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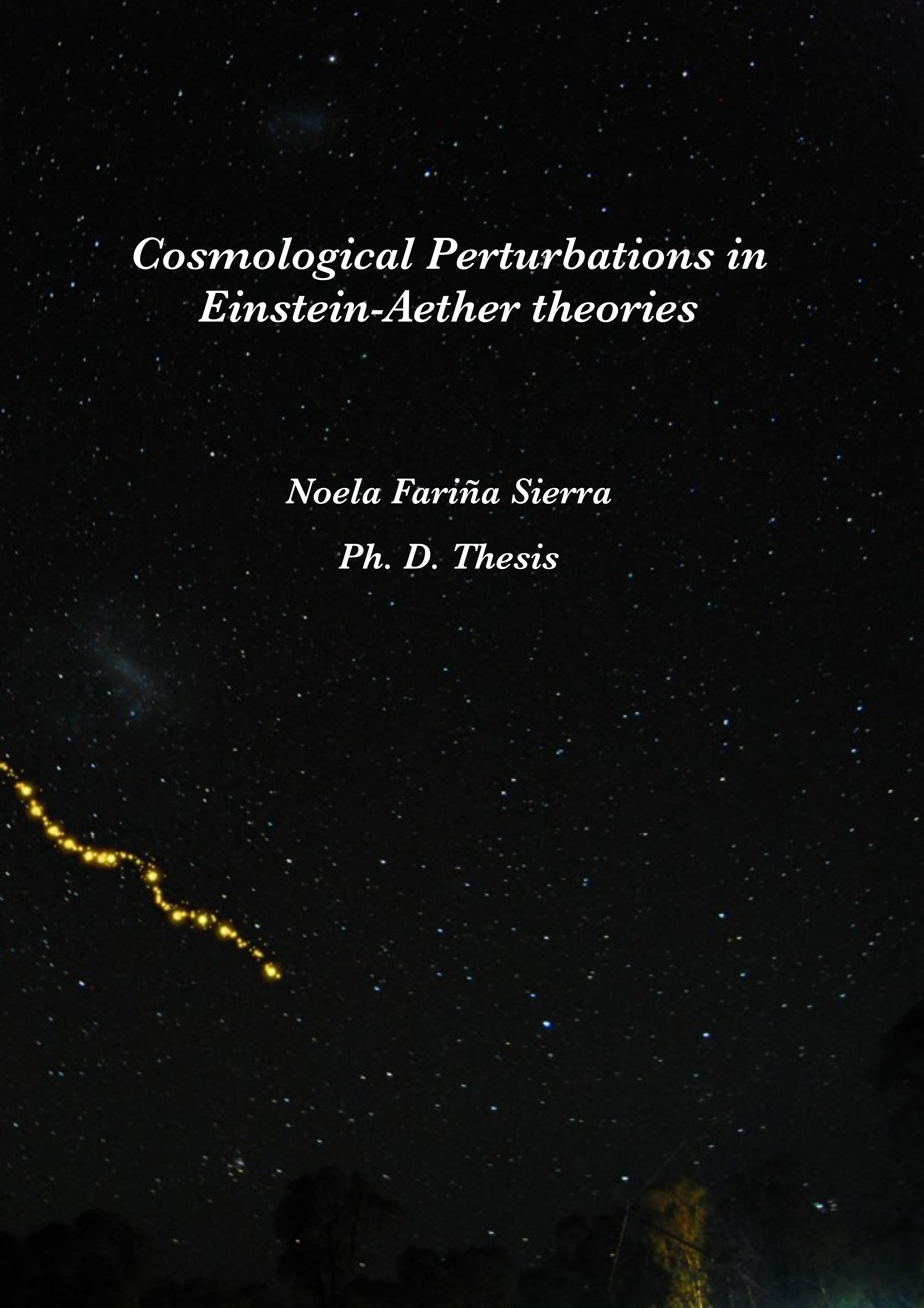
Cosmological Perturbations in Einstein-Aether Theories

Noela Fariña Sierra



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Ph. D. Thesis

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Tesi dirigida pel Dr. Jaume Garriga Torres

Departament de Física Fonamental i Institut de Ciències del Cosmos,
Universitat de Barcelona

Resum

La descripció estàndard de l'evolució cosmològica està basada en la Relativitat General i el seu èxit, descrivint l'evolució de l'univers, és excepcional. Però, en el nostre avanç en el coneixement dels detalls de l'univers, es fa necessari introduir nous elements per poder explicar totes les característiques que observem. Alguns dels problemes més coneguts, com el problema de l'horitzó o el de la planor, són resolts per la inflació. Hi ha altres problemes com el de l'expansió accelerada o el del contingut de matèria que necessiten una altra explicació. Aquests problemes s'acostumen a resoldre acudint a l'existència de dos components "foscos": l'energia fosca i la matèria fosca. Tot plegat es considera, usualment, el model cosmològic estàndard i és conegut com el model Λ-CDM.

La gravetat és una força atractiva que tendeix a apropar les coses. Però l'any 1998 dues col·laboracions, el *Supernova Cosmological Project* i el *High-z Supernova Search Team* [PAG+99, RFC+98], van descobrir que l'univers estava en expansió accelerada. Això suggeria l'existència d'un component amb una força que compensava la gravetat, una energia que ocupava la major part de l'univers i que va ser anomenada "energia fosca" degut a la seva naturalesa desconeguda. A l'inici, aquesta energia va ser associada a la constant cosmològica, com l'energia del buit quàntic, però això portava a una mida estimada molt més gran que la quantitat observada. Així, l'origen i la forma d'aquesta energia són encara problemes que s'han de resoldre en la cosmologia actual.

El fons còsmic de radiació de microones és una altra observació que, encara que proporciona un suport molt important a la cosmologia del *Big Bang*, necessita d'altres elements complementaris, que serien algun tipus d'energia i matèria fosques. La detecció inicial del fons còsmic de radiació de microones va ser feta al 1965 per Penzias i Wilson [PW65] i la seva significació cosmològica va ser explicada a [DPRW65]. Les mesures originals van proporcionar una estimació de la temperatura de radiació del cos negre d'uns 3.5 K. L'espectre de Planck del fons de microones va ser establert finalment amb gran precisió gràcies al satèl·lit COBE als anys 90 [SBK+92] i la seva temperatura fixada als $2.725 \pm 0.002\,\mathrm{K}$. Des de llavors, l'experiment WMAP [KDN+09] ha continuat proporcionant més i més detalls de l'espectre i de les anisotropies del fons còsmic de radiació de microones.

D'altra banda, es van proposar moltes alternatives per explicar aquestes observacions. Totes aquestes propostes comparteixen algunes característiques comunes. En primer lloc, totes són modificacions de la gravetat a llargues distàncies. En segon lloc, estan totes fortament restringides pel requeriment de covariància general. Per aquesta raó, la majoria de les alternatives a la Relativitat General proposades poden ser formulades com una Relativitat General acoblada a nous camps[1].

L'acceleració còsmica pot ser deguda a un camp escalar que descendeix lentament per un potencial [PR88][2] o pot ser simplement un camp escalar situat al mínim local del potencial [BP00]. Alternativament, pot ser conduïda per un terme cinètic no mínim d'un camp escalar de *k-essence* amb un Lagrangià de la forma $p(X, \phi)$, on $X = \partial_\mu\phi\partial^\mu\phi$. Aquesta forma és molt versàtil i pot ser usada per imitar fluids còsmics, amb un ample rang de possibilitats per la seva equació d'estat efectiva i la seva velocitat del so, incloent aquelles que són característiques de l'energia fosca i de la matèria fosca freda [APDM99, APMS00, APMS01, GM99].

El gradient del camp de *k-essence*, $\partial_\mu\phi$, és un vector de tipus temporal que trenca espontàniament la invariància de Lorentz, d'una forma que és paramètricament independent del seus efectes en l'evolució temporal de la geometria de fons. Concretament, la invariància de Lorentz pot ser trencada espontàniament per $\partial_\mu\phi$ mentre que l'espaitemps de fons es manté màximament simètric, una situació que es coneguda com *ghost condensation* [AHCLM04]. Tot i això, el "fluid" respon a l'atracció gravitatòria de la matèria ordinària, provocant modificacions dels potencials de llarg abast.

De manera més general, les teories amb un gravitó massiu poden ser escrites de forma covariant com Relativitat General acoblada a un conjunt de camps escalars de "Stückelberg", ϕ^A, amb termes cinètics no mínims i amb gradients que tenen valors esperats no nuls [AHGS03, Dub04, BCNP09]. Depenent de les interaccions i dels valors esperats dels condensats, això pot descriure diverses fases de la gravetat massiva. A més de la fase de Fierz-Pauli, en la qual es preserva la invariància de Lorentz [AHGS03] (veure també [CM10]), hi ha altres fases que es van estudiar a [Dub04, BCNP09] on la invariància Lorentz està trencada. Algunes d'aquestes fases tenen una fenomenologia molt interessant, com pot ser el fet que no tinguin "fantasmes" en l'espectre linealitzat, el tenir un gravitó massiu amb només dues polaritzacions transversals, o tenir potencials gravitatoris febles que es diferencien d'aquells de la Relativitat General estàndard per termes proporcionals al quadrat de

[1] Un contraexemple és l'escenari del món de branes DGP, on la gravetat es modifica a l'infraroig degut a un continu de gravitons de Kaluza-Klein [DGP00]. Per això, DGP no pot ser formulada com una versió estàndard de la Relativitat General en quatre dimensions amb camps addicionals.

[2] Això inclou el cas on modifiquem l'acció d'Einstein-Hilbert fent servir una funció arbitrària de l'escalar de Ricci [Cap02, CDTT04], ja que la teoria resultant pot ser reformulada com una teoria escalar-tensor estàndard [Wil93].

la massa del gravitó [Rub07, Dub04, BCNP09, DTT05, DFST09, BM09].

La inclusió de camps addicionals de spin 2 va ser considerada en teories de bigravetat (o multigravetat) [DK02], on l'espaitemps està dotat de diverses mètriques que interaccionen entre elles de forma no derivativa. Degut a la covariància general només un dels gravitons de l'espectre linealitzat es manté sense massa, mentre que la resta adquireix masses proporcionals als termes d'interacció no derivatius. La invariància Lorentz pot ser trencada espontàniament, fins i tot, quan totes les mètriques són planes, sempre que els seus cons de llum tinguin velocitats límit diferents. Això porta a una fenomenologia [BCNP07, BCNP08, BDG07] semblant a certes fases de la gravetat massiva que trenca la invariància Lorentz mencionada anteriorment [Rub07, Dub04, BCNP09, DTT05, DFST09, BM09].

Finalment, podem considerar l'addició de camps vectorials, que són l'objecte d'aquesta tesi. Les teories efectives pels vectors estan fortament restringides pels requeriments d'estabilitat. Habitualment, les teories d'aquest tipus que tenen una dinàmica cosmològica no trivial contenen un "fantasma" massiu [APDT09], que pot ser eliminat de l'espectre portant la seva massa cap a l'infinit. L'efecte és el mateix que si s'imposés una restricció fixa en la norma del vector, produint un valor esperat del buit que trenca la invariància Lorentz. Aquest fet va dur a Jacobson i Mattingly a anomenar aquests models com a teories d'Einstein-Aether [JM01][3]. Les seves excitacions de baixa energia són els bosons de Goldstone de la simetria Lorentz trencada[4], que participaran en la dinàmica de les interaccions de llarg abast.

Un desenvolupament recent interessant és la proposta feta per Hořava [Hoř09b, Hoř09a] que planteja que una teoria de gravetat que trenca la simetria Lorentz podria ser renormalitzable i completa a l'UV. El trencament de la invariància Lorentz es produeix en aquest cas introduint una foliació privilegiada de l'espaitemps, sense estructura addicional. Com va ser assenyalat en [BPS09], qualsevol teoria amb una foliació privilegiada pot ser escrita d'una forma covariant si tractem el paràmetre temporal que etiqueta les diferents superfícies com un camp escalar de Stückelberg, $\mathcal{T}$. La foliació és considerada física però no la parametrització i, per tant, la teoria covariant hauria de ser invariant sota redefinicions del camp $\mathcal{T} \to f(\mathcal{T})$. En altres paraules, el Lagrangià pot tenir una dependència respecte a la normal unitària a les hipersuperfícies, però no pot dependre de la magnitud del gradient $\mathcal{T}_{,\mu}$ (en contrast amb els exemples de *k-essence* i *ghost condensati-*

[3]Un camp vectorial amb norma fixa determinant un sistema de referència privilegiat es va ser fer servir, també, en les versions relativistes de MOND [Mil83], com en TeVeS [Bek04], que fa un intent per explicar les corbes de rotació de les galàxies sense utilitzar matèria fosca freda

[4]En les teories amb simetries d'espaitemps espontàniament trencades, el nombre de bosons de Goldstone no és, en general, el mateix que el nombre de generadors trencats. Però, si el paràmetre d'ordre que trenca la simetria d'espaitemps és independent de l'espaitemps (com el camp d'Aether constant), llavors, els dos nombres coincideixen [LM02].

on mencionats anteriorment). Partint d'aquesta observació, Blas, Pujolàs i Sibiryakov van mostrar [BPS10b, BPS10a] que la gravetat de Hořava podia estendre's incloent en l'acció tots els termes compatibles amb la simetria de reparametrització, i que era consistent amb la renormalització per *power counting*. Aquesta extensió és molt interessant, ja que permet solucionar alguns problemes en el sector escalar de la proposta original (com inestabilitats i acoblament fort a baixes energies [BPS09]). Jacobson [Jac10], ha clarificat la relació entre la teoria d'Einstein-Aether i aquesta versió estesa de la gravetat de Hořava, que va anomenar gravetat BPSH. Concretament, va assenyalar que qualsevol solució d'Einstein-Aether on el camp vectorial sigui ortogonal respecte a la hipersuperfície és també solució del límit de baixes energies de la gravetat de BPSH.

Ja que l'Aether només interacciona gravitatòriament qualsevol senyal d'aquest ha de ser proporcional a una potència de $(E/M_P)^2$, on M_P és la massa de Planck reduïda, i E l'escala d'energies. Per tant, encara que l'Aether conté camps sense massa, la seva presència és difícil de detectar. Això fa que la inflació proporcioni una finestra interessant per provar l'existència de l'Aether i les seves implicacions. Durant el temps d'inflació, les fluctuacions de petita escala del buit dels camps lleugers són transferides a distàncies cosmològiques, on poden deixar una empremta observable. Per això, és natural cercar empremtes d'Einstein-Aether a l'espectre de pertorbacions primordials, que és el tema al qual dediquem aquesta tesi.

Treballs previs sobre aquest tema [Lim05, LMB08], exploren una regió de l'espai de paràmetres més estreta i amb conclusions en certa mesura diferents. Al sector escalar hi trobem un mode d'isocurvatura primordial que pot ser interpretat com el potencial de velocitat de l'Aether respecte a la matèria. Segons siguin els paràmetres de l'Aether, aquest mode pot créixer a escales de superhoritzó, portant a un camp de velocitats aleatori per a l'Aether de magnitud apreciable. Resultats similars són aplicables al sector transversal del vector. Aquestes pertorbacions podrien ser d'interès fenomenològic. També observem que el mode d'isocurvatura està fortament correlacionat amb el mode adiabàtic usual, que es correspon amb les pertorbacions de la curvatura en el tall comòbil.

Treball previ sobre l'impacte de les pertorbacions escalars adiabàtiques en el fons de radiació de microones i les estructures a escales llargues en les teories de l'Aether (generalitzades) es pot trobar a [ZFZ08, ZZB$^+$10].

Durant l'elaboració d'aquesta tesi va sortir un article de gran interès de Kobayashi, Urakawa i Yamaguchi [KUY10] que analitzava l'evolució després d'inflació del mode escalar adiabàtic en la teoria BPSH. Parlarem més d'aquest article a la Secció 4.1.5. En aquells aspectes en els que els nostres treballs es superposen, les nostres conclusions coincideixen. Hi ha un altre article recent que estudia la polarització del mode B en el cas de l'Aether [NK11], en la seva majoria numèricament, però també fan un intent d'aproximació analítica. Parlarem més d'aquests resultats a la Secció 5.7.

L'objectiu d'aquesta tesi és estudiar la teoria d'Einstein-Aether des del punt de vista de les pertorbacions cosmològiques. Estem interessats en les restriccions pels paràmetres de la teoria que es poden obtenir d'aquesta anàlisi i en les característiques particulars que pot generar el mode vectorial, absent en Relativitat General. L'organització de la tesi és la següent.

Després de contextualitzar el tema que volem estudiar i les raons que ens porten a fer aquesta investigació, podem entrar en matèria. Començarem fixant la notació que farem servir al llarg de la tesi (Cap. 2). A continuació, (Cap. 3), farem una introducció a les teories d'Einstein-Aether, i ens aturarem per analitzar la seva dinàmica cosmològica i les restriccions fenomenològiques existents. Finalment, introduirem la teoria de la gravetat de Hořava i explicarem la relació que existeix entre aquesta teoria i les teories d'Einstein-Aether.

El següent pas serà l'estudi de les pertorbacions cosmològiques en les teories d'Einstein-Aether (Cap. 4). La teoria de pertorbacions lineals en un univers en expansió és una teoria realista per descriure el creixement de les inhomogeneïtats a escales subhoritzó després de recombinació. Podem separar les pertorbacions en modes escalar, vectorial i tensorial i estudiar cada un d'ells separadament. En el cas de l'escalar, aplicarem el formalisme canònic per expressar el Lagrangià escalar en termes de les variables invariants de gauge. La raó d'aquest estudi és obtenir la normalització dels modes escalars. Fet això, estudiarem els límits de longitud d'ona curta i llarga i obtindrem l'espectre de potències per aquests modes. Per finalitzar, calcularem les solucions de subhoritzó durant les èpoques de radiació i matèria i compararem els nostres resultats amb els obtinguts per la gravetat de Hořava en l'article [KUY10]. En segon lloc, analitzarem l'estabilitat i l'espectre de potències del mode vectorial durant una fase d'inflació amb llei de potències. Finalment, estudiarem el cas del tensor, que ens proporcionarà una restricció dels paràmetres degut als requeriments d'estabilitat clàssics.

Un cop estudiades les pertorbacions cosmològiques, examinarem l'impacte que la teoria d'Einstein-Aether té en les anisotropies del fons còsmic de microones (Cap. 5). Durant l'època de recombinació, la radiació còsmica de fons ens mostra que l'univers era molt homogeni i isòtrop, però avui podem veure a l'univers una estructura no lineal produïda per la inestabilitat gravitatòria. Això fa que la matèria sigui atreta a les regions d'alta densitat, amplificant les inhomogeneïtats presents prèviament. Per aquest motiu, l'estudi de les propietats de la radiació còsmica de fons proporciona informació de gran interès sobre el comportament de la teoria de gravetat. Resumirem el marc general de les anisotropies del fons còsmic de microones i discutirem el càlcul de les anisotropies en Relativitat General per després aplicar-lo als modes vectorials. Ens centrarem en la contribució dels modes vectorials a les anisotropies del fons còsmic de microones, inexistent pel cas de Relativitat General. Analitzarem les solucions per a radiació i matèria i calcularem l'espectre de potències angular tant a escales angulars petites com a escales angulars grans. Compararem els resultats amb la contribució procedent del mode tensorial i amb les observacions. Per últim, estudiarem els efectes que té en la polarització del fons còsmic de radiació.

Als dos primers apèndixs revisarem breument la teoria general de les pertorbacions cosmològiques i inclourem les fórmules detallades per calcular l'acció gravitatòria pertorbada i les equacions d'Einstein pertorbades en un univers amb una mètrica plana de Friedmann-Robertson-Walker (Ap. A i B). A l'Apèndix C hi incloem el conjunt complet d'equacions de moviment per a la teoria d'Einstein-Aether del sector escalar en el cas del gauge longitudinal. Els dos Apèndixs següents (D, E) comprenen l'estudi de les solucions del sector escalar, en primer lloc per un escenari amb una pertorbació d'inflató i, en segon lloc, pel cas de les solucions subhoritzó a les èpoques de dominació de radiació i matèria, tot en el cas del gauge longitudinal. L'últim Apèndix, F, inclou la derivació per un contingut general de matèria en el límit de longituds d'ona llargues.

Les conclusions de la tesi es recullen al Cap. 6. En aquesta teoria es destaca l'existència de dos camps dinàmics addicionals, un al sector escalar i un altre al vectorial, i que la inflació indueix pertorbacions mesurables a ambdós camps. Els resultats són també aplicables al límit de baixes energies de la gravetat BPSH.

Podem assumir que els paràmetres de l'Aether c_i $(i = 1, \dots, 4)$ són petits, fet justificat en què poden ser considerats com a proporcionals al quadrat de la raó entre l'escala de trencament de simetria M i l'escala de Planck $c_i \sim (M/M_P)^2 \ll 1$.

Les conclusions més destacades són les següents. Trobem, en el sector escalar, que a més del mode adiabàtic estàndard ζ (que es correspon a la curvatura de les superfícies de densitat de matèria constant), hi ha un mode addicional d'isocurvatura que podria tenir importància fenomenològica. Geomètricament, el mode d'isocurvatura pot ser descrit com el nombre d'*e-foldings* diferencials que separen les superfícies de densitat de matèria constant de les superfícies ortogonals a l'Aether. Això juga el paper d'un potencial de velocitat v per l'Aether respecte a la matèria. En el moment de la sortida de l'horitzó durant inflació, les amplituds de δN i v són comparables a les del mode adiabàtic estàndard ζ

$$v \sim \delta N \sim \zeta \sim \frac{H}{M_P}\epsilon^{-1/2} \quad \text{(sortida de l'horitzó)}.$$

Aquí H és el paràmetre de Hubble i $\epsilon \ll 1$ és el paràmetre de *slow-roll* durant inflació, que és independent dels paràmetres de l'Aether.

Una vegada creuat l'horitzó, la pertorbació de curvatura ζ es manté constant, mentre que el comportament de δN depèn del paràmetre $\tilde{\kappa} \equiv -\left(1 + \frac{\alpha}{c_{14}}\right)$. En el cas $\tilde{\kappa} < 0$ la pertorbació d'isocurvatura cau lentament a escales grans, mentre que per $\tilde{\kappa} > 0$ creix. D'altra banda, la pertorbació de velocitat ve donada per $v \sim (k/\dot{a})\delta N$, on k és el nombre d'ona comòbil i $\dot{a}$ és la derivada del factor d'escala respecte del temps propi. Per tant durant inflació quan $\dot{a}$ creix, el camp de velocitat de longituds d'ona llargues cau aproximadament en proporció a l'invers del factor d'escala. Després d'inflació, l'univers desaccelera i el camp de velocitat torna a créixer. En el moment de reentrada a l'horitzó, per escales cosmològicament rellevants, tenim

$$v \sim \delta N \sim e^{N\tilde{\kappa}}\zeta \sim e^{N\tilde{\kappa}}\, 10^{-5} \lesssim 1, \quad \text{(reentrada a l'horitzó)}$$

on $N \sim 60$ és el nombre de *e-foldings* d'inflació des del temps quan l'escala cosmològica creua primer l'horitzó. L'última desigualtat indica el límit de validesa de l'aproximació lineal. És necessari assenyalar que, per $\tilde{\kappa} = 0$, la pertorbació d'isocurvatura i el camp de velocitat de l'Aether són comparables a $\zeta \sim 10^{-5}$ en el moment de la reentrada a l'horitzó. Tot i així, amb $\tilde{\kappa} \lesssim 10/N$ tenim $\delta N \lesssim 1$. Si $\tilde{\kappa}$ és suficientment gran per saturar la desigualtat, encara permet velocitats moderadament relativistes per el camp de l'Aether $v \sim 1$ dins l'univers observable.

Pel sector vectorial trobem resultats semblants. Si denotem per V el component transversal del camp de velocitat de l'Aether respecte la matèria, trobem que a escales de superhoritzó

$$V \sim \left(\frac{\epsilon}{c_{14}}\right)^{1/2} v.$$

Per tant, si $c_{14} < \epsilon$ (que sembla natural si l'escala del trencament de la simetria Lorentz és baixa), la contribució al camp de velocitat serà dominat respecte el component logitudinal. D'altra banda, en una teoria com BPSH, el component transversal del vector no existeix i la part escalar v és la dominant.

També trobem que els potencials gravitatoris en el gauge longitudinal ϕ i ψ poden ser diferents, fins i tot, pel mode adiabàtic. En escales superhoritzó, trobem que aquest efecte (que pot ser atribuït a l'*anisotropic stress* del tensor d'energia-impuls de l'Aether) és de l'ordre

$$(\phi - \psi)_{\text{adiab}} \sim \phi_{\text{adiab}}\, c_{13} \sim \zeta\, c_{13} \sim 10^{-5} c_{13},$$

on $c_{13} \sim (M/M_P)^2$ és una combinació dels paràmetres de l'Aether ($c_{13} = c_1 + c_3$). Físicament, aquest paràmetre es pot expressar en termes de la velocitat de propagació dels modes tensorials $c_{13} = c_t^{-2} - 1$. El mode d'isocurvatura contribueix màximament a l'*anisotropic stress* però el potencial degut al mode d'isocurvatura està suprimit per c_{13}

$$(\phi - \psi)_{\text{isoc}} \sim \phi_{\text{isoc}} \sim c_{13}\, \delta N.$$

Ja que δN pot ser més gran que ζ, l'*anisotropic stress* pot estar dominat pel mode d'isocurvatura. L'*anisotropic stress* a escales observables està suprimit respecte al seu valor en el moment de creuar l'horitzó degut a la dinàmica de l'Aether a escales subhoritzó. Per $\tilde{\kappa} = 0$ l'efecte escala com k^{-2} per modes que van creuar l'horitzó durant l'època de matèria i com k^{-1} per modes que el van creuar a l'època de radiació. Les restriccions actuals per $\phi - \psi$ a escales cosmològiques no són molt restrictives i el cas $|c_{13}| \lesssim 1$ sembla estar permès per les observacions.

L'Aether apareix als paràmetres PPN a través dels efectes que depenen del sistema de referència, produint anisotropies al camp gravitatori dels cossos que es mouen respecte de l'Aether. D'aquesta manera el camp de velocitat generat durant inflació podria ser detectable. És necessari assenyalar, però, que sembla difícil que amb la tecnologia actual sigui possible observar les propietats estadístiques del camp aleatori per aquest tipus particular d'observacions. Fins i tot, encara que el camp de velocitats fos relativista a escales cosmològiques, $v \sim 1$, decau com k^{-2}. Concretament, el component que varia en escales de l'ordre dels 100 Mpc estaria per sota de la velocitat del virial, $v_{\rm vir} \sim 10^{-3}$, dels objectes lligats en galàxies i sembla poc probable que poguéssim mostrar efectes dependents del sistema de referència en objectes que es troben a distancies més llunyanes. D'altra banda, a les distàncies relativament petites on l'observació dels efectes dependents del sistema de referència és accessible, encara podem detectar un gran però bastant homogeni camp de velocitat, fins i tot, un de molt més gran que la velocitat del virial dels objectes lligats.

