4 (Shelah). $\mathsf{SPFA} \Longrightarrow (\dagger)$. *Therefore,* SPFA *is equivalent to* MM.

With these ingredients, we are now ready for the following.

Theorem 5.5. *If there is an extendible cardinal, then there is a poset $\mathbb{P}$ such that $\mathbb{P} \Vdash \mathsf{MM}^{++} + \mathsf{RA}(\text{stat. pres.})$. Hence, if the theory $\mathsf{ZFC} + $ "$\exists\, \kappa\, (\kappa$ is extendible$)$" is consistent, then so is the theory $\mathsf{ZFC} + \mathsf{MM}^{++} + \mathsf{RA}(\text{stat. pres.})$.*

Proof. Fix κ extendible and let $\ell : \kappa \longrightarrow V_\kappa$ be an extendibility Laver function. We shall define the forcing $\mathbb{P}$ exactly as in [20], using this fixed function ℓ. In particular, $\mathbb{P}$ will be a κ-iteration with RCS, the properties of which are explicitly stated in [20]. For the sake of completeness, let us repeat the definition of the forcing iteration here.

We initially let $\mathbb{P}_0 = \{\mathbb{1}\}$. Given $\alpha < \kappa$ and $\mathbb{P}_\alpha$, non-trivial forcing is done at the next stage only if $\alpha \in dom(\ell)$, and we consider the following alternatives:

1. $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a poset and $\mathbb{P}_\alpha \Vdash$ "$\ell(\alpha)$ is $\aleph_1$-semi proper", in which case we let
$$\dot{\mathbb{Q}}_\alpha = \ell(\alpha) * Col^{\mathbb{P}_\alpha * \ell(\alpha)}(\omega_1\,,\, 2^{|\mathbb{P}_\alpha * \ell(\alpha)|})$$
and we then define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

2. $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a poset and $\mathbb{P}_\alpha \nVdash$ "$\ell(\alpha)$ is $\aleph_1$-semi proper", in which case we let $\delta = \sup^{V^{\mathbb{P}_\alpha}}(2^{2^{|\ell(\alpha)|^+}}, 2^{|\mathbb{P}_\alpha|})$ and
$$\dot{\mathbb{Q}}_\alpha = Col^{\mathbb{P}_\alpha}(\omega_1\,,\,\delta)\,,$$
from which we then define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

Otherwise, i.e., if $\alpha \notin dom(\ell)$ or $\ell(\alpha)$ is not a $\mathbb{P}_\alpha$-name for a poset, trivial forcing is done at stage $\alpha + 1$. At limit stages, we use revised countable support. We finally let $\mathbb{P} = \mathbb{P}_\kappa$ which, in particular, is the direct limit of the $\mathbb{P}_\alpha$'s, for $\alpha < \kappa$.

By the arguments in [20], $\mathbb{P}$ is $\aleph_1$-semi proper and κ-c.c., $\mathbb{P} \Vdash$ "$\kappa = \aleph_2$" and, moreover, $\mathbb{P} \Vdash \mathsf{SPFA}^{++}$. Let us now check that $\mathbb{P} \Vdash \mathsf{RA}$(stat. pres.) as well. As we shall soon see, it is crucial for our argument that extendibility embeddings are involved, as opposed to supercompactness ones.

Fix $G \subseteq \mathbb{P}$-generic over V and suppose that $\mathbb{Q} \in V[G]$ is a stationary preserving poset (equivalently, $\aleph_1$-semi proper, by Theorem 5.4). Fix some $\mathbb{P}$-name $\dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$ and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-semi proper".

Now fix some $\lambda > rank(\dot{\mathbb{Q}})$ with $\lambda \in C^{(2)}$ and let $j : V \longrightarrow M$ be some λ-supercompact and λ-superstrong elementary embedding for κ, with $j(\kappa)$ inaccessible and $j(\ell)(\kappa) = \dot{\mathbb{Q}}$. Recall that in such a case $M \models \lambda \in C^{(2)}$ and, consequently, $M \models \mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\aleph_1$-semi proper" by the Σ_2-expressibility of the latter statement.

Since $\mathbb{P}_\alpha \in V_\kappa$ for any $\alpha < \kappa$, by elementarity and the definition of our iteration, we get that, in M,
$$j(\mathbb{P}) = \mathbb{P} * \dot{\mathbb{Q}} * Col^{\mathbb{P} * \dot{\mathbb{Q}}}(\omega_1\,,\, 2^{|\mathbb{P} * \dot{\mathbb{Q}}|}) * \dot{\mathbb{R}}\,,$$
where note that case (1) of the definition is employed at stage $\kappa + 1$, with $\dot{\mathbb{R}}$ being the tail forcing for stages in the interval $(\kappa\,,\, j(\kappa))$. We clarify our intentions

by mentioning that, it is our aim to show that $Col^{\mathbb{P}*\dot{\mathbb{Q}}}(\omega_1\,,\,2^{|\mathbb{P}*\dot{\mathbb{Q}}|})*\dot{\mathbb{R}}$ is the desired (name for the) further poset which achieves resurrection in $V[G]$ (i.e., $Col^{\mathbb{P}*\dot{\mathbb{Q}}}(\omega_1\,,\,2^{|\mathbb{P}*\dot{\mathbb{Q}}|})*\dot{\mathbb{R}}$ will play the rôle of the "$\dot{\mathbb{R}}$" in the definition of the resurrection axiom).

We now force to add (any) appropriate generics for the factors of $j(\mathbb{P})$ as displayed above, in order to lift the ground model embedding through the forcing $\mathbb{P}$. First, let $g\subseteq\mathbb{Q}$ be any $\mathbb{Q}$-generic over $V[G]$. Then, fix some

$$h*h'\subseteq\Big(Col^{\mathbb{P}*\dot{\mathbb{Q}}}(\omega_1\,,\,2^{|\mathbb{P}*\dot{\mathbb{Q}}|})*\dot{\mathbb{R}}\Big)_{G*g}$$

which is generic for this poset over $V[G][g]$. We let $\widetilde{G}=G*g*h*h'$ be the whole generic filter for $j(\mathbb{P})$, over V. It now follows that the lifting criterion $j''G\subseteq\widetilde{G}$ is satisfied and thus, the ground model embedding lifts to

$$j:V[G]\longrightarrow M[\widetilde{G}],$$

a lift which takes place in the enlarged universe $V[\widetilde{G}]$.

As κ and $j(\kappa)$ are inaccessible cardinals in V, we clearly have $V_\kappa=H_\kappa$ and $V_{j(\kappa)}=H_{j(\kappa)}$ (computed equivalently either in V or in M, since j was superstrong). It is also evident that, by the elementarity of the ground model embedding and the fact that $cp(j)=\kappa$, $H_\kappa\prec H_{j(\kappa)}$. Similarly, for the lifted version of the embedding, $H_\kappa^{V[G]}\prec H_{j(\kappa)}^{M[\widetilde{G}]}$. Now, towards verifying the resurrection axiom, we want to show that, in fact, $H_{j(\kappa)}^{M[\widetilde{G}]}=H_{j(\kappa)}^{V[\widetilde{G}]}$. This will immediately imply, in addition, that $j(\kappa)=\aleph_2^{M[\widetilde{G}]}=\mathfrak{c}^{M[\widetilde{G}]}=\aleph_2^{V[\widetilde{G}]}=\mathfrak{c}^{V[\widetilde{G}]}$.

In order to show that $H_{j(\kappa)}^{M[\widetilde{G}]}=H_{j(\kappa)}^{V[\widetilde{G}]}$, we use that the ground model embedding was λ-superstrong, i.e., that $V_{j(\lambda)}\subseteq M$. First, since M and V have the same (maximal) antichains of the poset $j(\mathbb{P})$ and the latter is $j(\kappa)$-c.c. in M, it follows that $j(\mathbb{P})$ is $j(\kappa)$-c.c. in V as well. In particular, $j(\kappa)$ remains a regular cardinal in both $M[\widetilde{G}]$ and $V[\widetilde{G}]$. Consequently, for any given $X\in H_{j(\kappa)}^{V[\widetilde{G}]}$, exactly as in Proposition 4.5, X can be coded in $V[\widetilde{G}]$ by a subset of $\alpha\times\alpha$, for some $\alpha<j(\kappa)$, so that X can be then retrieved by (the transitive collapse of) its code. But then, noticing again that any nice name for such a code belongs M, we get that $X\in H_{j(\kappa)}^{M[\widetilde{G}]}$ as desired. Moreover, observe that by the same coding arguments and the fact that the ground model embedding was λ-superstrong, we also obtain that, for every inaccessible $\beta\in(j(\kappa)\,,\,j(\lambda))$,

$$H_\beta^{M[\widetilde{G}]}=H_\beta[\widetilde{G}]=H_\beta^{V[\widetilde{G}]},$$

with β remaining inaccessible in both generic extensions. This remark is not relevant for the current proof but it will be relevant in the following section.

Hence, we can conclude at this point that $H_\kappa^{V[G]} \prec H_{j(\kappa)}^{V[\widetilde{G}]}$ or, equivalently, that

$$H_{\mathfrak{c}}^{V[G]} \prec H_{\mathfrak{c}}^{V[G][g][h*h']}.$$

Therefore, towards verifying resurrection, the only remaining thing which we have to show is that

$$V[G] \models \mathbb{Q} \Vdash \text{“} Col(\omega_1\,,\, 2^{|\mathbb{P}*\mathbb{Q}|}) * \dot{\mathbb{R}} \text{ is stationary preserving”},$$

where $Col(\omega_1\,,\, 2^{|\mathbb{P}*\mathbb{Q}|}) = \left(Col^{\,\mathbb{P}*\dot{\mathbb{Q}}}(\omega_1\,,\, 2^{|\mathbb{P}*\dot{\mathbb{Q}}|})\right)_G$. In fact, we will show that $Col(\omega_1\,,\, 2^{|\mathbb{P}*\mathbb{Q}|}) * \dot{\mathbb{R}}$ is forced to be $\aleph_1$ - semi proper by the poset $\mathbb{Q}$, in $V[G]$.

By the properties of RCS iterations listed in [20] and by elementarity, we have that, on the M side,

$$M \models j(\mathbb{P})_{\kappa+1} \Vdash \text{“} \dot{\mathbb{R}} \text{ is } \aleph_1\text{ - semi proper”},$$

where $j(\mathbb{P})_{\kappa+1} = \mathbb{P} * \dot{\mathbb{Q}} * Col^{\,\mathbb{P}*\dot{\mathbb{Q}}}(\omega_1\,,\, 2^{|\mathbb{P}*\dot{\mathbb{Q}}|})$. But now, again by the fact that $V_{j(\lambda)} \subseteq M$, where $j(\lambda)$ is $C^{(2)}$ in M, it follows that M faithfully verifies this statement, i.e., it is also true in V; thus,

$$V[G] \models \mathbb{Q} * Col(\omega_1\,,\, 2^{|\mathbb{P}*\mathbb{Q}|}) \Vdash \text{“} \dot{\mathbb{R}} \text{ is } \aleph_1\text{ - semi proper”}.$$

Finally, as it is easy to see that, in $V[G]$,

$$\mathbb{Q} \Vdash \text{“} Col(\omega_1\,,\, 2^{|\mathbb{P}*\mathbb{Q}|}) \text{ is } \aleph_1\text{ - semi proper”},$$

it now follows that, by a two – step combination of $\aleph_1$ - semi proper posets,

$$V[G] \models \mathbb{Q} \Vdash \text{“} Col(\omega_1\,,\, 2^{|\mathbb{P}*\mathbb{Q}|}) * \dot{\mathbb{R}} \text{ is } \aleph_1\text{ - semi proper”},$$

as desired. Recalling that the generic filters $g\,,\, h\,,\, h'$ which we chose were arbitrary, we finally get that $V[G] \models \mathsf{RA}$(stat. pres.) which completes the proof. □

Evidently, the previous proof shows that $V[G] \models \mathsf{RA}$(semi proper), with the latter being the resurrection axiom for the class of $\aleph_1$ - semi proper posets. In fact, since (†) holds in $V[G]$, it is easily seen that, in this case, RA(semi proper) implies RA(stat. pres.). However, there is a substantial difference between these axioms in terms of consistency strength: we shall see in Section 5.4 that RA(stat. pres.) implies that every set has a sharp, whereas, by results of Hamkins and Johnstone, RA(semi proper) can be forced from the existence of an uplifting cardinal.

The exact relation between the axioms MM (or MM^{++}) and RA(stat. pres.) has not been clarified yet. In particular, an important question is whether one can produce a model in which the former holds, while the latter fails. If this is possible at all, it would be highly desirable to produce such a model from a large cardinal assumption which is strictly weaker than that of extendibility; e.g., starting from the least supercompact. This would indeed give more value to Theorem 5.5.

On the other hand, following the suggestions in [29], we now show that the axiom RA(stat. pres.) is consistent with CH; consequently, RA(stat. pres.) certainly does not imply MM, since the latter entails $\mathfrak{c} = \aleph_2$.

Proposition 5.6. RA(stat. pres.) $+\,\mathsf{CH}$ *is relatively consistent.*

Proof. Suppose that we are given a model V of RA(stat. pres.) $+\neg\,\mathsf{CH}$, a situation which is consistent relative to (the existence of) an extendible cardinal, as we just saw in the proof of Theorem 5.5. We now force with the canonical poset $\mathbb{P} = Add(\omega_1\,,\,1)$, which forces CH and is clearly stationary preserving. Let us fix G, a $\mathbb{P}$-generic filter over V, and we now show that $V[G]$ satisfies RA(stat. pres.).

For this, let $\mathbb{Q} \in V[G]$ be a stationary preserving poset and fix some $\mathbb{P}$-name $\dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$ and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is stationary preserving". By a two-step combination, $\mathbb{P} * \dot{\mathbb{Q}}$ is stationary preserving in V and thus, by the resurrection axiom, there is a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a poset $\dot{\mathbb{R}}$, with $\mathbb{P} * \dot{\mathbb{Q}} \Vdash$ "$\dot{\mathbb{R}}$ is stationary preserving" and so that, for every $g * h \subseteq \mathbb{Q} * \dot{\mathbb{R}}$-generic over $V[G]$, we have

$$H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[G][g][h]}.$$

Notice that $\aleph_1$ is preserved in all the intermediate steps of the forcing constructions and so, without causing any confusion, we may drop the superscript from this particular symbol. Since CH fails, $\mathfrak{c}^V > \aleph_1$ and then $\mathfrak{c}^{V[G][g][h]} > \aleph_1$ as well. Hence, by relativizing any first-order formula to the $H_{\aleph_1}$ (which is a definable subclass of each) of the above structures, we also get

$$H_{\aleph_1} \prec H_{\aleph_1}^{V[G][g][h]}.$$

Note that, by the closure of the forcing $\mathbb{P}$, $H_{\aleph_1} = H_{\aleph_1}^{V[G]}$.

We now further force over $V[G][g][h]$ to make CH true, using the canonical poset $Add(\omega_1\,,\,1)$ of this model. It again follows that, if h' is $Add(\omega_1\,,\,1)$-generic over $V[G][g][h]$, then $H_{\aleph_1}^{V[G][g][h]} = H_{\aleph_1}^{V[G][g][h][h']}$. Summarizing, we so far have established that

$$H_{\aleph_1}^{V[G]} \prec H_{\aleph_1}^{V[G][g][h][h']}.$$

Finally observe that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} * Add(\omega_1\,,\,1)$ is stationary preserving" in $V[G]$ and thus, we can conclude at this point that

$$V[G] \models \mathsf{RA}(\text{stat. pres.}) + \mathsf{CH},$$

as desired. $\square$

It is clear that the same argument can be used in order to obtain the relative consistency of $\mathsf{RA}(\text{semi proper}) + \mathsf{CH}$ as well.

Remark. David Asperó has pointed out that, for the consistency of the axiom $\mathsf{RA}(\text{stat. pres.})$, it is sufficient to assume CH and that there is a proper class of Woodin cardinals; in fact, under such assumptions, if Γ is any (definable) class of posets with the property that, for every $\mathbb{P} \in \Gamma$ there is a $\mathbb{P}$-name for a poset $\dot{\mathbb{Q}}$ such that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}} \in \Gamma$" and $\mathbb{P} * \dot{\mathbb{Q}} \Vdash \mathsf{CH}$, then the axiom $\mathsf{RA}(\Gamma)$ holds. The reason is that, in the presence of a proper class of Woodin cardinals, a classical result of Woodin shows that $H_{\aleph_1}$ is an elementary substructure of the $H_{\aleph_1}$ of any (set) forcing extension (see [6] or [34]).

Hence, if one is satisfied with the consistency of $\mathsf{RA}(\text{stat. pres.}) + \mathsf{CH}$, then there is no need to invoke (the consistency of) extendibility; a proper class of Woodin cardinals suffices. $\perp$

Returning to the proof of Theorem 5.5, one might wonder how much more resurrection we can get in the obtained model of $\mathsf{MM}^{++} + \mathsf{RA}(\text{stat. pres.})$; let us now focus our attention on this very fruitful – as it turns out – question $^{\circlearrowleft}$.

5.2 Unbounded Resurrection

Looking closer at the proof of Theorem 5.5, we argued that the ground model elementarity $H_\kappa \prec H_{j(\kappa)}$ lifts to the elementarity $H_\kappa^{V[G]} \prec H_{j(\kappa)}^{V[\widetilde{G}]}$ in the generic extension, witnessing the resurrection axiom in $V[G]$.

Now, using the extendibility of κ, one is tempted to apply similar reasoning for the corresponding H_β and $H_{j(\beta)}$, for various $\beta > \kappa$. Of course, in such a case, we do not expect to have a fully elementary substructure, but we now argue that an elementary embedding between $H_\beta^{V[G]}$ and $H_{j(\beta)}^{V[\widetilde{G}]}$ may be found (and it will be, as it should be anticipated, the restriction of the lifted embedding j to $H_\beta^{V[G]}$).

Going into the details, for any inaccessible $\beta > \kappa$ and any stationary preserving poset $\mathbb{Q} \in H_\beta^{V[G]} = H_\beta[G]$, we now start with a (different) ground model embedding $j : V \longrightarrow M$ that anticipates (a $\mathbb{P}$-name for) $\mathbb{Q}$ and that is sufficiently extendible, i.e., sufficiently supercompact and superstrong, above β (in particular, $j(\beta)$ is inaccessible); we then likewise lift j through the forcing $\mathbb{P}$. Next, we show that there exists a $\mathbb{Q}$-name for a stationary preserving poset $\dot{\mathbb{R}}$ (which is just an appropriate tail forcing of the $j(\mathbb{P})$-iteration, exactly as in the

$^{\circlearrowleft}$ This question was asked by David Asperó and quickly led to the development of the material appearing in the three final sections of the current chapter. I am truly indebted to David for the discussion we had during that evening of June.

proof of Theorem 5.5) and, if $V[\widetilde{G}]$ again denotes the fully enlarged universe, we have $H_{j(\beta)}^{M[\widetilde{G}]} = H_{j(\beta)}[\widetilde{G}] = H_{j(\beta)}^{V[\widetilde{G}]}$. Consequently, there exists an elementary embedding $h \in V[\widetilde{G}]$ (namely, $h = j \restriction H_\beta^{V[G]}$) with $h : H_\beta^{V[G]} \longrightarrow H_{j(\beta)}^{V[\widetilde{G}]}$, $cp(h) = \omega_2^{V[G]} = \omega_2^{V[\widetilde{G}]}$ and $h(\omega_2^{V[G]}) > \beta$.

Taking one step further, given the fact that κ is extendible, the inaccessibility assumption on β is not really necessary. Indeed, for any fixed cardinal $\beta > \kappa$ and any $\mathbb{Q} \in H_\beta^{V[G]}$, we may work with any inaccessible $\lambda > \beta$ and produce, as in the previous paragraph, a $\mathbb{Q}$ - name for a poset $\dot{\mathbb{R}}$ and an elementary embedding $h : H_\lambda^{V[G]} \longrightarrow H_{j(\lambda)}^{V[\widetilde{G}]}$ with $cp(h) = \omega_2^{V[G]}$, $h(\omega_2^{V[G]}) > \lambda$ and $j(\lambda)$ inaccessible in $V[\widetilde{G}]$. Then, by just relativizing any first – order formula to $H_\beta^{V[G]}$ (which is definable in the structure $H_\lambda^{V[G]}$), we get that $h \restriction H_\beta^{V[G]} : H_\beta^{V[G]} \longrightarrow H_{j(\beta)}^{V[\widetilde{G}]}$ is of the desired form. These observations seem to suggest the introduction of the following axiom of *unbounded* resurrection, for various classes Γ of posets.

Definition 5.7. *For any fixed* (*definable*) *class* Γ *of posets, the* **Unbounded Resurrection Axiom** *for* Γ*, denoted by* $\mathsf{UR}(\Gamma)$*, is the assertion that for every cardinal* $\beta > max\{\omega_2, \mathfrak{c}\}$ *and every poset* $\mathbb{Q} \in H_\beta$ *with* $\mathbb{Q} \in \Gamma$*, there exists a* $\mathbb{Q}$ *- name for a poset* $\dot{\mathbb{R}}$ *such that* $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$"*, and there is an elementary embedding*

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}},$$

with $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$*,* $cp(j) = max\{\omega_2, \mathfrak{c}\}$ *and* $j(cp(j)) > \beta$*.*

Of course, in the discussion preceding Definition 5.7, one could have also looked at embeddings between the V_β 's instead of the H_β 's . We choose to work with the latter over the former structures because the H_β 's are more tailored for expressing instances of generic absoluteness, such as the characterizations mentioned at the beginning of this chapter (also, recall that for regular β, H_β is a model of ZFC^- and so it is a more suitable structure in the context of forcing). This way, we adhere to the motivational background which led to the consideration of the resurrection axioms.

In what follows, we focus our attention on the classes of c.c.c., σ - closed, proper, and of stationary preserving posets; at the relevant places, we shall also briefly comment on the case of $\aleph_1$ - semi properness. The apparent ambiguity regarding the value of $cp(j)$ in the above definition is included in order to account for the general setting; as we shall see, for the particular class of c.c.c. posets $cp(j) = \mathfrak{c}$, whereas, for the other classes of posets just mentioned, we necessarily have that $cp(j) = \omega_2$. We can now state our first consistency result for the class of stationary preserving posets.

Theorem 5.8. *If the theory* $\mathsf{ZFC} +$ "$\exists\,\kappa\,(\kappa$ *is extendible*)" *is consistent, then so is the theory* $\mathsf{ZFC} + \mathsf{MM}^{++} + \mathsf{UR}(\text{stat. pres.})$.