Finalment, hem calculat la contribució del camp vectorial transversal V a l'espectre de potències angular del fons de radiació de microones. Trobem que per $\tilde{\kappa} = 0$ l'espectre de coeficients multipolars C_ℓ^V té la mateixa forma que el dels modes tensorials. L'amplitud està relacionada amb l'espectre C_ℓ^t pels modes tensorials i amb C_ℓ^ζ pel mode escalar adiabàtic com

$$C_\ell^V \sim \frac{c_{13}^2}{c_{14}} e^{2N\tilde{\kappa}} C_\ell^t \sim \frac{\epsilon\, c_{13}^2}{c_{14}} e^{2N\tilde{\kappa}} C_\ell^\zeta.$$

Això vol dir que els modes vectorials en la teoria d'Einstein-Aether poden dominar de manera senzilla el senyal dels modes tensorials. A més, sabem que el fons de radiació de microones s'ajusta bé amb un espectre primordial de pertorbacions escalars adiabàtiques. Això imposa restriccions fenomenològiques addicionals entre els paràmetres c_{13} i $\tilde{\kappa}$ de la teoria d'Einstein-Aether de la forma

$$\tilde{\kappa} \lesssim \frac{1}{2N} \ln \left| \frac{c_{14}}{\epsilon\, c_{13}^2} \right|.$$

L'anàlisi de la polarització induïda pels modes vectorials és de gran interès fenomenològic. En particular, els modes vectorials contribueixen a la polarització de tipus B i podrien ser distingibles de la contribució procedent dels modes tensorials. Això pot proporcionar una manera de diferenciar les teories d'Einstein-Aether de la Relativitat General. L'enfocament que hem seguit aquí ens proporciona informació només de la dependència en l de l'espectre fins l'època de recombinació. Per poder obtenir més informació és necessari utilitzar el càlcul numèric. L'estudi fet en [NK11] (basat en part en el treball d'aquesta tesi) segueix aquest camí i suggereix que el mode B podria ser detectable pels futurs projectes d'observació del fons de radiació de microones i, també, que l'amplitud pot ser més gran que la procedent de les ones gravitatòries primordials.

Per concloure, els resultats presentats aquí mostren que el sistema de referència privilegiat seleccionat pel camp de l'Aether A^{μ} (o per la foliació privilegiada de la teoria de BPSH) pot haver adquirit una velocitat aleatòria gran, alimentada per les fluctuacions quàntiques durant l'època d'inflació. Depenent dels paràmetres, això pot ser lleument relativista a escales cosmològiques. Els efectes d'aquest camp de velocitat poden ser detectables en observacions del efectes als PPN dependents del sistema de referència o en característiques pròpies de l'espectre de la radiació còsmica de fons, com pot ser una contribució mesurable procedent dels modes vectorials a les anisotropies de la temperatura i a la polarització.

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Contents

Chapter 1

Introduction: Motivation and Outline

The standard description of cosmological evolution is based on General Relativity (GR), whose success explaining the evolution of the universe is overwhelming. However, as we advance in the knowledge of the details of the universe we have to introduce new elements in order of explain all the features observed. Some of the most well-known problems, as the horizon or flatness problems, are addressed by inflation. Other open problems are the accelerated expansion of the universe and the missing matter. The common explanation for these is to require the existence of two "dark" components, dark energy and dark matter. All together, this is usually considerer the fiducial model for cosmology, known as the Λ-CDM model.

Gravity is an attractive force that tends to pull things together. However, in 1998 two collaborations, the Supernova Cosmological Project and the High-z Supernova Search Team [PAG+99, RFC+98], discovered that the universe is in an accelerated expansion. That suggested the existence of a component with repulsive force that counteracts gravity, an energy that fills most of the universe and that, due to its unknown nature, was called "dark energy". Initially, this was identified with the cosmological constant, as the quantum vacuum-energy, but this leads to an estimated size much higher than the observed value. Thus, the origin and form of this energy is still a problem to solve in present cosmology.

The Cosmic Microwave Background (CMB) is the other observation that although it provides an important support for the Big Bang Cosmology it also confirms the need of some kind of dark energy and matter. The initial detection of the CMB was made on 1965 by Penzias and Wilson [PW65], and its cosmological significance was explained in [DPRW65]. The original measurements provide an estimate of the temperature of the black-body radiation of about 3.5 K. The Planck spectrum of the microwave background was finally established with great precision with the COBE satellite in the

90s [SBK+92], giving a temperature of $2.725 \pm 0.002\,\mathrm{K}$. Since then, the WMAP experiment [KDN+09] has continued to provide more and more details about the spectrum and the anisotropies of the CMB.

On the other hand, a large class of alternative proposals have been explored in the literature in order to explain these observations. These have some common characteristics. First of all, they are modifications of gravity at large distances. Second, they are strongly constrained by the requirement of general covariance. For that reason, most of the proposed alternatives to General Relativity can in fact be cast as GR coupled to new fields[1].

Cosmic acceleration may be due to a scalar field slowly rolling down a potential [PR88][2], or simply sitting in one of its local minima [BP00]. Alternatively, it can be driven by the non-minimal kinetic term of a k-essence scalar field with a Lagrangian of the form $p(X, \phi)$, where $X = \partial_\mu \phi \partial^\mu \phi$. This form is quite versatile, and can be used to mimic cosmic fluids with a wide range of possibilities for the effective equation of state and speed of sound, including those which are characteristic of dark energy and cold dark matter [APDM99, APMS00, APMS01, GM99].

The gradient of the k-essence field, $\partial_\mu \phi$, is a time-like vector which spontaneously breaks Lorentz invariance, in a way that is parametrically independent of its effects on the time evolution of the background geometry. In particular, Lorentz invariance can be spontaneously broken by $\partial_\mu \phi$ while the background spacetime remains maximally symmetric, a situation which is known as ghost condensation [AHCLM04]. Still, the "fluid" responds to the gravitational pull of ordinary matter, leading to modifications of the long range potentials.

More generally, theories with a massive graviton can be written in a covariant form as GR coupled to a set of "Stückelberg" scalar fields ϕ^A with non-minimal kinetic terms, whose gradients have non-vanishing expectation values [AHGS03, Dub04, BCNP09]. Depending on the interactions and the expectation values of the condensates, this can describe different phases of massive gravity. Aside from the Lorentz preserving Fierz-Pauli phase [AHGS03] (see also [CM10]), Lorentz breaking phases have been investigated in [Dub04, BCNP09]. Some of these have interesting phenomenology, such as the absence of ghosts in the linearized spectrum, a massive graviton with just two transverse polarizations, and weak gravitational potentials which differ from those in standard GR by terms proportional to the square of the graviton mass [Rub07, Dub04, BCNP09, DTT05, DFST09, BM09].

Additional fields of spin 2 have been considered in bigravity (or multi-

[1] A counterexample is the DGP brane-world scenario, where gravity is modified in the infrared by a continuum of Kaluza-Klein gravitons [DGP00]. Because of the continuum, DGP cannot be formulated as a standard four dimensional GR with additional fields.

[2] This includes the case where we "modify" the Einstein-Hilbert action to an arbitrary function of the Ricci scalar [Cap02, CDTT04], since the resulting theory can be reformulated as a standard scalar-tensor theory [Wil93].

gravity) theories [DK02], where space-time is endowed with several metrics interacting with each other non-derivatively. Due to general covariance, only one of the gravitons in the linearized spectrum stays massless, while the remaining ones acquire masses proportional to the non-derivative interaction terms. Lorentz invariance can be broken spontaneously even in cases where all metrics are flat, provided that their light-cones have different limiting speeds. This leads to phenomenology [BCNP07, BCNP08, BDG07] similar to that of certain phases of Lorentz breaking massive gravity referred to above [Rub07, Dub04, BCNP09, DTT05, DFST09, BM09], of which multigravity can be seen as a particular realization.

Finally, additional vector fields have received considerable attention in cosmology. Effective field theories for vectors are strongly constrained by stability requirements. Typically, those with non-trivial cosmological dynamics contain a massive ghost [APDT09], which can be removed from the spectrum by sending its mass to infinity. This amounts to imposing a fixed-norm constraint on the vector, which in turn forces a Lorentz-breaking vacuum expectation value. This led Jacobson and Mattingly to dub this type of models Einstein-Aether theories [JM01][3]. Their low-energy excitations are the Goldstone bosons of the broken Lorentz symmetry[4], which will participate in the dynamics of the long range gravitational interactions.

An interesting recent development is the proposal by Hořava [Hoř09b, Hoř09a] that a Lorentz-breaking theory of gravity may be renormalizable and UV complete. The breaking of Lorentz invariance in this case is implemented by introducing a preferred foliation of space-time, but no additional structure. As pointed out in [BPS09], any theory with a preferred foliation can be written in a generally covariant form by treating the time parameter which labels the different hypersurfaces as a Stückelberg scalar field $\mathcal{T}$. The foliation is considered to be physical, but not the parameterization, and therefore the covariant theory should be invariant under field redefinitions $\mathcal{T} \to f(\mathcal{T})$. In other words, the Lagrangian can have a dependence on the unit normal to the hypersurfaces, but not on the magnitude of the gradient $\mathcal{T}_{,\mu}$ (in contrast with the examples of k-essence and ghost condensation mentioned above). From this observation, Blas, Pujolàs and Sibiryakov showed [BPS10b, BPS10a] that Hořava gravity could be extended by including in the action all terms compatible with reparameterization symmetry, and consistent with power counting renormalizability. Interestingly enough, this extension also cured certain problems in the scalar sector of the original

[3] A fixed norm vector field determining a preferred frame has also been used in relativistic versions of MOND [Mil83], such as TeVeS [Bek04], which attempt to explain the rotation curves of galaxies without introducing cold dark matter

[4] In theories with spontaneously broken spacetime symmetries, the number of Goldstone bosons does not generally agree with the number of broken generators. However, if the order parameter that breaks the spacetime symmetry is spacetime-independent (as the constant Aether field), then both numbers do agree [LM02].

proposal (such as instabilities and strong coupling at low energies [BPS09]). Jacobson [Jac10], has recently clarified the relation between the Einstein-Aether theory and this extended version of Hořava gravity, which he dubbed BPSH gravity. In particular, he pointed out that any solution of Einstein-Aether where the vector field is hypersurface orthogonal is also a solution of the low energy limit of BPSH gravity.

Since the Aether only interacts gravitationally, any signal of it must be proportional to a power of $(E/M_P)^2$, where M_P is the reduced Planck mass, and E is an energy scale. Thus, even though the Aether contains massless fields, its presence is hard to detect. In that respect, inflation provides an interesting window to probe the Aether and its implications. During inflation, short-scale vacuum fluctuations of light fields are transferred to cosmological distances, where they may leave an observable imprint. It is thus natural to look for signatures of Einstein-Aether on the spectrum of primordial perturbations, which is the subject to which we devote this thesis.

Previous work on this subject has been done in Refs. [Lim05, LMB08], although in a somewhat narrower region of parameter space and with somewhat different conclusions. In the scalar sector, we find that there is a primordial isocurvature mode, which can be interpreted as the velocity potential for the Aether with respect to matter. Depending on the Aether parameters, this mode can grow on superhorizon scales, leading to a large random velocity field for the Aether. Similar results apply to the transverse vector sector. These perturbations may thus be of phenomenological interest. We also find that the isocurvature mode is strongly correlated with the usual adiabatic mode, which corresponds to curvature perturbations in the co-moving slicing.

For previous work on the impact of adiabatic scalar perturbations on the cosmic microwave background radiation (CMB) and large scale structure in (generalized) Aether theories, see [ZFZ08, ZZB+10].

While this thesis was being prepared, an interesting related paper by Kobayashi, Urakawa and Yamaguchi appeared [KUY10], which analyzes the post-inflationary evolution of the adiabatic scalar mode in BPSH theory. We will comment more about these article in Section 4.1.5, but summarizing we can say that where we overlap, our conclusions agree. There has also been a recent article that studies the B-mode polarization of the Aether [NK11], mostly numerically, but they also make an analytical approach. We will discuss more about these results when we talk about polarization in Section 5.7.

In this thesis we are going to study the Einstein-Aether (E-A) theory from the point of view of its cosmological perturbations. We are interested in the constraints that may be obtained from this analysis and the special features that can arise from the vector mode, absent in General Relativity. The plan of the thesis is the following.

First of all, in Chapter 2 we are going to fix the notation that we are going to follow in the text. After that, in Chapter 3 we will present the Einstein-Aether theory, its general properties and present constraints; also, we will introduce Hořava gravity and explain the connection that can be done between these two theories.

Next, we will start with the study of cosmological perturbations in Einstein-Aether theory (Chap. 4). The theory of linear perturbations in an expanding universe is a realistic theory to describe the growth of inhomogeneities on subhorizon scales after recombination. We can split the perturbations in scalar, vector and tensor modes, and we will study each of them separately. In the scalar case, we are going to apply the canonical formalism to express the scalar Lagrangian in terms of the gauge invariant variables in order to obtain the normalization for the scalar modes. Then, we will study the short and long wavelength limits, and obtain the power spectrum for these modes. Finally, we will calculate the subhorizon solutions during the radiation and matter domination epochs and compare these results with the ones obtained in [KUY10] for Hořava gravity. For the vector we will study the stability and the power spectrum during power-law inflation. The last section of this chapter will be devoted to the tensor modes, where we will check the conditions for classical stability and calculate the primordial power spectrum.

In Chapter 5 we will examine the impact of Einstein-Aether theory in the CMB anisotropies. At the epoch of recombination, the CMB shows us that the universe was very homogeneous and isotropic, however, today we can see a nonlinear structure on the universe produced by gravitational instability, making the matter to be attracted to high density regions amplifying the already existing inhomogeneities. Thus, the study of the properties of the CMB provides very useful information about the behavior of the gravity theory. We will overview the general framework of Cosmic Microwave Background anisotropies and discuss the calculation of the anisotropies in GR and then we will apply the same procedure to the vector modes in the E-A theory. We will focus on the contribution of the vector modes to the CMB anisotropies, as in GR there is no such contribution. We will analyze the solutions for radiation and matter and calculate the angular power spectrum both at large and small angular scales. We will compare these results with the contribution coming from the tensor modes and with observations. Finally, we will look at the effects in the polarization of the CMB.

We will summarize the results obtained in the previous sections and present the conclusions and outlook of this thesis in Chapter 6.

In the Appendices we will briefly review the general theory of cosmological perturbations (App. A) and include the detailed formulas for calculate the perturbed gravitational action and Einstein equations in a flat Friedmann-Robertson-Walker (FRW) universe (App. B). In Appendix C we include the complete set of equations of motion for Einstein-Aether theory for the scalar sector in the longitudinal gauge. The following two Appendices

D, E comprise the study of the scalar solutions for an inflaton perturbation and the subhorizon solutions during radiation and matter in the longitudinal gauge. The last Appendix F contains the derivation for long wavelengths for a generic matter content.

Chapter 2

Notation and Conventions

Unless otherwise stated, we shall use the space-time metric $g_{\mu\nu}$ with signature $(-,+,+,+)$. Geometric tensors are defined as follows

Covariant derivative

$$U_{\mu;\nu} \equiv U_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu} U_{\lambda},$$

$$V^{\mu}_{\ ;\nu} \equiv V^{\mu}_{\ ,\nu} + \Gamma^{\mu}_{\nu\lambda} V^{\lambda}.$$

Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} [g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}]. \tag{2.1}$$

Riemann tensor

$$R^{\alpha}_{\ \mu\beta\nu} = \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\rho\beta}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\rho}_{\mu\beta}, \tag{2.2}$$

Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\ \mu\alpha\nu}, \tag{2.3}$$

Curvature scalar

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{2.4}$$

Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{2.5}$$

In the **Lagrangian formulation**, we use

$$S_g = \frac{1}{2} M_P^2 \int d^4x \sqrt{-g} R = \frac{1}{2} M_P^2 \int L \sqrt{-g} d^4x = \int d^4x \mathcal{L}, \tag{2.6}$$

where M_P is the reduced Planck mass, $M_P^2 = \frac{1}{8\pi G}$.

Dots indicate derivatives with respect to time t and primes derivatives with respect to conformal time η.

The perturbed Friedmann-Robertson-Walker (FRW) metric for a flat universe is

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.7}$$

where $\bar{g}_{\mu\nu}$ is the background metric and $\delta g_{\mu\nu}$ the perturbation. More explicitly, the components will be written as

$$\begin{aligned} g_{\mu\nu} = a^2 \big[&-(1+2\phi)d\eta^2 + 2(B_{,i} + S_i)d\eta dx^i \\ &+ (\delta_{ij} - 2\psi\delta_{ij} + E_{,ij} + 2F_{(i,j)} + t_{ij})dx^i dx^j \big] . \end{aligned} \tag{2.8}$$

The scalar gauge-invariant variables are defined by

$$\begin{aligned} \Psi &= \psi - \mathcal{H}\left(B - \frac{1}{2}E'\right), \\ \Phi &= \phi + \left(B' - \frac{1}{2}E''\right) + \mathcal{H}\left(B - \frac{1}{2}E'\right), \end{aligned} \tag{2.9}$$

and the vector gauge-invariant variables are given by

$$\mathbf{Q} = \mathbf{F}' - \mathbf{S} \tag{2.10}$$

The boldface on vector quantities indicates that these are 3-component objects. The tensor sector is already gauge invariant.

In the Einstein-Aether theory we will have two extra modes, arising from the decomposition of the Aether field perturbation (see Eq. (4.2)), one scalar (C) and one vector mode ($\mathbf{V}$). The vector mode is already gauge-invariant, while the gauge-invariant variable for C is $Z = C + \frac{1}{2}E'$.

Chapter 3

Einstein-Aether Theories

The Einstein-Aether theories are, as we already mentioned in the introduction, gravity theories with additional vector fields. These type of theories are strongly constrained by stability requirements. To avoid instabilities a fixed-norm constraint is imposed to the vector field and thus Lorentz invariance is broken. In curved spacetime the vector must have dynamics in order to preserve general covariance. The original proposal of gravity with a dynamical preferred frame defined by a unit timelike vector was developed by [JM01].

As we said before, there are different motivations to study modified theories of gravity. First of all, it is useful to have a viable theory against which to compare gravitational observations, and this can also be seen as a way of testing General Relativity. The possibility that the vacuum in quantum gravity may determine a preferred rest frame was also a motivation in the studies of these theories. Finally, the presence of the dark components, dark energy and dark matter, and its role in explaining cosmic acceleration, galaxy rotation curves and structure formation, motivates the exploration of long-distance modifications of gravity.

In this thesis, we are going to focus in the effect that this modification has in primordial perturbations and the consequences for CMB anisotropies, but firstly, we need to introduce the theory and its general properties. We will consider the cosmological dynamics in an unperturbed universe and summarize the principal constraints for the parameters of the theory. In the last section of this chapter we will briefly comment on Hořava gravity and the connection between both theories.

The Einstein-Aether is described by the most general Lagrangian with two derivatives acting on a vector field of constrained norm [JM01],

$$\begin{aligned} L_A = {} & c_1 \nabla_\alpha A^\gamma \nabla^\alpha A_\gamma + c_2 \nabla_\alpha A^\alpha \nabla_\gamma A^\gamma + c_3 \nabla_\alpha A^\gamma \nabla_\gamma A^\alpha \\ & - c_4 A^\alpha A^\beta \nabla_\alpha A^\gamma \nabla_\beta A_\gamma + \lambda (A^\alpha A_\alpha + 1). \end{aligned} \tag{3.1}$$

Here, the c_i are dimensionless coefficients, and λ is a Lagrange multiplier that enforces the constraint

$$A^\mu A_\mu = -1. \tag{3.2}$$

The unit timelike vector field is dimensionless as is the metric. We restrict to two derivatives as higher derivatives would be suppressed by one power of a small length scale for each extra derivative, as usual in effective field theories.

The Lagrangian (3.1) can be thought of as the low-energy description of a theory in which boost invariance is spontaneously broken by the expectation value of A_μ, while spatial rotations and translations remain unbroken. The fixed-norm constraint eliminates the "radial" degree of freedom in field space, which is typically a ghost. We assume that A_μ is "minimally coupled" to gravity and to the rest of matter, so the total action is of the form,

$$S = \frac{M_P^2}{2}\int d^4x\sqrt{-g}\,[R + L_A] + \int d^4x\sqrt{-g}\,L_m. \tag{3.3}$$

Here M_P is the reduced Planck mass, and L_m is the Lagrangian of ordinary matter, which we assume does not contain couplings to the aether field.

The gravitational equations involve the energy-momentum tensor of the vector, $T_{\mu\nu} = (-1/\sqrt{-g})(\delta S_A/\delta g^{\mu\nu})$. This is given by

$$\begin{aligned} T_{\mu\nu} = \nabla_\sigma \left(J_{(\mu}{}^{\sigma} A_{\nu)} - J^{\sigma}{}_{(\mu} A_{\nu)} - J_{(\mu\nu)} A^\sigma \right) + Y_{\mu\nu} \\ + \frac{1}{2} g_{\mu\nu} L_A + \lambda A_\mu A_\nu - c_4 A^\alpha A^\beta (\nabla_\alpha A_\mu)(\nabla_\beta A_\nu), \end{aligned} \tag{3.4}$$

where

$$J^\alpha{}_\sigma = c_1 \nabla^\alpha A_\sigma + c_2 \delta^\alpha_\sigma \nabla_\beta A^\beta + c_3 \nabla_\sigma A^\alpha - c_4 A^\alpha A^\beta \nabla_\beta A_\sigma, \tag{3.5}$$

and

$$Y_{\alpha\beta} = c_1 \left[(\nabla_\gamma A_\alpha)(\nabla^\gamma A_\beta) - (\nabla_\alpha A_\gamma)(\nabla_\beta A^\gamma) \right]. \tag{3.6}$$

Variation of the Lagrangian (3.1) with respect to A leads to the field equation

$$\nabla_\alpha (J^\alpha{}_\beta) + c_4 A^\alpha (\nabla_\alpha A^\gamma)(\nabla_\beta A_\gamma) = \lambda A_\beta, \tag{3.7}$$

whilst variation of the Lagrangian density with respect to the Lagrange multiplier λ imposes the fixed norm constraint (3.2).

The coefficients c_i are subject to both theoretical and phenomenological restrictions, we will derive here the constraints coming from classical and quantum stability, and from phenomenological considerations related to primordial perturbations. There are other set of constraints we will detail below. All of them are summarized in Table (3.1). Their magnitude, relative to the symmetry breaking scale, can be estimated from dimensional analysis. The field redefinition $A_\mu = \tilde{A}_\mu / M$ leads to the fixed norm constraint $\tilde{A}_\mu \tilde{A}^\mu = -M^2$, from which we may interpret M as the scale at which Lorentz symmetry is spontaneously broken. In terms of the coefficients $\tilde{c}_i$ that would multiply the action for the rescaled field $\tilde{A}$, the original coefficients are given by $c_{1,2,3} = (M/M_P)^2 \tilde{c}_{1,2,3}$ and $c_4 = (M/M_P)^2 M^2 \tilde{c}_4$. We expect the dimensionless $\tilde{c}_{1,2,3}$ to be of order one, and the dimensionful $\tilde{c}_4$ to be of order M^{-2}, which leads to

$$c_i \sim \frac{M^2}{M_P^2}. \tag{3.8}$$

For convenience, in what follows we use the abbreviations

$$c_{13} = c_1 + c_3, \quad c_{14} = c_1 + c_4, \tag{3.9a}$$

$$\alpha = c_1 + 3c_2 + c_3, \quad \beta = c_1 + c_2 + c_3. \tag{3.9b}$$

Note that $\alpha = 3\beta - 2c_{13}$, so these abbreviations are not supposed to be an independent parameterization. Note also that our coefficients c_i and those of other works in the Aether literature may have opposite signs [Jac07].

3.1 Cosmological Dynamics

Let us consider the dynamics of a spatially flat unperturbed FRW universe in the presence of the Aether. Homogeneity and isotropy constrain the form of the metric and of the Aether. With the line element given by

$$ds^2 = a^2(\eta) \left[-d\eta^2 + d\vec{x}^2 \right] \tag{3.10}$$

we have, from Eq. (3.2),

$$A^\mu = (a^{-1}, 0, 0, 0). \tag{3.11}$$

Substituting into the expression for the energy-momentum tensor (3.4), we find that the energy density and pressure of the vector field are respectively given by

$$\rho_A = \frac{3\alpha}{16\pi G a^2} \mathcal{H}^2, \quad p_A = -\frac{\alpha}{16\pi G a^2} \left(\mathcal{H}^2 + 2\mathcal{H}' \right), \tag{3.12}$$

where $G = 1/8\pi M_P^2$, $\mathcal{H} = a'/a$ and a prime denotes a derivative with respect to conformal time. Thus, Einstein's equations read

$$\mathcal{H}^2 = \frac{8\pi G_{\rm cos}}{3} a^2 \rho, \tag{3.13a}$$

$$\mathcal{H}' = -\frac{4\pi G_{\rm cos}}{3} a^2 (\rho + 3p), \tag{3.13b}$$

where ρ and p are the energy density and pressure of the remaining matter fields (Aether excluded) and

$$G_{\cos} = \left(1 - \frac{\alpha}{2}\right)^{-1} G. \tag{3.14}$$

A comparison with the same equations in the absence of the Aether shows that the effect of the vector field is merely to "renormalize" the value of Newton's gravitational constant [CL04]; the energy density and pressure of the vector field mimic that of the remaining components in the universe.

On the other hand, the gravitational field created by isolated bodies is not exactly the same as that of General Relativity, and in that sense the Aether is a bona-fide modification of gravity. To lowest order in a post-Newtonian expansion, the potential sourced by a static and spherically symmetric body satisfies the Poisson equation $\Delta\phi = 4\pi G_N \rho$, but with a modified gravitational constant [Jac07]

$$G_N = \left(1 + \frac{c_{14}}{2}\right)^{-1} G. \tag{3.15}$$

Hence, the Aether also renormalizes the gravitational constant measured in "local" experiments, but by a different amount than in the cosmological case. Post-Newtonian corrections lead to further deviations of General Relativity, which place severe constraints on the Aether parameters. A summary of these and other constraints is given below. Nucleosynthesis, in particular, places a bound on the relative magnitude of the two Newton constants, of the form [CL04]

$$\left|\frac{G_{\cos}}{G_N} - 1\right| < 10\%. \tag{3.16}$$

Note that, for positive matter energy density and positive Newton's constant G, Eq. (3.13a) can only be solved if[1]

$$\alpha < 2. \tag{3.17}$$

Remarkably, this condition does not follow from any of the perturbative stability arguments which we shall consider below, but merely from the existence of a cosmological solution with positive energy density for ordinary matter. Note also that the Lagrange multiplier has a finite value along the cosmological solutions. Contracting the vector field equations of motion (3.7) with A^β we have

$$\lambda = \frac{3}{a^2}\left(\beta\mathcal{H}^2 - c_2\mathcal{H}'\right). \tag{3.18}$$

[1] We could have $\alpha > 2$ if we allow $G < 0$. However, this leads to instabilities in the tensor modes, as we shall discuss in Section 4.3.