Proof. By the arguments described just before Definition 5.7, it follows that the unbounded resurrection axiom $\mathsf{UR}(\text{stat. pres.})$ holds in the model of MM^{++} which we obtained in the proof of Theorem 5.5. □

Note that, exactly as for the corresponding RA axioms, the proof of Theorem 5.5 actually shows that, in the resulting model $V[G]$, $\mathsf{UR}(\text{semi proper})$ holds.

We now show that the explicit mention of MM^{++} in the conclusion of the previous result is redundant, since it is already implied by the unbounded resurrection axiom for stationary preserving posets. In addition, we argue that, under an assumption strictly weaker than (the consistency of) extendibility, MM^{++} can be separated from $\mathsf{UR}(\text{stat. pres.})$.

Proposition 5.9. $\mathsf{UR}(\text{stat. pres.})$ *implies* MM^{++}. *Moreover, if there is a model in which there exists a supercompact cardinal with a unique inaccessible above it, then there is a model of* MM^{++} *in which* $\mathsf{UR}(\text{stat. pres.})$ *fails.*

Proof. Suppose that $\mathsf{UR}(\text{stat. pres.})$ holds in V. We verify that MM follows; we leave it to the reader to check that a mild modification of the argument produces MM^{++} as well. For this, let us fix some stationary preserving poset $\mathbb{Q}$ and let $\langle\, D_\alpha : \alpha < \omega_1 \,\rangle$ be a collection of dense subsets of $\mathbb{Q}$.

We now fix a large enough regular $\beta > max\{\omega_2, \mathfrak{c}\}$, with $\mathbb{Q} \in H_\beta$. Clearly, $\langle\, D_\alpha : \alpha < \omega_1 \,\rangle \in H_\beta$ as well. Then, by $\mathsf{UR}(\text{stat. pres.})$, there exists some $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$, such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}}$ is stationary preserving", and an elementary embedding $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$ of the form

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}},$$

with $cp(j) = max\{\omega_2, \mathfrak{c}\}$ and $j(cp(j)) > \beta$. Finally, we fix any filter $g \subseteq \mathbb{Q}$-generic over V and any filter $H \subseteq \dot{\mathbb{R}}_g$-generic over $V[g]$. Hence, we get that $j : H_\beta \longrightarrow H_{j(\beta)}^{V[g][H]}$ is elementary with $cp(j) = max\{\omega_2, \mathfrak{c}\}$ and $j(cp(j)) > \beta$. Now, since ω_1 is fixed by the embedding,

$$j(\langle\, D_\alpha : \alpha < \omega_1 \,\rangle) = \langle\, j(D_\alpha) : \alpha < \omega_1 \,\rangle \in H_{j(\beta)}^{V[g][H]}$$

and, also, the pointwise image $j\,''g$ belongs to $H_{j(\beta)}^{V[g][H]}$ as well, since it is constructible in $V[g][H]$ from j and g, with the latter having size less than β. But then, as g is $\mathbb{Q}$-generic over V, it follows that, in $H_{j(\beta)}^{V[g][H]}$, $j\,''g$ generates a filter of $j(\mathbb{Q})$ which intersects every $j(D_\alpha)$, for $\alpha < \omega_1$. Hence, by elementarity, there exists, in H_β, a filter $G \subseteq \mathbb{Q}$ such that $G \cap D_\alpha \neq \varnothing$, for all $\alpha < \omega_1$. MM now follows.

To separate the axioms MM^{++} and UR(stat. pres.), we fix some model in which there is a supercompact κ and a unique inaccessible $\lambda > \kappa$, and we let $\mathbb{P}$ be the standard κ-iteration which forces MM^{++}, as in [20]. Let us then fix a forcing extension V in which MM^{++} holds and λ remains inaccessible. We show that, in the model V, UR(stat. pres.) fails. Towards a contradiction, suppose otherwise and consider the poset $\mathbb{Q} = Col(\omega_1, \lambda)$, which is certainly stationary preserving and it clearly belongs to H_{λ^+}.

Then, there must be a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$, such that $\mathbb{Q}$ forces that $\dot{\mathbb{R}}$ is stationary preserving, and an elementary embedding $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$ of the form $j : H_{\lambda^+} \longrightarrow H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\lambda^+)}$, with $cp(j) = \kappa = \omega_2 = \mathfrak{c}$ and $j(\kappa) > \lambda^+$. But now, by elementarity, we must have that $j(\lambda)$ is inaccessible in $H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\lambda^+)}$, hence in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$ as well, which is impossible since in the latter model there are no inaccessibles at all. This is the desired contradiction which concludes the proof. □

Similar reasoning shows that, if UR(stat. pres.) holds and there exists some inaccessible, then there must exist proper class many inaccessibles. In fact, the same is true if in place of inaccessibles we consider any object which cannot be created by stationary preserving forcing.

A moment's inspection shows that we may immediately generalize the proof of Proposition 5.9 to other classes Γ of posets, obtaining the following.

Corollary 5.10. *For any (definable) class Γ of posets,* $\mathsf{UR}(\Gamma)$ *implies the forcing axiom* $\mathsf{FA}^{++}(\Gamma)$. □

Recalling that SPFA implies (†), we immediately get that UR(semi proper) implies UR(stat. pres.). The following argument due to Asperó shows that, if we are granted enough large cardinals, then the converse holds as well.

Proposition 5.11 (Asperó). *Assume* UR(stat. pres.) *and suppose that there is a proper class of supercompact cardinals. Then,* UR(semi proper) *holds.*

Proof. We use the fact that, if κ is supercompact and $\mathbb{P} = Col(\omega_1, {<}\kappa)$ is the Lévy collapse to make $\kappa = \aleph_2$, then $\mathsf{MA}^+(\sigma\text{-closed})$ holds in $V^{\mathbb{P}}$ (this follows, essentially, from the fact that any σ-closed poset of cardinality $< \kappa$ can be completely embedded in $Col(\omega_1, {<}\kappa)$; see § 14 in [13]). Thus, by results in [20], the (†) principle holds in $V^{\mathbb{P}}$ as well. Towards verifying UR(semi proper), suppose that $\mathbb{Q}$ is $\aleph_1$-semi proper and let $\beta > \omega_2$ be a given cardinal with $\mathbb{Q} \in H_\beta$.

Let κ be supercompact with $\kappa > \beta$; clearly, κ remains supercompact in $V^{\mathbb{Q}}$. Consider the ($\mathbb{Q}$-name for the) poset $\dot{\mathbb{Q}}_0$ which is the Lévy collapse $Col(\omega_1, {<}\kappa)$

as computed in $V^{\mathbb{Q}}$. Now, $\mathbb{Q} * \dot{\mathbb{Q}}_0$ is stationary preserving in V and, therefore, by the axiom UR(stat. pres.), there exists a $\mathbb{Q} * \dot{\mathbb{Q}}_0$ - name for a poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} * \dot{\mathbb{Q}}_0 \Vdash$ "$\dot{\mathbb{R}}$ is stationary preserving", and there is an elementary embedding

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{(\mathbb{Q}*\dot{\mathbb{Q}}_0)*\dot{\mathbb{R}}}}$$

with $j \in V^{(\mathbb{Q}*\dot{\mathbb{Q}}_0)*\dot{\mathbb{R}}}$, $cp(j) = \omega_2$ and $j(\omega_2) > \beta$. Now recall that (†) holds in $V^{\mathbb{Q}*\dot{\mathbb{Q}}_0}$ and so we actually have that $\mathbb{Q} * \dot{\mathbb{Q}}_0 \Vdash$ "$\dot{\mathbb{R}}$ is $\aleph_1$ - semi proper". Observe that $\dot{\mathbb{Q}}_0 * \dot{\mathbb{R}}$ is of the form "σ - closed $*\, \aleph_1$ - semi proper" and thus it is $\aleph_1$ - semi proper in $V^{\mathbb{Q}}$. Hence, since $V^{(\mathbb{Q}*\dot{\mathbb{Q}}_0)*\dot{\mathbb{R}}} = V^{\mathbb{Q}*(\dot{\mathbb{Q}}_0*\dot{\mathbb{R}})}$, the conclusion follows. □

Given this result, the undermentioned question suggests itself.

Question 5.12 (Asperó). *Are the axioms* UR(stat. pres.) *and* UR(semi proper) *equivalent in general?*

Remark. Shortly after the current dissertation was deposited, Asperó announced to us that, using techniques related to Woodin's stationary tower forcing, he can prove that the assumption of MM^{++} together with a proper class of Woodin cardinals actually implies UR(stat. pres.). If his argument works, then such a result will constitute a substantial improvement regarding the consistency (upper) bound of the axiom UR(stat. pres.). $\perp$

It should have been clear by now that, for the cases of proper, of $\aleph_1$ - semi proper, and of stationary preserving posets, the critical point of the generic embeddings given by the corresponding unbounded resurrection axiom will always be ω_2, since both PFA and MM imply that the continuum is equal to $\aleph_2$.

On the other hand, for c.c.c. posets, as $\mathsf{MA}_{\aleph_1}$ implies $\neg\,\mathsf{CH}$, the generic embeddings given by the axiom UR(c.c.c.) will have, in general, $cp(j) = \mathfrak{c}$. As we shall see in the next section, UR(c.c.c.) actually implies that $\mathfrak{c}$ is weakly inaccessible; in the other extreme, we shall also show that UR(σ - closed) implies CH.

Yet, before dealing with the various implications of the UR axioms, we focus on obtaining their relative consistency from that of (the existence of) an extendible cardinal. As we have already mentioned, we shall consider the classes of c.c.c., of σ - closed, and of proper posets. Our treatment of the three cases will follow a unified pattern in the sense that, starting from an extendible cardinal, we will define a forcing iteration guided by an extendibility Laver function, taking into account only the posets which are relevant to the axiom at hand. Clearly, different supports have to be used in order to ensure that the (iterations of the) posets defined, belong to the class in question.

Modulo these modifications, the rest of the proof(s) will be identical in each case, following the lines of the "prototype" arguments which we employed in order to show that UR(stat. pres.) holds in the model produced for Theorem 5.5.

In all three cases we shall state the theorems in terms of the relative consistency results, although what we really prove is that, given an extendible cardinal, the corresponding forcing iterations which we define work as intended.

Let us begin by considering the case of proper posets. As countable supports will be used in this iteration, Baumgartner's proof regarding the consistency of PFA is relevant.

Theorem 5.13. *If the theory* ZFC + "$\exists\,\kappa\,(\kappa$ *is extendible*$)$" *is consistent, then so is the theory* $\mathsf{ZFC} + \mathsf{UR}(\text{proper})$.

Proof. Fix κ extendible and fix $\ell : \kappa \longrightarrow V_\kappa$ an extendibility Laver function. Our forcing $\mathbb{P}$ will be a κ-iteration of proper posets, using countable support. Formally, we have the following.

Let $\mathbb{P}_0 = \{\mathbb{1}\}$. Given $\alpha < \kappa$ and $\mathbb{P}_\alpha$, if $\alpha \in dom(\ell)$ and $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a poset with $\mathbb{P}_\alpha \Vdash$ "$\ell(\alpha)$ is proper", we let $\dot{\mathbb{Q}}_\alpha = \ell(\alpha)$ and we then define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Otherwise, trivial forcing is done at stage $\alpha+1$. At every limit stage $\alpha \leqslant \kappa$, we use countable support and we finally let $\mathbb{P} = \mathbb{P}_\kappa$.

It is clear that $|\mathbb{P}| = \kappa$, and, for every $\alpha < \kappa$, $\mathbb{P}_\alpha \in V_\kappa$. By standard facts (see [1] and [13] for more details), $\mathbb{P}$ is proper and it has the κ-c.c. Now fix a $G \subseteq \mathbb{P}$-generic over V. The usual arguments show that PFA^{++} holds in $V[G]$ (and, of course, $\kappa = \mathfrak{c} = \aleph_2$). We now verify that, in $V[G]$, UR(proper) holds as well (which actually implies PFA^{++}, by Corollary 5.10).

Let $\mathbb{Q} \in V[G]$ be a proper poset and let $\beta > \kappa$ be a cardinal with $\mathbb{Q} \in H_\beta^{V[G]}$. Fix some $\mathbb{P}$-name $\dot{\mathbb{Q}}$ such that $\dot{\mathbb{Q}}_G = \mathbb{Q}$ and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is proper". Finally, fix some sufficiently large inaccessible cardinal λ (in V) with $\lambda > \beta$, $\dot{\mathbb{Q}} \in H_\lambda$ and $H_\lambda \models \mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is proper"; now let $j : V \longrightarrow M$ be λ-supercompact and λ-superstrong for κ, with $j(\ell)(\kappa) = \dot{\mathbb{Q}}$ (and with $j(\kappa)$ and $j(\lambda)$ inaccessibles in V). In this situation, it follows that $M \models \mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is proper" as well, since the latter is a Σ_2-statement.

We now have that, on the M side, the image poset $j(\mathbb{P})$ factors as $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$, with $\dot{\mathbb{R}}$ being the name of the tail forcing of the $j(\kappa)$-iteration. Let us fix any filter $g \subseteq \mathbb{Q}$-generic over $V[G]$ and any $H \subseteq \dot{\mathbb{R}}_g$-generic over $V[G][g]$. We then let $\widetilde{G} = G * g * H$ be the whole generic for $j(\mathbb{P})$ over V. Consequently, as the lifting criterion is satisfied, the ground model embedding lifts to

$$j : V[G] \longrightarrow M[\widetilde{G}],$$

a lift which takes place in the enlarged universe $V[\widetilde{G}]$. Now, notice that for the restricted map $h = j \restriction H_\lambda^{V[G]} \in V[\widetilde{G}]$,

$$h : H_\lambda^{V[G]} \longrightarrow H_{j(\lambda)}^{M[\widetilde{G}]}$$

is an elementary embedding with $cp(h) = \kappa = \omega_2^{V[G]} = \mathfrak{c}^{V[G]}$ and $h(\kappa) > \beta$.

Hence, towards verifying UR(proper) in $V[G]$, it is sufficient to check that $H_{j(\lambda)}^{M[\widetilde{G}]} = H_{j(\lambda)}^{V[\widetilde{G}]}$ and that $V[G] \models \mathbb{Q} \Vdash$ "$\dot{\mathbb{R}}$ is proper". Given these two facts, it is then immediate that we may further restrict the embedding h to the H_β of $V[G]$, so that the (newly) restricted embedding together with the (name of the) poset $\dot{\mathbb{R}}$ jointly witness the unbounded resurrection axiom for $\mathbb{Q}$ and β, in $V[G]$.

For $H_{j(\lambda)}^{M[\widetilde{G}]} = H_{j(\lambda)}^{V[\widetilde{G}]}$, we use the inaccessibility of $j(\lambda)$, the λ-superstrongness of j and the fact that $j(\mathbb{P})$ is $j(\kappa)$-c.c. (both in V and in M) and we check, as remarked in the proof of Theorem 5.5, that both these structures are actually equal to $H_{j(\lambda)}[\widetilde{G}]$. Finally, it is also easy to see that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}}$ is proper" holds in $V[G]$, since this is a Σ_2-statement (in the parameter $\mathbb{Q}$) which is true in $M[G]_{j(\lambda)} = V[G]_{j(\lambda)}$, and $j(\lambda)$ is inaccessible (and thus, Σ_1-correct) in $V[G]$. This completes the proof. $\square$

By arguments analogous to the ones used in the proof of Proposition 5.9, the (consistency of the) existence of a supercompact with a single (equivalently, with boundedly many) inaccessible(s) above it, implies that one may separate the axiom UR(proper) from PFA^{++}, i.e., under such assumption(s), there is a model in which the latter holds but the former fails.

Let us now consider the consistency of UR(c.c.c.) and that of UR(σ-closed). Rather than repeating the same arguments all over again, we shall restrict ourselves to just defining the appropriate forcing iterations. The reader will then gladly verify that, along the lines of our earlier proof(s), these unbounded resurrection axioms hold in the corresponding generic extensions.

Theorem 5.14. *If the theory* ZFC + "$\exists\,\kappa\,(\kappa$ *is extendible*)" *is consistent, then so is each one of the theories*: $\mathsf{ZFC} + \mathsf{UR}$(c.c.c.) *and* $\mathsf{ZFC} + \mathsf{UR}$($\sigma$-closed).

Proof. Fix κ extendible and fix $\ell : \kappa \longrightarrow V_\kappa$ an extendibility Laver function. We define two forcings $\mathbb{P}$ and $\mathbb{P}'$ which will produce the two models respectively. The first one will be a finite support κ-iteration of c.c.c. posets; the second, a reverse Easton support κ-iteration of σ-closed posets. Both these iterations will be guided by ℓ. Formally, we have the following definitions (which we present jointly due to their similarities, although they should be really thought of as independent).

Let $\mathbb{P}_0 = \mathbb{P}'_0 = \{\mathbb{1}\}$. Given $\alpha < \kappa$ and $\mathbb{P}_\alpha$ (resp. $\mathbb{P}'_\alpha$), if $\alpha \in dom(\ell)$ and $\ell(\alpha)$ is a $\mathbb{P}_\alpha$-name for a poset with $\mathbb{P}_\alpha \Vdash$ "$\ell(\alpha)$ is c.c.c." (resp. $\ell(\alpha)$ is a $\mathbb{P}'_\alpha$-name for a poset with $\mathbb{P}'_\alpha \Vdash$ "$\ell(\alpha)$ is σ-closed"), we let $\dot{\mathbb{Q}}_\alpha = \ell(\alpha)$ (resp. $\dot{\mathbb{Q}}'_\alpha = \ell(\alpha)$) and we then define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ (resp. $\mathbb{P}'_{\alpha+1} = \mathbb{P}'_\alpha * \dot{\mathbb{Q}}'_\alpha$). Otherwise, trivial forcing is done at stage $\alpha + 1$.

For $\mathbb{P}$, at every limit stage $\alpha \leqslant \kappa$ we use finite support. For $\mathbb{P}'$, we use Easton support, i.e., we take direct limits at inaccessibles $\alpha \leqslant \kappa$ and inverse limits everywhere else. Finally, we let $\mathbb{P} = \mathbb{P}_\kappa$ and $\mathbb{P}' = \mathbb{P}'_\kappa$. It is clear $|\mathbb{P}| = |\mathbb{P}'| = \kappa$ and that, for every $\alpha < \kappa$, $\mathbb{P}_\alpha$ and $\mathbb{P}'_\alpha$ belong to V_κ. Also, $\mathbb{P}$ is c.c.c. while $\mathbb{P}'$ is σ-closed and κ-c.c. We fix $G \subseteq \mathbb{P}$ and $G' \subseteq \mathbb{P}'$ which are respective generic filters over V. Let $V_1 = V[G]$ and $V_2 = V[G']$. As in our earlier proof(s), one now checks both that $V_1 \models \mathsf{UR}$(c.c.c.), and that $V_2 \models \mathsf{UR}(\sigma\text{-closed})$. □

In the last proof, it is easy to see that $\kappa = \mathfrak{c}$ is a weakly inaccessible cardinal in the model V_1, whereas in V_2, CH holds and $\kappa = \aleph_2$. These observations are in accordance with the already advertized (but not yet proven) effects of the axioms UR(c.c.c.) and $\mathsf{UR}(\sigma\text{-closed})$ on the continuum.

We shall shortly see that, as a by – product of such effects, it is possible to separate $\mathsf{MA}^+(\sigma\text{-closed})$ from $\mathsf{UR}(\sigma\text{-closed})$, like in the cases of proper and of stationary preserving posets. Hence, without further ado, let us proceed to the next section where, apart from taking up issues of this sort, we also discuss the connection of the UR axioms with the resurrection axioms RA.

5.3 Effect on $\mathfrak{c}$ and relation to RA axioms

As we have already remarked, both UR(stat. pres.) and UR(proper) imply that $\mathfrak{c} = \aleph_2$, a direct corollary to their connection with the axioms MM and PFA, respectively.

On the other hand, let us now see that, in the case of σ-closed posets, the unbounded resurrection axiom has the ultimate bounding effect on the size of the continuum: it implies the Continuum Hypothesis. In particular, the generic embeddings given by $\mathsf{UR}(\sigma\text{-closed})$ will necessarily have critical point ω_2.

Apropos, this also shows that, in general, the unbounded resurrection axioms, just like the resurrection axioms, are not monotonous; i.e., if $\Gamma \subseteq \Gamma'$ are given classes of posets, then $\mathsf{UR}(\Gamma')$ does not necessarily imply $\mathsf{UR}(\Gamma)$.

Lemma 5.15. $\mathsf{UR}(\sigma\text{-closed}) \Longrightarrow 2^{\aleph_0} = \aleph_1$.

Proof. Assume $\mathsf{UR}(\sigma\text{-closed})$ and suppose, towards a contradiction, that $\mathfrak{c} \geqslant \aleph_2$ in V. Let $\mathbb{Q} = Add(\omega_1, 1)$ be the canonical σ-closed poset which forces CH and fix a regular $\beta > \mathfrak{c}$ with $\mathbb{Q} \in H_\beta$.

Then, by the axiom $\mathsf{UR}(\sigma\text{-closed})$, there is a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}}$ is σ-closed", and there is an elementary embedding $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$ of the form $j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$, with $cp(j) = \mathfrak{c}$ and $j(\mathfrak{c}) > \beta$. Finally, we fix any filter $g \subseteq \mathbb{Q}$-generic over V and any filter $H \subseteq \dot{\mathbb{R}}_g$-generic over $V[g]$. Hence, we get that $j : H_\beta \longrightarrow H_{j(\beta)}^{V[g][H]}$ is elementary with $cp(j) = \mathfrak{c}$ and $j(\mathfrak{c}) > \beta$. It now follows that $H_{j(\beta)}^{V[g][H]} \models$ "$\mathfrak{c} > \beta > \aleph_1$".

On the other hand, as the poset $\dot{\mathbb{R}}_g$ does not add any new subsets of ω, we have that $\mathcal{P}(\omega)^{V[g]} = \mathcal{P}(\omega)^{V[g][H]}$ and thus, $\mathfrak{c}^{V[g][H]} \leqslant \mathfrak{c}^{V[g]} = \aleph_1^{V[g]}$. But since $\aleph_1$ is the same in all three models V, $V[g]$ and $V[g][H]$, we have a contradiction. □

Observe that the exact same proof shows that $\mathsf{UR}(\Gamma)$ implies CH, for any class Γ of posets with the property that, CH can be forced by some $\mathbb{Q} \in \Gamma$ and the posets in Γ do not add reals.

An immediate consequence of Lemma 5.15 is that, as mentioned at the end of the prevous section, we may easily separate the axioms $\mathsf{MA}^+(\sigma\text{-closed})$ and $\mathsf{UR}(\sigma\text{-closed})$, if we are granted (the consistency of) a supercompact cardinal.

Corollary 5.16. *If there is a model in which there exists a supercompact cardinal, then there is a model satisfying* $\mathsf{MA}^+(\sigma\text{-closed}) + \neg\,\mathsf{UR}(\sigma\text{-closed})$.

Proof. By a theorem of Shelah, MM implies $\mathsf{MA}^+(\sigma\text{-closed})$ (see Theorem 37.26 in [28]). Hence, starting from a supercompact cardinal, if we force MM in the usual way, then, in the resulting model, the Continuum Hypothesis fails and thus, by Lemma 5.15, $\mathsf{UR}(\sigma\text{-closed})$ cannot possibly hold. □

As advertized, and unlike the other classes of posets which we have considered, the effect of $\mathsf{UR}(\text{c.c.c.})$ on the continuum is more dramatic: it implies that $\mathfrak{c}$ is weakly inaccessible.

This will be obtained as a direct corollary to results in [29], once we have said something regarding the relation between the UR and the RA axioms. The following lemma is true for any (definable) class Γ of posets, although we again focus on the classes of c.c.c., σ-closed, proper, and of stationary preserving posets.

Lemma 5.17. $\mathsf{UR}(\Gamma) \Longrightarrow \mathsf{RA}(\Gamma)$.

Proof. We deal with the class of σ-closed posets and leave the general case to the reader to verify. So assume that $\mathsf{UR}(\sigma\text{-closed})$ holds and let $\mathbb{Q}$ be any fixed σ-closed poset. Fix some regular $\beta > \omega_2$ with $\mathbb{Q} \in H_\beta$.

By the unbounded resurrection axiom, we may find a $\mathbb{Q}$-name for a poset $\dot{\mathbb{R}}$ with $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}}$ is σ-closed", and an elementary embedding $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$, such that $j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$, $cp(j) = \omega_2$ and $j(\omega_2) > \beta$.

But now notice that $cp(j) = \omega_2$ implies that $j \restriction H_{\aleph_2}$ is the identity, i.e., for every $x \in H_{\aleph_2}$, $j(x) = x$ (however, this does not imply surjectivity). It is now easily seen that, in the presence of the embedding,

$$H_{\aleph_2} \prec H_{\aleph_2}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}.$$

From the latter, as CH holds by Lemma 5.15, the axiom $\mathsf{RA}(\sigma\text{-closed})$ follows by just relativizing first–order formulas to the corresponding (definable subclass) $H_{\aleph_1}$ of each of these two structures. □

Corollary 5.18. $\mathsf{UR}(\text{c.c.c.}) \Longrightarrow \mathsf{MA} +$ "$\mathfrak{c}$ *is weakly inaccessible*".

Proof. By Lemma 5.17, we know that $\mathsf{UR}(\text{c.c.c.})$ implies the resurrection axiom $\mathsf{RA}(\text{c.c.c.})$. But now, by results in [29], we get both that MA holds and that $\mathfrak{c}$ is weakly inaccessible; in fact, $\mathfrak{c}$ is a limit of weakly inaccessibles, a limit of limits of weakly inaccessibles, etc. □

It also follows that, if Γ is the class of proper, of $\aleph_1$-semi proper, or of stationary preserving posets, then the unbounded resurrection axioms can be separated from the corresponding resurrection ones.

Corollary 5.19. *For* Γ *as above,* $\mathsf{RA}(\Gamma) + \neg\,\mathsf{UR}(\Gamma)$ *is relatively consistent.*

Proof. By Proposition 5.6, we may force over any model of $\mathsf{RA}(\Gamma)$ in order to obtain a model of $\mathsf{RA}(\Gamma) + \mathsf{CH}$. On the other hand, in all three cases of classes of posets, $\mathsf{UR}(\Gamma)$ implies that the continuum is equal to $\aleph_2$. □

For σ-closed posets, we may not argue likewise, since the axiom $\mathsf{UR}(\sigma\text{-closed})$ actually implies the Continuum Hypothesis. Nevertheless, we now show that $\mathsf{RA}(\sigma\text{-closed})$ can indeed be separated from $\mathsf{UR}(\sigma\text{-closed})$, if one grants the mild assumption of the (consistency of the) existence of an uplifting cardinal with a Mahlo above it. For the proof, we once again use results obtained in [29].

Lemma 5.20. *If there is a model in which there exists an uplifting cardinal with a Mahlo above it, then there is a model of* $\mathsf{RA}(\sigma\text{-closed}) + \neg\,\mathsf{UR}(\sigma\text{-closed})$.

Proof. Given a model in which there exists an uplifting cardinal with a Mahlo above it, we may pass to an inner model of $V = L$ in which there exists a unique Mahlo; namely, as Mahloness is downwards absolute to L, we either work with the whole constructible universe, or with that initial segment of it which is "cut off" at exactly the second L-Mahlo cardinal. Let us call this model V; let λ be the unique Mahlo and let $\kappa < \lambda$ be the least uplifting cardinal.

By Theorem 1.6, V_κ has a definable class function which is a miniature Laver function for κ, i.e., $f : \kappa \longrightarrow V_\kappa$ is such that, for every set $x \in V$, there exists an inaccessible cardinal γ with $\langle V_\kappa, f \rangle \prec \langle V_\gamma, f^* \rangle$ and $f^*(\kappa) = x$. Recall that f^* is the corresponding function for γ, defined exactly the same but over the structure V_γ.

Now, following [29], we may define an Easton support κ-iteration guided by the function f, taking into account only σ-closed posets. This produces a forcing notion $\mathbb{P}$ of size κ which forces $\mathsf{RA}(\sigma\text{-closed})$ together with CH and $\kappa = \aleph_2$ (see also the definition of the posets in the proof of Theorem 5.14). Let us now see that, if G is $\mathbb{P}$-generic over V, then $\mathsf{UR}(\sigma\text{-closed})$ fails in $V[G]$.

Obviously, λ remains Mahlo in $V[G]$. Towards a contradiction, assume that unbounded resurrection holds in $V[G]$ and let $\mathbb{Q} = Coll(\omega_1, \lambda)$ be the σ-closed poset which collapses the Mahlo to ω_1. Let us fix an appropriate name $\dot{\mathbb{R}}$ and a generic elementary embedding of the sort $j : H_{\lambda^+}^{V[G]} \longrightarrow H_{j(\lambda^+)}^{V[G]^{\mathbb{Q}*\dot{\mathbb{R}}}}$ given by the axiom $\mathsf{UR}(\sigma\text{-closed})$. Then, $j(\lambda)$ is Mahlo in $H_{j(\lambda^+)}^{V[G]^{\mathbb{Q}*\dot{\mathbb{R}}}}$ and hence it is Mahlo in $V[G]^{\mathbb{Q}*\dot{\mathbb{R}}}$ as well. But now, just as we did in the proof of Proposition 5.9, we get a contradiction since there are no Mahlo cardinals in $V[G]^{\mathbb{Q}*\dot{\mathbb{R}}}$. $\square$

Question 5.21. *Does the same situation occur in the case of c.c.c. posets? i.e., can we separate* RA(c.c.c.) *from* UR(c.c.c.)?

We shall see that the answer to this question is indeed "yes"; this will follow from considerations regarding consistency lower bounds of the various resurrection axioms, to which we now turn our attention.

5.4 On consistency lower bounds

By results of Hamkins and Johnstone, the resurrection axioms for the classes of c.c.c., of σ-closed, of proper, and of $\aleph_1$-semi proper posets, either follow from, or are actually equiconsistent with the existence of an uplifting cardinal; hence, they are all strictly weaker than a Mahlo cardinal in consistency strength.

On the other hand, for the consistency of the unbounded versions, the assumption of extendibility which we used is outrageously stronger than that of an uplifting cardinal. For RA(stat. pres.), the gap remains very large even if one assumes a proper class of Woodin cardinals. Thus, enquiries regarding the exact consistency strength of these axioms cannot be avoided.

One easy observation which can be made right away is that, since the unbounded resurrection axioms for proper, for $\aleph_1$-semi proper and for stationary preserving posets imply PFA (or even MM), it follows that they must have consistency strength at least at the level of infinitely many Woodin cardinals: it is known that PFA implies that the *Axiom of Determinacy* (AD) holds in $L(\mathbf{R})$ (cf. [45]) and thus, each one of the axioms: UR(proper), UR(semi proper), and UR(stat. pres.) entails that there is an inner model with infinitely many Woodin cardinals.

Of course, the exact consistency strength of the forcing axioms PFA and MM is an important open problem which we do not intend to tackle here. In contrast, we again focus on the classes of c.c.c. and of σ - closed posets. Even for the latter cases, we do not give (provably) optimal answers but we instead provide consistency lower bounds by deriving failures of (weak forms of) square principles. Finally, and via a different method, we also give a lower bound for the case of RA(stat. pres.), together with a comment on how one might initially improve it.

To begin with, we may already observe the failure of squares for the class of σ - closed posets. For this, recall that the axiom $\mathsf{MA}^+(\sigma\text{-closed})$ implies both the SCH and several reflection principles (see § 37 in [28]). In particular, it implies that for every regular $\kappa \geqslant \omega_2$, every stationary $S \subseteq \kappa$ consisting of ordinals of countable cofinality reflects below κ, i.e., there exists a $\gamma < \kappa$ (with $cf(\gamma) = \omega_1$) such that $S \cap \gamma$ is stationary in γ. Thus, by Corollary 5.10 and the remarks from Section 1.5, we immediately have the following.

Corollary 5.22. $\mathsf{UR}(\sigma\text{-closed}) \Longrightarrow \mathsf{SCH}$ + "$\square_\lambda$ *fails, for every* $\lambda \geqslant \omega_1$". $\square$

We will return to the case of σ - closed posets later on in this section, obtaining some failures of weak squares as well.

Let us now concentrate on c.c.c. posets, for which we show that UR(c.c.c.) implies the non – existence of good scales (cf. Definition 1.28) above the continuum. In particular, this implies the SCH and that various weak versions of square fail. For the proof, we use an argument due to Bagaria and Magidor (cf. [7]), which they apply in the context of ω_1 - strongly compact cardinals.

Theorem 5.23. *Assume* UR(c.c.c.). *Then, for every cardinal* $\lambda > \mathfrak{c}$ *such that* $cf(\lambda) = \omega$, *there is no good* λ^+ *- scale.*

Proof. Fix a cardinal $\lambda > \mathfrak{c}$ with $cf(\lambda) = \omega$ and fix some sequence $\langle \lambda_n : n \in \omega \rangle$ of regular cardinals so that $\sup_n \lambda_n = \lambda$. Towards a contradiction, assume that $\langle f_\alpha : \alpha < \lambda^+ \rangle$ is a good λ^+ - scale with respect to this sequence.

Let us also fix some regular $\beta > \lambda^+$ with $\langle f_\alpha : \alpha < \lambda^+ \rangle \in H_\beta$. If $\mathbb{Q} = \{\mathbb{1}\}$ is the trivial poset then, by unbounded resurrection, there exists a c.c.c. poset $\mathbb{R}$ and an elementary embedding

$$j : H_\beta \longrightarrow H_{j(\beta)}^{V^{\mathbb{R}}},$$

such that $j \in V^{\mathbb{R}}$, $cp(j) = \mathfrak{c}$ and $j(\mathfrak{c}) > \beta$. By elementarity, we have that $j(\langle f_\alpha : \alpha < \lambda^+ \rangle) = \langle f^*_\alpha : \alpha < j(\lambda^+) \rangle$ is a good $j(\lambda^+)$-scale, with respect to the sequence $\langle j(\lambda_n) : n \in \omega \rangle$.

Let $\delta = \sup(j''\lambda^+)$ and note that $\delta < j(\lambda^+)$ and $cf(\delta)^{V^{\mathbb{R}}} = \lambda^+$, with the latter being regular in $V^{\mathbb{R}}$. Hence, by definition of a good scale, there exists, in $H^{V^{\mathbb{R}}}_{j(\beta)}$, some $D \subseteq \delta$ cofinal in δ and some $n \in \omega$ so that, for every $\gamma < \gamma'$ in D and every $m > n$, we have the inequality:

$$f^*_\gamma(m) < f^*_{\gamma'}(m).$$

We now define, recursively for $\xi < \lambda^+$, an increasing sequence of ordinals of the form $D^* = \{\gamma_\xi : \xi < \lambda^+\} \subseteq D$, while keeping on the side an auxiliary sequence $\{\alpha_\xi : \xi < \lambda^+\} \subseteq \lambda^+$. Initially, we let $\gamma_0 = \min D$. Given γ_ξ for some $\xi < \lambda^+$, we let $\alpha_\xi < \lambda^+$ be least such that $\gamma_\xi < j(\alpha_\xi)$ and define $\gamma_{\xi+1}$ as the least ordinal in the set D with $j(\alpha_\xi) < \gamma_{\xi+1}$. At limit stages $\xi < \lambda^+$, we let γ_ξ be the least ordinal in D above the supremum of all the γ_ζ's defined so far. It is easy to see that, for every $\xi < \lambda^+$, $\alpha_\xi < \lambda^+$.

Furthermore, for each $\xi < \lambda^+$, there exists an $n_\xi \in \omega$ so that for all $m \geqslant n_\xi$, the following inequalities hold:

$$f^*_{\gamma_\xi}(m) < f^*_{j(\alpha_\xi)}(m) < f^*_{\gamma_{\xi+1}}(m).$$

Now let $E \subseteq \lambda^+$ be of cardinality λ^+ and with the property that, for all $\xi \in E$, the corresponding n_ξ is the same; say equal to some fixed $k \in \omega$. Then, for every $\xi < \zeta$ in E, we have the inequalities which are shown below:

$$f^*_{\gamma_\xi}(k) < f^*_{j(\alpha_\xi)}(k) < f^*_{\gamma_{\xi+1}}(k) \leqslant f^*_{\gamma_\zeta}(k) < f^*_{j(\alpha_\zeta)}(k) < f^*_{\gamma_{\zeta+1}}(k).$$

At this point observe that, for any $\xi < \lambda^+$, $f^*_{j(\alpha_\xi)}(k) = j(f_{\alpha_\xi}(k))$ where, by definition of a scale, $j(f_{\alpha_\xi}(k)) \in j''\lambda_k$.

But this is a contradiction, since the sequence $\langle f^*_{j(\alpha_\xi)}(k) : \xi \in E \rangle$ has order type λ^+, whereas λ_k is a (regular) cardinal below λ. □

We remark that in the last proof, we made heavy use of the fact that the forcing $\mathbb{R}$, due to its countable chain condition, preserved cofinalities and cardinals. We also relied on the fact that the critical point of the generic embeddings given by UR(c.c.c.) is $\mathfrak{c}$. A moment's reflection shows that we may easily modify the previous argument in order to account for any cofinality below $\mathfrak{c}$.

Corollary 5.24. *Assume* UR(c.c.c.). *Then, for every cardinal* $\lambda > \mathfrak{c}$ *such that* $cf(\lambda) < \mathfrak{c}$, *there is no good* λ^+-*scale.* □

We now draw more conclusions regarding the effect of UR(c.c.c.) on the universe.

Corollary 5.25. UR(c.c.c.) *implies the following*:

(i) SCH.

(ii) $\square^*_\lambda$ *fails, for every* $\lambda > \mathfrak{c}$ *with* $cf(\lambda) < \mathfrak{c}$.

Proof. For (i), first notice that the SCH holds (vacuously) at every singular $\lambda < \mathfrak{c}$. Additionally, it follows from Theorem 5.23 and Theorem 1.30 that, for any singular $\lambda > \mathfrak{c}$ with $cf(\lambda) = \omega$, the SCH holds at λ. Hence, by a result of Silver (see Theorem 8.13 in [28]), the SCH holds everywhere.

For (ii), we invoke Proposition 1.29 and combine it with Corollary 5.24; then, for every $\lambda > \mathfrak{c}$ with $cf(\lambda) < \mathfrak{c}$, we have that $\square^*_\lambda$ fails. $\square$

As we have pointed out in the Prelude, failures of (weak) squares imply inner models with large cardinals. For example, recall that as we mentioned at the (very) end of Section 1.5, if $\square_\lambda$ fails at some singular strong limit λ, then AD holds in $L(\mathbf{R})$. Therefore, UR(c.c.c.) implies, consistency – wise, the existence of inner models with infinitely many Woodin cardinals.

On the other hand, recalling the results of [29], the axiom RA(c.c.c.) is equiconsistent with the existence of an uplifting cardinal and, therefore, we indeed have a substantial gap in consistency strength between RA(c.c.c.) and UR(c.c.c.). Moreover, and as another consequence of Corollary 5.25, we may actually separate the two axioms as long as we are granted the (consistency of the) existence of an uplifting cardinal.

To see this, arguing as in [29], we may force RA(c.c.c.) by a c.c.c. poset starting from an uplifting cardinal in a model of $V = L$ (where global square holds). This produces a model of RA(c.c.c.) $+ \neg\,\mathsf{UR}$(c.c.c.) and gives an (anticipated) affirmative answer to Question 5.21.

We now turn to the case of σ - closed posets. To begin with, we introduce the notion of a *generically extendible* cardinal; this is in accordance with other notions of "generic large cardinals", such as generically supercompact and generically huge cardinals (see, for example, [17] or [18]).

Definition 5.26. *Let* $\kappa = \mu^+$, *where* μ *is regular, and fix some* (*definable*) *class* Γ *of posets which preserve cofinalities* $< \mu$. *We say that* κ *is* **generically extendible** *by* Γ *if for every cardinal* $\lambda > \kappa$, *there exists a poset* $\mathbb{P} \in \Gamma$ *and there is an elementary embedding*

$$j : H_\lambda \longrightarrow H^{V^{\mathbb{P}}}_{j(\lambda)},$$

with $j \in V^{\mathbb{P}}$, $cp(j) = \kappa$ *and* $j(\kappa) > \lambda$.

A direct consequence of the definition of the UR axioms is that $\mathsf{UR}(\Gamma)$ implies that ω_2 is generically extendible by Γ, for the classes Γ of σ-closed and of proper posets. On the other hand, recall that, in general, forcing with stationary preserving posets (in fact, $\aleph_1$-semi properness suffices) may drop the cofinality of an uncountable regular cardinal to ω.

Furthermore, and again for the classes Γ of σ-closed and of proper posets, the axiom $\mathsf{UR}(\Gamma)$ actually implies that ω_2 is *indestructibly* generically extendible, a notion which is stronger than generic extendibility and parallel to the definition of unbounded resurrection.

Definition 5.27. *Let $\kappa = \mu^+$, where μ is regular, and fix some (definable) class Γ of posets which preserve cofinalities $< \mu$. We say that κ is* **indestructibly generically extendible** *by Γ if for every cardinal $\lambda > \kappa$ and every $\mathbb{Q} \in \Gamma$ with $\mathbb{Q} \in H_\lambda$, there exists a (name for a) poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", and there is an elementary embedding*

$$j : H_\lambda \longrightarrow H_{j(\lambda)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}},$$

with $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $cp(j) = \kappa$ and $j(\kappa) > \lambda$.

We also consider the (known) notion of an indestructibly generically supercompact cardinal (cf. [14] and Definition 11.4 in [18]). The following is a slight modification of the corresponding definition in [18], as we are taking into account various classes of posets.

Definition 5.28. *Let $\kappa = \mu^+$, where μ is regular, and fix some (definable) class Γ of posets which preserve cofinalities $< \mu$. We say that κ is* **indestructibly generically supercompact** *by Γ if for every regular $\lambda > \kappa$ and every $\mathbb{Q} \in \Gamma$ with $\mathbb{Q} \in H_\lambda$, there is a (name for a) poset $\dot{\mathbb{R}}$ such that $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$", and there is an elementary embedding*

$$j : V \longrightarrow M \subseteq V^{\mathbb{Q}*\dot{\mathbb{R}}},$$

where M is transitive, j is a definable subclass of $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, $cp(j) = \kappa$, $j(\kappa) > \lambda$, $j''\lambda \in M$, $\sup(j''\lambda) < j(\lambda)$ and $cf(\lambda)^M = \mu$.

We now establish a connection which might already be expected.

Proposition 5.29. *Let $\kappa = \mu^+$, where μ is regular, and fix a (definable) class Γ of posets which preserve cofinalities $< \mu$. If κ is indestructibly generically extendible by Γ, then it is also indestructibly generically supercompact by Γ.*

Proof. Fix a cardinal $\kappa = \mu^+$, where μ is regular, let Γ be a class of posets which preserve cofinalities $< \mu$ and suppose that κ is indestructibly generically extendible by Γ. Fix a regular $\lambda > \kappa$ and some $\mathbb{Q} \in H_\lambda$ with $\mathbb{Q} \in \Gamma$. We shall find a (name for a) poset $\dot{\mathbb{R}}$ and a (class) elementary embedding in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$ witnessing the indestructible generic supercompactness of κ.

For this, further fix some $\beta = \beth_\beta > \lambda$ and let $\dot{\mathbb{R}}$ and j witness the indestructible generic extendibility of κ with respect to β^+, i.e., $\mathbb{Q} \Vdash$ "$\dot{\mathbb{R}} \in \Gamma$" and $j \in V^{\mathbb{Q}*\dot{\mathbb{R}}}$ is an elementary embedding of the form

$$j : H_{\beta^+} \longrightarrow H_{j(\beta^+)}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}},$$

with $cp(j) = \kappa$ and $j(\kappa) > \beta^+$. We now extract, in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, an appropriate (long) extender from j, measuring sets in V, and we then argue that the corresponding extender ultrapower witnesses the indestructible generic supercompactness of κ, as desired. At this point, the reader might want to recall the proof of Theorem 2.32, where an analogous strategy was followed.

So, let $E = \langle E_a : a \in [j(\beta)]^{<\omega} \rangle$ where, each E_a is a V-ultrafilter on $[\beta]^{|a|}$ defined as usual: for $X \in \mathcal{P}([\beta]^{|a|}) \cap V$,

$$X \in E_a \Longleftrightarrow a \in j(X).$$

It is easy to check that each E_a is also sufficiently complete: namely, for every $\alpha < \kappa$ and every α-sequence $\langle X_\xi : \xi < \alpha \rangle$, where $X_\xi \in E_a \cap V$ for all $\xi < \alpha$, we have that $\bigcap_{\xi<\alpha} X_\xi \in E_a$.

Note that, in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, κ has cardinality μ and so it is no longer a cardinal; in fact, since $V^{\mathbb{Q}*\dot{\mathbb{R}}} \models j(\kappa) = \mu^+$, the same is true for every ordinal in the interval $[\kappa\,, j(\kappa))$. This is the reason for not using the term "κ-complete" for the ultrafilters E_a or the term "$(\kappa, j(\beta))$-extender" for E. Still, we argue that E indeed has extender properties and that a corresponding ultrapower may be formed.