For later reference, let us consider the case where the matter sector consists of a scalar field with an exponential potential,

$$L_m = -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - V_0 \exp\left[-\mu \frac{\varphi}{M_P}\right]. \tag{3.19}$$

It is well-known that this potential leads to power-law inflation [LM85], with a constant equation of state parameter $w \equiv p_\varphi/\rho_\varphi$ determined by the coefficient μ in the exponential. With a constant equation of state w the solution of Eqs. (3.13a) and (3.13b) is then

$$a \propto (-\eta)^q, \quad \text{with} \quad q = \frac{2}{1+3w} = \frac{1}{\epsilon - 1}, \tag{3.20}$$

where

$$\epsilon \equiv -\frac{H'}{aH^2} = \frac{2-\alpha}{4}\mu^2 \tag{3.21}$$

is the conventional slow-roll parameter. Note that if $2 - \alpha$ is sufficiently small, inflation may be de Sitter-like even if μ is of order one. This broadens the class of "natural" inflationary models that do not require particularly flat potentials, though we shall not explore this possibility here.

3.2 General Constraints

As mentioned above there are several conditions that the c_i's have to satisfy, which arise from the Post-Newtonian limit of the theory, Big-Bang Nucleosynthesis, and from the arrival of high-energy cosmic rays to earth. An extensive summary of these constraints can be found in [Jac07].

3.2.1 Post-Newtonian limits

In any metric theory, the gravitational field created by non-relativistic bodies can be characterized beyond the Newtonian limit by a set of post-Newtonian PPN parameters, whose values are tightly constrained by solar system tests of gravity [Wil93]. The parameters β and γ in Aether theories agree with those of General Relativity, and also agree with the measured ones [EJ04]. But because the Aether defines a preferred frame, it also introduces additional departures from General Relativity, which manifest themselves as gravitational potentials that depend on the velocity of the interacting bodies with respect to the Aether. These preferred-frame effects are encoded in the PPN parameters α_1 and α_2. One of the most stringent limits on the value of α_1 comes from measurements of the eccentricity of the binary pulsar J2317+1439 (which would change if α_1 were non-zero) [BCD96], while one of the most stringent limits on α_2 stems from the alignment of the sun spin with the solar system angular momentum (a non-zero α_2 would lead to a misalignment) [Nor87]. These limits lead to the conditions

$$\alpha_1 = \frac{-8(c_1 c_4 + c_3^2)}{2c_1 - c_1^2 + c_3^2} \leqslant 1.7 \times 10^{-4}, \tag{3.22a}$$

$$\alpha_2 = \frac{\alpha_1}{2} - \frac{(2c_{13} - c_{14})(\alpha + c_{14})}{\beta(2 - c_{14})} \leqslant 1.2 \times 10^{-7}. \tag{3.22b}$$

It is important to stress that both limits assume that the velocity of the sun with respect to the Aether ω is the velocity with respect to the frame in which CMB dipole vanishes, $\omega \sim 10^{-3}$. Roughly speaking, the limit on α_1 is actually a limit on $\alpha_1\omega$, while the limit on α_2 actually constraints $\alpha_2\omega^2$. Thus, if ω were larger than assumed, as the results of our work seem to allow, the limits on α_1 and α_2 would be actually tighter. In other words, the constraints on the PPN parameters α_1 and α_2 actually are

$$\alpha_1 \lesssim 10^{-4} \times \left(\frac{10^{-3}}{\omega}\right), \tag{3.23a}$$

$$\alpha_2 \lesssim 10^{-7} \times \left(\frac{10^{-3}}{\omega}\right)^2, \tag{3.23b}$$

where ω is again the velocity of the sun or binary pulsar with respect to the preferred frame. The constraints (3.22) are typically satisfied if the norm of the $\tilde{A}$, defined above, is of order $M \sim 10^{-4} M_P$.

3.2.2 Big-Bang Nucleosynthesis

The agreement between the predicted light element abundances, and those actually observed (or indirectly measured using Cosmic Microwave Background observations [KDN+09]) constrains the value of the Hubble constant at the time the light elements formed, at temperatures of about $T \approx 10^9$ K. Because the expansion rate depends on the value of the renormalized Newton constant $G_{\text{cos}} = 2G/(2-\alpha)$ through Eq. (3.13a), and because the latter is related to the "Newtonian" gravitational constant by Eq. (3.15), given the measured value of G_N on small scales and the number of relativistic species during nucleosynthesis, one can determine how light element abundances depend on the parameters α and c_{14}. Agreement of such a prediction with observations then implies

$$\frac{c_{14} + \alpha}{2 - \alpha} \lesssim 10\%. \tag{3.24}$$

3.2.3 Cherenkov radiation

If any of the propagation speeds of tensor, vector or scalar modes we have discussed were sufficiently smaller than the speed of light, highly relativistic particles traveling close to the speed of light would loose energy into these modes by a process analogous to Cherenkov radiation (this is kinematically possible only if the dispersion relation of the emitted quanta is subluminal). Although the emission amplitude is inversely proportional to the Planck mass, the fact that we detect these cosmic rays, and that they must originate at astrophysical distances, allows one to place quite stringent limits on the parameters of the Aether [EMS05],

$$c_{13} < 1 \times 10^{-15}, \tag{3.25a}$$

$$\frac{c_{13}^2(c_{13}^2 + 2c_4)}{c_1^2} < 1.4 \times 10^{-31}, \tag{3.25b}$$

$$\frac{(c_3 - c_4)^2}{|c_{14}|} < 1 \times 10^{-30}, \tag{3.25c}$$

$$\frac{c_4 - c_2 - c_3}{c_1} < 3 \times 10^{-19}. \tag{3.25d}$$

It is important to realize though that these constraints are "one sided", they only apply if the different Aether modes propagate subluminally. Under this assumption conditions (3.25) can also be taken to imply the bound $M \leqslant 10^{-7} M_P$ on the norm of the Aether field $\tilde{A}_\mu$ defined above.

3.2.4 Propagation speed

Some authors impose further conditions on the parameters of the Aether, namely, that the propagation speed of the perturbations be subluminal. The origin of this requirement goes back to the violations of causality that appear in *Lorentz-invariant theories* with superluminal signals. As far as we know, there is no link however between superluminality and violations of causality in backgrounds like the ones we are considering. The cosmic Aether breaks Lorentz invariance and defines a preferred reference frame. Signals always travel forward in time in this frame, so no closed timelike curves can arise. Even the construction of [AAHD+06], in which due to the nature of the background closed timelike curves may appear seems difficult to realize here, because the Aether satisfies a fixed-norm constraint. Hence, we shall not require subluminal propagation, though because this is a somewhat controversial issue, we collect the appropriate conditions here for completeness. They easily follow from Eqs. (4.97), (4.24) and (4.84). In the limit of small coefficients, $c_i \ll 1$ they read

$$c_{13} \geqslant 0, \tag{3.26a}$$

$$c_1 - c_4 - c_{13}^2 \geqslant 0, \tag{3.26b}$$

$$\beta - c_{14} \geqslant 0. \tag{3.26c}$$

Note that because we have taken metric perturbations into account, these conditions differ from those derived in the limit of a decoupled Aether

[Lim05]. For alternative views on superluminal propagation we refer the reader to the references [AAHD+06, Bru07, BMV08].

3.3 Hořava Gravity

In this section we briefly summarize the main aspects of the Hořava gravity and its extended version, BPSH gravity. The relation between the BPSH gravity and Einstein-Aether theory has been recently clarified by Jacobson [Jac10]. He shows that any hypersurface orthogonal solution to E-A theory is a solution to the IR limit of BPSH gravity. This implies that FRW cosmological solutions to E-A theory are also solutions to BPSH gravity. So, we can apply many of the conclusions for the results obtained for E-A theory to BPSH. Note that, on the contrary to general E-A theories, vector perturbations are absent in BPSH.

Hořava gravity was proposed in [Hoř09b, Hoř09a] as a new approach to the theory of quantum gravity. At short distances describes interacting nonrelativistic gravitons. It is a Lorentz-breaking theory in which the breaking is implemented by a foliation by space-like surfaces. This way, the coordinates split and general covariance is also broken. This preferred foliation of spacetime defines a global causal structure. In order to improve the UV behavior of the graviton propagator and make the theory power-counting renormalizable higher spatial derivative terms are added to the GR action. It is thus a candidate for a UV completion of GR, as well as an infrared modification. Despite everything there were still doubts about the consistency of this proposal [BPS09], such as problems with instabilities and strong coupling at low energies. However, the extended proposal made by Blas, Pujolàs and Sibiryakov [BPS10b, BPS10a] cured these problems. In the original proposal by Hořava one considers the ADM decomposition of the metric in the preferred foliation

$$ds^2 = N^2 dt^2 - h_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \tag{3.27}$$

and writes the action of the form

$$S = \frac{M_P^2}{2} \int d^3x dt \sqrt{h} N (K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}(h_{ij})), \tag{3.28}$$

where M_P is the Planck mass and K_{ij} is the extrinsic curvature tensor

$$K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{3.29}$$

with trace K, h is the determinant of the spatial metric h_{ij}, N is the lapse function and N^i is the shift vector, and λ is a dimensionless constant. The "potential" term depends only on the 3-dimensional metric and its spatial derivatives and is invariant under 3-dimensional diffeomorphisms. Restricting to operators of dimensions up to 6 is sufficient to make the theory naïvely renormalizable by power-counting. It is possible to restrict N to depend only on time and in this case one obtains the "projectable" version of the theory. The unrestricted N case, in which it can be a generic function of the spacetime, constitutes the "non-projectable" version of the theory. At low energies, the presence of the λ parameter is the only difference with GR, suggesting that the theory might have GR as its low-energy limit. However, the explicit breaking of general covariance by the preferred foliation introduces a new scalar degree of freedom that continues to be present at low energies generating a pathological behavior of the theory. The proposal of [BPS10a] to avoid this pathological behavior consists in the addition of a certain type of new terms in the action. Starting from the non-projectable version of the theory they consider a 3-vector $a_i \equiv \frac{\partial_i N}{N}$. It describes the proper acceleration of the unit normal vector field to the foliation surfaces, which is covariant under the transformation

$$\mathbf{x} \mapsto \tilde{\mathbf{x}}(\mathbf{t}, \mathbf{x}) \quad t \mapsto \tilde{t}(t). \tag{3.30}$$

Now the potential will include terms depending on a_i. The existence of the preferred foliation structure allows to introduce into the action terms with higher derivatives in spatial directions which improve the UV behavior of the graviton propagator but, at the same time, the theory remains second order in time derivatives avoiding problems with unitarity. The explicit breaking of 4-dimensional diffeomorphisms gives rise to the presence of a new scalar gravitational degree of freedom. This mode is free of pathologies at all energies given the following restrictions in the parameters

$$\frac{\lambda - 1}{3\lambda - 1} > 0, \quad 0 < \alpha < 2. \tag{3.31}$$

The linear dispersion relation at low energies,

$$\omega^2 = \frac{\lambda - 1}{3\lambda - 1} \frac{2 - \alpha}{\alpha} p^2,$$

shows that, in general, the propagation velocity of this scalar mode is different to the one from gravitons. This is an indication of the breaking of Lorentz invariance at low energies. These properties imply a different phenomenology at low energies with respect to the one of GR, and allow to obtain constraints for the parameters.

Writing the action in a covariant form the connection with E-A theories becomes clear

$$S = -\frac{M_P^2}{2} \int d^4x \sqrt{-g} \left({}^{(4)}R + (\lambda - 1)(\nabla_\mu u^\mu)^2 + \alpha\, u^\mu a^\nu \nabla_\mu u^\rho \nabla_\nu u_\rho \right), \tag{3.32}$$

where $u_\mu \equiv \dfrac{\nabla_{mu}\phi}{\sqrt{\nabla_{nu}\phi\nabla^\nu\phi}}$. This would be a special case of E-A theory where the vector field u_μ is hypersurface orthogonal, thus characterized by a single scalar field. In this case, therefore, there is just one degree of freedom instead of the 3 degrees of general E-A theory. If we make the exchange $(1-\lambda) \rightarrow c_2$ and $-\alpha \rightarrow c_4$, and taking the E-A theory parameters $c_1 = c_3 = 0$, we can reproduce the results obtained in [BPS10a] about the effective Newton constant and the effective gravitational constant. The other point to emphasize is that, contrary to the case of E-A theory, in this case there is no effect on tensor modes compared with GR, vector propagation is absent and the deflection of light is the same as in GR ($c_1 = c_3 = 0$ in E-A theories implies $\phi = \psi$). However, more details and subtleties about this connection are studied in [Jac10], and in principle, when matter is present and if it is coupled minimally to the spacetime metric, this choice of parameter it is not possible.

Here we are mainly interested in the results for scalar modes in E-A theories that can be applied to the BPSH theory, as we will comment in the corresponding section (4.1.5). In this case, we can apply the relation between parameters mentioned above in order to compare our results.

Condition	Constraint	Equation
Solution of Einstein's equations	$\alpha < 2$	(3.17)
Stability of Tensors	$c_{13} > -1$	(4.98)
Stability of Scalars	$-2 \leqslant c_{14} < 0, \quad \beta < 0$	(4.25)
Stability of Vectors	$2c_1 \leqslant c_{13}^2(1 + c_{13})$	(4.85)
PPN Limits	*see* Equation	(3.22)
Big-Bang Nucleosynthesis	$c_{14} + \alpha \lesssim 0.2$	(3.24)
Cherenkov radiation (assumes subluminality)	*see* Equation	(3.25)
Superluminal Tensors	$c_{13} \leqslant 0$	(4.97)
Superluminal Scalars	$(2 + c_{14})\beta \leqslant (2 - \alpha)(1 + c_{13})c_{14}$	(4.24)
Superluminal Vectors	$2c_4 \geqslant -\dfrac{c_{13}^2}{1 + c_{13}}$	(4.84)
Anisotropic stress of long wavelength adiabatic modes	$\lvert c_{13}\rvert \lesssim 1$	(4.45)
Non-growing scalar isocurvature modes	$\dfrac{\alpha}{c_{14}} \geqslant -1$	(4.30)
Subdominant contribution of vectors to CMB	$C_\ell^V \lesssim C_\ell^\zeta$	(6.7)

Table 3.1: Summary of the theoretical and phenomenological conditions on the parameters of Aether theories. We use the abbreviations α, β, c_{13} and c_{14}, which are related to the standard Aether parameters c_i through Eqs. (3.9).

Chapter 4

Perturbations in Einstein-Aether Theories

In the previous chapter we have reviewed the general concepts about the E-A theories as well as their general constraints. Here we are going to focus in the analysis of the cosmological perturbations on this theory and the constraints that come from this analysis. We are going to treat separately tensor, vector and scalar. Vector perturbations are not present in General Relativity so they provide the most differentiating feature. We want to analyze their contribution to the temperature anisotropies and the polarization on the CMB.

Cosmological Perturbations

The background vector field (3.11) preserves rotational invariance, and so it is still convenient to use the standard decomposition of perturbations in scalars, vector and tensors under spatial rotations. The perturbed FRW metric can be written as

$$ds^2 = a^2(\eta)\Big[-(1+2\phi)d\eta^2 + 2(B_{,i}+S_i)d\eta dx^i + \left(\delta_{ij} - 2\psi\delta_{ij} + E_{,ij} + F_{i,j} + F_{j,i} + t_{ij}\right) dx^i dx^j \Big], \quad (4.1)$$

and the vector field as

$$A^0 = \frac{1}{a} + \delta A^0, \quad A^i = \frac{1}{a}\left(C_{,i} + V_i - S_i\right). \quad (4.2)$$

Since the metric and vector fields are related to the Lagrange multiplier by Eq. (3.7), we also need to perturb the Lagrange multiplier,

$$\lambda = \lambda_0 + \delta\lambda, \quad (4.3)$$

where λ_0 is the background value, given by Eq. (3.18). Variation of the second order action with respect to $\delta\lambda$ leads to the linearized form of the constraint (3.2),

$$\delta A^0 = -\frac{\phi}{a}. \tag{4.4}$$

Here, ϕ, B, ψ, E, C are scalars, S_i, F_i, V_i are transverse vectors, and t_{ij} is a transverse and traceless tensor. Note that t_{ij} and V_i are gauge-invariant. Scalars, vectors and tensors decouple from each other in the linearized theory, so we consider each sector separately. In momentum space, our convention for the Fourier components is

$$f_{\mathbf{k}}(\eta) \equiv f(\eta, \mathbf{k}) = \int \frac{d^3x}{(2\pi)^{3/2}} f(\eta, \mathbf{x}) \exp\left(-i\mathbf{k}\cdot\mathbf{x}\right). \tag{4.5}$$

We will start studying scalar perturbations, then the vector sector and, finally, the tensor one.

4.1 Scalar Perturbations

The scalar sector of Einstein-Aether theories consists of the five scalars ϕ, ψ, B, E, C defined in Eqs. (4.1) and (4.2)[1]. Thus, the Aether enlarges the scalar sector by the Aether perturbation C.

It is convenient to introduce a gauge-invariant description of the dynamical degrees of freedom. To this end, following [BPS10b, BPS10a, Jac10] we note that the scalar part of the Aether field A_μ can be represented by means of an auxiliary scalar field $\mathcal{T}$ through the identification

$$A_\mu \equiv \frac{-\mathcal{T}_{,\mu}}{(-\mathcal{T}_{,\nu}\mathcal{T}_{,}{}^{\nu})^{1/2}},$$

where it is assumed that the gradient of $\mathcal{T}$ is everywhere time-like. Surfaces of constant $\mathcal{T}$ define a foliation of space-like surfaces, and we can think of $\mathcal{T}$ as a time variable. Since the background A_μ is aligned with the FRW temporal coordinate, the background field is given by $\mathcal{T} = \mathcal{T}(\eta)$. The perturbations $\delta\mathcal{T}(\eta, \mathbf{x})$ lead to the linearized spatial components $A_i = -(a/\mathcal{T}')\delta\mathcal{T}_{,i}$. From Eq. (4.2) we have $A_i = a\,\partial_i(B + C)$, so it follows that

$$\frac{\delta\mathcal{T}}{\mathcal{T}'} = -(B + C). \tag{4.6}$$

In addition to the Einstein-Aether sector, we must also include the matter sector. When the dominant matter component is the inflaton field φ, a

[1] The perturbation in the Lagrange multiplier $\delta\lambda$ disappears from the Lagrangian after substituting the constraint to which it leads. To linearized order, this constraint is $\delta A^0 = -\phi/a$, which we use to eliminate the scalar δA^0 in favor of the potential ϕ.

convenient set of gauge-invariant variables is given by:

$$\zeta_a \equiv \psi - \mathcal{H}(B + C), \tag{4.7a}$$

$$\delta N \equiv \frac{\mathcal{H}}{\varphi'}\delta\varphi + \mathcal{H}(B + C). \tag{4.7b}$$

Geometrically, these can be interpreted as follows (see Fig. (4.1)). Using (4.6), it is clear that the variable ζ_a is the curvature perturbation on surfaces of constant field $\mathcal{T}$ (i.e. on hypersurfaces orthogonal to the Aether field A^μ). From the definition of ζ_a and δN it also follows that

$$\zeta \equiv \zeta_a + \delta N \tag{4.8}$$

is the curvature perturbation on surfaces of constant inflaton φ. At the end of inflation and afterwards, ζ will describe the curvature perturbation on hypersurfaces comoving with matter (excluding the Aether). On the other hand,

$$\delta N = \mathcal{H}\left(\frac{\delta\varphi}{\varphi'} - \frac{\delta\mathcal{T}}{\mathcal{T}'}\right) = \mathcal{H}\delta\eta, \tag{4.9}$$

where $\delta\eta$ is the amount of conformal time separating the surfaces of constant φ from the surfaces of constant $\mathcal{T}$. Hence δN can be interpreted as the differential e-folding number between these two types of surfaces. The velocity of Aether with respect to the matter is given by

$$v_i = \delta\eta_{,i} = \mathcal{H}^{-1}\delta N_{,i}\,. \tag{4.10}$$

Hence, we can also think of the isocurvature perturbation $\mathcal{H}^{-1}\delta N$ as a velocity potential for the Aether with respect to matter.

Initially we consider the case of an exponential inflaton potential, Eq. (3.19). This somewhat simplifies the analysis because the background solutions have a constant equation of state parameter $p = w\rho$.

In addition, the behavior of long wavelength perturbations of such a scalar field can mimic those of radiation and matter dominated eras for $w = 1/3$ and $w = 0$ respectively. The "equivalence" applies only *on large scales*, because scalar perturbations and fluid perturbations have different sound speeds. Nonetheless, in Appendix F we derive the form of the long wavelength adiabatic and isocurvature scalar modes for generic matter content and expansion history.

In Appendix C we derive the equations of motion considering the energy-momentum tensor of a perfect fluid. In Appendix E we write this equations in the longitudinal gauge for the cases of radiation and matter. We will use these equations in Section 4.1.5 in order to study the subhorizon solutions during radiation and matter dominated epochs in the longitudinal gauge, and we will compare our results with the ones obtained in [KUY10] for Hořava gravity.

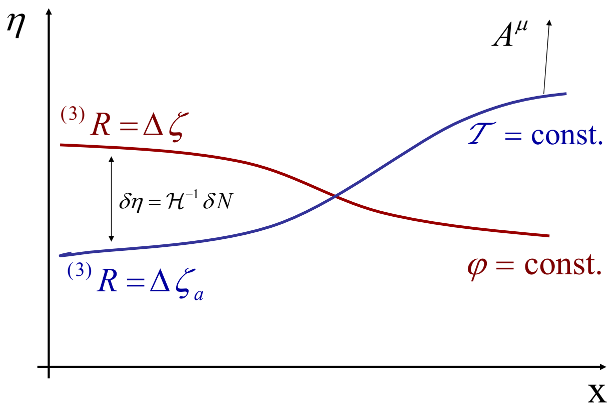


Figure 4.1: Geometrical interpretation of different perturbation variables. On hypersurfaces of constant inflaton φ, the curvature perturbation is ζ, while on hypersurfaces of constant Aether $\mathcal{T}$ the perturbation of the spatial curvature is ζ_a. In the presence of isocurvature modes, both hypersurfaces do not agree. Their distance in conformal time is the variable $\delta\eta$, which measures departures from adiabaticity.

4.1.1 Canonical reduction of the scalar sector

The normalization of the spectrum of scalar perturbations follows from the normalization of the action for the corresponding physical degrees of freedom. Here, we find the reduced set of gauge-invariant dynamical variables, and express the second order Lagrangian in terms of them. This Lagrangian can also be used, of course, to re-derive the scalar equations of motion (D.6) and (D.7).

The starting point is the Lagrangian for scalar perturbations in an arbitrary gauge, which is obtained by substituting the metric (4.1) into the action (3.3), and expanding to second order in the scalar perturbations. Using the constraint (3.2) which is obtained from variation with respect to $\delta\lambda$, we arrive at

$$
\begin{aligned}
\mathcal{L}_s^{(2)} = \frac{M_P^2}{2}a^2 \Bigg[& 2k^2\psi^2 - 3(2-\alpha)\psi'^2 - 4k^2\psi\phi + 4k^2\psi' B \\
& - (2-\alpha)k^2\psi' E' + 2\alpha k^2\psi' C + \beta k^4 \left(C + \frac{1}{2}E'\right)^2 \\
& - c_{14}k^2\left(\phi + C' + B'\right)^2 - 6(2-\alpha)\mathcal{H}\phi\psi' \\
& - 2(c_{14}-2)\mathcal{H}k^2\phi B - (2-\alpha)\mathcal{H}k^2\phi E'
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(\alpha - c_{14})\mathcal{H}k^2\phi C - (2-\alpha)(2\mathcal{H}^2 + \mathcal{H}')\phi^2 \\
&+ \left(\alpha(\mathcal{H}^2 - \mathcal{H}') + c_{14}(\mathcal{H}^2 + \mathcal{H}')\right) k^2 (C+B)^2 \\
&+ M_P^{-2}\left(\delta\varphi'^2 - k^2\delta\varphi^2 + 2\varphi'\delta\varphi\left(3\psi' - k^2 B + \frac{k^2 E'}{2}\right)\right. \\
&\qquad \left.\left. - 2\varphi'\delta\varphi'\phi - a^2 V_{,\varphi\varphi}\delta\varphi^2 - 2a^2 V_{,\varphi}\delta\varphi\phi\right)\right]. \qquad (4.11)
\end{aligned}
$$

Not all variables in this Lagrangian are dynamical. Some linear combinations are gauge modes, while others are constrained. We would like to find a Lagrangian that contains dynamical gauge-invariant variables only.

The identification of constraints and the reduction of phase space is best performed in the canonical formalism, where the equations of motion are at most of first order in time. Constraints are equations of motion without any time derivatives, and can be substituted back into the first order Lagrangian. Here, we closely follow Fadeev and Jackiw's method for dealing with constrained systems [Jac93]. For a discussion of cosmological perturbation theory in this framework, see [GMST98].

We begin by introducing new variables U and W through

$$2W = B + C, \qquad (4.12a)$$

$$2U = B - C. \qquad (4.12b)$$

The conjugate momenta of the system are given by

$$
\begin{aligned}
\Pi_\psi \equiv \frac{\mathcal{L}_s^{(2)}}{\partial\psi'} = M_P^2 a^2 \Big[&\alpha k^2 (W - U) + 2k^2(U + W) \\
&- \frac{(2-\alpha)}{2} k^2 E' - 3(2-\alpha)(\psi' + \mathcal{H}\phi) \\
&+ 3M_P^{-2}\varphi'\delta\varphi \Big], \qquad (4.13a)
\end{aligned}
$$

$$
\begin{aligned}
\Pi_E \equiv \frac{\mathcal{L}_s^{(2)}}{\partial E'} = \frac{1}{2} M_P^2 a^2 k^2 \Big[&\beta k^2 \left(W - U + \frac{1}{2} E' \right) \\
&- (2-\alpha)(\psi' + \mathcal{H}\phi) + M_P^{-2}\varphi'\delta\varphi \Big], \qquad (4.13b)
\end{aligned}
$$

$$\Pi_{\delta\varphi} \equiv \frac{\mathcal{L}_s^{(2)}}{\partial\delta\varphi'} = a^2(\delta\varphi' - \phi\varphi'), \qquad (4.13c)$$

$$\Pi_W \equiv \frac{\mathcal{L}_s^{(2)}}{\partial W'} = -2c_{14} M_P^2 a^2 k^2 \left(\phi + 2W'\right), \qquad (4.13d)$$

and we can write the first order Lagrangian

$$
\begin{aligned}
\mathcal{L}_s^{(1)} = \; &\Pi_E E' + \Pi_W W' + \Pi_{\delta\varphi}\delta\varphi' + \Pi_\psi \psi' \\
&- \frac{3\Pi_E^2}{M_P^2 a^2 k^4 (1 + c_{13})} + \frac{\Pi_W^2}{8c_{14}k^2 M_P^2 a^2} - \frac{\Pi_{\delta\varphi}^2}{2a^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\beta\Pi_\psi^2}{4M_{PL}^2a^2(2-\alpha)(1+c_{13})}-\frac{2k^2\beta W\Pi_\psi}{(2-\alpha)(1+c_{13})}\\
&+\frac{\Pi_E\Pi_\psi}{M_P^2a^2k^2(1+c_{13})}-2U\Pi_E-\frac{2(1-c_{13})W\Pi_E}{(1+c_{13})}\\
&-\frac{\varphi'\delta\varphi\Pi_\psi}{M_P^2(2-\alpha)}+M_P^2a^2k^2\psi^2+2M_P^2a^2k^2W^2\\
&\times\left(\frac{2k^2\beta}{(2-\alpha)(1+c_{13})}+(c_{14}+\alpha)\mathcal{H}^2+(c_{14}-\alpha)\mathcal{H}'\right)\\
&-\frac{1}{2}a^2\delta\varphi^2\left(k^2+a^2V_{,\varphi\varphi}-\frac{3\varphi'^2}{M_P^2(2-\alpha)}\right)\\
&+\frac{2\alpha a^2k^2\varphi'\delta\varphi W}{(2-\alpha)}+\phi\Big[-2c_{14}M_P^2a^2k^2\mathcal{H}W-2M_P^2a^2k^2\psi\\
&\qquad+\frac{1}{2}\Pi_W+\mathcal{H}\Pi_\psi-\varphi'\Pi_{\delta\varphi}-a^2\delta\varphi(a^2V_{,\varphi}+3\mathcal{H}\varphi')\Big]. \qquad (4.14)
\end{aligned}
$$

Variation with respect to the *independent* variables ψ, Π_ψ, E, Π_E, $\delta\varphi$, $\Pi_{\delta\varphi}$, W, Π_W, U and ϕ leads to the same equations of motion as those derived from the variation of (4.11) with respect to ψ, E, $\delta\varphi$, B, C and ϕ.

Note that the time derivatives of U and ϕ do not appear in Eq. (4.14), so variation with respect to these variables leads to the two constraints

$$\Pi_E=0, \qquad (4.15a)$$

$$
\begin{aligned}
\Pi_{\delta\varphi}=&\frac{-4c_{14}M_P^2a^2k^2\mathcal{H}W-2a^4V_{,\varphi}\delta\varphi-4M_P^2a^2k^2\psi}{2\varphi'}\\
&+\frac{\Pi_W+2\mathcal{H}\Pi_\psi-6a^2\mathcal{H}\varphi'\delta\varphi}{2\varphi'}. \qquad (4.15b)
\end{aligned}
$$

Substitution of these constraints also causes E, ϕ and U to drop from the Lagrangian, which therefore depends only on the five independent canonical variables ψ, Π_ψ, $\delta\varphi$, W, Π_W. Five is one too many, since we expect two canonical pairs only. Indeed, one of the variables is redundant, and it corresponds to the residual gauge invariance of the Lagrangian. Let us introduce the gauge-invariant combinations

$$\zeta\equiv\psi+\frac{\mathcal{H}}{\varphi'}\delta\varphi, \qquad (4.16a)$$

$$\delta N\equiv 2\mathcal{H}\Omega\equiv 2\mathcal{H}W+\frac{\mathcal{H}}{\varphi'}\delta\varphi. \qquad (4.16b)$$

Geometrically, these can be interpreted as follows. The variable ζ is the curvature perturbation on surfaces of constant inflaton field φ. The variable δN is the same as the one introduced in (4.9), and can be interpreted as the differential e-folding number between hypersurfaces of constant inflaton field and surfaces orthogonal to the Aether field.

The momenta conjugate to the gauge-invariant variables ζ and Ω are given by

$$\Pi_\zeta \equiv \Pi_\psi + \frac{2M_P^2 a^2 k^2}{\varphi'}\delta\varphi, \tag{4.17a}$$

$$\Pi_\Omega \equiv \Pi_W + \frac{2c_{14} M_P^2 a^2 k^2 \mathcal{H}}{\varphi'}\delta\varphi. \tag{4.17b}$$

In terms of the new variables, the field perturbation $\delta\varphi$ disappears from the Lagrangian (4.14) and we have

$$\begin{aligned}
\mathcal{L}_s^{(1)GI} = M_P^2 a^2 k^2 \Bigg[2 \Bigg(& \frac{2\beta k^2}{(2-\alpha)(1+c_{13})} + (c_{14}+\alpha)\mathcal{H}^2 \\
& +(c14-\alpha)\mathcal{H}' \Bigg) \Omega^2 + \zeta^2 - \frac{2M_P^2 k^2}{\varphi'^2}(\zeta + c_{14}\mathcal{H}\Omega)^2 \Bigg] \\
& + \left(\frac{1}{8c_{14} M_P^2 a^2 k^2} - \frac{1}{8a^2\varphi'^2} \right) \Pi_\Omega^2 \\
& + \left(\frac{\beta}{4M_P^2 a^2 (2-\alpha)(1+c_{13})} - \frac{\mathcal{H}^2}{2a^2\varphi'^2} \right) \Pi_\zeta^2 - \frac{\mathcal{H}}{2a^2\varphi'^2}\Pi_\zeta \Pi_\Omega \\
& + \frac{c_{14} M_P^2 \mathcal{H} k^2}{\varphi'^2} (2\mathcal{H}\Pi_\zeta + \Pi_\Omega)\,\Omega - \frac{2\beta k^2}{(2-\alpha)(1+c_{13})}\Pi_\zeta \Omega \\
& + \frac{2M_P^2 \mathcal{H} k^2}{\varphi'^2}\zeta \Pi_\zeta + \frac{M_P^2 k^2}{\varphi'^2}\zeta \Pi_\Omega + \Pi_\zeta \zeta' + \Pi_\Omega \Omega'.
\end{aligned} \tag{4.18}$$

Expression (4.18) gives the first order Lagrangian we have been looking for, since it is a function of two canonical pairs, corresponding to two field degrees of freedom.

To see this more explicitly, we may vary with respect to Π_Ω and Π_ζ, and plug the resulting equations back into Eq. (4.18) to obtain the second order Lagrangian. For reference we just reproduce the leading terms in the limit $k|\eta| \gg 1$ (the full expression is cumbersome and not very illuminating). In terms of ζ and δN, this is given by

$$\begin{aligned}
\mathcal{L}_s^{(2)GI} = \frac{M_P^2 a^2}{2} \Bigg[& -\frac{4(2+c_{14})k^2}{c_{14}}\zeta(\delta N) + \frac{2(2+c_{14})k^2}{c_{14}}\zeta^2 \\
& + \frac{k^2(2(4+c_{14}\alpha) + c_{14}(\alpha-2)(1+3w))}{2c_{14}}(\delta N)^2 \\
& + \frac{4(2-\alpha)(1+c_{13})}{\beta}(\delta N)'\zeta' - \frac{2(2-\alpha)(1+c_{13})}{\beta}\zeta'^2 \\
& - \frac{(2-\alpha)(4(1+c_{13}) - 3\beta(1+w))}{2\beta}(\delta N)'^2 + \cdots \Bigg],
\end{aligned} \tag{4.19}$$

where the ellipsis denote terms which are subleading in the momentum expansion.

Variation of (4.19) with respect to ζ and δN (including the terms that we do not explicitly write down) yields two second-order differential equations for ζ and δN. These equations of motion are valid in any gauge. To find their form in the longitudinal gauge, we may use Eq. (D.11) to express the inflaton perturbations in terms of metric and Aether perturbations. Substituting in (4.16a) and (4.16b), we can cast the equations of motion for ζ and Ω as two third-order differential equations for the longitudinal gauge variables ψ and C. The latter happen to be precisely linear combinations of Eqs. (D.6), (D.7) and the time derivative of (D.7)[2].

In Eq. (4.19), the curvature perturbation ζ on surfaces of constant inflaton field is coupled to the variable δN. However, if we replace ζ by the curvature perturbation $\zeta_a = \zeta - \delta N$ on hypersurfaces orthogonal to the Aether, this leads to a Lagrangian for two decoupled variables, ζ_a and δN

$$\mathcal{L}_{k\eta\gg 1} = \frac{1}{2Z_N}\left[(\delta N)'^2 - k^2(\delta N)^2\right] + \frac{1}{2Z_a}\left(\zeta_a'^2 - c_a^2 k^2 \zeta_a^2\right) + \cdots , \quad (4.20)$$

where the ellipsis denote terms which are subleading in the momentum expansion, and we have introduced

$$Z_N^{-1} = \frac{3(1+w)(2-\alpha)}{2} M_P^2\, a^2, \quad (4.21)$$

and

$$Z_a^{-1} = \frac{2(1+c_{13})(\alpha-2)}{\beta} M_P^2\, a^2,\ c_a^2 = -\frac{(2+c_{14})\beta}{c_{14}(\alpha-2)(1+c_{13})}. \quad (4.22)$$

This form of the Lagrangian will be used in order to normalize the positive frequency modes associated with the initial vacuum fluctuations.

4.1.2 Short wavelength Lagrangian and stability

Once that we have the Lagrangian for the two gauge-invariant degrees of freedom $(\zeta_a, \delta N)$ given by Eqs. (4.7a, 4.7b) in the short wavelength limit (Eq. (4.20)), we can analyze the stability requirements. We can rewrite

$$Z_N = \frac{4\pi G_{\cos}}{\epsilon}, \quad Z_a = -2\pi c_t^2 \beta G_{\cos}, \quad (4.23)$$

and

$$c_a^2 = \frac{G_{\cos}}{G_N}\frac{\beta}{c_{14}} c_t^2. \quad (4.24)$$

[2] Note in particular that Eqs. (D.6) and (D.7) cannot follow from a variational principle from a reduced Lagrangian depending quadratically on C and ψ. If $\alpha = -c_{14}$, the evolution of C decouples from that of ψ, while the evolution of the latter does depend on the evolution of the former.

Here we have introduced the slow roll parameter $\epsilon = (3/2)(1+w)$ and G_N as given in Eq. (3.15). For constant scale factor a, the residue Z_a and sound speed c_a agree with the corresponding quantities in a perturbed flat space, as discussed in [JM04][3]. Quantum stability requires $Z_a > 0$, and classical stability requires $c_a^2 \geqslant 0$. As we will see, the stability of tensors demands $c_t^2 > 0$, and recalling that (3.17) requires $G_{\cos} > 0$, we are led to the conditions

$$-2 \leqslant c_{14} < 0 \quad \text{and} \quad \beta < 0, \tag{4.25}$$

which in turn guarantees $G_N > 0$ (The case $c_{14} = 0$ is singular, and has to be treated separately).

From Eq. (4.20) we can read off the normalization of the positive frequency modes associated with the "in" vacuum in the limit $k|\eta| \to \infty$, corresponding to wavelengths well within the horizon. The two independent mode functions are given by

$$\zeta_a^{(\varphi)} \to 0, \qquad \delta N^{(\varphi)} \to \frac{Z_N^{1/2}}{a} \frac{e^{-ik\eta}}{\sqrt{2k}}, \tag{4.26a}$$

$$\zeta_a^{(a)} \to \frac{Z_a^{1/2}}{a} \frac{e^{-ic_a k\eta}}{\sqrt{2c_a k}}, \qquad \delta N^{(a)} \to 0. \tag{4.26b}$$

In the first mode, where $\zeta_a \to 0$, the surfaces of constant Aether field coincide initially with the so-called flat slicing, and δN is the number of e-folds separating the surfaces of constant inflaton field from the flat slicing. This mode survives in the limit when there is no Aether field (since one can still define the flat slicing surfaces). Hence, we may call this the inflaton perturbation. In the second mode, where $\delta N \to 0$, the inflaton is initially aligned with the Aether, so that there is no inflaton perturbation in the Aether frame. This mode survives in the flat space limit even when there is no inflaton field. Hence, we call this the Aether perturbation. This can be thought of as one of the Goldstone bosons of the spontaneous Lorentz symmetry breaking.

In the previous discussion we have assumed that the action (3.3) gives an accurate description of the Aether up to sufficiently high momenta, so that (4.26) applies to scales well within the horizon. Hence, we require that the Einstein-Aether as an effective theory should be valid at least up to some spatial cut-off $\Lambda \gg H$, where H is the Hubble rate during inflation. We expect the corrections introduced by the unknown physics above the cut-off scale to be at most of order of H/Λ (see for instance [APL03]).

It should be noted that, on large scales, the fluctuations due to the Aether will mix with those due to the inflaton. Hence, while in single field inflation

[3] The above expressions are singular for $\beta = 0$, but it is easy to show, following the derivation in the previous Section 4.1.1 that ζ_a is not dynamical in this case.

perturbations have to be adiabatic, in Einstein-Aether theories there should exist additional non-adiabatic modes, as we discuss next.

4.1.3 Long wavelength modes

The full Lagrangian in terms of the gauge-invariant variables ζ_a and δN is somewhat cumbersome away from the short wavelength limit, because the two modes are no longer decoupled. However, for long wavelengths the Lagrangian can be easily obtained from (4.18) and diagonalized, but now in terms of a new pair of gauge-invariant variables $(\zeta, \delta N)$, where ζ is the curvature perturbation on hypersurfaces of constant inflaton field, which we shall also refer to as comoving hypersurfaces, defined in Eq. (4.8). The long wavelength Lagrangian is given by

$$\mathcal{L}_{k\eta\ll 1} = \mathcal{L}_\zeta + \mathcal{L}_{\delta N} + \cdots . \tag{4.27}$$

Here the ellipsis denotes subdominant terms and the dominant ones are given by

$$\mathcal{L}_\zeta = \left(\frac{4}{3(1+w)} - \beta c_t^2\right)^{-1} \frac{a^2}{4\pi G_{\text{cos}}} \zeta'^2, \tag{4.28}$$

and

$$\mathcal{L}_{\delta N} = -c_{14}(1+3w)^2 \, \frac{a^2 (k\eta)^2}{64\pi G} \left[(\delta N)'^2 + \frac{\kappa}{\eta^2}(\delta N)^2\right], \tag{4.29}$$

where we have introduced

$$\kappa = -6\left(1+\frac{\alpha}{c_{14}}\right)\frac{1+w}{(1+3w)^2}. \tag{4.30}$$

As we shall see, there are a total of four independent long wavelength modes, which we derive in Appendix D.2. Two of them have the property that $\delta N = 0$. For these, matter and Aether are mutually at rest, and so we call these modes adiabatic. The other two have $\delta N \neq 0$ and $\zeta = 0$, so we call them isocurvature, since there is no curvature perturbation on co-moving hypersurfaces.

Adiabatic modes ($\delta N = 0$)

In standard single field inflation, the non-decaying solution for the "adiabatic" perturbation ζ, which we denote by ζ_1, stays constant on superhorizon scales. In Appendix F we show that the same is true in the presence of the Aether:

$$\zeta_1 = const. \tag{4.31}$$

for any expansion history (including the case where the equation of state changes abruptly in time). The corresponding gravitational potentials in the longitudinal gauge are given by Eqs. (D.19a):

$$\phi_1 = \frac{3(1+w)}{(5+3w)}\, c_t^2\, \zeta_1, \tag{4.32a}$$

$$\psi_1 = \phi_1 + c_{13}\, c_t^2\, \zeta_1. \tag{4.32b}$$

The form of the two adiabatic modes (non-decaying and decaying) for an arbitrary expansion history and matter content is derived in Appendix F. The decaying adiabatic mode is given by (F.8), and it is characterized by

$$\phi_2 = \psi_2 \propto \mathcal{H} a^{-2}, \tag{4.33a}$$

$$\zeta_2 = 0. \tag{4.33b}$$

It is worth mentioning that, although these adiabatic modes have the properties described in [Wei03], they do not share the properties postulated in [Wei04b, Wei04a, Wei08]. In particular, for the first adiabatic mode ζ_1, the anisotropic stress is non-vanishing ($\phi_1 \neq \psi_1$).

Isocurvature modes ($\delta N \neq 0, \zeta = 0$)

As shown in Appendix D.2, for the case where the background equation of state parameter w is constant, the two isocurvature modes behave as powers of conformal time:

$$\delta N \propto (-\eta)^t, \tag{4.34}$$

where the exponents t are given by

$$t_\pm = -\frac{1}{2}\left(\frac{5+3w}{1+3w}\right) \pm \sqrt{\frac{1}{4}\left(\frac{5+3w}{1+3w}\right)^2 + \kappa}, \tag{4.35}$$

and κ is given in Eq. (4.30). Note that for $\kappa > 0$, there is always a growing isocurvature mode. If we don't want this mode to grow out of control, then κ should not be too large and positive,

$$-\infty < \kappa \ll 1. \tag{4.36}$$

In the following subsection we shall be more precise about the upper limit of this range (after discussing the overall normalization of the corresponding power spectrum). Note that for $\alpha = -c_{14}$, we have $\kappa = 0$ and the dominant isocurvature mode stays constant on large scales, just like the adiabatic one. Hence, from the point of view of observability of isocurvature modes, the interesting range of parameters is around $\alpha \approx -c_{14}$.

From Eq. (D.22c), the gravitational potentials for the isocurvature modes are given in terms of δN by

$$\psi = -c_{13}\, c_t^2\, \delta N, \tag{4.37}$$

and

$$\frac{\psi - \phi}{\phi} \sim 1. \tag{4.38}$$

Hence, isocurvature modes have sizable anisotropic stress. It is also straightforward to check from the relations in the Appendix D.2 that for the long wavelength isocurvature mode, the velocity of the Aether with respect to matter is given by

$$v_i = \mathcal{H}^{-1}\delta N_{,i} = c_t^{-2} C_{,i}\,, \tag{4.39}$$

where in the first equality, we use Eq. (4.10) and C is the scalar Aether perturbation in the longitudinal gauge. From Eq. (F.11) in Appendix F, it is clear that at the time of a sudden transition in the equation of state parameter, the variable C and its derivative remain continuous for the long wavelength isocurvature mode. This means that the velocity field matches trivially:

$$[v_i] = [v_i'] = 0, \tag{4.40}$$

where the square brackets indicate the discontinuity at the time of the transition. On the other hand, since the pressure changes abruptly at the transition, so does $\mathcal{H}'$, and therefore the matching conditions for δN are $[\delta N] = 0$, $[\delta N'] = (3/2)[w]\ \mathcal{H}\delta N$.

4.1.4 Power spectra

As shown in Subsection 4.1.2, the variables ζ_a and δN are uncorrelated on subhorizon scales. Hence, from Eqs. (4.8) and (4.26), it is clear that, at *short wavelengths*, the power spectra associated to δN and ζ are given by

$$\mathcal{P}_{\delta N} = \frac{Z_N}{(2\pi)^2}\left(\frac{k}{a}\right)^2, \tag{4.41a}$$

$$\mathcal{P}_{\zeta} = \mathcal{P}_{\zeta_a} + \mathcal{P}_{\delta N} = \frac{Z_a + Z_N}{(2\pi)^2}\left(\frac{k}{a}\right)^2. \tag{4.41b}$$

These spectra are valid for $k\eta \gg 1$[4].

[4]Here, and for the rest of this section, we shall assume that the speed of propagation of Aether is larger than or comparable to 1. This is convenient so we do not have to introduce the scale of sound horizon crossing in the discussion of the adiabatic mode. Also, this assumption avoids the need of imposing the constraints due to Cherenkov radiation discussed in 3.2.3.

Adiabatic modes

For wavelengths comparable to the cosmological horizon, δN and ζ are coupled to each other, and their evolution will not have a simple form. Nonetheless, the evolution of δN and ζ is again simple in the long wavelength limit, as we saw in the previous subsection. In particular, ζ stays constant at long wavelengths. The power spectrum for ζ will be approximately equal to its value at the time of horizon crossing, which we can estimate from (4.41a) by setting $k/a = H$,

$$\mathcal{P}_\zeta \sim \frac{Z_a + Z_N}{(2\pi)^2} H^2, \tag{4.42}$$

where H^2 is evaluated at the time of horizon crossing.

From Eqs. (4.23) and (4.24) we have

$$Z_N + Z_a = \left(1 - \frac{\beta\epsilon c_t^2}{2}\right) Z_N. \tag{4.43}$$

Note that Z_a is parametrically suppressed with respect to Z_N by one power of Aether parameters $c_i \sim (M/M_p)^2 \ll 1$ and by one power of the slow roll parameter ϵ. Hence

$$\mathcal{P}_\zeta(k|\eta| \ll 1) \approx \left.\frac{8G_{\text{cos}}^2\rho}{3\epsilon}\right|_{\eta_k} \left[1 + O(\beta\epsilon c_t^2)\right], \tag{4.44}$$

where ρ is the energy density and η_k is the time of horizon exit during inflation. Up to the small corrections introduced by the fluctuation of the Aether, which are controlled by Z_a, this expression is the same as the one in Einstein gravity, with Newton's constant G replaced with the effective Newton's constant G_{cos} which appears in the Friedmann equation (3.13a).

In summary, due to the smallness of the Aether parameters c_i, the spectrum of primordial adiabatic modes does not change significantly with respect to the case of standard Einstein gravity. As we saw in the previous subsection, in the presence of the Aether the adiabatic modes do not have the properties generally attributed to adiabatic perturbations. In particular, from Eq. (4.32b), on super-horizon scales the non-decaying adiabatic mode has a non-vanishing anisotropic stress

$$\frac{\psi_{\text{adiab}} - \phi_{\text{adiab}}}{\phi_{\text{adiab}}} \sim c_{13} \tag{4.45}$$

both in the matter and radiation dominated era. It is easy to see from Eq. (D.6) that for $\alpha + c_{14} = 0$, the Aether perturbation C behaves exactly like a massless field which propagates at the speed c_a. Hence C oscillates while its amplitude decays as the inverse of the scale factor, $C \propto (1/a)e^{-ic_a k\eta}$. It then follows from (D.4) that $\phi - \psi$ also decays in inverse proportion to the scale factor, and hence it is suppressed by a factor of $a(t_k)/a(t_0)$, where t_k is the time of horizon crossing. For modes which cross the horizon during the matter era, this means that the effect is suppressed with distance as[5]

$$\frac{\psi - \phi}{\phi} \sim c_{13}\ (kt_0)^{-2} \quad (t_{eq} \ll k^{-1} \ll t_0, \quad \alpha \approx -c_{14}), \tag{4.48}$$

where we adopt the standard convention $a(t_0) = 1$. On the other hand, for modes that crossed the horizon during the radiation era, the scaling is with k^{-1}. This is in agreement with the result that at small scales the post-Newtonian parameter ψ/ϕ equals one, as in General Relativity [EJ04]. Nonetheless, as a matter of principle, there could still be a distinct phenomenological signatures in the adiabatic sector imprinted on large scales.

Constraints on the ratio ψ/ϕ on cosmological scales have been derived under several different assumptions, using combinations of different large scale structure probes [KM06, DCC$^+$09, GMSM10, BT10]. At present, however, the constraints are quite weak, and it appears that values of ψ/ϕ of order one are still consistent with the data.

Isocurvature modes

Next, let us consider the spectrum of long wavelength isocurvature modes $\mathcal{P}_{\delta N}$. The phenomenological situation depends on whether $\alpha + c_{14}$ is positive or negative. If $\alpha < -c_{14}$, then δN decays on superhorizon scales, during and after inflation. Hence, these modes will remain insignificant with respect to the adiabatic ones. If $\alpha = -c_{14}$, then there is a constant non-decaying isocurvature mode, and δN stays constant on superhorizon scales. Finally, for $\alpha > -c_{14}$ there is a growing mode and δN can be very large at the time of re-entry even if it was small at the time of horizon exit.

Phenomenologically, the most interesting case seems to be the limit $|\alpha + c_{14}| \ll |c_{14}|$, in which the supercurvature mode δN stays approximately constant on large scales. Otherwise, either the mode is too suppressed to be

[5] Assuming that the Aether parameters are small, these conclusions are easily extended to the case $\alpha \neq -c_{14}$. In this case, Eq. (D.6) can be solved as the sum of the "homogeneous" equation which is obtained by ignoring terms proportional to ϕ, plus the contribution of a particular "inhomogeneous" solution. The first one takes the form $C_h \propto a^{-(1+d/2)} e^{ic_a k\eta}$, where

$$d = c_{13}\ c_t^2 \left(1 + \frac{\alpha}{c_{14}}\right). \tag{4.46}$$

This leads to

$$\frac{\psi - \phi}{\phi} \sim c_{13} \left[(kt_0)^{-(2+d)} + O(kt_0)^{-2}\right] \quad (t_{eq} \ll k^{-1} \ll t_0). \tag{4.47}$$

This applies to modes that entered the horizon during the matter era. For those which crossed the horizon during the radiation era, the scaling is with one less power of k in the denominator.

of any significance, or it grows too fast to be compatible with observations. In this case, the exponent $t_\pm$ for the dominant mode can be approximated by

$$\hat{t} \approx \frac{1+3w}{5+3w}\,\kappa, \tag{4.49}$$

where κ is given in (4.30), and we have $|\hat{t}| \ll 1$ both during inflation and afterwards. At the time of horizon crossing, the adiabatic and isocurvature modes have comparable amplitudes, $\mathcal{P} \sim (2\pi)^{-2} Z_N H^2$, and these will remain roughly comparable throughout cosmic history up to the present time provided that $\hat{t}$ is sufficiently close to zero. In order to assess how small it would have to be, we can make a rough estimate of the evolution of the amplitude of δN from the time of horizon crossing during inflation to the time of equality:

$$(\delta N)_{\text{eq}} \sim Z_N^{1/2} H e^{-\hat{t}_i N} \left(\frac{\eta_{\text{eq}}}{\eta_{\text{rh}}}\right)^{\hat{t}_r} \sim Z_N^{1/2} H e^{(\hat{t}_r - \hat{t}_i)N} \lesssim 1. \tag{4.50}$$

Here, the subindices i and r refer to inflation and radiation era respectively. Assuming $Z_N^{1/2} H \sim 10^{-5}$, as follows from the normalization of the adiabatic modes, we find that for

$$\hat{t}_r - \hat{t}_i \approx \frac{1}{3}\kappa_r \lesssim \ln(10^5)/N \tag{4.51}$$

the perturbation δN remains within the linear regime up to the time of equality of matter and radiation. Here, we have neglected κ_i, which is suppressed with respect to κ_r by a slow roll factor (we are assuming that the Aether parameters are the same today than they are during inflation), and

$$N \sim 60$$

is the number of e-foldings of inflation after the mode with co-moving wavenumber $k \sim \eta_{\text{eq}}$ first crossed the horizon. Note also that, according to (4.37), the contribution of the isocurvature mode to the gravitational potential is suppressed by c_{13},

$$\psi_{\text{isoc}} = -c_{13}\, c_t^2\, \delta N \lesssim 10^{-5},$$

where the last inequality is the observational bound on the gravitational potentials. For $-c_{13}\ c_t^2 \lesssim 10^{-5}$, ψ_{isoc} can remain small enough even if the inequality (4.51) is saturated, so that $\delta N \sim 1$. Also ψ_{isoc} becomes comparable to the contribution of the adiabatic mode ψ_{adiab}, when the inequality $\kappa \leqslant -3\ln|c_{13}\ c_t^2|/4N$ is saturated, and combining with (4.51), we require

$$\kappa \lesssim \frac{3}{N}\min\{\ln(10^5), -\ln|c_{13}\ c_t^2|\}. \tag{4.52}$$

Let us estimate what the physical implications of the isocurvature perturbations might be. On one hand, they would induce maximal anisotropic stress on large scales, as can be seen from Eq. (4.38),

$$\frac{\psi_{\text{isoc}} - \phi_{\text{isoc}}}{\psi_{\text{isoc}}} \sim 1,$$

which means that there would be a sizable difference between the two gravitational potentials ϕ and ψ provided that the contribution of the isocurvature mode is comparable to that of the adiabatic mode. For $\kappa = 0$ we have $\psi_{\text{isoc}} \sim c_{13}\, c_t^2\, \delta N \sim c_{13}\, c_t^2\, \psi_{\text{adiab}}$, and so from (4.45) both adiabatic and isocurvature modes contribute to the anisotropic stress in a similar amount (unless c_t^2 is very large). However, if κ is small and positive, then the isocurvature mode grows on superhorizon scales, and will contribute more to the anisotropic stress than the adiabatic one. As argued in the previous subsection, the difference $1 - (\phi/\psi)$ decays after horizon crossing, and so does its magnitude as a function of co-moving scale, which roughly goes as k^{-2} for modes which crossed the horizon during the matter era, and as k^{-1} for modes that crossed before the time of equality (see Eqs. (4.47), (4.48)).