For this, one patiently verifies that the defining clauses of an extender (see Definition A.4 in the Appendix) are satisfied, with respect to the $\kappa = \mu^+$ of V. For example, we check that $E_{\{\kappa\}}$ is not $(\kappa^+)^V$-complete: just define, in V and for each $\xi < \kappa$, the set $X_\xi = \{\{\alpha\} : \xi < \alpha < \beta\}$ and note that $X_\xi \in E_{\{\kappa\}}$. Then, $\bigcap_{\xi<\kappa} X_\xi \in V$ while $\{\kappa\} \notin j(\bigcap_{\xi<\kappa} X_\xi)$ (the latter is verified, of course, in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$). Similarly, one checks that the rest of the proof of Lemma A.6 in the Appendix goes through; we only comment on the issue of well-foundedness.

Let us assume, towards a contradiction, that E is not well-founded, i.e., the well-foundedness condition of Definition A.4 fails. Then, following the proof (namely, the "converse" direction) of Proposition A.5, we obtain a sequence of elements $[a_n, [F_n]]$ witnessing the ill–foundedness of $\widetilde{M_E}$. But recall that in the aforementioned proof, for each $n \in \omega$, we have that $a_n \in [j(\beta)]^{<\omega}$ and $F_n : [\beta]^{|a_n|} \longrightarrow \beta^+$, where the F_n 's are defined in terms of an appropriate rank function. As all the elements $[a_n, [F_n]]$ belong to the $H_{j_E(\beta^+)}$ of $\widetilde{M_E}$, it follows that a counterexample to its well–foundedness may already be found there. Now we reach a contradiction, since the $H_{j_E(\beta^+)}$ of $\widetilde{M_E}$ is embeddable into the $H_{j(\beta^+)}$ of $V^{\mathbb{Q}*\dot{\mathbb{R}}}$, via the restrictions of the usual commuting embeddings.

Having established that E is a legitimate extender, we may now let

$$j_E : V \longrightarrow M_E \subseteq V^{\mathbb{Q}*\dot{\mathbb{R}}}$$

be the extender embedding, which is of course a definable subclass of $V^{\mathbb{Q}*\dot{\mathbb{R}}}$ and where M_E is transitive. By standard arguments regarding extenders, $cp(j_E) = \kappa$. We now check the rest of the clauses towards establishing the indestructible generic supercompactness of κ, as witnessed by ($\dot{\mathbb{R}}$ and) j_E; i.e., we check that $j_E(\kappa) > \lambda$, $j_E{}''\lambda \in M_E$, $\sup(j_E{}''\lambda) < j_E(\lambda)$ and $cf(\lambda)^{M_E} = \mu$.

We argue as in the proof of Theorem 2.32 and we get a restricted version of the usual commutative diagram, defining $k_E^* : H^{M_E}_{j_E(\beta)} \longrightarrow H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$, by letting

$$k_E^*([\,a, [f]\,]) = j(f)(a),$$

for all $[\,a, [f]\,] \in H^{M_E}_{j_E(\beta)}$, where $a \in [j(\beta)]^{<\omega}$ and $f : [\beta]^{|a|} \longrightarrow H_\beta$ with $f \in V$. We point out that this definition makes sense since any such function, representing an element in $H^{M_E}_{j_E(\beta)}$, indeed belongs to H_{β^+}. Moreover, it is easily checked that k_E^* is a well–defined $\{\in\}$-embedding and so, in particular, injective. We then get the commutative diagram

$$\begin{tikzcd}
H_\beta \arrow[r, "j \restriction H_\beta"] \arrow[d, "j_E \restriction H_\beta"'] & H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)} \\
H^{M_E}_{j_E(\beta)} \arrow[ur, "k_E^*"'] &
\end{tikzcd}$$

where $j \restriction H_\beta = k_E^* \circ (j_E \restriction H_\beta)$. Next, we show that k_E^* is in fact the identity. Since $\beta = \beth_\beta$, we fix some bijection $g : [\beta]^1 \longrightarrow H_\beta$, where $g \in H_{\beta^+}$. Then, by elementarity, we have that $j(g) : [j(\beta)]^1 \longrightarrow H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$ is also a bijection and $j(g) \in H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta^+)}$. Thus, for every $x \in H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$, there is some $\xi < j(\beta)$ such that $x = j(g)(\{\xi\})$. But this means that for every $x \in H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$,

$$x = k_E^*([\,\{\xi\}, [g]\,]),$$

where $[\{\xi\},[g]]$ is an element of $H^{M_E}_{j_E(\beta)}$, i.e., k^*_E is also surjective. Therefore, it must be the identity, since its domain and range are transitive sets. Hence, we conclude that

$$H^{M_E}_{j_E(\beta)} = H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)},$$

i.e., $H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)} \subseteq M_E$ and, also, for every ordinal $\alpha \leqslant \beta$, $j_E(\alpha) = j(\alpha)$. In particular, $j_E(\kappa) = j(\kappa) > \lambda$, $j_E(\lambda) = j(\lambda)$ and $j_E{}''\lambda = j''\lambda \in H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)}$; therefore, we obtain that $j_E{}''\lambda \in M_E$ and $\sup(j_E{}''\lambda) = \sup(j''\lambda) < j(\lambda) = j_E(\lambda)$ as well.

To finish the proof, by $V^{\mathbb{Q}*\dot{\mathbb{R}}} \models |\lambda| = \mu$ and the fact that $H^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}_{j(\beta)} \subseteq M_E$, it follows that $cf(\lambda)^{M_E} = cf(\lambda)^{V^{\mathbb{Q}*\dot{\mathbb{R}}}} \leqslant \mu$. But notice that the latter inequality cannot be strict, because the posets in the class Γ are supposed to preserve cofinalities $<\mu$. The proof is now complete. □

As we have already remarked, in the context of the axioms $\mathsf{UR}(\Gamma)$ for the classes of σ-closed or of proper posets, we may regard ω_2 as being (indestructibly) generically extendible (resp. supercompact, by the last proposition).

With this picture in mind, let us depart from full generality in order to draw conclusions related to the specific unbounded resurrection axiom $\mathsf{UR}(\sigma\text{-closed})$. Towards this goal, we give the following generalization of an argument due to Foreman and Magidor (cf. § 5 in [19]). This is a result already quoted in [14], of which we now provide a proof.

Theorem 5.30. *Suppose that ω_2 is indestructibly generically supercompact by the class of σ-closed posets. Then, for every (uncountable) strong limit λ with $cf(\lambda) = \omega$, $\square^*_\lambda$ fails.*

Proof. Assume that ω_2 is indestructibly generically supercompact by the class of σ-closed posets. Fix some strong limit $\lambda > \omega_2$ with $cf(\lambda) = \omega$ and suppose, towards a contradiction, that

$$\mathscr{C} = \langle\, \mathcal{C}_\alpha : \alpha \in Lim(\lambda^+)\,\rangle$$

is a $\square^*_\lambda$-sequence, i.e., $\mathscr{C}$ satisfies the following conditions for every $\alpha \in Lim(\lambda^+)$:

(*i*) $\mathcal{C}_\alpha \subseteq \mathcal{P}(\alpha)$ and $1 \leqslant |\mathcal{C}_\alpha| \leqslant \lambda$.

(*ii*) Every $C \in \mathcal{C}_\alpha$ is a club in α, with $\mathrm{ot}(C) < \lambda$.

(*iii*) For every $C \in \mathcal{C}_\alpha$ and every $\beta \in Lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

(*iv*) There is some $C \in \mathcal{C}_\alpha$ with $\mathrm{ot}(C) = cf(\alpha)$.

(*v*) For every $C \in \mathcal{C}_\alpha$ and every club $D \subseteq C$, $D \in \mathcal{C}_\alpha$ as well.

Recall that by Definition 1.26 and its subsequent remarks, the ordinary defining clauses of a $\square^*_\lambda$ - sequence are (i) - (iii) (note that in condition (ii) the singularity of λ is used) and, without loss of generality, we may also assume condition (iv). As for condition (v), it can be assumed in the light of the fact that, in the current situation, λ is a (singular) strong limit cardinal.

Now let $\mathbb{Q} = \{\mathbb{1}\}$ be the trivial poset. By indestructible generic supercompactness of ω_2, there is some σ - closed poset $\mathbb{R}$ and an elementary embedding

$$j : V \longrightarrow M \subseteq V^{\mathbb{R}},$$

where M is transitive, j is a definable subclass of $V^{\mathbb{R}}$, $cp(j) = \omega_2$, $j(\omega_2) > \lambda^+$, $j\,''\lambda^+ \in M$, $\sup(j\,''\lambda^+) < j(\lambda^+)$ and $cf(\lambda^+)^M = \omega_1$. Let $\gamma = \sup(j\,''\lambda^+)$ and let us denote by

$$\langle\, \mathcal{C}^*_\alpha : \alpha \in Lim(j(\lambda^+))\,\rangle$$

the image $j(\mathscr{C}) \in M$ of the weak square sequence. Then, working temporarily in the model M, there is some club $D_\gamma \subseteq \gamma$ with $D_\gamma \in \mathcal{C}^*_\gamma$. Using the fact that $cf(\gamma) = cf(\lambda^+) = \omega_1$, we may assume by condition (iv) that $\mathrm{ot}(D_\gamma) = \omega_1$. Moreover, since $j\,''\lambda^+$ is an ω - club in γ, we may further assume (by intersecting D_γ with $j\,''\lambda^+$ and using condition (v) if necessary) that $D_\gamma \subseteq j\,''\lambda^+$.

By condition (iii), for every $\delta \in Lim(D_\gamma)$, we have that $D_\delta = D_\gamma \cap \delta \in \mathcal{C}^*_\delta$. But then, from the perspective of $V^{\mathbb{R}}$ now, D_δ is a countable set of ordinals, subset of the range of j. Thus, by the σ - closure of $\mathbb{R}$, there exists some (countable) $x \in V$ with $j(x) = D_\delta$. So, if $\alpha_\delta < \lambda^+$ is chosen so that $\delta = j(\alpha_\delta)$, we moreover get – by elementarity – that $x \in \mathcal{C}_{\alpha_\delta}$.

In other words, the preceding discussion shows that the forcing $\mathbb{R}$ has added a so – called *thread* of order type ω_1 and cofinal in λ^+, through the ground model weak square sequence; that is, we have added a set $E \in V^{\mathbb{R}}$ which has order type ω_1 and is cofinal in λ^+ and with the property that, for every $\alpha \in Lim(E)$, $E \cap \alpha \in \mathcal{C}_\alpha$. Namely, the thread E is the pre – image of D_γ under the embedding j. Note that, clearly, such a thread cannot exist in V although, by the σ - closure of $\mathbb{R}$, all of its initial segments do.

We now use the closure of $\mathbb{R}$ in order to derive a contradiction. By standard abuse of notation, we dispense with the "dots" and the "checks" when referring to elements of $V^{\mathbb{R}}$; in particular, E is really an $\mathbb{R}$ - name. Without loss of generality, E may be chosen so that $\mathbb{R}$ forces that E has order – type ω_1 and is cofinal in λ^+. We first need the following claim, showing that for any condition $p \in \mathbb{R}$, there exists some $\alpha < \lambda^+$ so that many ground model elements are forced (below p) to be equal to $E \cap \alpha$.

Claim. *For every $p \in \mathbb{R}$ there exists some $\alpha < \lambda^+$ such that*

$$|\{z \in V \cap [\lambda^+]^\omega : \textit{there is } r \leqslant p \textit{ s.t. } r \Vdash z = E \cap \alpha\}| \geqslant \lambda.$$

Proof of claim. Towards a contradiction, fix some $p \in \mathbb{R}$ which is a counterexample to the claim. Moreover, fix $\langle \lambda_n : n \in \omega \rangle$ a sequence of regular cardinals which is cofinal in λ. Now, for every $\alpha < \lambda^+$, consider the (non-empty) ground model set

$$T_\alpha = \{z \in V \cap [\lambda^+]^\omega : \text{there is } r \leqslant p \text{ s.t. } r \Vdash z = E \cap \alpha\}.$$

By assumption on p, there is some $n_\alpha \in \omega$ with $|T_\alpha| < \lambda_{n_\alpha}$. Moreover, for any $\alpha < \alpha' < \lambda^+$, $|T_\alpha| \leqslant |T_{\alpha'}|$. Hence, there exists some fixed $n \in \omega$ such that, for every $\alpha < \lambda^+$, $|T_\alpha| < \lambda_n$.

From the latter bound and the regularity of λ_n, we get that for every $\alpha < \lambda^+$ with $cf(\alpha) = \lambda_n$, there is some $\beta < \alpha$ with the property that, for every pair of initial segments $z, z' \in T_\alpha$, if $z \neq z'$ then $z \cap \beta \neq z' \cap \beta$. But this produces a regressive function on the stationary set of ordinals below λ^+ which have cofinality λ_n; thus, there is a stationary $S \subseteq \lambda^+$ and there is some fixed $\beta < \lambda^+$ so that

$$\forall \alpha \in S \;\; \forall z, z' \in T_\alpha \, (z \neq z' \longrightarrow z \cap \beta \neq z' \cap \beta).$$

Now fix some $z \in T_\beta$; that is, $z \in V \cap [\lambda^+]^\omega$ and, for some condition $r \leqslant p$, $r \Vdash z = E \cap \beta$. But then, for every $\alpha \in S$ (most interestingly for $\alpha > \beta$), there must be exactly one $z_\alpha \in T_\alpha$ which is "compatible" with z, in the sense that $z = z_\alpha \cap \beta$ and $r_\alpha \Vdash z_\alpha = E \cap \alpha$, for some r_α below r. Hence, the whole thread E can already be decided in the ground model, i.e., $E \in V$. This is a clear contradiction which proves the claim. $\square$

Given the claim, we now build a tree of conditions in $\mathbb{R}$, indexed by finite sequences $s \in {}^{<\omega}\lambda$. We perform the construction recursively based on the length of s, aiming at producing, for each $n \in \omega$, a set of conditions

$$A_n = \{q_s : s \in {}^n\lambda\} \subseteq \mathbb{R}$$

and an ordinal $\delta_n < \lambda^+$ such that, for every $s \in {}^n\lambda$, $q_s \in A_n$ determines the initial segment $E \cap \delta_n$ of the thread. We initialize the construction by letting $A_0 = \{\mathbb{1}_{\mathbb{R}}\}$ and $\delta_0 = \varnothing$.

Now, suppose that A_n and δ_n are given, for some $n \in \omega$. For any fixed $s \in {}^n\lambda$ and $q_s \in A_n$, we show how to extend q_s to q_t, for every $t \in {}^{n+1}\lambda$ with $s \sqsubseteq t$. By the claim, there exists some $\alpha_s < \lambda^+$ so that the set

$$T_{\alpha_s} = \{z \in V \cap [\lambda^+]^\omega : \text{there is } r \leqslant q_s \text{ s.t. } r \Vdash z = E \cap \alpha_s\}$$

has cardinality at least λ. Hence, by choosing for each such z some condition $r \leqslant q_s$ witnessing the fact that $z \in T_{\alpha_s}$, we produce an antichain D_s of size

λ, consisting of conditions below q_s which force incompatible information about $E \cap \alpha_s$. We index these incompatible conditions using finite sequences $t \in {}^{n+1}\lambda$ with $s \sqsubseteq t$, i.e., we write the produced antichain as:

$$D_s = \{r_t : t \in {}^{n+1}\lambda,\, s \sqsubseteq t\}.$$

In other words, this procedure gives, for any $s \in {}^{n}\lambda$ and $q_s \in A_n$, some ordinal $\alpha_s < \lambda^+$ and some antichain D_s of size λ, consisting of conditions indexed by the finite sequences in ${}^{n+1}\lambda$ which extend s, as above. Recall that each such $r_t \in D_s$ corresponds to some $z_t \in V$ with $r_t \Vdash z_t = E \cap \alpha_s$.

We now let $\delta_{n+1} = \sup\{\alpha_s : s \in {}^{n}\lambda\}$ (where notice that $\delta_{n+1} < \lambda^+$). In order to define A_{n+1}, for every $t \in {}^{n+1}\lambda$ with $s \sqsubseteq t$, we choose some q_t which is an extension of $r_t \in D_s$ and such that q_t decides $E \cap \delta_{n+1}$, i.e., for some $z_t \in V$, we have $q_t \Vdash z_t = E \cap \delta_{n+1}$. Then, we let A_{n+1} be the collection of those chosen extensions q_t's. It is now immediate that the resulting set $A_{n+1} = \{q_t : t \in {}^{n+1}\lambda\}$ along with δ_{n+1} satisfy the construction requirement.

Furthermore, by the σ-closure of $\mathbb{R}$, we may assume, by enlarging δ_{n+1} and extending each q_t ω-many times if necessary, that every q_t forces that its corresponding $z_t = E \cap \delta_{n+1}$ is unbounded in δ_{n+1}. Finally, let $\delta = \sup_n \delta_n < \lambda^+$.

For each function $f : \omega \longrightarrow \lambda$, let $q_f \in \mathbb{R}$ be a lower bound of the descending chain $\{q_{f\restriction n} : n \in \omega\}$. Observe that for every such function, q_f determines $E \cap \delta$; namely, if $z_f = \bigcup_{n\in\omega} z_{f\restriction n}$, then z_f is countable, $z_f \in V$ and $q_f \Vdash z_f = E \cap \delta$. Moreover, q_f forces that z_f is unbounded in δ. In particular, as E is supposed to be a thread, we have that $z_f \in \mathcal{C}_\delta$.

Consequently, if $f \neq g$ are distinct functions from ${}^{\omega}\lambda$, then $z_f \neq z_g$ and so $\mathcal{C}_\delta$ must have cardinality at least $\lambda^\omega \geqslant \lambda^+$. But we have just arrived at a contradiction since, by condition (i) of the weak square sequence, $|\mathcal{C}_\delta| \leqslant \lambda$. This completes the proof. □

Recalling that $\mathsf{UR}(\sigma\text{-closed})$ implies that ω_2 is indestructibly generically extendible by the class of σ-closed posets, Theorem 5.30 combined with Proposition 5.29 immediately give the following (adding to Corollary 5.22).

Corollary 5.31. $\mathsf{UR}(\sigma\text{-closed})$ *implies that, for every (uncountable) strong limit cardinal λ with $cf(\lambda) = \omega$, $\square^*_\lambda$ fails.* □

Let us point out that, regarding Theorem 5.30, the assumption "$cf(\lambda) = \omega$" became important at the very final step of the proof: we used it to conclude that, as a consequence of König's Theorem, $\lambda^\omega > \lambda$. Clearly, this would not have been granted if we had assumed that $cf(\lambda) > \omega$. At any rate, we may ask:

Question 5.32. *Can we dispense with the "strong limit" assumption in the previous result(s)? Moreover, does* $\mathsf{UR}(\sigma\text{-closed})$ *imply failure of even weaker principles, such as the approachability property?*

As a concluding result of this chapter, let us consider anew the resurrection axiom for stationary preserving posets. We shall show that, unlike the other resurrection axioms appearing in [29], RA(stat. pres.) has consistency strength beyond the realm of large cardinals compatible with $V = L$; namely, it already implies that every set has a sharp. For this, we shall use some of the techniques developed by R. Schindler in order to get lower bounds for the consistency strength of BMM (cf. [40] and [41]). Before we proceed to the actual theorem, we need some background material from [40].

Let $r \subseteq \omega$. We describe a recursive construction (of length at most ω_1) which will produce an ordinal $\xi_r \leqslant \omega_1$, a function $f_r : \xi_r \longrightarrow \omega_1$, a sequence of the form $d^{(r)} = \langle\, d_i^{(r)} : i < \xi_r \,\rangle$ and some $A_r \subseteq \xi_r$. Intuitively, the ordinal ξ_r will indicate the length for which the recursive construction can be carried out, for our fixed r.

Suppose that, for some $\nu \leqslant \omega_1$, we have already defined $f_r \restriction \nu$, $\langle\, d_i^{(r)} : i < \nu \,\rangle$, and $A_r \cap \nu$. If $\nu = \omega_1$ or if $\nu < \omega_1$ and ν is uncountable in $L[A_r \cap \nu]$, we then set $\xi_r = \nu$ and finish the construction. Otherwise, we define $f_r(\nu)$ to be the least ordinal $\beta < \omega_1$ such that $L_{\beta+1}[A_r \cap \nu] \models$ "ν is countable"; moreover, we let $d_\nu^{(r)}$ be the $L[A_r \cap \nu]$-least $d \subseteq \omega$ which is almost-disjoint from all the $d_i^{(r)}$'s, for $i < \nu$. Finally, we put ν into A_r if and only if $d_\nu^{(r)} \cap r$ is finite. This concludes the description of the construction. Then, our final definition is the following.

Definition 5.33 ([40]). *We say that* $r \subseteq \omega$ **codes a reshaped subset of** ω_1 *if the above construction can be carried out for all ordinals up to* ω_1*, i.e., if* $\xi_r = \omega_1$*.*

Obviously, if $r \subseteq \omega$ codes a reshaped subset of ω_1, then $\omega_1^{L[r]} = \omega_1^V$. Moreover, if we are given a real $r \subseteq \omega$, then r codes a reshaped subset of ω_1 if and only if this is witnessed in the structure $H_{\aleph_1}$; that is, $H_{\aleph_1}$ uses r as a parameter and faithfully verifies that the aforementioned recursive construction can be carried out for all ordinals in ω_1. In addition, by absoluteness of the computations, in such a case the associated witnessing triple $\langle\, f_r\,,\, d^{(r)}\,,\, A_r \,\rangle$ is the same, whether it is computed in $H_{\aleph_1}$ or in V. We are now ready for the theorem.

Theorem 5.34. RA(stat. pres.) $\Longrightarrow$ *For every* $X \in V$*,* $X^\#$ *exists.*

Proof. First, note that by Theorem 1.24, RA(stat. pres.) $+ \neg\,\mathsf{CH}$ entails BMM. Hence, in this case the conclusion follows from Theorem 1.3 in [40] (in fact, [41] provides even better bounds for BMM). It is therefore sufficient to consider the situation in which we have RA(stat. pres.) together with CH.

Towards a contradiction, assume that for some set X, $X^{\#}$ does not exist. Then, again by results of Schindler (cf. [39]), there is a stationary preserving poset $\mathbb{P}$ which adds a real $r \subseteq \omega$ coding a reshaped subset of ω_1; the latter fact, by our previous remarks, is actually witnessed in $H_{\aleph_1}^{V^{\mathbb{P}}}$. But then, if $\dot{\mathbb{R}}$ is a further (name for a) stationary preserving poset which achieves resurrection, i.e., such that

$$H_{\aleph_1} \prec H_{\aleph_1}^{V^{\mathbb{P}*\dot{\mathbb{R}}}},$$

and since ω_1 is preserved, we have that r still codes a reshaped subset of ω_1 in $V^{\mathbb{P}*\dot{\mathbb{R}}}$ and thus, the same is true in the structure $H_{\aleph_1}^{V^{\mathbb{P}*\dot{\mathbb{R}}}}$. Hence, by elementarity, there must exist reals $r \in V$ which code reshaped subsets of ω_1.

Now, let $<^*$ be the ordering relation on functions in ${}^{\omega_1}\omega_1$ defined by:

$$f <^* g \iff \exists\, C \subseteq \omega_1 \,(\text{“}C \text{ is a club”} \wedge \forall\, \alpha \in C\,(f(\alpha) < g(\alpha)))\,.$$

By the well–foundedness of $<^*$, let us fix a real $r \in V$, coding a reshaped subset of ω_1, with its associated f_r being $<^*$-minimal among functions f_x associated with reals $x \in V$ coding reshaped subsets of ω_1. Let $d^{(r)} = \langle\, d^{(r)}_\alpha : \alpha < \omega_1 \,\rangle \in L[r]$ be the sequence of almost–disjoint subsets of ω associated with this r.

Then, by Lemma 3.3 in [40], there is a stationary preserving poset $\mathbb{Q}_1$ forcing the existence of some real r' and of some club $C \subseteq \omega_1$, so that r' codes a reshaped subset of ω_1 and the club C witnesses that $f_{r'} <^* f_r$. In $V^{\mathbb{Q}_1}$, let $\dot{\mathbb{Q}}_2$ be the (name for the) c.c.c. poset which codes C by a real z, relative to the sequence $d^{(r)}$. That is, $\dot{\mathbb{Q}}_2$ is the standard (Jensen–Solovay) almost–disjoint coding forcing, which produces a $z \subseteq \omega$ with the property that, in $V^{\mathbb{Q}_1 * \dot{\mathbb{Q}}_2}$, for every $\alpha < \omega_1$,

$$\alpha \in C \iff |z \cap d^{(r)}_\alpha| < \aleph_0.$$

Let $\mathbb{Q} = \mathbb{Q}_1 * \dot{\mathbb{Q}}_2$ and notice that $\mathbb{Q}$ is stationary preserving in V. Hence, by the resurrection axiom RA(stat. pres.), there exists some further (name for a) stationary preserving poset $\dot{\mathbb{R}}$ giving that

$$H_{\aleph_1} \prec H_{\aleph_1}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}.$$

Clearly, ω_1 is preserved throughout these forcing constructions. We now derive a contradiction by arguing that, in $H_{\aleph_1}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$, we may express the statement “there is a real r' coding a reshaped subset of ω_1 with $f_{r'} <^* f_r$”. This will be enough since in this case, by elementarity, such a real must already exist in V, which would contradict the $<^*$-minimality of f_r.

For this, first note that both the fact that r' codes a reshaped subset of ω_1 and that C is a club witnessing $f_{r'} <^* f_r$ remain true in $V^{\mathbb{Q}*\dot{\mathbb{R}}}$. The former fact, as we have already remarked, can be indeed witnessed in $H_{\aleph_1}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$. As for the latter, since

$z \in H_{\aleph_1}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}}$ and the computations of $f_{r'}$, f_r and of $d^{(r)}$ are absolute once the reals r' and r are given, we may express the statement "$f_{r'} <^* f_r$" by saying:
"*there exists a* $z \subseteq \omega$ *such that, the ordinals* α *for which* $|z \cap d_\alpha^{(r)}| < \aleph_0$*, where* $d_\alpha^{(r)}$ *is the* α^{th} *element appearing in the recursive construction of* $d^{(r)}$ *associated with* r*, form a closed and unbounded class*; *moreover, for every such* α*, we have that* $f_{r'}(\alpha) < f_r(\alpha)$". Hence, it now follows that

$$H_{\aleph_1}^{V^{\mathbb{Q}*\dot{\mathbb{R}}}} \models \exists\, r' \subseteq \omega\,(\text{“}r'\text{ codes a reshaped subset of }\omega_1\text{”} \wedge \text{“}f_{r'} <^* f_r\text{”})$$

and then, by elementarity, we get the desired contradiction in V. $\square$

The previous result was evidently one final digression from the general spirit of the rest of the dissertation. It should be thought of as an *en passant* theorem, rather than an indication of our expertise in all of the techniques involved.

It is worth mentioning that, as pointed out to us by Ralf Schindler himself, the arguments showing that BMM implies strong cardinals in the *core model* (cf. [41]) are similarly applicable to the case of RA(stat. pres.). For the time being, we take his word for it and conveniently avoid delving into the subtleties, and the laborious details of inner model theory.

Conclusions & Questions

We share with the (remaining) readers some concluding thoughts, together with a non – exhaustive list of open questions which have arisen along the way. Many of these questions, rather than precise formal statements, are given in the form of general enquiries.

Starting with the various $C^{(n)}$ - cardinal hierarchies, we have established consistency upper bounds via constructions of elementary chains. In particular, we have shown that, consistency – wise, the assumption of almost hugeness is an adequate upper bound for all the $C^{(n)}$ - cardinals considered in Chapter 2. This bound is an improvement of the results appearing in [5], where the consistency of the aforementioned $C^{(n)}$ - cardinals was obtained from that of the existence of rank – into – rank elementary embeddings. A natural question, then, concerns the optimality of our results, accompanied with considerations regarding (non – trivial) consistency lower bounds for such notions.

Related to the latter, and as already discussed at the end of Chapter 2, an important and apparently delicate open problem is whether one can *separate* the various large cardinal notions from their corresponding $C^{(n)}$ - versions, with the case of supercompactness being of central interest. Hence, as a first step, we may ask the following.

Question 1. *Assume that κ is $C^{(1)}$ - supercompact. Is there a forcing notion (even class forcing) which kills the $C^{(1)}$ - supercompactness while preserving the supercompactness of κ?*

Additionally, we may also wonder about indestructibility results, while keeping in mind the limitations observed in Chapter 4.

Question 2. *Can the $C^{(n)}$ - supercompactness be made indestructible under various classes of forcing notions?*

In the same context, we can indeed reconsider some of our definitions if it becomes necessary; for instance, in the case of $C^{(n)}$ - Woodin cardinals.

Question 3. *What happens if we drop the requirement "$j(\delta) = \delta$" in the definition of a $C^{(n)}$ - Woodin cardinal?*

We have pointed out in Chapter 3 that there are certain obstacles one is facing when trying to study the behaviour of the Σ_n - correct ordinals with respect to forcing. A closely related query has to do with the possibilities left unanswered by Lemma 3.1.

Question 4. *For $n \geqslant 2$, under which circumstances, if any, can a $C^{(n)}$ - cardinal be created in some forcing extension?*

In the other direction, and concretely for $n = 2$, we may also ask:

Question 5. *Is there a general criterion or, even, a restricted family of forcing notions which preserve the "$C^{(2)}$ - ness" of any particular ordinal?*

In Chapter 3, we also gave some results regarding the preservation of $C^{(n)}$ - tall and of $C^{(n)}$ - supercompact cardinals by posets which are (sufficiently) distributive; however, the cases of superstrongness and of extendibility were left untackled.

Question 6. *Are $C^{(n)}$ - extendible (resp. $C^{(n)}$ - superstrong) cardinals preserved by sufficiently distributive forcing?*

More specifically, and in connection with Theorem 4.8 from Chapter 4:

Question 7. *Are $C^{(n)}$ - extendible cardinals preserved by the canonical forcing for the global* GCH?

Lastly, as far as $C^{(n)}$ - cardinals are concerned, one could further investigate even stronger notions (e.g., hugeness, superhugeness, rank – into – rank embeddings), building up on the results obtained in [5].

Moving on to the resurrection axioms which we considered in Chapter 5, there are various issues which have not been confronted yet. Initially, there is a pending question related to Theorem 5.5.

Question 8. *Does* MM (*or even* MM^{++}) *imply* RA(stat. pres.)? *Can we separate the two axioms from an assumption weaker than extendibility?*

Furthermore, for the case of the unbounded resurrection axioms, given that they constitute a newly introduced category of principles (at least in this form), there is a wide variety of related open problems. Our treatment of the subject gave a general picture which is – in our opinion – coherent, but far from complete.

To begin with, one may study the axioms $\mathsf{UR}(\Gamma)$ when Γ ranges over all sorts of different classes of posets; this would give rise to a large number of relevant questions. It should be emphasized that one has to be careful though, since the UR axiom for some of these classes may already lead to a contradiction. For example, if Γ is the class of $\leqslant \omega_1$ - closed posets, then an easy adaptation of the argument used in the proof of Lemma 5.15 shows that UR(Γ) is inconsistent.

Still in the general setting, one could also consider amplifications of the unbounded resurrection axioms in the following sense.

Question 9. *What happens if we allow additional predicates in the structures* $\langle H_\beta , \in , \dots \rangle$ *and redefine the* UR *axioms appropriately*?

Clearly, such a question applies to the case of RA axioms as well.

Regarding indestructibility matters, it is known that forcing axioms like PFA and MM are (partially or fully) preserved by appropriate forcing notions (see, e.g., [31] and [32]). An obvious question is whether similar results hold for the resurrection axioms.

Perhaps of even greater importance are the issues which touch on consistency lower bounds; this is definitely the case for axioms like PFA, whose consistency strength is conjectured to be exactly that of a supercompact cardinal. Given the fact that the unbounded resurrection axioms (for the appropriate classes of posets) are apparently stronger postulates, we may thus ask if UR(stat. pres.), or even UR(proper), implies (some degree of) supercompactness. Let us also recall Question 5.12 from Section 5.2 which addresses the relation between the axioms UR(stat. pres.) and UR(semi proper).

On the other hand, for the classes of c.c.c. and of σ - closed posets, our approach yields questions which are connected with further failures of combinatorial principles.

Question 10. *Does the failure of principles weaker than* $\square^*$*, such as the approachability property, follow from* UR(σ - closed) *or from* UR(c.c.c.)?

As we remarked at the very end of Chapter 5, R. Schindler's techniques apparently give that RA(stat. pres.) implies, consistency – wise, strong cardinals. Can we (first check and then) improve this bound?

Finally, enquiries regarding possible connections between the UR axioms and W.H. Woodin's $(*)$ axiom (cf. Definition 5.1 in [46]) cannot be resisted.

Question 11. *What is the relationship between* UR(stat. pres.) *and the* $(*)$ *axiom*? *In particular, does the former imply the latter*?

Although we do not intend to insinuate any unjustified optimism, a positive answer to the latter question would indeed be a remarkable result.

APPENDIX A

Notes on Extenders

As a postlude, we give a brief (?) presentation of the basic theory of extenders. Our exposition is based on acclaimed text references, such as [30] (§ 26) and the Martin – Steel classic [36], and should be ideally read in parallel with these excellent sources. Regarding the present treatment, and at least as far as the introductory material is concerned, we adhere to the standard results with the tendency to fill in details wherever we feel that this might help the non – expert reader become more accustomed to the underlying concepts and techniques. It should be emphasized that none of these results and techniques are due to the author.

Historically, extenders were introduced by Anthony Dodd and Ronald Jensen, who built on some earlier work done by William Mitchell. The basic motivation for considering such objects was the need to "combinatorially approximate" a given elementary embedding $j : V \longrightarrow M$ between inner models, much like the way in which usual ultrapowers capture measurability embeddings. Indeed, the notion of an extender – which proved to be rather central in the study of inner model theory – generalizes that of a normal measure and is devised for embeddings which have strength (typically) at the level of strong, superstrong and Woodin cardinals.

As it turns out, elementary embeddings for the aforementioned large cardinals can be approximated via suitable *sequences of measures* that are extracted from the given $j : V \longrightarrow M$. Any such sequence, which is called an *extender* and is usually denoted by E, enables us to construct a model M_E and an elementary embedding $j_E : V \longrightarrow M_E$ in a way which is nicely definable from E; moreover, if E is chosen carefully, then j_E and M_E closely resemble the initial j and M (in particular, they witness the same large cardinal strength for $\kappa = cp(j) = cp(j_E)$).

The structure of the appendix is as follows. In Section A.1, we begin our study of extenders by describing in more detail the setting of the previous paragraph; that is, we look at extenders which are derived from a given elementary embedding. Subsequently, in Section A.2, we abstract the essential features of the previous procedure and we give the general definition of an extender along with some of its properties. Finally, in Section A.3, we conclude with a not – so – typical discussion of Martin – Steel extenders and their connection with supercompactness.

A.1 Extenders derived from embeddings

Suppose that $j : V \longrightarrow M$ is an elementary embedding with $cp(j) = \kappa$ and M being a transitive class. Let us pick some $\lambda > \kappa$ and let ζ be the least ordinal for which $\lambda \leqslant j(\zeta)$ (where $\kappa \leqslant \zeta$). Now, for each $a \in [\lambda]^{<\omega}$, we define an ultrafilter E_a on $[\zeta]^{|a|}$ by: $X \in E_a \iff a \in j(X)$.

Observe that if $X \subseteq [\zeta]^{|a|}$ then $j(X) \subseteq [j(\zeta)]^{|a|}$, so (by choice of ζ) this definition makes sense. It is easy to check that each E_a is a κ - complete ultrafilter and that E_a is principal when $a \in [\kappa]^{<\omega}$.

Definition A.1. *In the setting described above,* $E = \langle\, E_a : a \in [\lambda]^{<\omega} \,\rangle$ *is called* **the** (κ, λ) - **extender derived from** j*.*

Using the extender ultrafilters, we may construct the familiar corresponding ultrapowers. Note that κ - completeness implies that these ultrapowers will be well – founded and thus, for each $a \in [\lambda]^{<\omega}$, we let $M_a \cong \mathrm{Ult}_{E_a}(V)$ be the transitive collapse. Each such construction comes along with a pair of elementary embeddings which make the following diagram to commute:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_a"'] & M \\ M_a \arrow[ur, "k_a"'] & \end{tikzcd}$$

$$j_a(x) = [c^a_x]_{E_a}, \text{ for each } x \in V$$

$$k_a([f]_{E_a}) = j(f)(a), \text{ for each } f \in V \cap {}^{[\zeta]^{|a|}}V$$

where $c^a_x : [\zeta]^{|a|} \longrightarrow \{x\}$ is the constant function. The power of the extender concept comes from the way in which the M_a 's are interrelated and to which we now turn our attention.

For every $a\,, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, we consider a *projection* function π_{ba} which can be described as follows: let $b = \{\xi_1, \ldots, \xi_n\}$, where we always assume that $\xi_1 < \ldots < \xi_n$, and let $a = \{\xi_{i_1}, \ldots, \xi_{i_m}\}$, with $1 \leqslant i_1 < \ldots < i_m \leqslant n$. Now define $\pi_{ba} : [\zeta]^{|b|} \longrightarrow [\zeta]^{|a|}$ by letting, for all $\{\alpha_1, \ldots, \alpha_n\} \in [\zeta]^{|b|}$,

$$\pi_{ba}(\{\alpha_1, \ldots, \alpha_n\}) = \{\alpha_{i_1}, \ldots, \alpha_{i_m}\}.$$

That is, we project down to a subset according to the relation between the finite sets a and b.

Using these projections, we are about to establish the way in which the ultrapowers interact. Not surprisingly, something about the corresponding ultrafilters has to be said first.

Coherence property

For all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, $X \in E_a \iff \{s \in [\zeta]^{|b|} : \pi_{ba}(s) \in X\} \in E_b$.

To see that this holds, it is enough to notice that $j(\pi_{ba})(b) = a$, which follows from the definition of the projection functions. We are now in position to give the elementary embeddings which relate the M_a's. So, for every $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, define $i_{ab} : M_a \longrightarrow M_b$ by letting, for every $f : [\zeta]^{|a|} \longrightarrow V$ with $f \in V$,

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}.$$

It is easy to check that the i_{ab}'s are well–defined, elementary and that make the following diagram to commute:

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[d, "j_a"'] \arrow[ddr, "j_b"] & & M \\
M_a \arrow[urr, "k_a"] \arrow[dr, "i_{ab}"'] & & \\
& M_b \arrow[uur, "k_b"'] &
\end{tikzcd}$$

$$k_a \circ j_a = k_b \circ j_b = j$$

$$k_b \circ i_{ab} = k_a$$

$$i_{ab} \circ j_a = j_b$$

At this point we form $\langle\langle\, M_a : a \in [\lambda]^{<\omega} \rangle ; \langle\, i_{ab} : a \subseteq b \in [\lambda]^{<\omega} \rangle\rangle$, which is easily seen to be a *directed system*. Consequently, we can construct the corresponding *direct limit*, $\widetilde{M_E} = \langle\, D_E \,,\, \in_E \rangle$. This is a standard procedure and may be described in the following manner:

- We define the equivalence relation $\sim_E$ on $\displaystyle\bigcup_{a \in [\lambda]^{<\omega}} \{a\} \times M_a$ by:

  $$\langle\, a, [f]_{E_a} \rangle \sim_E \langle\, b, [g]_{E_b} \rangle \iff \exists\, c \supseteq a \cup b \text{ s.t. } i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b}).$$

  The (Scott) equivalence class of $\langle\, a, [f]_{E_a} \rangle$ will be denoted by $[\langle\, a, [f]_{E_a} \rangle]_E$. We now consider the quotient of $\displaystyle\bigcup_{a \in [\lambda]^{<\omega}} \{a\} \times M_a$ by $\sim_E$, which we call D_E and is the domain of the constructed direct limit.

- In a parallel manner, we define membership $\in_E$ in D_E by:

$$[\langle\, a, [f]_{E_a} \rangle]_E \in_E [\langle\, b, [g]_{E_b} \rangle]_E \iff \exists\, c \supseteq a \cup b \text{ s.t. } i_{ac}([f]_{E_a}) \in i_{bc}([g]_{E_b}).$$

It is obvious from the construction that every element $x \in \widetilde{M_E}$ is of the form $x = [\langle\, a, [f]_{E_a} \rangle]_E$, for some $a \in [\lambda]^{<\omega}$ and some $[f]_{E_a} \in M_a$, where $f : [\zeta]^{|a|} \longrightarrow V$ is a function in V. Let us also remark that, by already established facts, one obtains the following equivalents for equality and membership in the direct limit structure. We have that $[\langle\, a, [f]_{E_a} \rangle]_E =_E [\langle\, b, [g]_{E_b} \rangle]_E$ if and only if there exists some $c \supseteq a \cup b$ such that $i_{ac}([f]_{E_a}) = i_{bc}([g]_{E_b})$ or, equivalently,

$$[f \circ \pi_{ca}]_{E_c} = [g \circ \pi_{cb}]_{E_c} \iff j(f \circ \pi_{ca})(c) = j(g \circ \pi_{cb})(c) \iff j(f)(a) = j(g)(b)$$

and similarly, $[\langle\, a, [f]_{E_a} \rangle]_E \in_E [\langle\, b, [g]_{E_b} \rangle]_E \iff j(f)(a) \in j(g)(b)$.

In what follows, we avoid unnecessary formalistic complication by supressing continued brackets and subscripts of the form "E_a" and "E". Thus, when we write, e.g., $[\, a, [f]\,] \in [\, b, [g]\,]$ in $\widetilde{M_E}$, what we really mean is that $[f] = [f]_{E_a} \in M_a$, $[g] = [g]_{E_b} \in M_b$ and $[\langle\, a, [f]_{E_a} \rangle]_E \in_E [\langle\, b, [g]_{E_b} \rangle]_E$. Our next goal is to define elementary embeddings interconnecting all the models we have considered so far and then, establish some basic properties of the direct limit structure.

Before this, though, we now argue that the constructed direct limit is well–founded, a fact which enables us to work with its transitive collapse and, at the same time, justifies some of the formalistic simplifications mentioned above.

For this, suppose that in $\widetilde{M_E}$ there exist elements $x_n = [\, a_n, [f_n]\,]$ which form an $\in_E$-descending chain, i.e., $x_{n+1} \in_E x_n$, for all $n \in \omega$. By the equivalent of membership which we just stated, $j(f_{n+1})(a_{n+1}) \in j(f_n)(a_n)$ for all $n \in \omega$, which contradicts the well–foundedness of the model M. Therefore, we may work with M_E, the transitive collapse of the direct limit structure. We now define the desired elementary embeddings k_{aE}, j_E and k_E, as shown in the following commutative diagrams:

$$\begin{tikzcd}
V \arrow[rr, "j"] \arrow[d, "j_a"'] \arrow[ddr, "j_E"] & & M \\
M_a \arrow[urr, "k_a"] \arrow[dr, "k_{aE}"'] & & \\
& M_E \arrow[uur, "k_E"'] &
\end{tikzcd}$$

$j_E(x) = [\, a, [c^a_x]\,]$, for some (any) $a \in [\lambda]^{<\omega}$ and for each $x \in V$

$k_{aE}([f]) = [\, a, [f]\,]$, for each $a \in [\lambda]^{<\omega}$ and $f \in V \cap {}^{[\zeta]^{|a|}}V$

$k_E([\, a, [f]\,]) = j(f)(a)$, for each $a \in [\lambda]^{<\omega}$ and $f \in V \cap {}^{[\zeta]^{|a|}}V$

$$\begin{tikzcd} M_a \arrow[rr, "k_{aE}"] \arrow[dr, "i_{ab}"'] & & M_E \\ & M_b \arrow[ur, "k_{bE}"'] & \end{tikzcd} \qquad k_{bE} \circ i_{ab} = k_{aE}\,, \text{ for all } a \subseteq b \in [\lambda]^{<\omega}$$

Let us remark that in the definition of $j_E(x)$ it does not matter which $a \in [\lambda]^{<\omega}$ we choose, since, if $a \neq \hat{a}$ and we let $b = a \cup \hat{a}$, then $[c_x^a \circ \pi_{ba}] = [c_x^{\hat{a}} \circ \pi_{b\hat{a}}] = [c_x^b]$ in M_b and thus, $[\,a, [c_x^a]\,] = [\,\hat{a}, [c_x^{\hat{a}}]\,]$ (equivalently, the latter equality holds because $j(c_x^a)(a) = j(c_x^{\hat{a}})(\hat{a}) = j(x)$).

The previous remarks show, in addition, that the embeddings k_E and k_{aE} are well – defined (recall here the definition of the ultrafilter E_a). The fact that all the aforementioned embeddings commute, comes from straightforward computations which we omit.

Let us now check, inductively, that each k_{aE} is elementary; we leave it to the reader to verify that from this fact, the elementarity of j_E and of k_E easily follows.