Another possible signature might be due to preferred-frame effects due to the motion of matter with respect to the Aether [DEF94, GJW05], such as a dipole anisotropy in the gravitational potential of massive bodies. The primordial perturbations cause the Aether to point in different directions at different places in the observable universe. Hence, the velocity of matter with respect to the Aether (and the corresponding gravitational dipole, for instance) would have a random distribution. From (4.10) an isocurvature perturbation with wave-number k, induces a relative speed of the Aether with respect to matter given by

$$v = \frac{k}{\mathcal{H}} \delta N.$$

When the mode reenters the horizon during the radiation or matter era, at time $t_k \sim a/k$, we have

$$v \sim \delta N(t_k).$$

This has to be compared with the peculiar velocities in bound objects at the same scale, which is of order $\zeta^{1/2} \sim 10^{-3}$. Hence, the effect of the peculiar velocity of the Aether will be subdominant unless δN has grown from the time of horizon exit, in such a way that at the time of reentry it is at least of the order $\zeta^{1/2}$. This possibility exists, since we have seen that δN has a growing mode for $\alpha > -c_{14}$. Because of that, the velocity of the Aether at the time of horizon crossing could even approach moderately relativistic speeds without compromising the validity of the linear approximation and without contradicting current observations [Note from Eqs. (4.37) and

(4.38), that the gravitational potentials along the isocurvature mode, and hence their effects on the CMB, are suppressed by a factor c_{13}, which can be very small].

The velocity field of the Aether is strongly correlated with the amplitude of adiabatic modes, since both have a common origin in the amplitude of the short wavelength mode δN when it first crosses the horizon during inflation. Should the velocity field of the Aether be detected, such correlation would indicate that the velocity field has a primordial inflationary origin.

We may define a power spectrum $\mathcal{P}_v$ for the longitudinal velocity field of the Aether through the equation

$$\langle v_i(\eta,\mathbf{k}) v_j(\eta,\mathbf{k}')\rangle \equiv \frac{2\pi^2}{k^3}\mathcal{P}_v(\eta,k)\,\frac{k_i k_j}{k^2}\,\delta(\mathbf{k}-\mathbf{k}'). \tag{4.53}$$

Note that at the time of horizon exit during inflation, we have

$$\mathcal{P}_v \sim \mathcal{P}_{\delta N} \sim \mathcal{P}_\zeta \sim 10^{-5}, \tag{4.54}$$

where the last estimate follows from observations. However, since the perturbation δN grows on large scales for $0<\kappa\ll 1$, we can have $\mathcal{P}_v \sim \mathcal{P}_{\delta N} \gg \mathcal{P}_\zeta$ at the time of horizon reentry. As we shall see in the next section, vector perturbations can give an additional contribution to the velocity field (which can of course be disentangled from the scalar isocurvature contribution from the fact that the corresponding velocity field is transverse). It turns out that the scalar component and transverse vector component of the velocity field obey the same equation of motion on large scales. Hence, we defer the discussion of the spectral properties of $\mathcal{P}_v$ on currently observable scales to the next section.

4.1.5 Subhorizon solutions

In the following, we are going to study in detail the solutions of the scalar equations during radiation- and matter-dominated epochs. We have already estimated this behavior but now we are going to obtain the analytic solutions and also to make a numerical study in order to compare with the results obtained for the special BPSH case in [KUY10]. The general equations for the scalar modes and their form in the longitudinal gauge for both radiation and matter epochs are included in Appendix E.

Solutions during radiation-dominated epoch

In the short wavelength limit the equations (E.1, E.2) reduce to

$$\psi'' + \frac{2+c_{14}}{3(2-\alpha)}k^2\psi + \frac{(c_{14}(1+c_{13})-3\beta)}{3(2-\alpha)}k^2 C' = 0, \tag{4.55}$$

and

$$C'' + \frac{\beta}{c_{14}(1+c_{13})} k^2 C + \frac{\alpha + c_{14}}{c_{14}(1+c_{13})} \psi' = 0. \tag{4.56}$$

Using the ansatz

$$\psi = \widetilde{\psi}(k\eta) \exp(-ic_s k\eta), \quad C = \widetilde{C}(k\eta)\, \psi, \tag{4.57}$$

we get the two positive frequency solutions

$$\psi_1 = \exp\left(-\frac{ik\eta}{\sqrt{3}}\right), \tag{4.58a}$$

$$C_1 = \frac{i}{k} \frac{\sqrt{3}\,(\alpha + c_{14})}{3\beta - c_{14}(1+c_{13})}\, \psi_1, \tag{4.58b}$$

and

$$\psi_2 = \exp\left(-i\sqrt{\frac{\beta\,(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta\right), \tag{4.59a}$$

$$C_2 = \frac{i}{k} \frac{\alpha - 2}{\beta} \sqrt{\frac{\beta\,(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}\, \psi_2. \tag{4.59b}$$

We can get the normalization matching the solutions at $k\eta = 1$ with the ones obtained for the long wavelength modes for both the adiabatic

$$\tilde{\psi}_{ad} = \frac{2 + 3c_{13}}{3(1+c_{13})} \zeta_0, \tag{4.60a}$$

$$\tilde{C}_{ad} = -\frac{1}{3(1+c_{13})} \eta\, \zeta_0, \tag{4.60b}$$

and isocurvature cases

$$\tilde{\psi}_{iso} = -\frac{c_{13}}{1+c_{13}} \left(\frac{\eta}{\eta_{eq}}\right)^{t_+} \delta \mathrm{N}_0, \tag{4.61a}$$

$$\tilde{C}_{iso} = \frac{1}{1+c_{13}} \left(\frac{\eta}{\eta_{eq}}\right)^{t_+} \eta\, \delta \mathrm{N}_0, \tag{4.61b}$$

where

$$\zeta_0 \simeq \delta \mathrm{N}_0 \simeq \frac{H}{2\pi} \sqrt{\frac{4\pi G}{2-\alpha}}, \tag{4.62}$$

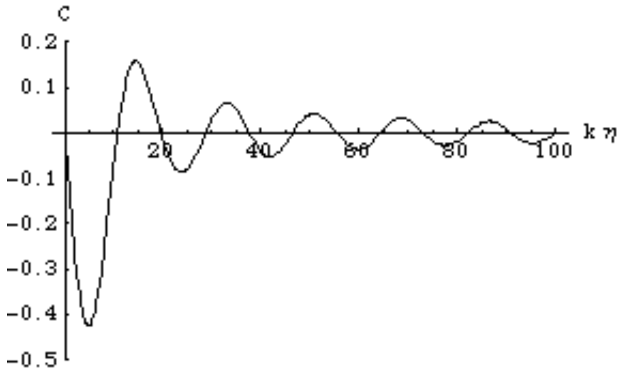
and

$$t_+ \simeq -\frac{3}{2}\left(1 + \frac{\alpha}{c_{14}}\right). \tag{4.63}$$

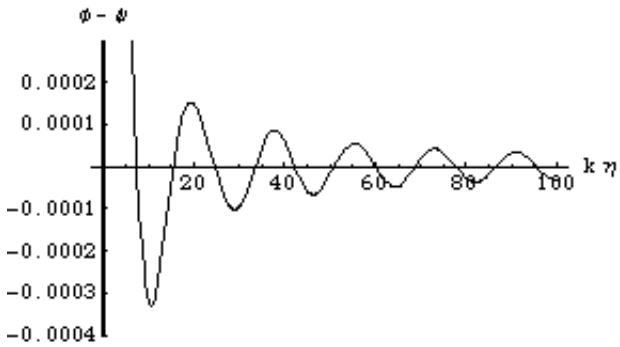
We want to solve the equations (E.1, E.2) numerically. The initial conditions are given by Eqs. (4.60, 4.61) for the adiabatic and the isocurvature modes respectively. We use the values for the parameters

$$\begin{aligned} \alpha &= 0.005, \\ \beta &= -0.001, \\ c_{14} &= -0.008, \\ c_{13} &= -0.004. \end{aligned} \tag{4.64}$$

Adiabatic mode In Fig. (4.2(a)) we can see the behavior of C and in Fig. (4.2(b)) the difference among the potentials. As we can expect from the analytical solutions they have an oscillatory behavior. The difference $\phi - \psi \propto c_{13} C$, as expected. We can check that both modes decays approximately as $1/k$, in agreement with the estimation made in the previous section (Eq. (4.48)).



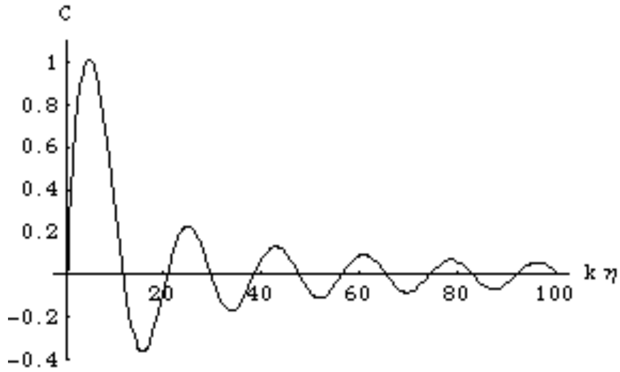
(a) Evolution of C



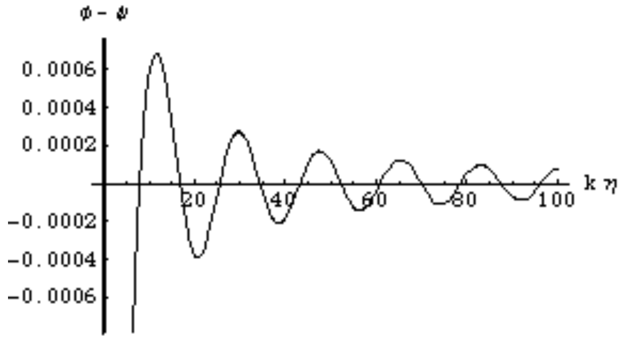
(b) Evolution of $\phi - \psi$

Figure 4.2: Behavior of the perturbations in a radiation dominated universe for the adiabatic modes.

Isocurvature mode In Fig. (4.3(a)) we have the behavior of C. In Fig. (4.3(b)) we have plotted $\phi-\psi$. Both of them are for the limit $|c_{14}+\alpha| \ll |c_{14}|$, where $t_+ \sim 0$. The isocurvature modes seem to be around two times bigger than the adiabatic ones, but they are fairly of the same order, as expected for the regime in which we are. They decay approximately in the same way as the adiabatic modes (Eq. (4.48)).



(a) Evolution of C for $t_+ \sim 0$



(b) Evolution of $\phi - \psi$ for $t_+ \sim 0$

Figure 4.3: Behavior of the perturbations in a radiation dominated universe for the isocurvature modes.

Solutions during matter-dominated epoch

In the short wavelength limit the equations (E.10, E.11) reduce to

$$\psi'' - \frac{\beta}{2-\alpha} k^2 C' = 0, \tag{4.65}$$

$$C'' + \frac{\beta}{c_{14}(1+c_{13})} k^2 C + \frac{\alpha + c_{14}}{c_{14}(1+c_{13})} \psi' = 0. \tag{4.66}$$

From the first equation we get

$$\psi' = A + \frac{\beta}{2-\alpha}k^2 C, \tag{4.67}$$

and plugging this into the second equation we get a second-order differential equation for C,

$$C'' + \frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}k^2 C + \frac{\alpha + c_{14}}{c_{14}(1+c_{13})}A = 0, \tag{4.68}$$

with solution

$$\begin{aligned} C = -\frac{(\alpha+c_{14})(2-\alpha)}{\beta(2+c_{14})}\frac{A}{k^2} &+ B\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta \\ &+ D\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta. \end{aligned} \tag{4.69}$$

Now, integrating Eq. (4.67) we get the solution for ψ

$$\begin{aligned} \psi = F &+ \frac{2-\alpha}{2+c_{14}}A\eta \\ &+ \sqrt{\frac{\beta c_{14}(1+c_{13})}{(2+c_{14})(2-\alpha)}}k\left(B\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta\right. \\ &\left. -D\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta\right). \end{aligned} \tag{4.70}$$

The four constants (A, B, D, F) are obtained through the matching with the long wavelength solutions, both for the adiabatic

$$\tilde{\psi}_{ad} = \frac{3+5c_{13}}{5(1+c_{13})}\zeta_0, \tag{4.71a}$$

$$\tilde{C}_{ad} = -\frac{1}{5(1+c_{13})}\eta\,\zeta_0, \tag{4.71b}$$

and isocurvature cases

$$\tilde{\psi}_{iso} = -\frac{c_{13}}{1+c_{13}}\left(\frac{\eta}{\eta_{eq}}\right)^{t_+}\delta\mathrm{N}_0, \tag{4.72a}$$

$$\tilde{C}_{iso} = \frac{1}{2(1+c_{13})}\left(\frac{\eta}{\eta_{eq}}\right)^{t_+}\eta\,\delta\mathrm{N}_0, \tag{4.72b}$$

where

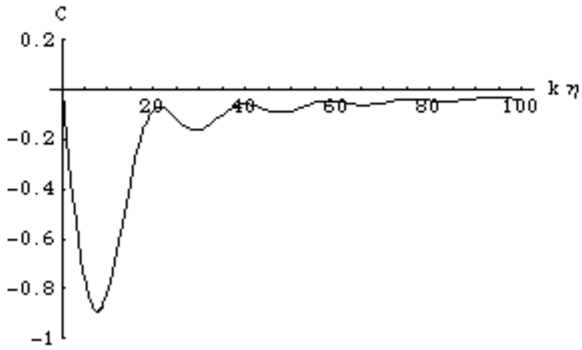
$$\zeta_0 \simeq \delta\mathrm{N}_0 \simeq \frac{H}{2\pi}\sqrt{\frac{16\pi G}{3(2-\alpha)}}, \tag{4.73}$$

and

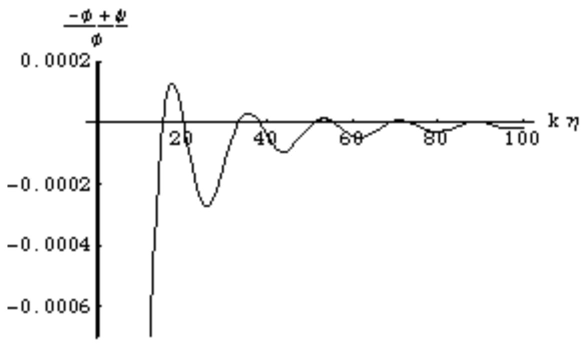
$$t_+ \simeq -\frac{6}{5}\left(1+\frac{\alpha}{c_{14}}\right). \tag{4.74}$$

We will repeat the same calculation that we have done before for radiation, now for the case of matter. The initial conditions are given by Eqs. (4.71, 4.72). The parameters are the same ones used in the radiation case (4.64). The situation is very similar to the case of radiation-dominated epoch, being the main difference that the two solutions decay now roughly as $1/k^2$ instead of as $1/k$ (Eq. (4.48)).

Adiabatic mode In figure 4.4 we see that the behavior observed agrees with the statement made before saying that C and $(\phi-\psi)/\phi$ decay in, approximately, the same way. In this case neither the Aether perturbation C nor the anisotropic stress oscillate around zero, contrary to the isocurvature case. The anisotropic stress is proportional to the parameter c_{13} (Eq. (4.48)).



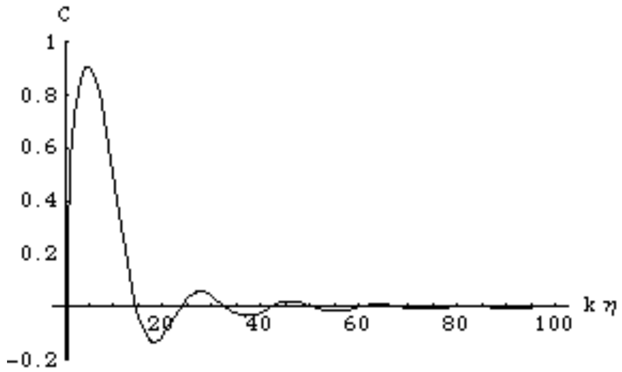
(a) Behavior of the Aether perturbation



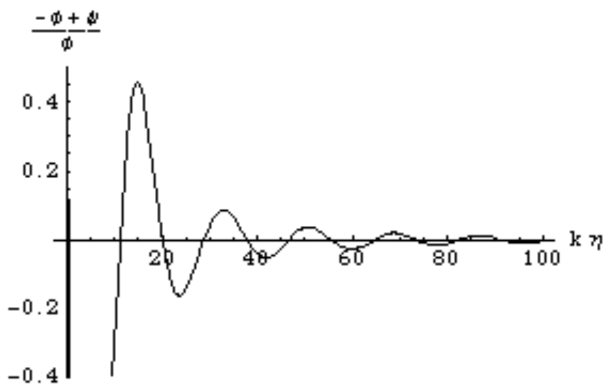
(b) Behavior of the anisotropic stress $\frac{\phi-\psi}{\phi}$

Figure 4.4: Evolution of the potentials for the adiabatic modes during matter dominated universe.

Isocurvature mode As we did for the radiation epoch, we show the plots for $t_+ \sim 0$. Both the Aether perturbation and the anisotropic stress behave in a similar way. The amplitude of the anisotropic stress, as happens during radiation epoch, is larger than the amplitude for the adiabatic modes, although the amplitude for the Aether perturbation is fairly of the same order. This is a consequence of the initial conditions, the anisotropic stress for adiabatic modes has a factor c_{13} that is absent for isocurvature modes (Eq. (4.45)).



(a) Evolution of C for $t_+ \sim 0$



(b) Evolution of the anisotropic stress $\frac{\phi-\psi}{\phi}$ for $t_+ \sim 0$

Figure 4.5: Evolution of the potentials for the isocurvature modes during matter dominated universe.

Comparison with Kobayashi et al.

We also want to compare these solutions with the ones plotted in [KUY10]. To do so we have to go to a different gauge. Their analysis corresponds to a choice of the parameters $c_1 = c_3 = c_{13} = 0$, that means our potentials verify $\phi = \psi$. Their variables (which we denote with the subscript U) are related to the ones in the longitudinal gauge in the following way

$$\begin{aligned} B_U &= -C, \\ \psi_U &= \psi - \mathcal{H}C, \\ \phi_U &= \phi + (C' + \mathcal{H}C). \end{aligned} \tag{4.75}$$

The values for the parameters are (the equivalents to the ones used in [KUY10])

$$\begin{aligned} c_2 &= -0.05, \\ c_4 &= -0.1. \end{aligned} \tag{4.76}$$

The initial conditions here are, for radiation

$$\tilde{\psi}_U^{(ad)} = \zeta_0, \tag{4.77a}$$

$$\tilde{B}_U^{(ad)} = \frac{1}{3}\eta\,\zeta_0, \tag{4.77b}$$

$$\tilde{\psi}_U^{(iso)} = -\left(\frac{\eta}{\eta_{eq}}\right)^{t_+^{(r)}} \delta N_0, \tag{4.77c}$$

$$\tilde{B}_U^{(iso)} = -\left(\frac{\eta}{\eta_{eq}}\right)^{t_+^{(r)}} \eta\,\delta N_0, \tag{4.77d}$$

and for matter

$$\tilde{\psi}_U^{(ad)} = \zeta_0, \tag{4.78a}$$

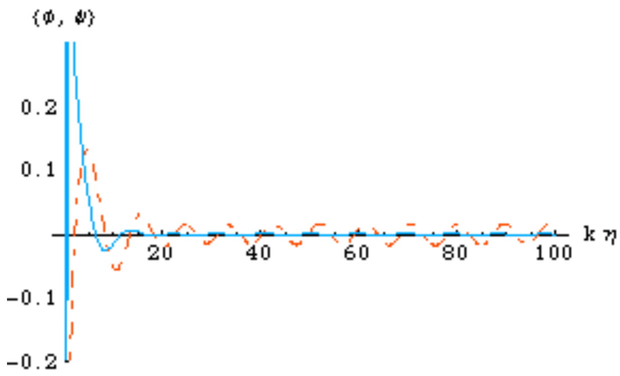
$$\tilde{B}_U^{(ad)} = \frac{1}{5}\eta\,\zeta_0, \tag{4.78b}$$

$$\tilde{\psi}_U^{(iso)} = -\left(\frac{\eta}{\eta_{eq}}\right)^{t_+^{(m)}} \delta N_0, \tag{4.78c}$$

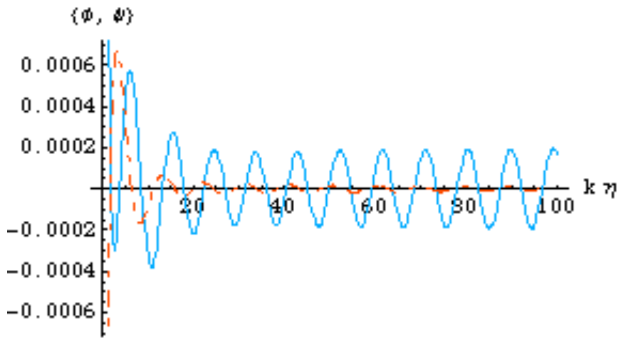
$$\tilde{B}_U^{(iso)} = -\frac{1}{2}\left(\frac{\eta}{\eta_{eq}}\right)^{t_+^{(m)}} \eta\,\delta N_0, \tag{4.78d}$$

where $\zeta_0 \simeq \delta \mathrm{N}_0 \simeq \frac{H}{2\pi}\sqrt{\frac{16\pi G}{3(2-3c_2)}}$, $t_+^{(r)} = -1.7$ and $t_+^{(m)} = -3$ for these values of the parameters.

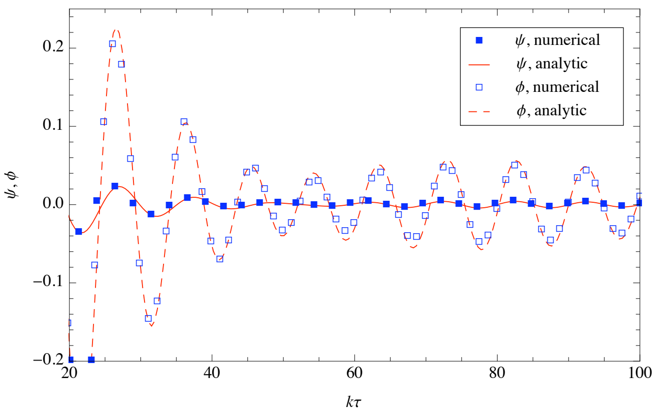
Radiation In Fig. (4.6) we compare our solutions with the ones plotted in [KUY10]. We can see that the behavior agrees with their results in the shape and size of the potentials for the adiabatic mode, but we get two solutions instead of just one. In their analysis they do not consider the presence of the isocurvature solutions, that we found, and it is two orders of magnitude smaller than the adiabatic mode. The isocurvature mode is initially suppressed by a factor of $\sim 10^{-3}$.



(a) Evolution of ψ_U and ϕ_U (Adiabatic mode)



(b) Evolution of ψ_U and ϕ_U (Isocurvature mode)

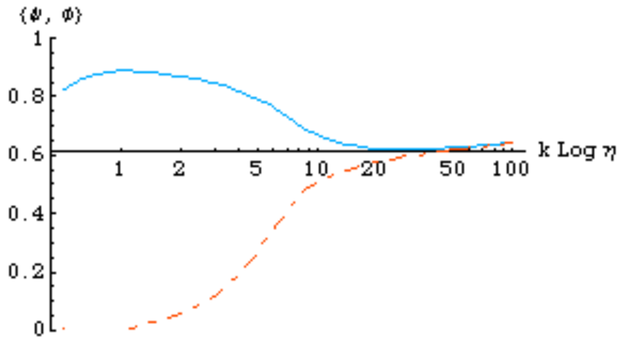


(c) Evolution of ψ_U, ϕ_U from [KUY10]

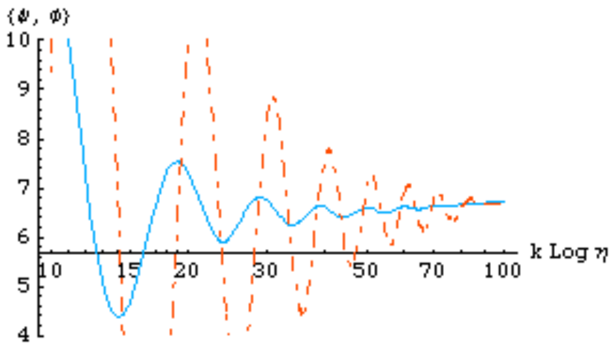
Figure 4.6: Evolution of ψ_U and ϕ_U (Radiation)

Matter In Figs. (4.7(a), 4.7(b),) we plot both the adiabatic and the isocurvature numerical solution we obtain. For comparison, we plot also the figure from [KUY10]. We can see that the behavior of the adiabatic mode agrees with their result in the shape of the potentials, there are some differences in the size, but an exact comparison cannot be made, as there are possible differences in the values of constants taken in the numerical analysis, together with the fact that they are plotting against $ln(a/a_c)$ instead of plotting against $ln(k\eta)$ as we do. As well as in the radiation case, we get a second solution, corresponding to the isocurvature mode. This mode has a decaying wave shape, and it is clearly suppressed compared to the adiabatic one. The difference in the potentials is proportional to the Aether mode, as this decays during the matter epoch the potentials tends to a common value. The isocurvature mode is initially suppressed by a factor of $\sim 10^{-5}$ and has a oscillatory behavior.

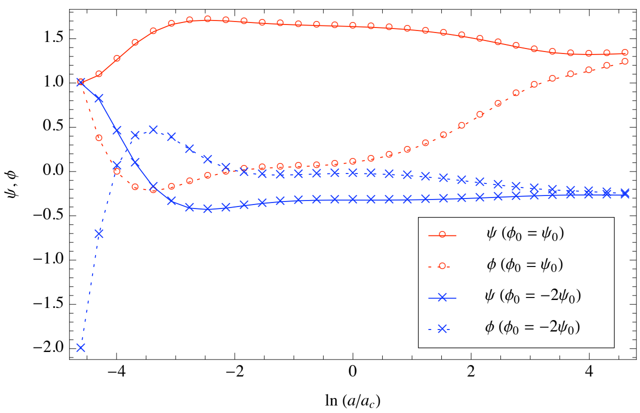
As we expected, the solutions for E-A theory and BPSH agree, although one mode was missing in the study made for the BPSH. Also, we confirm that the anisotropic stress decays on subhorizon scales according to Eq. (4.48).



(a) Evolution of ψ_U and ϕ_U (Adiabatic mode)



(b) Evolution of ψ_U and ϕ_U, scaled by 10^8 (Isocurvature mode)



(c) Evolution of ψ_U, ϕ_U from [KUY10]

Figure 4.7: Evolution of ψ_U and ϕ_U (Matter)

4.2 Vector Perturbations

In a universe dominated by a scalar field there are no vector perturbations. Perfect fluids do support vector perturbations, but they decay as the universe expands. By contrast, the Aether contains a massless vector field (under spatial rotations), which renders vector metric perturbations dynamical. For a certain range of parameters, these modes can grow on large scales, leading to potentially interesting signals, or to constraints on the parameters of Einstein-Aether theories.