Lemma A.2. *For every $a \in [\lambda]^{<\omega}$, k_{aE} is an elementary embedding.*

Proof. We proceed inductively on the complexity of formulas (dealing with all $a \in [\lambda]^{<\omega}$ simultaneously). Suppose that $[f]$ and $[g]$ belong to M_a, for some $a \in [\lambda]^{<\omega}$, where both functions f and g are (in V and) on $[\zeta]^{|a|}$. By the preceding discussion,

$$M_a \models [f] = [g] \iff j(f)(a) = j(g)(a) \iff M_E \models [\,a, [f]\,] = [\,a, [g]\,]$$

and analogously for membership, i.e., elementarity holds for the atomic formulas. Moreover, the cases of negation and of conjuction are immediate. So, suppose that elementarity holds for $\varphi(x, y)$ and let $[g] \in M_a$, where $a \in [\lambda]^{<\omega}$ and $g \in V \cap {}^{[\zeta]^{|a|}}V$. On the one hand,

$$\begin{aligned} M_a \models \exists\, x\, \varphi(x, [g]) \;&\Longrightarrow\; M_a \models \varphi([f], [g]), \text{ some } [f] \in M_a \\ &\overset{\text{I.H.}}{\Longrightarrow}\; M_E \models \varphi(k_{aE}([f]), k_{aE}([g])) \\ &\Longrightarrow\; M_E \models \exists\, x\, \varphi(x, k_{aE}([g])). \end{aligned}$$

Conversely, suppose that $M_E \models \exists\, x\, \varphi(x, k_{aE}([g]))$, i.e., there exists some element $[\,b, [f]\,] \in M_E$ such that $M_E \models \varphi([\,b, [f]\,], k_{aE}([g]))$. Let $c = a \cup b$. Recalling that $k_{cE} \circ i_{ac} = k_{aE}$ and $k_{cE} \circ i_{bc} = k_{bE}$, our inductive hypothesis and the fact that i_{ac}

is elementary give the following:

$$\begin{aligned} M_E \models \varphi([\,b,[f]\,],k_{aE}([g])) &\Longrightarrow M_E \models \varphi(k_{bE}([f]),k_{aE}([g])) \\ &\Longrightarrow M_E \models \varphi\,(k_{cE}\circ i_{bc}([f]),k_{cE}\circ i_{ac}([g])) \\ &\overset{\text{I.H.}}{\Longrightarrow} M_c \models \varphi\,(i_{bc}([f]),i_{ac}([g])) \\ &\Longrightarrow M_c \models \exists\, x\,\varphi(x,i_{ac}([g])) \\ &\Longrightarrow M_a \models \exists\, x\,\varphi(x,[g]). \end{aligned}$$

□

We are now in position to establish the basic properties of j_E and M_E, which will conclude this section.

Proposition A.3. [Properties of j_E, M_E]

(i) *$cp(k_E) \geqslant \lambda$. Thus, $cp(j_E) = \kappa$ and $j_E(\zeta) \geqslant \lambda$. Moreover, if $\lambda = j(\zeta)$ then $cp(k_E) > \lambda$ and so $j_E(\zeta) = \lambda$.*

(In particular, if $\lambda = j(\kappa)$ then $cp(k_E) > \lambda$ and $j_E(\kappa) = j(\kappa) = \lambda$).

(ii) $M_E = \{j_E(f)(a) : a \in [\lambda]^{<\omega},\ f \in V,\ f : [\zeta]^{|a|} \longrightarrow V\}$.

(iii) *If, for any ordinal γ, $(|V_\gamma| \leqslant \lambda)^M$, then $V_\gamma^M = V_\gamma^{M_E} \subseteq range(k_E)$ and $k_E \restriction V_\gamma^{M_E} = id$.*

Proof. As we have already seen, for every $x \in M_E$ there is some $a \in [\lambda]^{<\omega}$ and some $f : [\zeta]^{|a|} \longrightarrow V$ with $x = k_{aE}([f])$. Thus,

$$k_E(x) = k_E(k_{aE}([f])) = k_a([f]) = j(f)(a)$$

and then,

$$(\div) \quad range(k_E) = \{j(f)(a) : a \in [\lambda]^{<\omega},\ f \in V,\ f : [\zeta]^{|a|} \longrightarrow V\}.$$

(i) Let $\alpha < \lambda$. Put $a = \{\alpha\} \in [\lambda]^1$ and let us consider the identity function $f = id^1 : [\zeta]^1 \longrightarrow [\zeta]^1$. Obviously, $a = j(f)(a)$ and it now follows from $(\div)$ that $a = \{\alpha\} \in range(k_E)$. By similar computations, $\lambda \subseteq range(k_E)$ and $[\lambda]^{<\omega} \subseteq range(k_E)$. Hence, $cp(k_E) \geqslant \lambda$.

To see that $cp(j_E) = \kappa$, notice that if $cp(j_E) = \alpha < \kappa$, we would then have that $\alpha < j_E(\alpha) \leqslant k_E(j_E(\alpha)) = j(\alpha) = \alpha$. Furthermore, if $cp(j_E) > \kappa$, then, since $k_E(\kappa) = \kappa$, then $j(\kappa) = k_E(j_E(\kappa)) = k_E(\kappa) = \kappa$, contradicting $cp(j) = \kappa$. Hence, we conclude that $cp(j_E) = \kappa$. Finally, from $k_E \restriction \lambda = id$ and $j(\zeta) = k_E(j_E(\zeta)) \geqslant \lambda$, we get that $j_E(\zeta) \geqslant \lambda$.

For the "moreover" part, just observe that if $\lambda = j(\zeta) = k_E(j_E(\zeta))$ then $\lambda + 1 \subseteq range(k_E)$ and so $cp(k_E) > \lambda$ and $j_E(\zeta) = \lambda$.

(ii) As we mentioned above, $[\lambda]^{<\omega} \subseteq range(k_E)$. In fact, $k_E \restriction \lambda = id$ implies that for every $a \in [\lambda]^{<\omega}$, $k_E(a) = a$.

Using $(\div)$ and these observations, for any $x = [\, a, [f]\,] \in M_E$,

$$k_E(x) = j(f)(a) = k_E \circ j_E(f)(a) = k_E(j_E(f))(k_E(a)) = k_E(j_E(f)(a))$$

and, consequently, since k_E is injective, $x = j_E(f)(a)$ from which the desired representation of the model M_E follows.

(iii) Let us fix a function $g : [\zeta]^1 \longrightarrow V$ with the property that, for any ordinal α, if $|V_\alpha| \leqslant \zeta$ then $g \restriction [\,|V_\alpha|\,]^1 : [\,|V_\alpha|\,]^1 \longrightarrow V_\alpha$ is a bijection.

Now suppose that for some ordinal γ, $(|V_\gamma| \leqslant \lambda)^M$. Since $\lambda \leqslant j(\zeta)$, by choice of g and elementarity, $j(g) \restriction [\,|V_\gamma|^M\,]^1$ is a bijection between $[\,|V_\gamma|^M\,]^1$ and V_γ^M. Thus, for every $x \in V_\gamma^M$, there is $\xi < \lambda$ with $j(g)(\{\xi\}) = x$. But then, according to $(\div)$, $x \in range(k_E)$, i.e., we have just shown that $V_\gamma^M \subseteq range(k_E)$.

To conclude the proof, we have that $k_E^{-1} : range(k_E) \longrightarrow M_E$ is the collapsing isomorphism and hence $V_\gamma^M = V_\gamma^{M_E}$ and $k_E \restriction V_\gamma^{M_E} = id$.

□

Before turning to the general definition of an extender (i.e., regardless of a given ambient embedding j), let us point out one final thing. As we have already seen, for any $a \in [\lambda]^{<\omega}$, $k_E(a) = a$. From this, it follows that for any $X \subseteq [\zeta]^{|a|}$,

$$a \in j(X) \iff a \in k_E(j_E(X)) \iff k_E(a) \in k_E(j_E(X)) \iff a \in j_E(X),$$

i.e., if we try to define the *new* (κ, λ)-extender E' derived from j_E, we obtain that $E' = E$.

A.2 General theory of extenders

After having discussed extenders derived from a given elementary embedding, we now move on to the general definition of a (κ, λ)-extender which, as we shall see, bares the same essential features.

Following the formal definition, we will define a direct limit structure M_E along with an elementary embedding $j_E : V \longrightarrow M_E$, in a spirit resembling the one of the previous section. Finally, we shall give several properties of j_E and M_E and, also, establish the connection between the two kinds of extenders studied so far.

Let us now state the official definition.

Definition A.4. *Let κ be a cardinal, $\lambda > \kappa$ and $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$. We say that E is a (κ, λ)* - **extender** *if, for some $\zeta \geqslant \kappa$, the following conditions are satisfied:*

1. *(i) For all $a \in [\lambda]^{<\omega}$, E_a is a κ-complete ultrafilter on $[\zeta]^{|a|}$.*

   (ii) There is some $a \in [\lambda]^{<\omega}$ such that E_a is not κ^+-complete.

   (iii) For all $\gamma < \zeta$, there is $a \in [\lambda]^{<\omega}$ with $\{s \in [\zeta]^{|a|} : \gamma \in s\} \in E_a$.

2. (Coherence)

   For all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$,

   $$X \in E_a \Longleftrightarrow \{s \in [\zeta]^{|b|} : \pi_{ba}(s) \in X\} \in E_b$$

   (where $\pi_{ba} : [\zeta]^{|b|} \longrightarrow [\zeta]^{|a|}$ is the projection function defined exactly as in the previous section).

3. (Normality)

   If for some $a \in [\lambda]^{<\omega}$ and some $f : [\zeta]^{|a|} \longrightarrow V$

   $$\{s \in [\zeta]^{|a|} : f(s) \in max(s)\} \in E_a,$$

   then there is some $b \in [\lambda]^{<\omega}$ so that $b \supseteq a$ and

   $$\{s \in [\zeta]^{|b|} : f \circ \pi_{ba}(s) \in s\} \in E_b.$$

4. (Well – foundedness)

   For any countable families of $a_n \in [\lambda]^{<\omega}$ and $X_n \subseteq [\zeta]^{|a_n|}$ respectively, with $X_n \in E_{a_n}$ for all $n \in \omega$, there exists an order – preserving function $d : \bigcup_{n \in \omega} a_n \longrightarrow \zeta$ such that $d''a_n \in X_n$, for all $n \in \omega$.

We remark that κ is frequently referred to as the *critical point*, while λ is referred to as the *support* of the extender. Given such a (κ, λ)-extender E, we follow a similar route to the one we took in Section A.1. We summarize the procedure below, with appropriate references to our earlier discussion.

- Initially we construct, for all $a \in [\lambda]^{<\omega}$, the corresponding ultrapowers $\mathrm{Ult}_{E_a}(V)$. Note that both condition 1 (i) and condition 4 of Definition A.4 imply that each E_a is *countably complete*, i.e., given a collection of the form $\{X_n : n \in \omega\}$ consisting of subsets of $[\zeta]^{|a|}$,

$$\text{if, for all } n \in \omega,\ X_n \in E_a, \text{ then } \bigcap_{n\in\omega} X_n \neq \varnothing.$$

To see that this follows from condition 4, we just let $a_n = a$, for all $n \in \omega$. Countable completeness is sufficient in order to conclude that all the ultrapowers are well–founded and so, we can work with the transitive collapses $M_a \cong \mathrm{Ult}_{E_a}(V)$ as usual. Moreover, we have the elementary embeddings $j_a : V \longrightarrow M_a$, given by $j_a(x) = [c^a_x]_{E_a}$, for all $x \in V$.

- Next, for all $a, b \in [\lambda]^{<\omega}$ with $a \subseteq b$, we define – exactly as in Section A.1 – the embeddings $i_{ab} : M_a \longrightarrow M_b$ by:

$$i_{ab}([f]_{E_a}) = [f \circ \pi_{ba}]_{E_b}\,, \text{ for all } f \in V \cap {}^{[\zeta]^{|a|}}V.$$

Using the coherence condition of Definition A.4, one easily shows that these are well–defined elementary embeddings and that they commute with the j_a's, i.e., $i_{ab} \circ j_a = j_b$.

- Clearly, we can again form the directed system

$$\langle\langle\, M_a :\ a \in [\lambda]^{<\omega}\,\rangle\,;\,\langle\, i_{ab} :\ a \subseteq b \in [\lambda]^{<\omega}\,\rangle\rangle$$

from which we construct the direct limit $\widetilde{M_E} = \langle\, D_E\,, \in_E\,\rangle$. As before, this construction comes together with the embeddings $j_E : V \longrightarrow \widetilde{M_E}$ and $k_{aE} : M_a \longrightarrow \widetilde{M_E}$, defined by

$$j_E(x) = [\,a, [c^a_x]\,],\ \text{for some (any) } a \in [\lambda]^{<\omega} \text{ and } x \in V$$

and

$$k_{aE}([f]) = [\,a, [f]\,],\ \text{for each } f \in V \cap {}^{[\zeta]^{|a|}}V.$$

Using inductive arguments as in the previous section, one shows that k_{aE} is elementary and then, that the same is true for j_E. The only difference here is that there is no reference to an ambient elementary embedding j but despite of that, the remaining part of the arguments works just fine. This concludes the description of our construction.

At this point, one should expect that a commutative diagram encapsulating all the revelant embeddings might be formed. Yet, we refrain ourselves from stating this explicitly since there is one important thing which needs to be checked.

As the attentive reader might have already noticed, we have not said anything regarding the well–foundedness of the direct limit $\widetilde{M_E}$. Recall that, in Section A.1, the well–foundedness of $\widetilde{M_E}$ was established – essentially – from the elementarity of the map $k_E : M_E \longrightarrow M$ (and, at any rate, with reference to the given elementary embedding j). Since this is no longer the case, we have to modify our arguments. Evidently, condition 4 of Definition A.4 has to be exploited.

Proposition A.5. *Condition 4 is equivalent to the well–foundedness of $\widetilde{M_E}$.*

Proof. ($\Longrightarrow$) Suppose that $\widetilde{M_E}$ is ill–founded, i.e., there are $x_n = [\,a_n, [f_n]\,] \in \widetilde{M_E}$ so that, for all $n \in \omega$, $x_{n+1} \in_E x_n$. First, we claim that the a_n's can be chosen in such a way that $m \leqslant n$ implies $a_m \subseteq a_n$.

To see this, observe that by construction of the direct limit $\widetilde{M_E}$, for every $a \subseteq b$ and each $f \in V \cap {}^{[\zeta]^{|a|}}V$, we have that $[\,a, [f]\,] = [\,b, [f \circ \pi_{ba}]\,]$. Therefore, given a descending sequence $\langle\, x_n = [\,a_n, [f_n]\,] : n \in \omega \rangle$, we may define another sequence $\langle\, y_n = [\,b_n, [g_n]\,] : n \in \omega \rangle$ where, for each $n \in \omega$,

$$b_n = \bigcup_{k \leqslant n} a_k \text{ and } g_n = f_n \circ \pi_{b_n\, a_n}.$$

It is then easy to check that the y_n's form an $\in_E$ - descending chain as well. This proves our claim and we may thus assume that the sequence of the x_n's is as described above. We now let $X_0 = [\zeta]^{|a_0|}$ and define recursively, for $n \in \omega$:

$$X_{n+1} = \{ s \in [\zeta]^{|a_{n+1}|} : f_{n+1}(s) \in f_n \circ \pi_{a_{n+1}\, a_n}(s) \}.$$

Since $x_{n+1} \in_E x_n$ for every $n \in \omega$, it follows that $X_{n+1} \in E_{a_{n+1}}$. Obviously, $X_0 \in E_{a_0}$ and thus, $X_n \in E_{a_n}$ for all $n \in \omega$. Having said all that, we are about to contradict condition 4.

For, suppose that there is an order–preserving function $d : \bigcup_{n \in \omega} a_n \longrightarrow \zeta$ such that $d''a_n \in X_n$, for all $n \in \omega$. This means that, for all $n \in \omega$,

$$f_{n+1}(d''a_{n+1}) \in f_n \circ \pi_{a_{n+1}\, a_n}(d''a_{n+1}) = f_n(d''a_n),$$

i.e., the sequence $\langle\, f_n(d''a_n) : n \in \omega \rangle$ is an infinite $\in$ - descending chain in V, which is absurd.

($\Longleftarrow$) Conversely, suppose that condition 4 fails, i.e., fix some countable families of $a_n \in [\lambda]^{<\omega}$ and of $X_n \subseteq [\zeta]^{|a_n|}$, so that, for all $n \in \omega$, $X_n \in E_{a_n}$ but there is no order–preserving $d : \bigcup_{n \in \omega} a_n \longrightarrow \zeta$ with the property that, for all $n \in \omega$, $d''a_n \in X_n$. Our aim of course is to show that $\widetilde{M_E}$ is ill–founded. To begin with, we claim that the a_n's and the X_n's can be chosen so that the following hold:

(1) $m \leqslant n \Longrightarrow a_m \subseteq a_n$

(2) $s \in X_n \wedge m \leqslant n \Longrightarrow \pi_{a_n\, a_m}(s) \in X_m$.

To show both of them, a similar idea to the one we used in the first part of the proof is employed. For (1), we define, for each $n \in \omega$,

$$b_n = \bigcup_{k \leqslant n} a_k \text{ and } Y_n = \{ s \in [\zeta]^{|b_n|} : \pi_{b_n\, a_n}(s) \in X_n \}.$$

By coherence, we clearly have that, for every $n \in \omega$, $Y_n \in E_{b_n}$. Now, if there is an order – preserving $d : \bigcup_{n\in\omega} b_n \longrightarrow \zeta$ with $d''b_n \in Y_n$, for all $n \in \omega$, then this means that $\pi_{b_n\, a_n}(d''b_n) \in X_n$, i.e., $d''a_n \in X_n$, for all $n \in \omega$ and this contradicts our assumption.

For (2), since we can assume (1) by now, fix some $n \in \omega$ and define, for each $m \leqslant n$, the set

$$A_m = \{ s \in [\zeta]^{|a_n|} : \pi_{a_n\, a_m}(s) \in X_m \},$$

which belongs to E_{a_n} by coherence. By the finite intersection property of the measures, for each $n \in \omega$, the set

$$Z_n = \bigcap_{m \leqslant n} A_m = \{ s \in [\zeta]^{|a_n|} : (\forall\, m \leqslant n)\, \pi_{a_n\, a_m}(s) \in X_m \}$$

belongs to E_{a_n}. Now, it is evident that the Z_n's satisfy (2). Moreover, the exact same computation used for (1) shows that there cannot be an order – preserving function $d : \bigcup_{n\in\omega} a_n \longrightarrow \zeta$ such that $d''a_n \in Z_n$, for all $n \in \omega$. This concludes our two – part claim and we now proceed with the rest of the proof.

We define the following set:

$$T = \{ \langle\, s_i : i < n \,\rangle : n > 0 \wedge (\exists\, s \in X_{n-1})\, (\forall\, i < n)\, (\pi_{a_{n-1}\, a_i}(s) = s_i) \}$$

where a typical element $s^* \in T$ is a sequence $s^* = \langle\, s_0, s_1, \ldots, s_{n-2}, s_{n-1} \,\rangle$ or, equivalently,

$$s^* = \langle\, \pi_{a_{n-1}\, a_0}(s), \pi_{a_{n-1}\, a_1}(s), \ldots, \pi_{a_{n-1}\, a_{n-2}}(s), s \,\rangle,$$

where $s_{n-1} = s \in X_{n-1}$. Notice that by the second part of our claim, the latter implies that $s_i = \pi_{a_{n-1}\, a_i}(s) \in X_i$, for all $i < n$. We moreover define an order relation on T by:

$$s^* \prec t^* \Longleftrightarrow s^* \text{ properly extends } t^*.$$

Let us now show that, with respect to this ordering, T is a well – founded poset. Towards a contradiction, suppose there is an infinite $\prec$ - descending chain in T; that is, there exists a chain of the form

$$\ldots \prec \langle\, s_0, s_1, \ldots, s_{n-1} \,\rangle \prec \ldots \prec \langle\, s_0, s_1 \,\rangle \prec \langle\, s_0 \,\rangle,$$

where we can always assume that the $\prec$ - largest element of the chain is an one – element sequence. Then, it is readily seen that one can define appropriately an order – preserving $d : \bigcup_{n\in\omega} a_n \longrightarrow \zeta$ in such a way that $d''a_n = s_n$, for all $n \in \omega$. But this contradicts our initial assumption, since it implies that $d''a_n \in X_n$, for all $n \in \omega$.

The well – foundedness of the poset T allows us to consider a rank function on its elements, i.e., a function $\mathrm{rk}_T : T \longrightarrow \mathbf{ON}$ such that, for every $s^* \in T$,

$$\mathrm{rk}_T(s^*) = \sup\{\mathrm{rk}_T(t^*) + 1 : t^* \in T \wedge t^* \prec s^*\}.$$

Using the rank function, we are about to establish the ill – foundedness of $\widetilde{M_E}$. For this, define, for each $n \in \omega$, a function $F_n : [\zeta]^{|a_n|} \longrightarrow \mathbf{ON}$ by:

$$F_n(s) = \begin{cases} \mathrm{rk}_T(\underbrace{\langle\, \pi_{a_n\, a_0}(s), \pi_{a_n\, a_1}(s), \ldots, \pi_{a_n\, a_{n-1}}(s), s \,\rangle}_{s^*}) & , \quad s \in X_n \\ \\ \varnothing & , \quad \text{otherwise.} \end{cases}$$

We should point out that these functions are well – defined: if $s \in X_n \subseteq [\zeta]^{|a_n|}$, then $s^* = \langle\, \pi_{a_n\, a_0}(s), \pi_{a_n\, a_1}(s), \ldots, \pi_{a_n\, a_{n-1}}(s), s \,\rangle$ is uniquely determined by projecting down to all the previous indices and, moreover, $s^* \in T$ by definition of the latter.

But now it follows that, for every $s \in X_n$, $s^* \prec \big(\pi_{a_n\, a_{n-1}}(s)\big)^*$ and so, by the order – preserving property of the rank function, we get that, for every $n \geqslant 1$,

$$X_n \subseteq \{s \in [\zeta]^{|a_n|} : F_n(s) \in F_{n-1}(\pi_{a_n\, a_{n-1}}(s))\}.$$

Therefore, since $X_n \in E_{a_n}$, by definition of membership in the direct limit, we have that $[\, a_n, [F_n]\,] \in_E [\, a_{n-1}, [F_{n-1}]\,]$, for all $n \geqslant 1$, i.e., $\widetilde{M_E}$ is ill – founded. The proof is complete. $\square$

Having established the well – foundedness of $\widetilde{M_E}$, we also remark that the definition of the membership relation $\in_E$ (recall Scott's trick as well) implies its *set – likeness* (cf. Chapter III, Definition 5.2 in [33]) and hence we can work with the transitive collapse M_E of the direct limit structure. Thus, we get the anticipated commutative diagram of embeddings, as shown below:

$$\begin{tikzcd}
V \arrow[rr, "j_E"] \arrow[d, "j_a"'] \arrow[ddr, "j_b"] & & M_E \\
M_a \arrow[urr, "k_{aE}"] \arrow[dr, "i_{ab}"'] & & \\
& M_b \arrow[uur, "k_{bE}"'] &
\end{tikzcd}$$

$$k_{aE} \circ j_a = k_{bE} \circ j_b = j_E$$

$$k_{bE} \circ i_{ab} = k_{aE}$$

$$i_{ab} \circ j_a = j_b$$

Our goal now is to give the connection between the two kinds of extenders studied so far in this exposition and to present several properties of j_E and M_E. As an indication of what to expect, we first verify that any extender derived from an embedding satisfies the general definition.