4.2.1 Short wavelength stability

As in the case of the scalar modes, we can read off from the action for the vector perturbations whether the vector sector in Einstein-Aether theories is both quantum and classically stable on short scales. Inserting the expansions (4.1) and (4.2) into the action (3.3), expanding to quadratic order in the vectors $\mathbf{S}$ and $\mathbf{V}$, and using the background equations of motion, we obtain the following Lagrangian for the vector perturbations

$$\begin{aligned}\mathcal{L}_v^{(2)} = \frac{M_P^2 a^2}{2}\Big[& -c_{14}\mathbf{V}'^2 + \frac{1}{2}(1+c_{13})\partial_i\mathbf{Q}\cdot\partial^i\mathbf{Q} \\ & + c_1\partial_i\mathbf{V}\cdot\partial^i\mathbf{V} + c_{13}\partial_i\mathbf{Q}\cdot\partial^i\mathbf{V} \\ & + \alpha\left(\mathcal{H}^2-\mathcal{H}'\right)\mathbf{V}^2 + c_{14}(\mathcal{H}^2+\mathcal{H}')\mathbf{V}^2\Big],\end{aligned} \tag{4.79}$$

where we have introduced the gauge-invariant combination

$$\mathbf{Q} \equiv \mathbf{F}' - \mathbf{S} \tag{4.80}$$

(the vector perturbation $\mathbf{V}$ is also gauge-invariant). Note that $\mathbf{Q}$ is not a bona-fide Lagrangian variable, since its definition (4.80) relates it to the time derivative of $\mathbf{F}$. Hence, we shall merely think of it as shorthand for the right hand side of (4.80). Variation of Eq. (4.79) with respect to $\mathbf{S}$ gives the response of the metric to a given perturbation of the Aether field,

$$\mathbf{Q} = -c_{13}\, c_t^2\mathbf{V}. \tag{4.81}$$

In the canonical (first order) formalism, this equation corresponds to the vanishing of the canonical momentum conjugate to $\mathbf{F}$, $\Pi_\mathbf{F} = 0$. Upon substitution of this constraint back into the first order Lagrangian, one is left with a Lagrangian for the single canonical pair formed by $\mathbf{V}$ and its conjugate momentum $\Pi_\mathbf{V}$. Rewriting this reduced Lagrangian back in second order form gives

$$\mathcal{L}_v^{(2)} = \frac{M_P^2}{2}\left[-c_{14}\xi'^2 + \alpha(\mathcal{H}^2-\mathcal{H}')\xi^2 + c_1\left(1-\frac{c_{13}^2c_t^2}{2c_1}\right)\partial_i\xi\cdot\partial^i\xi\right], \tag{4.82}$$

where, for convenience, we have introduced the rescaled variable

$$\xi_i \equiv a V_i. \tag{4.83}$$

The absence of ghosts requires the coefficient in front of ξ'^2 in the Lagrangian (4.82) to be positive, and classical stability requires that the squared speed [JM04]

$$c_v^2 = \frac{c_1}{c_{14}} \left(1 - \frac{c_{13}^2 c_t^2}{2c_1}\right) \tag{4.84}$$

be non-negative. Therefore, stability in the vector sector demands both

$$c_{14} \leqslant 0 \quad \text{and} \quad c_1 \leqslant \frac{c_{13}^2}{2(1+c_{13})}. \tag{4.85}$$

In a Minkowski background, the two modes in the vector sector are massless fields, which we may interpret as two of the Goldstone modes of the broken boost invariance. The broken generators transform as a spatial vector under the unbroken group of spatial rotations, so the corresponding Goldstone bosons transform as a vector. This can be decomposed into a transverse part and a longitudinal part. The longitudinal Goldstone (with helicity zero) is of course part of the scalar sector, which we discussed in the previous section. It should be noted that Lorentz invariance is generically broken in any curved spacetime. For instance, if the spacetime curvature is non-constant, the gradient $\nabla_\mu R$ defines a non-zero vector field which is not invariant under Lorentz-transformations. What is particular about the Einstein-Aether is that the breaking of Lorentz-invariance has physical consequences, namely, the existence of Nambu-Goldstone bosons, whose dispersion relations approach non-relativistic expressions in the high-momentum limit, and whose masses vanish in flat space[6].

4.2.2 Solutions during power-law inflation

Variation of (4.82) with respect to ξ leads to the equation of motion for the vector perturbations,

$$\xi_i'' + c_v^2 \, k^2 \, \xi_i + \frac{\alpha}{c_{14}} \left(\mathcal{H}^2 - \mathcal{H}'\right) \xi_i = 0. \tag{4.86}$$

In terms of the original variable $V_i = \xi_i/a$, we have

$$V_i'' + 2\mathcal{H}V_i' + c_v^2 \, k^2 \, V_i + \left[\left(1 + \frac{\alpha}{c_{14}}\right)\mathcal{H}^2 + \left(1 - \frac{\alpha}{c_{14}}\right)\mathcal{H}'\right] V_i = 0. \tag{4.87}$$

[6]In an arbitrary spacetime, the "mass" of these bosons is non-zero, as illustrated in the case of a FRW universe by the contribution to the effective mass of the last term in the Lagrangian (4.82).

In a universe that undergoes power-law inflation (3.20) this can be solved in terms of Bessel functions and we have

$$V_i = \frac{1}{2M_P}\sqrt{\frac{\pi}{-c_{14}}}\frac{(-\eta)^{1/2}}{a}H_\nu^{(1)}(-c_v k\,\eta)\,e_i. \tag{4.88}$$

Here,

$$\nu = \frac{t_+ - t_-}{2} = \sqrt{\frac{1}{4} - \frac{\alpha q(q+1)}{c_{14}}}, \tag{4.89}$$

where $t_\pm$ is given in (4.35). The parameter q is defined in Eq. (3.20) and e_i is a normalized transverse polarization vector, $\mathbf{e}\cdot\mathbf{k} = 0$ and $\mathbf{e}^2 = 1$. For a given wave number $\mathbf{k}$, there are two such linearly independent polarizations, orthogonal to $\mathbf{k}$. We have chosen the amplitude of V_i in Eq. (4.88) so that the solution has the appropriate normalization of a positive frequency mode in the limit $\eta \to -\infty$. Note that the factor $q(q+1)$ is non negative if the null energy condition is satisfied ($w \geqslant -1$).

The long wavelength power spectrum of vector perturbations created during inflation is defined by

$$\langle V_i(\eta,\mathbf{k})V_j(\eta,\mathbf{k}')\rangle \equiv \frac{2\pi^2}{k^3}\mathcal{P}_V(k,\eta)\,\Pi_{ij}\,\delta(\mathbf{k}-\mathbf{k}'), \tag{4.90}$$

where $\Pi_{ij} = \delta_{ij} - k_i k_j/k^2$ projects onto the subspace orthogonal to $\mathbf{k}$. At the time of reheating η_{rh}, we have

$$\mathcal{P}_V^{\text{rh}}(k) = \mathcal{A}_V^2 \times \left(\frac{k}{k_N}\right)^{n_v}, \tag{4.91a}$$

$$n_v \equiv 3 - 2\nu, \tag{4.91b}$$

$$\mathcal{A}_V^2 \equiv \frac{H_{\text{rh}}^2}{M_P^2}\left(\frac{-1}{c_{14}q^2}\right)\frac{\Gamma^2(\nu)}{(2\pi)^3}\left(\frac{c_v}{2}\right)^{-2\nu}\exp\left(\frac{N n_v}{q}\right). \tag{4.91c}$$

In Eqs. (4.91) H_{rh} is the value of the Hubble constant at the end of inflation, and k_N is the mode that crossed the cosmic horizon N e-folds before the end of inflation ($|k_N\eta_N| = 1$). For the mode that is entering the horizon today, the value of N depends logarithmically on the unknown reheating temperature, and typically equals 50 to 70 e-folds (see for instance [DH03]). It is important to realize that the time at which the spectrum is evaluated matters, since the vector modes do not freeze out at horizon crossing. The superscript "rh" is meant to imply that the power spectrum describes the amplitude of the modes just before the end of inflation. Likewise, we may define the spectrum of the corresponding metric perturbation, which according to Eq. (4.81) is given by

$$\mathcal{P}_Q^{\text{rh}}(k) = \mathcal{A}_Q^2 \times \left(\frac{k}{k_N}\right)^{n_v}, \tag{4.92a}$$

$$\mathcal{A}_Q^2 = c_{13}^2 c_t^4 \mathcal{A}_V^2. \tag{4.92b}$$

In the limit in which de Sitter inflation is approached ($q \to -1$) the index ν tends to $1/2$, so the amplitude of long wavelength perturbations is proportional to $\exp(-2N)$. Hence, velocity perturbations on observationally accessible scales are very small in this limit [Lim05, LMB08]. On the other hand, in typical inflationary models q differs from -1 and the rate of decay can be smaller. In fact, if

$$\alpha > \frac{-2c_{14}}{q(q+1)} = \frac{-2c_{14}}{\epsilon}(1-\epsilon)^2, \tag{4.93}$$

the combination $3-2\nu$ would be negative, and long wavelength perturbations would be amplified exponentially with N, in stark contrast with the de Sitter case. It turns out, however, that we do not need to deviate much from $\nu = 1/2$ in order to have an observable signal. As we shall see, even if the long wavelength velocity field is very tiny at the end of inflation, it may resurface from obscurity during the radiation and matter era, so that it can be quite sizable at the moment of horizon reentry.

Indeed, the behavior of long wavelength vector perturbations is completely analogous to that of the scalar component of the velocity field v_i which we studied in the previous section. To see this, we note that in the long wavelength limit, Eq. (4.87) is the same as Eq. (F.11) for the isocurvature perturbation. The latter is written in terms of the variable C in the longitudinal gauge, which according to Eq. (4.39) is proportional to the longitudinal velocity field of the Aether v_i. Hence, on superhorizon scales, longitudinal and transverse velocity fields satisfy the same equation of motion Eq. (4.87). In particular,

$$V \propto v \propto (k/a)\mathcal{H}^{-1}\delta N \propto \eta^{1-q+t_\pm} \qquad (k\eta \ll 1), \tag{4.94}$$

where $t_\pm$ is given in (4.35). For $\alpha = -c_{14}$, the velocity field decays exponentially during inflation. Nonetheless, as we saw in the previous section, the isocurvature perturbation δN stays frozen on superhorizon scales (except at the transitions where the equation of state changes, where the dominant mode changes also by factors of order one). This can lead to a sizable velocity field v of order δN at horizon reentry. The overall normalization of v and V is different, but it is clear that the relative size of V and v at horizon reentry is determined by their relative size at the time when they exit the horizon during inflation. In other words, the spectra of long wavelength modes are related by

$$\frac{\mathcal{P}_v(\eta,k)}{\mathcal{P}_V(\eta,k)} = \frac{\mathcal{P}_v(\eta_k,k)}{\mathcal{P}_V(\eta_k,k)} \sim Z_N \frac{c_{14}}{M_P^2} \sim \frac{c_{14}}{\epsilon}, \tag{4.95}$$

where the relative normalization can be read off from the corresponding short wavelength actions, and ϵ is the slow roll parameter during inflation. Note that ϵ is of order of a few percent, while the Einstein-Aether parameters such as $c_{14} \sim (M/M_P)^4$ are suppressed by the square of the symmetry breaking scale over the Planck scale. Unless M and M_P are very close, we expect $c_{14} \ll \epsilon$. Therefore, parametrically, we expect that the transverse vectors may give a much bigger contribution than the scalar isocurvature modes. For that reason, it is very important to assess their impact on observables such as the CMB, as we do in Subsection 5.5.2.

One may worry that if the primordial amplitude of vectors due to quantum fluctuations generated at horizon crossing decays during inflation, then it may be insignificant compared with the contribution of non-linear effects which source the vector modes at later times. However, in order to construct a vector from a quadratic expression involving scalars and tensors, it is necessary to use at least one derivative. Because of that, the terms which may source the vectors from the scalar and tensor sector are momentum suppressed, and hence they also decay in inverse proportion to the scale factor. We conclude that if the initial amplitude of the vectors at horizon crossing is sizable, compared to that of scalars and tensors, then we can safely use linear evolution in order to determine its amplitude at the end of inflation (even if that amplitude is exceedingly small). Vector modes can still grow during radiation and matter domination from that initial tiny amplitude, so that their effect on cosmological observables may be important.

4.3 Tensor Perturbations

As discussed in [Lim05] the presence of the Aether modifies the propagator and the dispersion relation of the tensor modes. Substituting Eq. (4.1) into the action (3.3), with matter Lagrangian given by (3.19), expanding to quadratic order in t_{ij} and using the background equations of motion we obtain,

$$\mathcal{L}_t^{(2)} = \frac{M_P^2 a^2}{8}\left[(1+c_{13})\mathbf{t}' \cdot \mathbf{t}' - \partial_i \mathbf{t} \cdot \partial^i \mathbf{t}\right], \tag{4.96}$$

where $\mathbf{t}$ stands for a matrix with components t_{ij} and the dot indicates contraction of both indices (with the Euclidean metric). On short (subhorizon) scales, gravity waves propagate at a speed [JM04]

$$c_t^2 = \frac{1}{1+c_{13}}. \tag{4.97}$$

Classical stability of tensors thus imposes the condition

$$1 + c_{13} > 0, \tag{4.98}$$

since, otherwise, high frequency modes grow exponentially fast.

In the previous chapter we noted that the background solution only exists for $\alpha < 2$, implicitly assuming that the "bare" Newton's constant is positive $G > 0$. Here, we note that for $M_P^2 < 0$, the coefficient in front of the kinetic term of t_{ij} has the "wrong" sign, and the two independent transverse and traceless tensor modes are ghosts.

A theory with ghosts is quantum mechanically unstable. The vacuum can decay by emitting positive energy particle plus negative energy quanta while conserving energy. In a Lorentz-invariant theory, the phase space available for the decay of the vacuum would be infinite, and the lifetime of the vacuum is then infinitely short, which makes the theory unviable. In a non-Lorentz invariant theory, the decay rate may be finite, and the vacuum may be sufficiently long-lived (see for instance [CJM04]). In our case, Lorentz invariance is spontaneously broken, and the effective theory we are using is supposed to be valid only well below the symmetry breaking scale M. The decay rate is UV sensitive, so strictly speaking it is unclear whether the theory can be made sense of in the presence of ghosts. However, to be conservative, we shall systematically exclude from parameter space the cases when ghosts are present.

The equation of motion for the tensor modes just differs from the one of GR (5.39) in the propagation speed

$$h'' + \left(c_t^2 k^2 - \frac{a''}{a}\right) h = 0, \tag{4.99}$$

where $t_{ij} = \frac{h}{a} e_{ij}$.

The primordial spectrum of tensor modes seeded during inflation is immediately obtained from (4.96), and is inversely proportional to their propagation speed,

$$\mathcal{P}_t(k) = \frac{1}{\pi^2 c_t} \frac{H^2}{M_P^2}\bigg|_{c_s k = \mathcal{H}}. \tag{4.100}$$

Hence, the amplitude of the primordial tensor modes differs from that in General Relativity (for the same values of H and M_P).

Chapter 5

CMB Anisotropies

The Cosmic Microwave Background (CMB) provides us with a lot of useful information about the evolution of the universe and the behavior of gravity. What we know from these photons that have freely travelled along the universe since the last scattering surface is, first of all, that they all have the same temperature, around 2.7 K. But, once we can look further into the details of these background radiation, we can see that there are small anisotropies, of order 10^{-5}, and that these anisotropies can provide information about the cosmological perturbations.

The large angular scale anisotropies correspond to perturbations that haven't entered the horizon at recombination time. These scales haven't changed much and provide information about the primordial spectrum of perturbations. On the other hand, the small scale perturbations have entered the horizon before recombination and gravitational instabilities have a big effect on them. They provide information about the cosmological parameters that control the change of perturbation amplitudes. The large scales are dominated by gravitational effects and are characterized by a nearly scale invariant spectrum. At intermediate scales we encounter the acoustic oscillations, whose amplitude and distance between peaks strongly depends on cosmological parameters. Finally, anisotropies on small scales are erased by Silk damping and free streaming. Here, we overview the main characteristics of the CMB anisotropies in the case of General Relativity in order to better understand the differences we could observe in the E-A theories. For more details see for example [Muk05, Wei08, LL09, Dur01, Muk04]. To analyze large angular scales we may assume that the last scattering was an instantaneous process, but in the case of small scales it is necessary to be more careful, as the duration of recombination affects the anisotropies we will observe. Both temperature and polarization spectra can be calculated numerically in GR for different values of cosmological parameters using numerical codes publicly available such as CMBFAST [SZ96].

Once the basics of the CMB anisotropies were established we will focus on the vector perturbations for Einstein-Aether theories. As vector contribution in GR is absent, this constitutes the most interesting feature that these theories possess to differentiate from GR. We will analyze its contribution to the temperature anisotropies and to the polarization modes in the CMB and compare our results with the contribution coming from tensor modes.

5.1 Boltzmann equation and Sachs-Wolfe effect

Assuming that recombination was an instantaneous process, one can consider that before recombination radiation behaves as a perfect fluid and after, as an ensemble of free photons. The free propagating photons are described by the distribution function n

$$n = n\left(\frac{E}{T}\right), \tag{5.1}$$

where $E = -p_\mu u^\mu$ is the energy of a photon as measured by an observer at rest in the coordinates $\mathbf{x}$. The temperature depends not only on position but also on the direction of arrival of the photons and on the moment of time

$$T = T_0(\eta) + \delta T(\eta, x^\mu, l^i), \tag{5.2}$$

with T_0 the unperturbed temperature and $l_i \equiv p_i/p$ where $p \equiv (\delta^{ij} p_i p_j)$. Making a coordinate transformation $\tilde{\mathbf{x}} = \mathbf{x} + \xi$ we can see that the temperature fluctuation in the new coordinate system is

$$\tilde{\delta T} = \delta T - T_0'\xi^0 + T_0 {\xi^i}' l^i, \tag{5.3}$$

and so the monopole (l^i-independent) and the dipole (proportional to l^i) depend on the coordinate system in which the observer is at rest. Then, in our case, the monopole can be removed by a redefinition of the background, and the dipole depends of the motion of the observer with respect to a "preferred frame". These cannot provide much information about initial fluctuations and we will be interested in higher-order multipoles only.

If we consider scalar metric perturbations up to first order, we get

$$p^0 = \frac{p}{a^2}\frac{1+2\psi}{1+2\phi} \simeq \frac{p}{a^2}(1+\psi-\phi), \tag{5.4a}$$

$$p_0 = -\, a^2(1+2\phi)p^0 \simeq -p(1+\psi+\phi), \tag{5.4b}$$

using $p_\mu p^\mu = 0$, and

$$n = \frac{E}{T} = -\frac{p_0}{\sqrt{-g_{00}}T} = \frac{p}{aT_0}\frac{1+\psi+\phi}{\left(1+\frac{\delta T}{T_0}\right)(1+2\phi)^{1/2}}$$

$$= \frac{E_0}{T_0}\left(1 + \psi - \frac{\delta T}{T_0}\right). \tag{5.5}$$

The Boltzmann equation for the free gas of photons is

$$\frac{\partial n}{\partial \eta} + \frac{\partial n}{\partial x^i}\frac{dx^i}{d\eta} + \frac{\partial n}{\partial p_i}\frac{dp_i}{d\eta} = 0. \tag{5.6}$$

From the geodesic equations

$$\frac{dp^0}{d\eta} = -\, p(\phi' + \psi'), \tag{5.7a}$$

$$\frac{dp^i}{d\eta} = -\, p(\phi + \psi)_{,i}, \tag{5.7b}$$

$$\frac{dx^i}{d\eta} = \frac{p^i}{p^0} = \frac{(1+2\psi)a^{-2}\delta^{ij}p_j}{p^0} = \frac{(1+2\psi)\delta^{ij}l_j}{1+\psi-\phi} \sim (1+\psi+\phi)l^i, \tag{5.7c}$$

and substituting in (5.6)

$$\frac{\partial n}{\partial \eta} + (1+\psi+\phi)l^i\frac{\partial n}{\partial x^i} - p(\phi+\psi)_{,i}\frac{\partial n}{\partial p_i} = 0. \tag{5.8}$$

Taking into account that $\partial_\eta(E_0/T_0) = \partial_{x^i}(E_0/T_0) = 0$, and $\partial_{p_i}(E_0/T_0) = (l_i/p)(E_0/T_0)$, and up to first order in perturbations

$$\left(\frac{\partial}{\partial \eta} + l^i\frac{\partial}{\partial x^i}\right)\left(\frac{\delta T}{T_0} - \psi\right) + l_i(\phi+\psi)_{,i} = 0, \tag{5.9}$$

that can be rewritten as

$$\left(\frac{\partial}{\partial \eta} + l^i\frac{\partial}{\partial x^i}\right)\left(\frac{\delta T}{T_0} + \phi\right) = \frac{\partial}{\partial \eta}(\phi+\psi). \tag{5.10}$$

These equations simplify for General Relativity, where $\psi = \phi$,

$$\left(\frac{\partial}{\partial \eta} + l^i\frac{\partial}{\partial x^i}\right)\left(\frac{\delta T}{T_0} + \phi\right) = 2\frac{\partial}{\partial \eta}\phi. \tag{5.11}$$

If the universe is matter-dominated after recombination, we can consider that $\phi \sim$ *constant* and then

$$\left(\frac{\delta T}{T_0} + \phi\right) = constant. \tag{5.12}$$

This influence of the gravitational potential in the CMB fluctuations is what is known as the Sachs-Wolfe effect. In reality, the gravitational potential is not exactly constant. If we take into account its slowly time-varying character, then this change in the potential will produce a contribution to the fluctuations called early Integrated Sachs-Wolfe effect. The late Integrated

Sachs-Wolfe effect is the contribution due to variations of the potential at late times, during dark energy domination. Today, $\frac{\delta T}{T}(\eta_0, x_0^i, l^i)$ is given by

$$\frac{\delta T}{T}(\eta_0, x_0^i, l^i) = \frac{\delta T}{T}(\eta_{rec}, x_{rec}^i, l^i) + \phi(\eta_{rec}, x_{rec}^i) - \phi(\eta_0, x_0^i), \tag{5.13}$$

so the angular dependence is given by the initial temperature fluctuations at recombination time and the value of the gravitational potential at that time; the last term can be ignored as it only contributes to the monopole. The first contribution can be expressed in terms of the gravitational potential and the fluctuations of the energy density of photons. A more detailed calculation of this contribution is done for the vector case in the following section. We will just quote the result for the scalar sector (see for example [Muk05])

$$\left(\frac{\delta T}{T}\right)_{\mathbf{k}}(\eta_{rec}, \mathbf{l}) = \frac{1}{4}\left(\delta_{\mathbf{k}} + \frac{3\,i}{k^2}(k_m l^m)\delta'_{\mathbf{k}}\right). \tag{5.14}$$

The final expression for temperature fluctuations at present time is then

$$\frac{\delta T}{T}(\eta_0, x_0^i, l^i) = \int \left[\left(\phi + \frac{\delta}{4}\right)_{\mathbf{k}} - \frac{3\delta'_{\mathbf{k}}}{4k^2}\frac{\partial}{\partial \eta_0}\right]_{\eta_{rec}} e^{i\mathbf{k}(\mathbf{x}_0 + \mathbf{l}(\eta_{rec}-\eta_0))}\frac{d^3k}{(2\pi)^{3/2}}, \tag{5.15}$$

where δ are the fluctuations of the photon energy density, $k \equiv |\mathbf{k}|$, $\mathbf{k}\cdot\mathbf{l} \equiv k_m l^m$ and $\mathbf{k}\cdot\mathbf{x}_0 \equiv k_n x_0^n$.

5.2 CMB anisotropies in the vector sector

As we have just done for the scalar modes, we are going to apply a similar procedure to the vector perturbations. The effect of vector perturbations on the amplitude of CMB anisotropies is easily estimated in the same approximation of a sharp transition between thermal equilibrium and complete transparency at the moment of decoupling. Again, before the transition, photons and baryons are approximated as a perfect fluid, whereas after the transition the radiation will be described in terms of a distribution of free photons.

The number of photons in a phase space cell can be written as

$$dn = n(\mathbf{x}, \mathbf{p}) \prod_k dx^k \prod_i dp_i, \tag{5.16}$$

where x^k are space coordinates and p_k are the spatial components of the momentum. For a gas of free photons, the number density in phase space obeys the collisionless Boltzmann equation (5.6). Further, we assume that the distribution of photons traveling in a given direction $\mathbf{l}$ at any given point has the Planckian spectrum (5.1) where again

$$E = -p_\mu u^\mu = -a^{-1} p_0 \tag{5.17}$$

is the energy of a photon as measured by an observer at rest in the coordinates $\mathbf{x}$. The four-velocity of this observer is given by $u^\mu = (-g_{00})^{-1/2}\delta^\mu_0$, and in the last equality we have used that $-g_{00} = a^2$ is unperturbed in the linearized vector sector. Before decoupling, when the system is in thermal equilibrium, the temperature anisotropy is just a dipole, corresponding to the local motion of the photon fluid. This is characterized by the four-velocity δu^μ. Note that n is a scalar, and so T is defined in such a way that the ratio $y \equiv E/T$ transforms as a scalar. In the co-moving frame, where the fluid is at rest, we have

$$y = \frac{E_c}{T_c} = \frac{-(u^\mu + \delta u^\mu)p_\mu}{T_c} = \frac{E - \delta u^i p_i}{T_c}, \tag{5.18}$$

where the co-moving temperature $T_c = T_0(1 + \delta_0(\eta, \mathbf{x}))$ is isotropic, and we have used $\delta u^0 = 0$ (to linear order in δu^i). Since $y = E/T = E_c/T_c$, it follows from (5.18) that at the time of decoupling

$$\frac{\delta T}{T_0}(\eta_{\rm dec}, \mathbf{x}, \mathbf{l}) = \delta_0 + a\ \delta u^i\ l_i. \tag{5.19}$$

Later, after decoupling, the photons arriving from different directions at a given spacetime point have originated at different locations on the surface of last scattering, which leads to anisotropies also in the higher multipoles.

The monopole and dipole components in (5.19) are related to the perturbations in T^0_0 and T^0_i, which can be obtained from the expression

$$T^\mu_\nu = \frac{1}{\sqrt{-g}} \int n(y) \frac{p^\mu p_\nu}{p^0} d^3\mathbf{p}. \tag{5.20}$$

Here $\mathbf{p}$ stands for the spatial components of the momentum, with lower indices. Let us consider the perturbation in the energy density. This will be related to the monopole component in the temperature anisotropy. For vector perturbations,

$$a^{-2}\delta g_{0i} \;=\; a^2 \delta g^{0i} = S^i, \tag{5.21}$$
$$a^{-2}\delta g_{ij} \;=\; -a^2 \delta g^{ij} = (F^{i,j} + F^{j,i}), \tag{5.22}$$

the linearized metric determinant is $\sqrt{-g} = a^4$, and the condition $p_\mu p^\mu = 0$ leads to

$$p_0 = -p(1 - S^i l_i - F^{i,j}\ l_i l_j). \tag{5.23}$$

The energy density of photons is given by

$$\rho_\gamma = -T^0_{\ 0} = -\frac{1}{a^4}\int n(y)\, p_0\, p^2\, dp\, d^2\mathbf{l}. \tag{5.24}$$

We can now use that $p_0 = -a\, T_0\, y\, (1 + \delta T/T)$ and $p = a\, T_0\, y\, (1 + S^i l_i + F^{i,j}\, l_i l_j + \delta T/T)$ to eliminate p and p_0 in favor of y. After simple manipulations, one obtains

$$\rho_\gamma = \rho_\gamma^{(0)}\left[1 + 4\int \frac{d^2\mathbf{l}}{4\pi}\frac{\delta T}{T}\right]. \tag{5.25}$$

Since vector perturbations do not change the energy density, we have $\delta\rho_\gamma = 0$. Therefore, using (5.19) in (5.25) we find $\delta_0 = 0$. Hence, the temperature anisotropy (5.19) for vector perturbations in the perfect fluid is purely dipolar:

$$\frac{\delta T}{T_0}(\eta_{\rm dec}, \mathbf{x}, \mathbf{l}) = -S^i l_i + \frac{\delta u_i}{a}\, l_i. \tag{5.26}$$

For later convenience, here we have expressed the result in terms of the velocity perturbation with lower indices, which is gauge-invariant.

The evolution of the temperature anisotropy after decoupling can be inferred from the Boltzmann equation. Defining

$$E_0 = p/a, \tag{5.27}$$

we have $\partial_\eta(E_0/T_0) = \partial_{x^k}(E_0/T_0) = 0$, and $\partial_{p_k}(E_0/T_0) = (l_k/p)(E_0/T_0)$. Substituting (5.1) in (5.6), and linearizing in perturbations, it is straightforward to show that

$$\left(\frac{\partial}{\partial\eta} + l_k\frac{\partial}{\partial x^k}\right)\left(\frac{\delta E}{E_0} - \frac{\delta T}{T_0}\right) + \frac{l_k}{p}\frac{dp_k}{d\eta} = 0, \tag{5.28}$$

where $\delta E = E - E_0 = -(p_0 + p)/a$. The geodesic equation reads

$$\frac{dp_k}{d\eta} = \frac{1}{2p^0}\frac{\partial g_{\mu\nu}}{\partial x^k}p^\mu p^\nu = S^i{}_{,k}\, p_i + F^{i,j}{}_{,k}\,\frac{p_i p_j}{p}. \tag{5.29}$$

Using (5.23) and (5.29) in (5.28) we have

$$\frac{d}{d\eta}\left(\frac{\delta T}{T_0} + \mathbf{F}'\cdot\mathbf{l}\right) = \mathbf{Q}'\cdot\mathbf{l}. \tag{5.30}$$

Here, $d/d\eta = \partial_\eta + l_i\partial_{x^i}$ is the total derivative along the line of sight, and primes indicate partial derivatives with respect to η. The result is expressed in terms of the gauge-invariant combinations $(\delta T/T_0) + \mathbf{F}'\cdot\mathbf{l}$ and $Q^i \equiv F^{i\prime} - S^i$. Eq. (5.30) can be integrated along the trajectory of the photons

$\mathbf{x}(\eta) = (\eta - \eta_0)\mathbf{l}$, from the time of decoupling $\eta_{\rm dec}$ to the present time η_0, to obtain the temperature anisotropy which is observed at present:

$$\left(\frac{\delta T}{T_0} + \mathbf{F}' \cdot \mathbf{l}\right)_0 = \left(Q^i l_i + \frac{\delta u_i}{a}\, l_i\right)_{\rm dec} + \int_{\eta_{\rm dec}}^{\eta_0} d\eta\; \mathbf{Q}' \cdot \mathbf{l}. \tag{5.31}$$

Here we have used the initial condition determined by (5.26). As we mention in Subsection 5.5.1, a non-vanishing velocity perturbation δu_i cannot be generated as long as the perfect fluid description is valid, so we shall ignore δu_i in Eq. (5.31), and simply write

$$\left(\frac{\delta T}{T_0}\right)_0 = (\mathbf{Q} \cdot \mathbf{l})_{\rm dec} + \int_{\eta_{\rm dec}}^{\eta_0} d\eta\; \mathbf{Q}' \cdot \mathbf{l}. \tag{5.32}$$

Here we have also dropped the dipole term at the time of observation, since this is always subtracted.

5.3 Correlation functions

If the spectrum of temperature of the CMB is Gaussian, it can be characterized by the two-point correlation function

$$C(\theta) \equiv \left\langle \frac{\delta T}{T_0}(\mathbf{l}_1) \frac{\delta T}{T_0}(\mathbf{l}_2) \right\rangle, \tag{5.33}$$

where $l_1 \cdot l_2 = \cos\theta$. Substituting Eq. (5.15) we can rewrite this expression as a discrete sum over multipole moments C_l (the monopole and dipole are excluded)

$$C(\theta) = \frac{1}{4\pi} \sum_{l=2}^{\infty} (2l+1) C_l P_l(\cos\theta), \tag{5.34}$$

where $P_l(\cos\theta)$ are the Legendre polynomials. Finally, the multipole moments are (see [Muk05])

$$C_l = \frac{2}{\pi} \int \left| \left(\phi_k(\eta_{rec}) + \frac{\delta_k(\eta_{rec})}{4}\right) j_l(k\eta_0) - \frac{3\delta_k'(\eta_{rec})}{4} \frac{dj_l(k\eta_0)}{d(k\eta_0)} \right|^2 k^2 dk, \tag{5.35}$$

where $j_l(k\eta)$ are the spherical Bessel functions of order l.

At the time of recombination the Hubble radius corresponds to an angular size of 1°. The analysis of the modes that enter the horizon long before or after the time of recombination can be understood qualitatively by using suitable approximations.

Large angular scales

The large angular scales correspond to angles $\theta \gg 1^\circ$. As they exceed the Hubble radius at recombination, they have not evolved since the end of inflation, providing information about the primordial inhomogeneities. The perturbations at this scale are dominated by gravitational effects. They lead to a nearly flat plateau, $l(l+1)C_l \sim \textit{constant}$ for $l \ll 200$. As we approach this value of l we start to see the acoustic peaks. They are caused by the acoustic oscillations of the photon-baryon fluid.

Small angular scales

In the case of short angular scales we cannot consider recombination to be an instantaneous process. We have to take into account a certain duration for the process, that is, we place the photon last scattering in a range $1200 > z > 900$. This is the finite thickness effect, that leads to a suppression of the temperature fluctuations at small angular scales. The corresponding l's for these scales are $l > 200$ and we can use the approximation of $l \gg 1$ in the calculations of the multipole moments. In order to account to this we calculate the probability that the photon was scattered in the time interval $\Delta\eta_L$ at time η_L

$$dP(\eta_L) = \mu'(\eta_L) e^{-\mu(\eta_L)} d\eta_L, \tag{5.36}$$

where we have defined the optical depth

$$\mu(\eta_L) \equiv \int_{\eta_L}^{\eta_0} \sigma_T n_t X a(\eta) d\eta, \tag{5.37}$$

where n_t is the number density of electrons and X the ionization fraction. Then, the modified expression is

$$\frac{\delta T}{T} = \int \left[\left(\phi + \frac{\delta}{4} \right)_{\mathbf{k}} - \frac{3\delta'_{\mathbf{k}}}{4k^2} \frac{\partial}{\partial \eta_0} \right]_{\eta_L} e^{i\mathbf{k}(\mathbf{x}_0 + \mathbf{l}(\eta_L - \eta_0))} \mu' e^{-\mu} d\eta_L \frac{d^3k}{(2\pi)^{3/2}}. \tag{5.38}$$

Calculations of the spectrum are now more complicated and we are not interested here in the details of the calculations. In the references mentioned before it is possible to follow the complete analysis. At large l's the spectrum is also damped by the diffusion damping (Silk damping).

5.4 Gravitational waves

The equations of motion for tensor modes are quite simple and easy to analyze at all scales since they are decoupled. The equation

$$h'' + \left(k^2 - \frac{a''}{a} \right) h = 0, \text{ where } \quad t_{ij} = \frac{h}{a} e_{ij}, \tag{5.39}$$

has an exact solution at radiation domination epoch but we can just have a look at the two limits.

The non-decaying mode for $k\eta \ll 1$ (long wavelengths) is constant during this time. On the other side, for wavelengths smaller than the Hubble radius the amplitude of the mode decays in inverse proportion to the scale factor. Therefore, gravitational waves (GW) generated during inflation that enter the horizon after recombination, but well before present time, have a nearly flat spectrum at $\eta = \eta_{rec}$,

$$\left|t_k^2(\eta_{rec}k^3)\right| \sim B_{gw} \sim constant. \tag{5.40}$$

Inside horizon, for modes $k\eta_{eq} \gg 1$, the spectrum will be significantly modified,

$$\left|t_k^2(\eta_{rec}k^3)\right| \sim B_{gw}\left(\frac{1}{k\eta_{eq}}\right)^2\left(\frac{z_{rec}}{z_{eq}}\right)^2. \tag{5.41}$$

The gravitational Sachs-Wolfe effect is given by

$$\frac{\delta T}{T} = -\frac{1}{2}\int_{\eta_{rec}}^{\eta_0} e_i e_j \frac{\partial t_{ij}}{\partial \eta} d\eta, \tag{5.42}$$

and the correlation function (for a detailed calculation on how to arrive to this formula see e.g. [Dur01])

$$C_l^T = \frac{(l-1)l(l+1)(l+2)}{2\pi}\int dk k^2 \left\langle \int_{\eta_{rec}}^{\eta_0} \left| t'_k \frac{j_l(k(\eta_0-\eta)}{(k(\eta_0-\eta))^2} d\eta \right|^2 \right\rangle. \tag{5.43}$$

We will introduce the new variable $x = k(\eta_0 - \eta)$, and for $l \gg 1$ and $k\eta_0 \gg 1$ (see for example [Muk05])

$$C_l^T \simeq \frac{(l-1)l(l+1)(l+2)}{2\pi}\int_0^\infty \left|t_k^2(\eta_{rec}k^3)\right| \frac{j_l^2(x_0)}{x_0^5} dx_0, \tag{5.44}$$

where $j_l(x)$ is the spherical Bessel function and $x_0 = k\eta_0$. Considering the case where the spectrum is approximately constant we have

$$l(l+1)C_l^T \sim B_{gw}. \tag{5.45}$$

When we consider the modified spectrum for $k \gg \eta_{eq}^{-1}$ we see that

$$l(l+1)C_l^T \sim B_{gw}\left(\frac{l_{eq}}{l}\right)^2, \tag{5.46}$$

with $l_{eq} = \frac{\eta_0}{\eta_{eq}}$. This behavior can be clearly seen in Fig. (5.1), where the effect of the acoustic peaks for large l becomes visible. In the case of E-A theories, we will just have a different amplitude, that depends on the values of the parameters of the theory (see Eq. (4.100)).

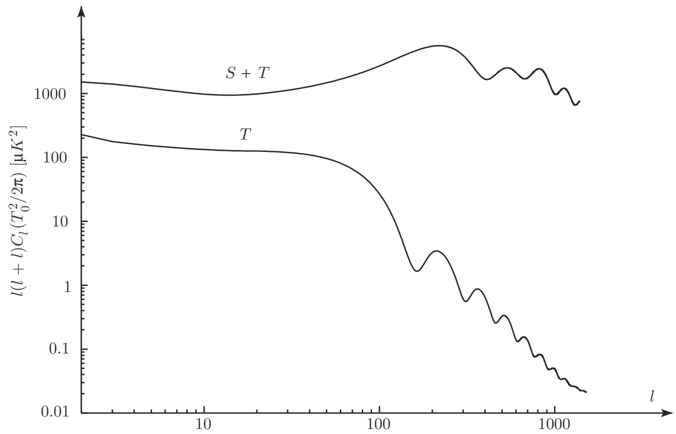


Figure 5.1: CMB temperature produced by tensor perturbations calculated using a numerical code for the case of the standard model (from [Muk05]).

5.5 Vector contribution to the power spectrum of the CMB

As we discussed above, the power spectrum of the transverse velocity field may easily dominate over that of the longitudinal component. It is therefore of interest to determine the imprint that this power spectrum may have on CMB observations, to which this section is devoted.

5.5.1 Solutions during radiation and matter domination

In order to find the power spectrum of the vector modes at the time of recombination, we must first evolve it from the time of thermalization through the radiation and matter dominated epochs. For a set of perfect fluids which do not interact with each other, the conservation equation $\nabla_\mu T_i^{\mu\ (k)} = 0$ holds for each fluid component, which we label by (k). This leads to a homogeneous equation for the gauge-invariant velocity perturbation $\delta u_i^{(k)}$ that does not contain metric perturbations,

$$\frac{\partial}{\partial\eta}\left[a^3(\rho_k + p_k)\delta u_i^{(k)}\right] = 0, \tag{5.47}$$

from where it follows that $\delta u_i/a \propto 1/[a^4(\rho + p)]$. Eq. (5.47) tells us that if $\delta u_i = 0$ initially, then it will not be generated as long as the perfect fluid description is valid. Furthermore, $\delta u_i/a$ decreases during cosmological evolution, except in the radiation dominated stage, where it stays constant. Hence, in what follows, we assume that $\delta u_i^{(k)} = 0$ for matter and radiation. Then, the only contribution to the metric perturbations stems from the

Aether, and it can be shown that the equations of motion in the vector sector are still given by Eqs. (4.81) and (4.86).

The general solution of Eq. (4.86) during a stage of cosmic expansion in which $a \propto \eta^q$ is proportional to a linear combination of Bessel functions. For our present purposes, it will suffice to work with the long wavelength approximations

$$\xi = \mathcal{A}_+ \left(\frac{\eta}{\eta_*}\right)^{\frac{1}{2}+\nu} + \mathcal{A}_- \left(\frac{\eta}{\eta_*}\right)^{\frac{1}{2}-\nu}, \tag{5.48}$$

where ν is given by Eq. (4.89) and $\xi_i \equiv \xi \cdot e_i$. For real values of ν, $\mathcal{A}_+$ is the amplitude of the dominant mode, and $\mathcal{A}_-$ is the amplitude of the subdominant mode.

In order to determine the mode amplitudes $\mathcal{A}^r_+$ and $\mathcal{A}^r_-$ during radiation domination, we simply demand that ξ and its time derivative be continuous at a sudden transition from inflation to radiation domination. We expect this approximation to be valid for scales much longer than the duration of reheating. Proceeding in this manner, and dropping the contribution from the subdominant mode we find that the amplitude of the dominant mode changes during reheating by a factor

$$\frac{\mathcal{A}^r_+}{\mathcal{A}^i_+} = \frac{\nu^i + \nu^r}{2\nu^r}, \tag{5.49}$$

where the superscripts label the expansion epoch (i for inflation and r for radiation domination), and the subscripts label the different modes ($+$ for the dominant mode and $-$ for the subdominant one.)

Eq. (4.86) has an exact solution at long wavelengths during radiation and matter domination, which we can use to determine the change in the mode amplitudes during the transition from radiation to matter domination. Since this change is typically of order one, we shall neglect it, and assume that the amplitude of the growing mode at the transition remains unchanged.

Once a mode enters the "sound horizon", $c_v k\eta = 1$, the field starts oscillating. In the limit $c_v k\eta \gg 1$, the solution of Eq. (4.86) that approaches the growing mode at early times is

$$\xi(\eta) = \mathcal{A}_+ \mathcal{C}_{\text{osc}} \cos\left[c_v k\eta - \varphi\right], \quad \text{where} \tag{5.50a}$$

$$\mathcal{C}_{\text{osc}} = \frac{c_{13}(\nu+1)}{\sqrt{\pi}} \left(\frac{2}{c_v k\eta_*}\right)^{\frac{1}{2}+\nu}, \tag{5.50b}$$

and φ is k-independent phase. Note that the amplitude of the oscillations $\mathcal{A}_+\mathcal{C}_{\text{osc}}$ is roughly the value of ξ at horizon entry.

Collecting then the results from Eqs. (4.83), (5.48) and (5.49) we find that during matter domination the transfer function for the vector perturbations $\mathcal{T}_k$, which we implicitly define by the relation $\mathbf{Q}(\eta) = \mathcal{T}_k(\eta)\mathbf{Q}(\eta_{\text{rh}})$

is

$$\mathcal{T}_k \approx T \times \begin{cases} \left(\frac{a_{\rm eq}}{a_{\rm rh}}\right)^{\frac{1}{2}+\nu^r} \left(\frac{\eta}{\eta_{\rm eq}}\right)^{\frac{1}{2}+\nu^m}, & k\eta_{\rm eq} \ll k\eta \ll c_v^{-1}, \\ \left(\frac{a_{\rm eq}}{a_{\rm rh}}\right)^{\frac{1}{2}+\nu^r} \mathcal{C}^m_{\rm osc} \cos(c_v k\eta), & k\eta_{\rm eq} \ll c_v^{-1} \ll k\eta, \\ \mathcal{C}^r_{\rm osc} \cos(c_v k\eta), & c_v^{-1} \ll k\eta_{\rm eq} \ll k\eta, \end{cases} \tag{5.51a}$$

where

$$T = \frac{a_{\rm rh}}{a} \frac{\nu^i + \nu^r}{2\nu^r}, \tag{5.51b}$$

and in $\mathcal{C}^p_{\rm osc}$ the transition time η_* equals $\eta_{\rm rh}$ for $p = r$ and $\eta_{\rm eq}$ for $p = m$. The first line in Eq. (5.51a) holds for those modes that have not entered the sound horizon at time η. The second applies to those which enter between the time $\eta_{\rm eq}$ of equality between matter and radiation densities and time η, and the third one to the ones which enter between reheating and the time of equality. The power spectrum of Q at any time after reheating is given by

$$\mathcal{P}_Q(\eta) = |\mathcal{T}_k(\eta)|^2 \mathcal{P}_Q^{\rm rh}. \tag{5.52}$$

5.5.2 Impact on the CMB

We derive in Section 5.2 the contribution of vector perturbations to the temperature anisotropies in the cosmic microwave background radiation. In order to determine the angular power spectrum and relate it to the primordial spectrum, it is convenient to rewrite Eq. (5.32) in Fourier space,

$$\left(\frac{\delta T}{T_0}\right)_0 = e^{-i\mathbf{k}\cdot\mathbf{l}\eta_0} \int d^3k \Bigg[\mathbf{l}\cdot\mathbf{Q}(\eta_{\rm dec},\mathbf{k}) \exp\left(i\mathbf{k}\cdot\mathbf{l}\,\eta_{\rm dec}\right) + \int_{\eta_{\rm dec}}^{\eta_0} d\eta\, \mathbf{l}\cdot\mathbf{Q}'(\eta,\mathbf{k}) \exp\left(i\mathbf{k}\cdot\mathbf{l}\,\eta\right)\Bigg]. \tag{5.53}$$

The contribution of the two terms on the right-hand-side of (5.53) is similar to that of scalar perturbations. The first term is the analogue of the Sachs-Wolfe effect, which relates the temperature anisotropies to the state of the perturbations at last scattering. The second term is the analogue of the integrated Sachs-Wolfe effect, which takes into account the change of the metric potentials along the line of sight, and vanishes if the latter are constant.

The angular power spectrum C_ℓ is defined by the relation

$$\left\langle \frac{\delta T}{T_0}(\hat{n}) \frac{\delta T}{T_0}(\hat{m}) \right\rangle \equiv \sum_\ell C_\ell \frac{2\ell+1}{4\pi} P_\ell(\hat{n}\cdot\hat{m}), \tag{5.54}$$

where $\hat{n}$ and $\hat{m}$ are two directions on the sky, and the P_ℓ are Legendre polynomials. Because scalar, vector and tensor perturbations are uncorrelated, their contributions to the temperature anisotropies add in quadrature, $C_\ell = C_\ell^s + C_\ell^V + C_\ell^t$. Inserting Eq. (5.53) into the left-hand-side of Eq. (5.54), using Eq. (4.90), and comparing to the right-hand-side of Eq. (5.54) we find after some work that the contribution of vector perturbations to the angular power spectrum is given by

$$C_\ell^V = 4\pi\ell(\ell+1)\int \frac{dk}{k}\,\mathcal{P}_Q^{\rm rh}\,|\mathcal{N}-\mathcal{I}|^2\,, \tag{5.55a}$$

$$\mathcal{N} \equiv \mathcal{T}_k\left(\eta_{\rm dec}\right)\frac{j_\ell(x_{\rm dec})}{x_{\rm dec}}, \tag{5.55b}$$

$$\mathcal{I} \equiv \int_0^{x_{\rm dec}} dx\,\frac{d\mathcal{T}_k}{dx}\frac{j_\ell(x)}{x}. \tag{5.55c}$$

In these equations $x \equiv k(\eta_0 - \eta)$, the j_ℓ are the spherical Bessel functions of the first kind, the primordial spectrum is given by Eqs. (4.92) and the transfer function $\mathcal{T}_k$ by Eqs. (5.51). After an integration by parts, Eqs. (5.55) agree with the expression derived in [HW97] by somewhat different methods. Note that x is the ratio of the comoving distance to time η divided by the wavelength of the perturbation $1/k$. Thus, x is the inverse of the angle that an object of comoving size $1/k$ at (comoving) distance $\eta_0 - \eta$ would subtend on the sky at time η_0. It will be useful to consider separately those modes that enter well before and well after decoupling. These contribute to the temperature anisotropies, respectively, on small and large angular scales.

The structure of expressions (5.55) still reflects the two contributions to the temperature anisotropies we mentioned above. The value of $\mathcal{N}$ captures the analogue of the Sachs-Wolfe effect, while the value of the integrated term $\mathcal{I}$ captures the analogue of the integrated Sachs-Wolfe effect. For scalar perturbations, the Sachs-Wolfe effect is dominant except on the largest scales, because the gravitational potential remains constant until relatively recently. For vector perturbations however, this is not always the case. To see this, it is useful to realize that we can employ the same approximations developed to study the contribution of tensor modes to the temperature anisotropies [Muk05]. We begin by noting that, for $\ell \gg 1$, the Bessel function can be approximated by [GR80]

$$j_\ell(x) \approx \begin{cases} 0, & x < \ell + \frac{1}{2} \\ \dfrac{\cos\left[\sqrt{y} - (\ell+\frac{1}{2})\arccos\left(\frac{\ell+\frac{1}{2}}{x}\right) - \frac{\pi}{4}\right]}{x^{1/2}y^{1/4}}, & x > \ell + \frac{1}{2} \end{cases} \tag{5.56}$$

where $y \equiv x^2 - (\ell+1/2)^2$. Because the integrand is negligible for $x < \ell+1/2$, only modes that have entered the horizon by today, $x_{\rm dec} \approx k\eta_0 \gtrsim \ell + 1/2$ can contribute to the temperature anisotropies on angular scales $\ell \gg 1$.

Large Angular Scales

Large angular scales correspond to modes that cross the sound horizon after decoupling. Let us estimate the contribution of the integrated term $\mathcal{I}$ on those scales first. From Eq. (5.51a), the derivative of the transfer function $d\mathcal{T}_k/dx$ is dominant during the interval $x_k \lesssim x \lesssim x_{\rm deq}$, where $x_k \equiv k(\eta_0 - \eta_k)$ corresponds to the time of sound horizon crossing $\eta_k \equiv 1/(c_v k)$. After horizon crossing, this function oscillates with period $2\pi/c_v$ and a slowly varying amplitude. On the other hand, the ratio $j_l(x)/x$ changes slowly in the interval $\Delta x \lesssim 1 \ll l \lesssim x$. Assuming that c_v is not much smaller than 1, we have $\Delta x = x_{\rm deq} - x_k \approx k\eta_k = 1/c_v \lesssim 1$, and we can pull the factor $j_l(x)/x$ out of the integral. What remains is a boundary term that can be readily evaluated,

$$\mathcal{I} \equiv \int_0^{x_{\rm dec}} dx \, \frac{d\mathcal{T}_k}{dx} \frac{j_\ell(x)}{x} \approx \frac{j_\ell(x_{\rm dec})}{x_{\rm dec}} \left[\mathcal{T}_k(\eta_{\rm dec}) - \mathcal{T}_k(\eta_*)\right], \tag{5.57}$$

where the effective lower limit of integration η_* is of order η_k.

The dominant term in the right hand side of Eq. (5.57) depends on whether the long wavelength modes are decaying (which happens for $\nu^m < 3/2$), or growing ($\nu^m > 3/2$) before horizon crossing. In the first case we have

$$\mathcal{I} \approx \frac{j_\ell(x_{\rm dec})}{x_{\rm dec}} \mathcal{T}_k(\eta_{\rm dec}), \tag{5.58}$$

while in the second case we have instead

$$\mathcal{I} \sim -\frac{j_\ell(x_{\rm dec})}{x_{\rm dec}} \mathcal{T}_k(\eta_k). \tag{5.59}$$

Therefore, comparison of Eqs. (5.58) and (5.59) with (5.55b) shows that $\mathcal{N}$ and $\mathcal{I}$ are of the same order if there is no growing mode during matter domination ($\frac{1}{2} + \nu^m \leqslant 2$), and that $\mathcal{I} \gg \mathcal{N}$ otherwise ($\nu^m > 3/2$).

We are ready now to calculate the angular power spectrum on large angular scales, which are dominated by modes that crossed the vector sound horizon after decoupling ($c_v k \eta_{\rm dec} < 1$). The relevant expression for the transfer function is given by the first line of Eq. (5.51a) at $\eta = \eta_{\rm dec}$. If there is no growing mode during matter domination, the contributions from the integrated and non-integrated terms in (5.55) are roughly equal, and substituting Eqs. (4.92) and (5.52) into (5.55) we get

$$C_\ell^V \approx 4\pi \mathcal{A}_Q^2 \, \mathcal{T}_k^2(\eta_{\rm dec}) \int \frac{dx_0}{x_0} \frac{\ell(\ell+1)}{x_0^2} x_0^{n_v} j_\ell^2(x_0), \tag{5.60}$$

where we have chosen k_N in Eq. (4.91) to be the mode that is crossing the horizon today, $k_N \eta_0 = 1$, and used that $x_{\rm dec} \approx x_0 \equiv k\eta_0$.

From reheating to the time of decoupling, the amplitude of the vector modes changes by $\mathcal{T}_k(\eta_{\rm dec})$. The spectrum is thus proportional to the square of the transfer function times the primordial amplitude $\mathcal{A}_Q^2$. This factor is independent of angular scale, since we are taking the long wavelength limit. The angular dependence in the last equation can be estimated as follows. The Bessel function is negligible for $x_0 \lesssim \ell$, and rapidly decays at $x_0 > \ell$, so the anisotropies are dominated by $x_0 \sim \ell$. In the integrand, the maximum of the Bessel function is of order $1/x_0$, and the two remaining factors of x_0 in the denominator "cancel" the enhancement proportional to $\ell(\ell+1)$ one would otherwise have. In summary, we have $C_\ell^V \propto \ell^{n_v - 2}$.