Lemma A.6. *Suppose that $j : V \longrightarrow M$ is an elementary embedding and that $cp(j) = \kappa$. Let $\lambda > \kappa$ and consider E, the (κ, λ) - extender derived from j. Then, E is a (κ, λ) - extender.*

Proof. We fix $\zeta \geqslant \kappa$, the least ordinal for which $\lambda \leqslant j(\zeta)$. Towards verifying all the clauses of Definition A.4, let us first point out that conditions $1\,(i)$ and 2 have already been checked in Section A.1. Moreover, condition 4 follows from Proposition A.5 since, as we saw in the previous section, $\widetilde{M_E}$ is well – founded. For the rest of the argument, we have the following.

$1\,(ii)$. We want some $a \in [\lambda]^{<\omega}$ so that E_a is not κ^+-complete. Let $a = \{\kappa\}$, where $\kappa = cp(j)$. Now, to see that κ^+-completeness fails for this a, define, for each $\alpha < \kappa$, $X_\alpha = \{\{\xi\} : \xi > \alpha\} \subseteq [\zeta]^1$. The fact that $j(\kappa) > \kappa$ immediately implies that, for all $\alpha < \kappa$, $X_\alpha \in E_{\{\kappa\}}$. On the other hand, we clearly have that $\{\kappa\} \notin j(\bigcap\limits_{\alpha<\kappa} X_\alpha)$.

$1\,(iii)$. Given any $\gamma < \zeta$, $j(\gamma) < \lambda$ by choice of ζ. Hence, since for any $a \in [\lambda]^{<\omega}$, $\{s \in [\zeta]^1 : \gamma \in s\} \in E_a$ if and only if $j(\gamma) \in a$, we may pick $a = \{j(\gamma)\}$, which is a legitimate element of $[\lambda]^{<\omega}$ and is readily seen to satisfy the desired condition.

3. To check normality, suppose that for some $a \in [\lambda]^{<\omega}$ and some f on $[\zeta]^{|a|}$, we have that $\{s \in [\zeta]^{|a|} : f(s) \in max(s)\} \in E_a$. Equivalently, this means that $j(f)(a) \in max(a)$.

   Now, since $j(f)(a) \in max(a) \in \lambda$, we also have $j(f)(a) \in \lambda$ by transitivity. Therefore, we may let $b = a \cup \{j(f)(a)\}$ and it is then easy to check that $\{s \in [\zeta]^{|b|} : f \circ \pi_{ba}(s) \in s\} \in E_b$.

□

We may now state the basic properties of j_E and M_E. As a matter of notation, for $n \in \omega$, let $id^n : [\zeta]^n \longrightarrow [\zeta]^n$ be the identity function. With this convention in mind, we have the following basic proposition, which should be compared with Proposition A.3. For the proof, the interested reader may consult [30] (in particular, Lemma 26.2 and Exercise 26.4).

Proposition A.7. [Properties of j_E, M_E]

(i) *For every* $a \in [\lambda]^{<\omega}$, $k_{aE}([id^{|a|}]) = a$.

(ii) $cp(j_E) = \kappa$ *and* ζ *is the least ordinal such that* $\lambda \leqslant j_E(\zeta)$.

(iii) $M_E = \{ j_E(f)(a) : a \in [\lambda]^{<\omega}, f : [\zeta]^{|a|} \longrightarrow V, f \in V \}$.

(iv) *For any* ξ, $j_E(\xi) < (|^{\zeta}\xi| \cdot |\lambda|)^+$.

(v) *If* $\mu > \lambda$ *is a strong limit cardinal with* $cf(\mu) > \zeta$, *then* $j_E(\mu) = \mu$.

(vi) *For any set* X *with* $|X| > \zeta$, $j_E{}''X \notin M_E$.

(vii) $E \notin M_E$.

We regret omitting this (rather long) proof, since it sheds more light on the importance and relevance of the various conditions which are included in Definition A.4. On the other hand, we do give one corollary to the above proposition which, together with Lemma A.6, completely describe the connection between the two kinds of extenders considered so far.

Corollary A.8. *Suppose that* $E = \langle E_a : a \in [\lambda]^{<\omega} \rangle$ *is a* (κ, λ) *- extender and let* $j_E : V \longrightarrow M_E$ *be the associated extender embedding. If* E' *is the* (κ, λ) *- extender derived from* j_E, *then* $E' = E$.

Proof. Suppose that $E' = \langle E'_a : a \in [\lambda]^{<\omega} \rangle$ is the (κ, λ) - extender derived from j_E. First, we remark that if each E_a is on $[\zeta]^{|a|}$, then by Proposition A.7 (ii) the same is true for each E'_a. Thus, for every $a \in [\lambda]^{<\omega}$ and every $X \subseteq [\zeta]^{|a|}$,

$$
\begin{aligned}
X \in E'_a &\iff a \in j_E(X) && \text{(definition of } E'_a\text{)} \\
&\iff k_{aE}([id^{|a|}]) \in k_{aE} \circ j_a(X) && \text{(Proposition A.7 } (i)\text{)} \\
&\iff [id^{|a|}] \in j_a(X) && \text{(elementarity)} \\
&\iff X \in E_a && \text{(definition of } M_a\text{)}
\end{aligned}
$$

and we therefore conclude that $E' = E$. □

Consequently, every extender E is derived from an embedding (namely j_E) and, conversely, every embedding gives rise to an extender. This means that the two kinds of extenders which we have considered are essentially the two opposite sides of the same coin, i.e., it all reduces to a matter of perspective.

We refrain ourselves from giving the well – known connections of extenders with several large cardinal notions such as strongs, superstrongs and Woodins; we only mention that the extender machinery renders these notions formalizable in first – order ZFC (see [30] for more details). Let us now proceed to the final section, where we discuss Martin – Steel extenders and their application to supercompactness.

A.3 Martin – Steel extenders and supercompactness

In their classic paper [36], D. Martin and J. Steel defined a generalized notion of an extender which served their purposes towards establishing (levels of) *Projective Determinacy* (PD) from appropriate large cardinal assumptions. Comparing it with the usual extender notion, the essential difference in their definition is the fact that one is allowed to use any arbitrary transitive set Y as the support of the extender, instead of using an ordinal λ which had been the case so far.

This new feature comes together with various issues which one needs to take into account when trying to adapt (one's arguments) to the newly defined objects; nevertheless, it will soon become clear that the underlying intuition and the ideas behind the new definition are parallel to those related to ordinary extenders.

As we have (hopefully) gained by now some insight into the extender machinery, let us take the opposite route to the one we took at the beginning of this exposition; that is, we shall first define the new extender notion by giving its general properties and, afterwards, we shall consider Martin – Steel extenders derived from elementary embeddings. Unless otherwise stated, all results in this section come from [36]. We start with the following, which is very much in the spirit of Definition A.4.

Definition A.9. (Martin & Steel)
Let κ be a cardinal and let Y be some transitive set. We say that the sequence $E = \langle\, E_a : a \in [Y]^{<\omega} \,\rangle$ is a (κ, Y)- **extender** *if, for some $\zeta \geqslant \kappa$, the following conditions are satisfied:*

1. *Each E_a is a κ-complete ultrafilter on ${}^{a}(V_\zeta)$, and at least one E_a is not κ^+-complete.*
2. *For every $a \in [Y]^{<\omega}$, $\{s \in {}^{a}(V_\zeta) : \langle a\,,\, \in\rangle \overset{s}{\cong} \langle range(s)\,,\, \in\rangle\} \in E_a$.*
3. (Coherence) *For all $a, b \in [Y]^{<\omega}$ with $a \subseteq b$,*
$$X \in E_a \Longleftrightarrow \{s \in {}^{b}(V_\zeta) : s \restriction a \in X\} \in E_b.$$
4. (Normality) *If for some $a \in [Y]^{<\omega}$ and some $f : {}^{a}(V_\zeta) \longrightarrow V_\zeta$*
$$\{s \in {}^{a}(V_\zeta) : f(s) \in \bigcup range(s)\} \in E_a,$$
then there is some $y \in Y$ such that
$$\{s \in {}^{a\cup\{y\}}(V_\zeta) : f(s \restriction a) = s(y)\} \in E_{a\cup\{y\}}.$$
5. *The direct limit $\widetilde{M_E}$ constructed from E is well-founded.*

Several remarks are in order, regarding this definition. First of all, our new extender sequences consist of ultrafilters which are on sets of (finite) functions of the form $s : a \longrightarrow V_\zeta$, instead of just (finite) subsets of ζ. This has some nice advantages as, for example, the fact that we do not have to deal with projection functions anymore; we just restrict any finite function on the relevant subset. This is evident in the coherence property.

On the other hand, the absence of a canonical well – ordering of the support set Y, dictates several (not only notational) changes in order to express the desired properties in the definition. This is the case in condition 2, which should be included if we would like to avoid sets of "degenerate" or of non – order – preserving (finite) functions.

In addition, let us point out that regarding the well – foundedness condition, we are referring to a direct limit structure $\widetilde{M_E} = \langle D_E , \in_E \rangle$ constructed in a totally analogous way to the one in Section A.2. The obvious changes which need to be made are along the lines of the (notational) modifications in the coherence property. Thus, for instance, after defining the (Scott) equivalence classes of the sort $[a, [f]]$ (where $a \in [Y]^{<\omega}$ and $f : {}^a(V_\zeta) \longrightarrow V$), we stipulate that

$$[a, [f]] \in_E [b, [g]] \iff \exists\, c \supseteq a \cup b \text{ s.t. } \{s \in {}^c(V_\zeta) : f(s \restriction a) \in g(s \restriction b)\} \in E_c.$$

By condition 5 of Definition A.9, after constructing $\widetilde{M_E}$ we may immediately consider its transitive collapse M_E and then, as one should expect, we have the extender elementary embedding $j_E : V \longrightarrow M_E$ where, for every $x \in V$, we let $j_E(x) = [a, [c^a_x]]$ for some (any) $a \in [Y]^{<\omega}$.

Let us also remark that there is an equivalent combinatorial characterization of the well – foundedness condition, resembling condition 4 in Definition A.4 (for details, we refer the interested reader directly to the source, i.e., to [36]).

This finishes (the sketch of) our description of the new situation, with the hope that one can easily fill in the missing details, most of which are straightforward adaptations of the previously discussed versions. Let us now give one basic lemma which establishes two important features of our new extender notion.

Lemma A.10. *Let* $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ *be a* (κ, Y) *- extender (for some transitive* Y*) and let* $j_E : V \longrightarrow M_E$ *be the extender embedding. Then:*

(i) $j_E \restriction V_\kappa = id$ *and* $cp(j_E) = \kappa$.

(ii) $Y \subseteq M_E$.

Proof. For the first part of (i), one employs a standard inductive argument to show that $j_E \restriction V_\alpha = id$, for every $\alpha < \kappa$. In particular, it follows that $cp(j_E) \geqslant \kappa$.

To see that $cp(j_E) = \kappa$, we fix some $a \in [Y]^{<\omega}$ so that the ultrafilter E_a fails to be κ^+-complete and we let $\{X_\xi : \xi < \kappa\}$ be a collection of subsets of ${}^a(V_\zeta)$ witnessing this failure, i.e., $X_\xi \in E_a$ for all $\xi < \kappa$ but, (un)fortunately, $\bigcap_{\xi<\kappa} X_\xi \notin E_a$. We may assume, without loss of generality, that $\bigcap_{\xi<\kappa} X_\xi = \varnothing$. We then define a function $f : {}^a(V_\zeta) \longrightarrow \kappa$ by letting, for every $s \in {}^a(V_\zeta)$, $f(s)$ to be equal to the least ordinal $\xi < \kappa$ for which $s \notin X_\xi$. It is now easy to check that, for every $\xi < \kappa$, $\xi < [\,a, [f]\,]$ whereas, on the other hand, $[\,a, [f]\,] < j_E(\kappa)$.

For the proof of the important property stated in (ii), we employ condition 2 and normality of Definition A.9. Initially, we define for every $y \in Y$ and every $a \in [Y]^{<\omega}$ with $y \in a$, the function $f_{a,y} : {}^a(V_\zeta) \longrightarrow V_\zeta$ by letting $f_{a,y}(s) = s(y)$, for every $s \in {}^a(V_\zeta)$ (these are essentially projection functions). We now show that, in fact, $y = [\,a, [f_{a,y}]\,] \in M_E$ from which the conclusion follows.

First of all, observe that if $y \in a \cap b$ for some $a, b \in [Y]^{<\omega}$, we clearly have that $[\,a, [f_{a,y}]\,] = [\,b, [f_{b,y}]\,]$ in M_E. Now, to show that $y = [\,a, [f_{a,y}]\,]$, we proceed inductively on the rank of y (taking care of all $a \in [Y]^{<\omega}$ with $y \in a$ at the same time).

The base case is $y = \varnothing$ (since by condition 1, $Y \neq \varnothing$ and so $\varnothing \in Y$ by transitivity). Consider any $a \in [Y]^{<\omega}$ with $\varnothing \in a$. In order to show that $[\,a, [f_{a,\varnothing}]\,] = \varnothing$ we argue as follows. Suppose, towards a contradiction, that $[\,a, [g]\,] \in [\,a, [f_{a,\varnothing}]\,]$ (where, by the above observation we may indeed assume that g is on ${}^a(V_\zeta)$) which means that

$$\{s \in {}^a(V_\zeta) :\, g(s) \in s(\varnothing)\} \in E_a.$$

In particular, $g(s) \in \bigcup range(s)$ for E_a-almost all $s \in {}^a(V_\zeta)$. Now, by applying normality, we may find some $z \in Y$ such that

$$\{s \in {}^{a\cup\{z\}}(V_\zeta) :\, g(s \restriction a) = s(z)\} \in E_{a\cup\{z\}},$$

where note that $[\,a, [g]\,] = [\,a \cup \{z\}, [f_{a\cup\{z\},z}]\,]$. Moreover, by applying coherence, we get

$$\{s \in {}^{a\cup\{z\}}(V_\zeta) :\, g(s \restriction a) \in s(\varnothing)\} \in E_{a\cup\{z\}}$$

and thus, $\{s \in {}^{a\cup\{z\}}(V_\zeta) :\, s(z) \in s(\varnothing)\} \in E_{a\cup\{z\}}$. But, by condition 2, it follows that

$$\{s \in {}^{a\cup\{z\}}(V_\zeta) :\, z \in \varnothing\} \in E_{a\cup\{z\}},$$

which is a contradiction. This shows the base case.

Next, assume that the desired property holds inductively, for all $y' \in Y$ with $\mathrm{rk}(y') < \mathrm{rk}(y) \neq 0$. To show that $y \subseteq [\,a, [f_{a,y}]\,]$, let us fix some $z \in y$ and some $a \in [Y]^{<\omega}$ with $\{z, y\} \subseteq a$. By condition 2, we consequently obtain that $\{s \in {}^{a}(V_\zeta) : s(z) \in s(y)\} \in E_a$ which, by the definition of the $f_{a,y}$'s, gives

$$\{s \in {}^{a}(V_\zeta) : f_{a,z}(s) \in f_{a,y}(s)\} \in E_a$$

and thus, $[\,a, [f_{a,z}]\,] \in [\,a, [f_{a,y}]\,]$. But now, the inductive hypothesis implies that $z \in [\,a, [f_{a,y}]\,]$ which shows the desired inclusion. Moreover, by our observation, the same holds for any $b \in [Y]^{<\omega}$ with $y \in b$.

For the converse inclusion, if for some element $[\,a, [g]\,] \in M_E$ we have that $[\,a, [g]\,] \in [\,a, [f_{a,y}]\,]$ (where $y \in a$), we argue exactly as in the base case to show that there is some $z \in Y$ such that

$$[\,a, [g]\,] = [\,a \cup \{z\}, [f_{a \cup \{z\}, z}]\,]$$

and

$$\{s \in {}^{a \cup \{z\}}(V_\zeta) : s(z) \in s(y)\} \in E_{a \cup \{z\}}.$$

But now the latter, by condition 2, gives $z \in y$ and then, the inductive hypothesis again implies that $z = [\,a \cup \{z\}, [f_{a \cup \{z\}, z}]\,]$ (since by our observation, the particular finite set is not important as long as it contains the element in question; in this case z). Therefore, $z = [\,a, [g]\,] \in y$ which means that $[\,a, [f_{a,y}]\,] \subseteq y$ and the proof is complete. $\square$

The importance of the property "$Y \subseteq M_E$" should be apparent: we are free to choose any transitive set as the support of our extender and then, this set will be included in the ultrapower structure M_E. The analogy with $[\lambda]^{<\omega}$ in the case of (κ, λ)-extenders is obvious, just by recalling Proposition A.7 (i). We now turn to a brief discussion of (κ, Y)-extenders derived from ambient elementary embeddings; comparisons with Section A.1 are inevitable.

Let $j : V \longrightarrow M$ be an elementary embedding into a transitive model M with $cp(j) = \kappa$. Let us pick some transitive $Y \subseteq M$ with $\kappa \in Y$ and let $\zeta \geqslant \kappa$ be the least ordinal for which $Y \subseteq V^M_{j(\zeta)} = j(V_\zeta)$. For each $a \in [Y]^{<\omega}$, we define an ultrafilter E_a on ${}^{a}(V_\zeta)$ by letting:

$$X \in E_a \Longleftrightarrow j^{-1} \restriction j(a) \in j(X).$$

Clearly, $j^{-1} \restriction j(a) : j(a) \longrightarrow a$ is an isomorphism (recall that a is finite) and so, if $X \subseteq {}^{a}(V_\zeta)$ then $j(X) \subseteq {}^{j(a)}(V^M_{j(\zeta)})$; in other words, this definition not only makes sense but also, it is arguably the obvious modification of Definition A.1 which we have to consider. It is again easy to check that, for every $a \in [Y]^{<\omega}$, E_a is in fact a κ-complete ultrafilter and that the coherence property is satisfied.

As one should expect, $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ is called *the* (κ, Y)-*extender derived from* j. We now check that extenders of this kind satisfy the general definition.

Lemma A.11. *If $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ is the (κ, Y) - extender derived from $j : V \longrightarrow M$, then E is a (κ, Y) - extender.*

Proof. We argue as in Lemma A.6, taking into account the necessary modifications which need to be made in our new context. First of all, we use the fact that $\kappa = cp(j)$ in order to show that $E_{\{\kappa\}}$ is not κ^+- complete.

For each $\alpha < \kappa$, we let $X_\alpha = \{s \in {}^{\{\kappa\}}(V_\zeta) : \alpha < s(\kappa) < \kappa\}$ and observe that $X_\alpha \in E_{\{\kappa\}}$ because $(j^{-1} \restriction j(\{\kappa\}))\,(j(\kappa)) = (j^{-1} \restriction \{j(\kappa)\})\,(j(\kappa)) = \kappa$. On the other hand though, it is clear that $\bigcap_{\alpha<\kappa} X_\alpha = \varnothing$.

Also, $j^{-1} \restriction j(a)$ being an isomorphism implies that condition 2 of Definition A.9 holds as well.

For normality now, suppose that for some $a \in [Y]^{<\omega}$ and for some function $f : {}^a(V_\zeta) \longrightarrow V_\zeta$, we have that

$$\{s \in {}^a(V_\zeta) : f(s) \in \bigcup range(s)\} \in E_a.$$

By definition of E_a, this means that $j(f)(j^{-1} \restriction j(a)) \in \bigcup a$ and so, by transitivity of Y, $y = j(f)(j^{-1} \restriction j(a)) \in Y$. It is now easy to check that for this particular $y \in Y$, the desired conclusion follows.

Finally, we let $j_E : V \longrightarrow \widetilde{M_E}$ and $k_E : \widetilde{M_E} \longrightarrow M$ be the usual embeddings, as depicted in the following diagram:

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "j_E"'] & M \\ \widetilde{M_E} \arrow[ur, "k_E"'] & \end{tikzcd}$$

$j_E(x) = [\,a, [c^a_x]\,]$, for some (any) $a \in [Y]^{<\omega}$ and for each $x \in V$

$k_E([\,a, [f]\,]) = j(f)(j^{-1} \restriction j(a))$, for each $a \in [Y]^{<\omega}$ and $f \in V \cap {}^{a(V_\zeta)}V$

Along the lines of Section A.1, one checks that these are well – defined elementary embeddings commuting with j and so, in particular, $\widetilde{M_E}$ is well – founded. $\square$

Having checked that such a derived E is a (κ, Y) - extender, we may conveniently work with the transitive collapse M_E of the direct limit structure. Recall that, by Lemma A.10, we have the inclusion $Y \subseteq M_E$. Knowing this, we may try to define the *new* (κ, Y) - extender E' derived from j_E. As it should be anticipated, it follows that $E' = E$. This, together with some other related properties, are summarized (and proved) below.

Proposition A.12. *Let $E = \langle E_a : a \in [Y]^{<\omega} \rangle$ be the (κ, Y) - extender derived from $j : V \longrightarrow M$ and consider $j_E : V \longrightarrow M_E$ and $k_E : M_E \longrightarrow M$, the elementary embeddings associated with E. Then:*

(*i*) $k_E \restriction Y = id$.

(*ii*) $M_E = \{j_E(f)(j_E^{-1} \restriction j_E(a)) : a \in [Y]^{<\omega},\ f : {}^a(V_\zeta) \longrightarrow V,\ f \in V\}$.

(*iii*) *If E' is the (κ, Y)-extender derived from j_E, then $E' = E$.*

Proof. For (*i*), we just recall that by the proof of Lemma A.10 (*ii*), for every element $y \in Y$ and for any $a \in [Y]^{<\omega}$ with $y \in a$,

$$k_E(y) = k_E([\,a, [f_{a,y}]\,]) = j(f_{a,y})(j^{-1} \restriction j(a)) = (j^{-1} \restriction j(a))(j(y)) = y.$$

For (*ii*), we initially observe that as a direct corollary to part (*i*), for every $a \in [Y]^{<\omega}$, $k_E(a) = a$ and, also, using the commutativity of the embeddings, it is readily seen that $k_E(j_E^{-1} \restriction j_E(a)) = j^{-1} \restriction j(a)$. So, let $x = [\,a, [f]\,] \in M_E$ be any element. Then,

$$k_E(x) = j(f)(j^{-1} \restriction j(a)) = k_E(j_E(f)(j_E^{-1} \restriction j_E(a))),$$

and since k_E is injective, the conclusion follows.

For (*iii*), we let $E' = \langle E'_a : a \in [Y]^{<\omega} \rangle$ be the (κ, Y)-extender derived from the embedding j_E where recall that, for every $a \in [Y]^{<\omega}$ and every $X \subseteq {}^a(V_\zeta)$, we have that $X \in E'_a \Longleftrightarrow j_E^{-1} \restriction j_E(a) \in j_E(X)$. If $a = \varnothing$, then it is obvious that $E_\varnothing = E'_\varnothing = \{\{\varnothing\}\}$.

If $a \neq \varnothing$, then for any $y \in a$, $j_E^{-1}(j_E(y)) = y = [\,a, [f_{a,y}]\,]$. Thus, if we consider the function F_a on ${}^a(V_\zeta)$ such that, for every s, $F_a(s)$ is a function on a with $F_a(s)(y) = s(y) = f_{a,y}(s)$, for all $y \in a$, after simple computations we obtain

$$k_E(j_E^{-1} \restriction j_E(a)) = j^{-1} \restriction j(a) = k_E([\,a, [F_a]\,])$$

and, therefore, $j_E^{-1} \restriction j_E(a) = [\,a, [F_a]\,]$ by the injectivity of k_E. But now, by definition of F_a,

$$[\,a, [F_a]\,] = [\,a, [\langle\, s : s \in {}^a(V_\zeta)\,\rangle]\,] = [\,a, [id^a]\,]$$

where $id^a : {}^a(V_\zeta) \longrightarrow {}^a(V_\zeta)$ is the identity function. Hence, it follows that

$$X \in E'_a \Longleftrightarrow j_E^{-1} \restriction j_E(a) = [\,a, [id^a]\,] \in [\,a, [c^a_X]\,] = j_E(X)$$

and so, $X \in E'_a \Longleftrightarrow \{s \in {}^a(V_\zeta) : s \in X\} \in E_a \Longleftrightarrow X \in E_a$. □

At this point we are ready for our basic application which will be to encode a λ-supercompact embedding via an appropriately derived Martin – Steel extender. Having benefitted from the material of [36], we now diverge from this source; the following results may also be found in § 5 of [5]. Let us first describe the ideas and motivation behind the several details with which we shall then proceed.

Given a λ-supercompact embedding, say $j : V \longrightarrow M$ with $cp(j) = \kappa$, the main issue is to pick the right transitive set $Y \subseteq M$ as the support of the derived extender E. Of course, our goal is to pick this set in a way that the extender embedding $j_E : V \longrightarrow M_E$ is also λ-supercompact, i.e., so that ${}^{\lambda}M_E \subseteq M_E$. The dominant idea for showing the latter is to include $j''\lambda$ in Y (and thus in M_E), and use it as a prototype sequence in order to encode every other λ-sequence. Recall here the similar methodology which we used in Section 2.5 of Chapter 2.

Specifically, fix some collection $\{x_i : i < \lambda\} \subseteq M_E$ where, by the representation of the structure M_E given by Proposition A.12, each element x_i is of the form $x_i = j_E(f_i)(j_E^{-1} \restriction j_E(b_i))$, for some $b_i \in [Y]^{<\omega}$ and some function f_i on ${}^{b_i}(V_\zeta)$. Our aim is to find an $A \in [Y]^{<\omega}$ and a function F on ${}^{A}(V_\zeta)$, so that the element $X = j_E(F)(j_E^{-1} \restriction j_E(A))$ encodes the λ-sequence of x_i's in M_E, i.e., for all $i < \lambda$, $X(i) = x_i$.

As we shall show below, if apart from including $j''\lambda$, we also choose the support set in such a way that it is closed under finite subsets, closed under λ-sequences and closed under j, then $j \restriction Y = j_E \restriction Y$ and we may encode the entire λ-sequence of the $j_E^{-1} \restriction j_E(b_i)$'s as a single element of Y. In fact, these conditions on Y, together with the requirement of it being transitive, are sufficient in order to define the A and the F that work. Let us now see how to do it, provided we are given such a Y. After that, we shall briefly comment on how to actually obtain the suitable $Y \subseteq M$; this will conclude the construction and accomplish our goal.

Proposition A.13. *Suppose that κ is λ-supercompact, witnessed by the embedding $j : V \longrightarrow M$. Suppose that $Y \subseteq M$ is transitive, $[Y]^{<\omega} \subseteq Y$, ${}^{\lambda}Y \subseteq Y$, $j''Y \subseteq Y$ and $j''\lambda \in Y$. Let E be the (κ, Y)-extender derived from j and let $j_E : V \longrightarrow M_E$ be the extender embedding. Then, j_E is λ-supercompact for κ.*

Proof. As we have already seen, $Y \subseteq M_E$ and $k_E \restriction Y = id$, where k_E is the embedding commuting with j and j_E. We first notice that $j \restriction Y = j_E \restriction Y$; this follows easily from commutativity, $k_E \restriction Y = id$ and $j''Y \subseteq Y$. Consequently, for every $a \in [Y]^{<\omega}$,

$$j_E^{-1} \restriction j_E(a) = j^{-1} \restriction j(a) \in Y$$

and so, in particular, in the representation of M_E given in Proposition A.12 (*ii*), we may replace j_E by j as shown below:

$$M_E = \{j_E(f)(j^{-1} \restriction j(a)) : a \in [Y]^{<\omega},\ f : {}^{a}(V_\zeta) \longrightarrow V,\ f \in V\}.$$

Recall here that ζ is the least ordinal for which $Y \subseteq j(V_\zeta)$. Moreover, if we let $\mu = \sup(\mathbf{ON} \cap Y)$, then $\lambda < \mu$ and $\mu \subseteq Y$ (because $\lambda < j(\kappa)$, $j''\lambda \in Y$ and Y

is transitive) and thus, since j_E and j agree on Y, $j_E{}''\lambda = j{}''\lambda$. In particular, $\lambda < j_E(\kappa) = j(\kappa)$, where, of course, $cp(j_E) = \kappa$. Hence, in order to establish the λ-supercompactness of the embedding j_E, it remains to check that ${}^{\lambda}M_E \subseteq M_E$. For this, we use the several closure properties of the given Y.

Fix throughout some $\{x_i : i < \lambda\} \subseteq M_E$ where, for each $i < \lambda$, we have that $x_i = j_E(f_i)(j^{-1} \restriction j(b_i))$ for some $b_i \in [Y]^{<\omega}$ and some function f_i on ${}^{b_i}(V_\zeta)$. As we have remarked, we want to find some $A \in [Y]^{<\omega}$ and some function F on ${}^{A}(V_\zeta)$, such that the element $X = j_E(F)(j^{-1} \restriction j(A)) \in M_E$ is the desired λ-sequence, i.e., for all $i < \lambda$, $X(i) = x_i$.

Let $f = \langle f_i : i < \lambda \rangle$ and $b = \langle j^{-1} \restriction j(b_i) : i < \lambda \rangle$. Observe that, by the closure of Y, both b and the function $j \restriction \lambda : \lambda \longrightarrow j{}''\lambda$ belong to Y. We now consider $A = \{j \restriction \lambda\,, b\} \in [Y]^{<\omega}$. By some trivial computations, we get that $j_E(A) = j(A) = \{j(j \restriction \lambda), j(b)\}$ and, then,

$$j^{-1} \restriction j(A) : \{j(j \restriction \lambda), j(b)\} \longrightarrow \{j \restriction \lambda, b\}$$

is the function whose values are $j(j \restriction \lambda) \mapsto j \restriction \lambda$ and $j(b) \mapsto b$. Also, it is clear that $j_E^{-1} \restriction j_E(A) = j^{-1} \restriction j(A)$ belongs to ${}^{j_E(A)}j_E(V_\zeta) \cap {}^{j(A)}j(V_\zeta)$ and, furthermore, any element $s \in {}^{A}(V_\zeta)$ is of the form

$$\{\langle\, j \restriction \lambda\,,\ s(j \restriction \lambda)\,\rangle\,,\ \langle\, b\,,\ s(b)\,\rangle\}.$$

Let us now turn to the definition of the desired F on ${}^{A}(V_\zeta)$. Given any $s \in {}^{A}(V_\zeta)$, we first define an auxiliary function g_s as follows:

- If both $s(j \restriction \lambda)$ and $s(b)$ are functions with domain the same ordinal, say α, then g_s is a function on α so that, for every $i < \alpha$,

$$g_s(i) = \begin{cases} f(s(j \restriction \lambda)(i))(s(b)(i)) & , \text{ if } s(j \restriction \lambda)(i) \in dom(f) \text{ and} \\ & \quad\ s(b)(i) \in dom(f(s(j \restriction \lambda)(i))) \\ \varnothing & , \text{ otherwise;} \end{cases}$$

- Otherwise, $g_s = \varnothing$.

We finally define F by letting, for every $s \in {}^{A}(V_\zeta)$, $F(s) = g_s$. Now, by elementarity, $j_E(F)$ is on ${}^{j_E(A)}j_E(V_\zeta)$ and so, $j_E(F)(j^{-1} \restriction j(A))$ makes sense, since for the particular element $s = j^{-1} \restriction j(A)$, we have that $s \in {}^{j_E(A)}j_E(V_\zeta)$. In this situation, it is clear that both $s(j(j \restriction \lambda)) = j \restriction \lambda$ and $s(j(b)) = b$ are functions with domain the same ordinal, namely, λ.

Thus, by the explicit definition of F and elementarity, $j_E(F)(s)$ is the (non–empty) auxiliary function g_s on λ, as described above. In fact, the second alternative in the definition g_s does not occur: for every $i < \lambda$,

$$s(j(j \restriction \lambda))(i) = (j \restriction \lambda)(i) = j(i) = j_E(i) \in dom(j_E(f))$$

and, then, $j_E(f)(s(j(j \restriction \lambda))(i)) = j_E(f)(j_E(i)) = j_E(f(i)) = j_E(f_i)$; hence,

$$s(j(b))(i) = b(i) = j^{-1} \restriction j(b_i) \in dom(j_E(f_i))$$

since f_i was a function on ${}^{b_i}(V_\zeta)$ and $j^{-1} \restriction j(b_i) = j_E^{-1} \restriction j_E(b_i)$. Therefore, we after all have that, for every $i < \lambda$,

$$X(i) = j_E(F)(j^{-1} \restriction j(A))(i) = j_E(f_i)(j^{-1} \restriction j(b_i)),$$

i.e., $X(i) = x_i$ as desired. This completes the proof. □

Towards the finale, let us now briefly describe a way in which, given any λ-supercompact embedding $j : V \longrightarrow M$, one may construct a $Y \subseteq M$ which meets all the requirements stated in the previous proposition.

The idea is simple and was actually used in the proof of Theorem 2.27: we start with $j''\lambda$ (which belongs to M) and we then recursively close under all properties of interest. After λ^+-many steps (taking unions at limit stages), the resulting set Y has all the desired features. See the proof of Theorem 2.27 for the formal recursive definition.

This construction together with Proposition A.13 jointly accomplish our goal of encoding λ-supercompact embeddings by Martin – Steel extenders. In fact, they provide us with the following characterization which has been repeatedly used in the present dissertation and which concludes its postlude.

Theorem A.14. *A cardinal κ is λ-supercompact if and only if there exists a (κ, Y)-extender E such that Y is transitive, $[Y]^{<\omega} \subseteq Y$, ${}^\lambda Y \subseteq Y$, $j_E''Y \subseteq Y$, $j_E''\lambda \in Y$ and $\lambda < j_E(\kappa)$.*

Proof. The forward direction follows immediately from Proposition A.13 and the construction mentioned above. For the converse, let such an extender be given and consider $j_E : V \longrightarrow M_E$. In order to see that ${}^\lambda M_E \subseteq M_E$, repeat the relevant arguments used in the proof of Proposition A.13, replacing j by j_E everywhere. □

Τέλος

Resumen

En la presente tesis doctoral trabajamos en el contexto de la teoría de conjuntos ZFC donde estudiamos, por un lado varias jerarquías de grandes cardinales y, por el otro, la categoría de axiomas llamados *Axiomas de Resurrección.* La tesis está organizada como sigue.

La base preliminar de los conocimientos necesarios se encuentra en el primer capítulo.

En el **Capítulo 2**, estudiamos las jerarquías de los cardinales $C^{(n)}$, tal y como fueron introducidos por J. Bagaria (cf. [5]). En el contexto de una inmersión elemental de un cardinal $C^{(n)}$ dado que ya hemos fijado, y bajo suposiciones adecuadas, estamos en condiciones de obtener la consistencia para tales nociones, a través de la construcción de cadenas elementales de subestructuras; en particular, nos ocupamos de los casos de cardinales *tall*, *superstrong*, *supercompact*, y *extendible.* Para los dos últimos conceptos, estudiamos la conexión entre ellos, obteniendo además una formulación equivalente para los *extendible.*

Se consideran también las versiones $C^{(n)}$ de los cardinales *Woodin* y de los *strongly compact* que no fueron estudiados en [5]. A pesar de que estas nociones no encajan en el marco metodológico descrito en el párrafo anterior, obtenemos caracterizaciones para ellos en términos de los cardinales correspondientes estándar.

En el **Capítulo 3**, analizamos brevemente la interacción de los cardinales $C^{(n)}$ con el método de *forcing*, dando algunas aplicaciones básicas de resultados bien conocidos.

En el **Capítulo 4**, dirigimos nuestra atención a los cardinales *extendible.* A través de una combinación de métodos y resultados del Capítulo 2, establecemos la existencia de *funciones de Laver* adecuadas para ellos. Aunque esto ya era conocido (cf. [11]), nos sirve de puente entre los resultados anteriores y el material que sigue en el Capítulo 5.

Contrariamente a lo que sucede en el caso de los cardinales *supercompact*, ar-

gumentamos que en el caso de los cardinales *extendible* no se pueden usar las *funciones de Laver* con el fin de obtener resultados de *indestructibilidad*. Durante el proceso, se obtiene además otra caracterización de extensibilidad y demostramos que se puede forzar la *Hipótesis del Continuo Generalizada* en el universo, mientras preservamos tales cardinales.

En el **Capítulo 5**, nos centramos en los axiomas de resurrección, tal y como fueron introducidos por J.D. Hamkins y T. Johnstone (cf. [29]). Inicialmente, consideramos la clase de ordenes parciales que *preservan subconjuntos estacionarios de* ω_1 y, suponiendo la (consistencia de la) existencia de un cardinal *extendible*, obtenemos un modelo en el que se cumple el axioma de resurrección correspondiente para esta clase.

Mediante el análisis de la demostración del resultado anterior, llegamos a versiones de resurrección más fuertes para las que introducimos axiomas adecuados, bajo el nombre general *Unbounded Resurrection*. A continuación, establecemos que su consistencia sigue la de un cardinal *extendible* y que, para las clases correspondientes de ordenes parciales, implican los axiomas de forcing PFA y MM bien conocidos.

Además, consideramos diversas implicaciones de los principios de *unbounded resurrection* (por ejemplo, su efecto en el continuo, para las clases de *c.c.c.* y de σ - *closed* ordenes parciales) así como su conexión con los axiomas de resurrección de [29]. Por último, también establecemos límites bajos de consistencia para tales axiomas, principalmente a través de la obtención de fallos (de versiones débiles) de *squares*.

Concluimos la parte matemática de la tesis actual con una lista de preguntas abiertas, seguida de un Apéndice sobre *extenders* y (algunas de) sus aplicaciones.

End Credits

[1] Abraham, U., *Proper forcing.* In Handbook of Set Theory, Editors M. Foreman & A. Kanamori, pp. 333 – 394, Springer, 2010.

[2] Apter, A.W., *Indestructibility and strong compactness.* In Proceedings of Logic Colloquium 2003, Lecture Notes in Logic 24, pp. 27 – 37, 2006.

[3] Bagaria, J., *A characterization of Martin's Axiom in terms of absoluteness.* In Journal of Symbolic Logic, Vol. 62 (2), pp. 366 – 372, 1997.

[4] Bagaria, J., *Bounded forcing axioms as principles of generic absoluteness.* In Archive for Mathematical Logic, Vol. 39 (6), pp. 393 – 401, 2000.

[5] Bagaria, J., $C^{(n)}$ – *cardinals.* In Archive for Mathematical Logic, Vol. 51 (3 – 4), pp. 213 – 240, 2012.

[6] Bagaria, J., *Generic Absoluteness.* Book manuscript.

[7] Bagaria, J., Magidor, M., *On* ω_1 *– strongly compact cardinals.* In preparation.

[8] Brooke – Taylor, A., *Large Cardinals and* L *– Like Combinatorics.* Ph.D. dissertation, 2007.

[9] Brooke – Taylor, A., *Indestructibility of Vopěnka's principle.* In Archive for Mathematical Logic, Vol. 50 (5), pp. 515–529, 2011.

[10] Brooke – Taylor, A., Friedman, S.D., *Large cardinals and gap – 1 morasses.* In Annals of Pure and Applied Logic, Vol. 159 (1 – 2), pp. 71 – 99, 2009.

[11] Corazza, P., *Laver sequences for extendible and super – almost – huge cardinals.* In Journal of Symbolic Logic, Vol. 64 (3), pp. 963 – 983, 1999.

[12] Cummings, J., *A model in which GCH holds at successors but fails at limits.* In Transactions of the AMS, Vol. 329 (1), pp. 1 – 39, 1992.

[13] Cummings, J., *Iterated forcing and elementary embeddings.* In Handbook of Set Theory, Editors M. Foreman & A. Kanamori, pp. 775 – 883, Springer, 2010.

[14] Cummings, J., Foreman, M., Magidor, M., *Squares, scales and stationary reflection.* In Journal of Mathematical Logic, Vol. 1 (1), pp. 35 – 98, 2001.

[15] Dobrinen, N., Friedman, S.D., *Homogeneous iteration and measure one covering relative to HOD.* In Archive for Mathematical Logic, Vol. 47 (7 – 8), pp. 711 – 718, 2008.

[16] Eisworth, T., *Successors of singular cardinals.* In Handbook of Set Theory, Editors M. Foreman & A. Kanamori, pp. 1229 – 1350, Springer, 2010.

[17] Foreman, M., *Potent axioms.* In Transactions of the AMS, Vol. 294 (1), pp. 1 – 28, 1986.

[18] Foreman, M., *Ideals and generic elementary embeddings.* In Handbook of Set Theory, Editors M. Foreman & A. Kanamori, pp. 885 – 1147, Springer, 2010.

[19] Foreman, M., Magidor, M., *A very weak square principle.* In Journal of Symbolic Logic, Vol. 62 (1), pp. 175 – 196, 1997.

[20] Foreman, M., Magidor, M., Shelah, S., *Martin's Maximum, saturated ideals, and non – regular ultrafilters. Part I.* In Annals of Mathematics, Vol. 127, pp. 1 – 47, 1988.

[21] Gitman, V., Hamkins, J.D., Johnstone, T., *What is the theory ZFC without power set?* Preprint (2011) available at: `http://arxiv.org/abs/1110.2430`

[22] Hamkins, J.D., *Fragile measurability.* In Journal of Symbolic Logic, Vol. 59 (1), pp. 262 – 282, 1994.

[23] Hamkins, J.D., *Canonical seeds and Prikry trees.* In Journal of Symbolic Logic, Vol. 62 (2), pp. 373 – 396, 1997.

[24] Hamkins, J.D., *The lottery preparation.* In Annals of Pure and Applied Logic, Vol. 101, pp. 103 – 146, 2000.

[25] Hamkins, J.D., *A class of strong diamond principles.* Preprint (2002) available at: `http://arxiv.org/abs/math/0211419`

[26] Hamkins, J.D., *Tall cardinals.* In Mathematical Logic Quarterly, Vol. 55 (1), pp. 68 – 86, 2009.

[27] Hamkins, J.D., *Forcing and Large Cardinals.* Book manuscript.

[28] Jech, T., *Set Theory, The Third Millennium Edition.* Springer – Verlag, 2002.

[29] Johnstone, T., *Notes to "The Resurrection Axioms".* Notes (2009) available at: `http://websupport1.citytech.cuny.edu/faculty/tjohnstone`

[30] Kanamori, A., *The Higher Infinite.* Springer – Verlag, 1994.

[31] König, B., *Forcing indestructibility of set – theoretic axioms.* In Journal of Symbolic Logic, Vol. 72 (1), pp. 349 – 360, 2007.

[32] König, B., Yoshinobu, Y., *Fragments of Martin's Maximum in generic extensions.* In Mathematical Logic Quarterly, Vol. 50 (3), pp. 297 – 302, 2004.

[33] Kunen, K., *Set Theory, An Introduction to Independence Proofs.* Studies In Logic and The Foundations of Mathematics, Vol. 102, North – Holland, 1980.

[34] Larson, P.B., *The Stationary Tower, Notes on a Course by W. Hugh Woodin.* University Lecture Series, Vol. 32, American Mathematical Society, 2004.

[35] Laver, R., *Making the supercompactness of κ indestructible under κ – directed closed forcing.* In Israel Journal of Mathematics, Vol. 29, pp. 385 – 388, 1978.

[36] Martin, D., Steel, J., *A proof of projective determinacy.* In Journal of the AMS, Vol. 2 (1), pp. 71 – 125, 1989.

[37] Moore, J., *Set mapping reflection.* In Journal of Mathematical Logic, Vol. 5 (1), pp. 87 – 97, 2005.

[38] Schimmerling, E., *A core model toolbox and guide.* In Handbook of Set Theory, Editors M. Foreman & A. Kanamori, pp. 1685 – 1751, Springer, 2010.

[39] Schindler, R., *Coding into K by reasonable forcing.* In Transactions of the AMS, Vol. 353, pp. 479 – 489, 2000.

[40] Schindler, R., *Semi – proper forcing, remarkable cardinals, and Bounded Martin's Maximum.* In Mathematical Logic Quarterly, Vol. 50 (6), pp. 527 – 532, 2004.

[41] Schindler, R., *Bounded Martin's Maximum and strong cardinals.* In *Set Theory*, Centre de recerca Matemàtica, Barcelona, 2003 – 2004, Editors J. Bagaria & S. Todorčević, pp. 401 – 406, Birkhäuser, 2006.

[42] Shelah, S., *Cardinal Arithmetic.* Clarendon Press, Oxford Logic Guides, Vol. 29, 1994.

[43] Shelah, S., *Proper and Improper Forcing.* Springer – Verlag, 1998.

[44] Stavi, J., Väänänen, J., *Reflection principles for the continuum.* In Logic and Algebra, Editor Yi Zhang, Contemporary Mathematics, Vol. 302, pp. 59 – 84, American Mathematical Society, 2002.

[45] Steel, J.R., *PFA implies* $AD^{L(\mathbb{R})}$. In Journal of Symbolic Logic, Vol. 70 (4), pp. 1255 – 1296, 2005.

[46] Woodin, W.H., *The Axiom of Determinacy, Forcing Axioms, and the Non – Stationary Ideal.* Second Edition, De Gruyter Series in Logic and its Applications, 2010.