More precisely, since the period of oscillations of the Bessel function is much shorter than any other characteristic scale in the integrand of Eq. (5.60), we may replace the oscillations with their average, 1/2. If the spectral index n_v is not too blue ($n_v < 4$), the dominant contribution to the integral is given by the value of the integrand at $x \approx \ell$, so the angular power spectrum becomes (for $\nu^m \leqslant 3/2$)

$$\ell(\ell+1)C_\ell^V \sim 2\pi\mathcal{A}_Q^2 \left(\frac{\nu^i+\nu^r}{2\nu^r}\right)^2 \left(\frac{a_{\rm dec}}{a_{\rm rh}}\right)^{2\nu^r-1} \left(\frac{a_{\rm dec}}{a_{\rm eq}}\right)^{\nu^m-2\nu^r-\frac{1}{2}} \ell^{\,n_v}. \quad (5.61)$$

This expression is valid for those scales that entered the vector horizon after recombination, which corresponds to $\ell \lesssim 50/c_v$ (for a ΛCDM model with $\Omega_\Lambda = 0.7$.)

If there is a growing mode during matter domination, the integrated term (5.55c) yields the dominant contribution to the temperature anisotropies. Proceeding along the same lines as above, we find that in this case the angular power spectrum is (for $\nu^m > 3/2$)

$$\ell(\ell+1)C_\ell^V \sim 2\pi\mathcal{A}_Q^2 \left(\frac{\nu^i+\nu^r}{2\nu^r}\right)^2 \left(\frac{a_{\rm eq}}{a_{\rm rh}}\right)^{2\nu^r-1} \left(\frac{a_0}{c_v a_{\rm eq}}\right)^{\nu^m-\frac{3}{2}} \ell^{\,n_v+3-2\nu^m}. \quad (5.62)$$

Of course, in the crossover case $\nu^m = 3/2$ the two angular power spectra (5.61) and (5.62) agree.

Small Angular Scales

For the scales that enter the vector horizon before decoupling a precise estimate of the integrated term in (5.55c) becomes more difficult. For these modes the derivative of the transfer function is an oscillatory function, whose amplitude decreases in time. Hence, it is most important at earlier times $x \approx x_{\rm dec}$ and sharply decays within an interval $\Delta x = k\eta_{\rm dec}$. Whereas the latter is small for modes that cross after decoupling, for those scales that enter the horizon well before that time $\Delta x = k\eta_{\rm dec}$ is large, and the approximation of a constant Bessel function that led to (5.58) breaks down. Nonetheless, if we are interested in the order of magnitude of the Bessel function, and not in the oscillations, we can still use Eq. (5.58), since the "amplitude" of $j_\ell(x)$ only changes significantly within $\Delta x = x_{\rm dec} \approx k\eta_0 \gg k\eta_{\rm dec}$. In that case, the integrated term is at most of the same order of magnitude as the non-integrated one, and Eqs. (5.55) imply that the temperature anisotropies at any given angular scale will depend on the vector anisotropies on the appropriate comoving distance at the time of decoupling.

Under the assumption that $\mathcal{N}$ and $\mathcal{I}$ are of the same order, the angular power spectrum on small angular scales can be now calculated as before. For simplicity, let us concentrate on relatively small scales, which cross the sound horizon before equality of matter and radiation densities. The relevant expression for the transfer function is given by the third line of Eq. (5.51a). Substituting Eqs. (4.92) and (5.51a) into (5.55) we obtain

$$C_\ell^V \approx 4\pi\,\mathcal{A}_Q^2 \mathcal{T}^2_{\eta_0^{-1}}(\eta_{\rm dec}) \times \int \frac{dx_0}{x_0}\frac{\ell(\ell+1)}{x_0^2} x_0^{n_v-2\nu^r-1}\cos^2\left(\frac{c_v\eta_{\rm dec}}{\eta_0}x_0\right) j_\ell^2(x_0). \quad (5.63)$$

As before, the power is proportional to the primordial contribution x^{n_v} times an additional factor $x^{-2\nu^r-1}$, which just reflects that modes enter the horizon at different times, and thus evolve differently. The cosine represents a snapshot of the "acoustic oscillations" of the vector perturbations at decoupling.

Before we proceed, we should mention an additional effect that influences the anisotropies on very small scales. So far, we have been assuming that the decoupling of the photons from the baryons is instantaneous. This is an accurate approximation for scales in which the argument of the cosine in Eq. (5.51a) does not change much during the duration of decoupling. On scales in which the cosine does change significantly, the spread in time at which a photon last scatters dampens the fluctuations by an exponential factor $\exp\left(-x_0^2/2\sigma^2\right)$ [Muk05]. A similar suppression is also due to Silk-damping, which originates from the breakdown of the tight-coupling approximation at scales of the order of the mean free path of photons in the plasma. For the observed values of the cosmological parameters, both effects yield an overall value of the suppression scale $\sigma \approx 500$.

Due to the exponential damping, the integral over modes converges for any power-law spectrum. As before, if the effective spectral index is not too blue, the dominant contribution to the integral is given by the value of the integrand at $x_0 \approx \ell$, so the angular power spectrum becomes

$$\ell(\ell+1)C_\ell^V \sim 2c_{13}^2(\nu_r+1)\mathcal{A}_Q^2\left(\frac{\nu^i+\nu^r}{2\nu^r}\right)^2\left(\frac{a_{\rm rh}}{a_{\rm dec}}\right)^2\left(\frac{2\eta_0}{c_v\eta_{\rm rh}}\right)^{2\nu^r-1} \times \ell^{\,n_v-2\nu^r-1}\cos^2\left(\frac{c_v\eta_{\rm dec}}{\eta_0}\ell\right)e^{-\ell^2/2\sigma^2}. \quad (5.64)$$

This equation is qualitatively valid at small angular scales, those corresponding to modes that crossed the horizon before equality, $\ell \gtrsim 120/c_v$. The acoustic oscillations subtend an angle $c_v\eta_{\rm dec}/\eta_0$ on the sky, the ratio of the comoving size of the sound horizon at decoupling to the comoving distance to the last scattering surface. A plot of the angular power spectrum for vector modes for a specific set of parameters is shown in Figure 5.2.

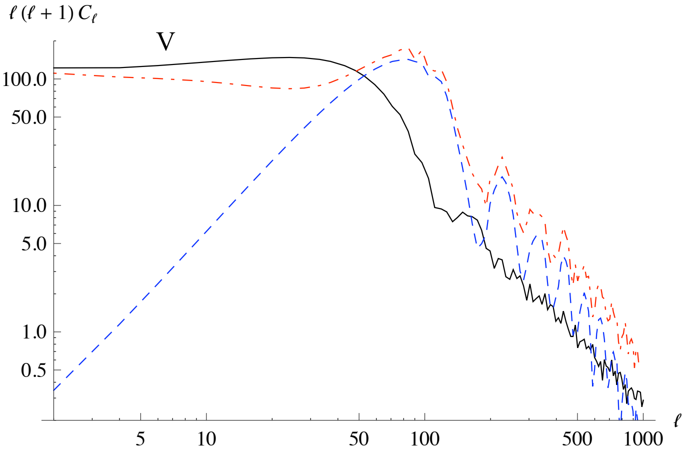


Figure 5.2: The contribution of vector modes to the temperature anisotropy power spectrum for models with $c_{14} = -\alpha$ and $c_v = 1$ (the normalization is arbitrary). The black (continuous) curve is the added contribution of the integrated and non-integrated terms, Eq. (5.55). The contribution of the integrated term alone is shown in red (dashed-dotted), while the contribution of the non-integrated term alone is shown in blue (dashed). On large angular scales, the spectrum is well approximated by Eq. (5.62). On small scales, Eq. (5.64) gives a qualitatively correct approximation.

Comparison with Tensor Modes

It is also illustrative to compare the contribution of the vector modes to the angular power spectrum (5.55) to that of the tensor modes [Sta85, TWL93, Muk05], which, in the limit of instantaneous decoupling, is given by

$$C_\ell^t = \pi \frac{(\ell+2)!}{(\ell-2)!} \int \frac{dk}{k}\, \mathcal{P}_t^{\rm rh} \left| \int_0^{x_{\rm dec}} dx\, \frac{d\mathcal{T}_k}{dx} \frac{j_\ell(x)}{x^2} \right|^2, \tag{5.65}$$

where this time $\mathcal{T}_k$ is the transfer function of the tensor modes. Up to a factor $\sim (\ell/x)^2$ this is just what the vector modes would contribute if the non-integrated term in (5.55) were negligible. In fact, because each power of x in the integral over momenta typically yields a factor ℓ, this scales with

ℓ in the same way as the contribution from the integrated term of the vector modes.

The amplitude of the tensor modes remains constant on superhorizon scales and decays also with $1/a$ inside the horizon. Thus, for modes that cross the horizon after decoupling but well before the present, the analogue of the large-scale approximation (5.58) is

$$\int_0^{x_{\rm dec}} dx\, \frac{d\mathcal{T}_k}{dx} \frac{j_\ell(x)}{x^2} \approx \mathcal{T}_k(\eta_{\rm dec}) \frac{j_\ell(x_{\rm dec})}{x_{\rm dec}^2}. \tag{5.66}$$

Substituting then Eq. (5.66) into (5.65) and following the same steps as before we obtain

$$\ell(\ell+1)C_\ell^t \approx \frac{\pi}{2} \mathcal{A}_t^2 \ell^{n_t}, \tag{5.67}$$

which, again, holds for $\ell \lesssim 50$. Thus, on large scales the shape of the spectrum for tensor and vector modes roughly agree if the spectral indices are the same, and there is no growing mode in the vector sector. On small scales, the situation is different though. Along the same lines as before, the contribution of the integrated term can be estimated qualitatively by Eq. (5.66). Carrying out the same approximations as for the vector modes, the angular power spectrum from the tensor modes then becomes

$$\begin{aligned} \ell(\ell+1)C_\ell^t \approx \frac{\pi}{2} \mathcal{A}_t^2 \left(\frac{a_{\rm eq}}{a_{\rm dec}}\right)^2 \left(\frac{\eta_0}{\eta_{\rm eq}}\right)^2 & \\ \times\, \ell^{n_t-2} \cos^2\left(\frac{\eta_{\rm dec}}{\eta_0}\ell\right) \exp\left(-\frac{1}{2}\frac{\ell^2}{\sigma^2}\right)&. \end{aligned} \tag{5.68}$$

This result agrees qualitatively well with numerical simulations [TWL93, Muk05], and lends further support to the approximation (5.57) for the vector modes on small scales. It shows that, at these scales, the angular power spectrum of vector modes with spectral index n_v is essentially the same as the angular power spectrum of tensor modes with spectral index

$$n_t = n_v + 1 - 2\nu^r. \tag{5.69}$$

A particularly relevant example of the equivalence arises for $\alpha + c_{14} = 0$, which leads to $\nu^r = 3/2$, $\nu^m = 5/2$. In this case, the spectral index of the vector modes is $n_v \approx 2$, which is just the spectral index of a massless field in flat spacetime. Even though the amplitude of this spectrum is negligible on large scales at the end of inflation, because there is a growing mode during radiation domination, the amplitude of the spectrum at decoupling may be sizable. In any case, during radiation domination the spectrum at time η is proportional to $(k\eta)^2$, so all modes enter the sound horizon $c_v k\eta = 1$ with the *same* amplitude. After horizon crossing the amplitude decays as $1/a$, as for tensor modes. Hence, up to the frequency of the acoustic oscillations, this case is almost equivalent to that of tensor modes with a nearly scale-invariant primordial spectrum, as stated by (5.69). The equivalence also extends to large angular scales. With $\nu^m = 5/2$, Eq. (5.62) yields a flat plateau in the contribution of vector modes to the temperature anisotropies, just as for a scale-invariant spectrum of gravitational waves.

Comparison with Observations

Present measurements of the CMB temperature anisotropies seem to be well-fit by a nearly scale-invariant spectrum of *scalar* perturbations [KDN+09]. Therefore, if vector modes do contribute to the temperature anisotropies at observable scales, their contribution must be subdominant. This requirement places constraints on the parameters of Aether theories, which follow from demanding

$$C_\ell^V \lesssim C_\ell^s. \tag{5.70}$$

Let us obtain a very rough estimate of the contribution of vector modes to the temperature anisotropies on large scales. It follows from Eqs. (4.91), (5.61) and (5.62) that the bulk part of the contribution stems from the four large factors

$$\begin{aligned} \ell(\ell+1)C_\ell^V &\sim \frac{c_{13}^2}{c_{14}}\frac{H_{\rm rh}^2}{M_P^2} \exp\left(\frac{N n_v}{q}\right)\left(\frac{a_{\rm eq}}{a_{\rm rh}}\right)^{2\nu^r-1} \\ &\sim \frac{c_{13}^2}{c_{14}}\frac{H_{\rm rh}^2}{M_P^2}\left(\frac{a_{\rm eq}}{a_{\rm rh}}\right)^{2\nu^r-n_v-1}. \end{aligned} \tag{5.71}$$

Here, we have ignored the (recent) stage of cosmic acceleration, the difference between equality and decoupling, and the redshift to the time of equality of matter and radiation densities. Using Eqs. (4.89) and (4.91b) we find

$$2\nu^r - n_v - 1 \approx -4 + \sqrt{1+4\kappa_i} + 3\sqrt{1+(4/9)\kappa_r} \approx \frac{2}{3}\kappa_r, \tag{5.72}$$

where κ is defined in (4.30), and the indices i and r refer to inflation and the radiation era respectively. We have also expanded for small κ in the last step, and neglected κ_i in front of κ_r, because of a relative slow roll suppression factor. The sign of κ is determined by the sign of $(1+\alpha/c_{14})$. Hence, if $(1+\alpha/c_{14}) > 0$, vector modes are primordially suppressed, and the subsequent growth during radiation domination cannot compensate for this suppression. On the other hand if $(1+\alpha/c_{14}) < 0$, the growth during radiation domination may bring the signal well above what is observationally allowed. Hence, it seems that the range which is the most interesting from the point of view of observation is when $|1+\alpha/c_{14}| \ll 1$, which corresponds to $\kappa \ll 1$. Therefore, it is not excluded that the present amplitude of these modes is quite sizable, producing detectable signals in the CMB, or dipole contributions to the gravitational potentials of massive bodies through the effect of the velocity field of the vector modes of the Aether with respect to matter, as discussed at the end of the previous section.

An interesting question is whether, in the range where $|1 + \alpha/c_{14}| \ll 1$, the contribution of the scalar isocurvature mode to observables will be larger or smaller than that of the vector modes. As we noted around Eq. (4.95), the relative amplitude of the longitudinal to the transverse velocity field power spectra is of order $c_{14}/(1 + w)$, which is likely to be quite small if the scale of Lorentz symmetry breaking is low. Nonetheless, depending on parameters, both situations seem possible. Furthermore, in a theory such as BPSH, the vector mode is completely absent, and we only have the scalar contribution. A full analysis of the CMB signatures for the scalar mode is left for further research.

5.6 CMB Polarization

The spectrum of the CMB is polarized so, the same way we can calculate the correlation functions for the temperature, we can do for the polarization. It is a consequence of the fact that recombination is not an instantaneous process, and it is proportional to the time it lasts. Its effect is of the order of ten per cent of the total temperature fluctuations for small angular scales and less than one per cent for large angular scales.

Although the effect is small, it is of great interest in the case of B polarization. This one is absent for scalar modes, so it provides a way of indirectly detecting primordial gravitational waves. The amplitude of GW decreases on subhorizon scales, being the maximum contribution at $l \sim 100$.

To characterize the polarization we are going to consider a plane wave E arriving at the observer's position from the z direction. The intensity measured by a detector of linearly polarized radiation in a plane with azimuthal angle θ is

$$\frac{dI}{d\omega} = \langle E_\theta \rangle = I + Q\cos 2\theta + U \sin 2\theta, \tag{5.73}$$

being

$$E_\theta = E_x \cos\theta + E_y \sin\theta, \tag{5.74}$$

and where

$$I \equiv \langle E_x^2 \rangle + \langle E_y^2 \rangle, \tag{5.75a}$$
$$Q \equiv \langle E_z^2 \rangle - \langle E_y^2 \rangle, \tag{5.75b}$$
$$U \equiv 2Re\langle E_x^* E_y \rangle, \tag{5.75c}$$

and ω is the angular frequency.

The Stokes parameters Q and U specify the plane polarization, and we can define the combination $Q_\pm \equiv Q \pm iU$. We can expand this in spherical harmonics

$$Q_\pm(\mathbf{e}) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} Q_{lm}^{\pm} Y_{lm}^{\pm}(\mathbf{e}), \tag{5.76}$$

where $\mathbf{e}$ is the incoming direction of the wave. Polarization multipoles are defined

$$Q_{lm}^{\pm} = E_{lm} \pm iB_{lm}. \tag{5.77}$$

Following the method used by [HW97], we have that the power spectra of temperature and polarization anisotropies is defined as $C_l^{TT} \equiv \langle |a_{lm}|^2 \rangle$ for $T = \sum a_{lm} Y_l^m$, and similarly for the other quantities, obtaining

$$(2l+1)^2 C_l^{X\widetilde{X}} = \frac{2}{\pi} \int \frac{dk}{k} k^3 X_l^{(m)*}(\eta_0, k) \widetilde{X}_l^{(m)}(\eta_0, k), \tag{5.78}$$

where X will be T, E or B. There is no cross correlation C_l^{TB} or C_l^{EB} and $C_l^{BB} = 0$ for scalars.

In this section we are going to change the convention for derivatives, and use dots for derivatives with respect to conformal time $\dot{} \equiv \frac{d}{d\eta}$ and $' \equiv \frac{d}{d\,x}$, where $x = k(\eta_0 - \eta)$, as it is generally used in the literature.

The Boltzmann equations can be written in integral form giving the solutions

$$\frac{E_l^{(m)}(\eta_0, k)}{2l+1} = -\sqrt{6} \int_0^\infty d\eta\, \dot{\tau} e^{-\tau} P^{(m)} \epsilon_l^{(m)}, \tag{5.79}$$

$$\frac{B_l^{(m)}(\eta_0, k)}{2l+1} = -\sqrt{6} \int_0^\infty d\eta\, \dot{\tau} e^{-\tau} P^{(m)} \beta_l^{(m)}, \tag{5.80}$$

with

$$\tau(\eta) = \int_0^{\eta_0} d\tilde{\eta}\, \dot{\tau}(\tilde{\eta}), \tag{5.81}$$

being the optical depth. The combination $\dot{\tau} e^{-\tau}$ is the probability that a photon last scattered within $d\eta$, therefore it is peaked at the last scattering epoch. $P^{(m)}$ is the anisotropic scattering source and $\epsilon_l^{(m)}$ and $\beta_l^{(m)}$ are combinations of spherical Bessel functions and are given in Table (5.1).

In the tight coupling limit $P^{(m)}$ is giving by

$$P^{(1)} = \frac{\sqrt{3}}{9} \frac{k}{\dot{\tau}} \mathbf{Q}, \tag{5.82}$$

mode	$\epsilon_l^m(x)$	$\beta_l^m(x)$
0	$\sqrt{\frac{3}{8}\frac{(l+2)!}{(l-2)!}}\frac{j_l(x)}{x^2}$	0
1	$\frac{1}{2}\sqrt{(l-1)(l+2)}\left[\frac{j_l(x)}{x^2}+\frac{j_l'(x)}{x}\right]$	$\frac{1}{2}\sqrt{(l-1)(l+2)}\frac{j_l(x)}{x}$
2	$\frac{1}{4}\left[j_l''(x)-j_l(x)+2\frac{j_l(x)}{x^2}+4\frac{j_l'(x)}{x}\right]$	$\frac{1}{2}\left[j_l'(x)+2\frac{j_l(x)}{x}\right]$

Table 5.1: Functions ϵ_l^m and β_l^m for the three modes

where $\mathbf{Q}$ is given by (4.80) for vectors, and

$$P^{(2)} = -\frac{1}{3}\frac{\dot{\mathbf{t}}}{\dot{\tau}}, \tag{5.83}$$

where $\mathbf{t}$ is the tensor perturbation.

5.7 Polarization in E-A Theories

We want to study the polarization for vector modes in the case of E-A theory. The procedure is similar to the one used to calculate the correlation function of the temperature fluctuations. We have already compared the correlation function of the temperature fluctuations for the vector and tensor modes in a previous section. Now, we want to do the same comparison for the polarization modes. The interesting mode, the one that can be detected providing useful information about tensor and vector modes, is the B mode, absent for the scalars. To do the calculations we will follow the method used by [HW97]. In a recent article [NK11] they calculate the contribution of the B-mode using a numerical code, and also give some analytical arguments to explain the spectrum. Our intention here is to do the complete analytical approach at large scales and, although not being of the same experimental interest, extend it to the E-modes.

We will start with the calculation of the B-mode polarization.

B mode The integral solution for the Boltzmann equation for B_l component, taking into account we are in the tight coupling limit, is given by (see Section 5.6)

$$\frac{B_l}{2l+1} = \sqrt{6}\int_0^{\eta_0} d\eta\dot{\tau}e^{-\tau}\frac{\sqrt{3}}{9}\frac{k}{\dot{\tau}}Q\frac{1}{2}\sqrt{(l-1)(l+2)}\frac{j_l(x)}{x}$$

$$= -\frac{\sqrt{2}}{6}\sqrt{(l-1)(l+2)}\int_0^{\eta_0} d\eta\dot{\tau}e^{-\tau}\frac{kQ}{\dot{\tau}}\frac{j_l(x)}{x}. \tag{5.84}$$

We can approximate $\dot{\tau}e^{-\tau} \sim \delta(\eta-\eta_{rec})$ and considering that $(\eta_0-\eta_{rec}) \sim \eta_0$

$$\frac{B_l}{2l+1} \simeq -\frac{\sqrt{2}}{6}\sqrt{(l-1)(l+2)}\frac{kQ(\eta_{rec})}{\dot{\tau}_{rec}}\frac{j_l(x_0)}{x_0}, \tag{5.85}$$

with $x_0 = k\eta_0$.

The angular power spectrum is given by

$$\begin{aligned} C_l^{BB} &\simeq \frac{1}{9\pi}(l-1)(l+2)\int\frac{dk}{k}k^3\left(\frac{kQ(\eta_{rec})}{\dot{\tau}_{rec}}\frac{j_l(x_0)}{x_0}\right)^2 \\ &\simeq \frac{1}{9\pi}(l-1)(l+2)\left(\frac{\mathcal{A}_Q|\mathcal{T}_k(\eta_{rec})|}{\dot{\tau}_{rec}\eta_0}\right)^2\int\frac{dx_0}{x_0}x_0^{n_v}j_l(x_0)^2, \end{aligned} \tag{5.86}$$

where we have used $\mathcal{P}_Q = k^3|Q|^2$, $\mathcal{P}_Q(\eta) = \mathcal{P}_Q^{\rm rh}|\mathcal{T}_k(\eta)|^2$, $\mathcal{P}_Q^{\rm rh} = \mathcal{A}_Q^2(\frac{k}{k_N})^{n_v}$ and $k_N\eta_0 \sim 1$. As $T_k(\eta_{rec})$ is independent of k we can take it out of the integral (first line of Eq. (5.51a)).

In order to evaluate the integral we ignore the oscillating part and consider only the amplitude of the Bessel function. We evaluate the integral for $x_0 \sim l$ ending with

$$\begin{aligned} C_l^{BB} &\simeq \frac{1}{9\pi}(l-1)(l+2)\left(\frac{\mathcal{A}_Q|\mathcal{T}_k(\eta_{rec})|}{\dot{\tau}_{rec}\eta_0}\right)^2 l^{n_v-2} \\ &\sim \frac{1}{9\pi}\left(\frac{\mathcal{A}_Q|\mathcal{T}_k(\eta_{rec})|}{\dot{\tau}_{rec}\eta_0}\right)^2 l^{n_v}, \end{aligned} \tag{5.87}$$

Using $\dot{\tau}_{rec} \sim \frac{1}{\Delta\eta} \sim \frac{10}{\eta_{rec}}$ and $\mathcal{A}_Q = c_{13}c_t^2\mathcal{A}_V$

$$C_l^{BB} \simeq \frac{1}{900\pi}\left(\frac{c_{13}}{1+c_{13}}\right)^2\mathcal{A}_V^2|\mathcal{T}_k(\eta_{rec})|^2\left(\frac{\eta_{rec}}{\eta_0}\right)^2 l^{n_v} \sim C_l^{TT}\times l^2, \tag{5.88}$$

Substituting $\mathcal{T}_k(\eta_{rec})$ from the first line of Eq. (5.51a)

$$\begin{aligned} l(l+1)C_l^{BB} &\simeq \frac{(l-1)l(l+1)(l+2)}{900\pi}\left(\frac{\eta_{rec}}{\eta_0}\right)^2\mathcal{A}_V^2|\mathcal{T}_k(\eta_{rec})|^2 l^{n_v-2} \\ &\sim \frac{\mathcal{A}_V^2}{900\pi}\left(\frac{\nu^i+\nu^r}{2\nu^r}\right)^2\left(\frac{a_{\rm rh}}{a_{rec}}\right)^2\left(\frac{a_{eq}}{a_{rh}}\right)^{1+2\nu^r} \\ &\quad\times\left(\frac{\eta_{rec}}{\eta_{eq}}\right)^{1+2\nu^m}\left(\frac{\eta_{rec}}{\eta_0}\right)^2 l^{n_v+2} \\ &\sim l^{n_v+2}. \end{aligned} \tag{5.89}$$

Taking $n_v = 1$ as it is done in [NK11], we arrive to the same dependence $l(l+1)C_l^{BB} \propto l^3$.

E mode We will apply the same procedure to the E-mode. The results we expect are similar to the ones for the B-mode, as the differences should appear out of the range of our calculation, where the peaks start to shown up.

$$\begin{aligned}\frac{E_l}{2l+1} &= -\sqrt{6}\int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau}\frac{\sqrt{3}}{9}\frac{k}{\dot{\tau}}Q\frac{1}{2}\sqrt{(l-1)(l+2)}\left(\frac{j_l(x)}{x^2}+\frac{j_l'(x)}{x}\right)\\ &= -\frac{\sqrt{2}}{6}\sqrt{(l-1)(l+2)}\int_0^{\eta_0} d\eta \dot{\tau} e^{-\tau}\frac{kQ}{\dot{\tau}}\left(\frac{j_l(x)}{x^2}+\frac{j_l'(x)}{x}\right).\end{aligned} \tag{5.90}$$

We can approximate $\dot{\tau}e^{-\tau} \sim \delta(\eta-\eta_{rec})$ and taking into account that $(\eta_0 - \eta_{rec}) \sim \eta_0$

$$\frac{E_l}{2l+1} \simeq -\frac{\sqrt{2}}{6}\sqrt{(l-1)(l+2)}\frac{kQ(\eta_{rec})}{\dot{\tau}_{rec}}\left(\frac{j_l(x_0)}{x_0^2}+\frac{j_l'(x_0)}{x_0}\right), \tag{5.91}$$

with $x_0 = k\eta_0$. Using the relations for derivatives of the Bessel functions and for large values of l

$$\begin{aligned}\frac{j_l(x)}{x^2}+\frac{j_l'(x)}{x} &= \frac{j_l(x)}{x^2}+\frac{j_{l-1}(x)}{x}-\frac{(l+1)j_l(x)}{x^2} = \frac{j_{l-1}(x)}{x}-\frac{lj_l(x)}{x^2}\\ &\sim \frac{j_l(x)}{x}-\frac{lj_l(x)}{x^2}\sim\frac{j_l(x)}{x},\end{aligned} \tag{5.92}$$

we end up with

$$\frac{E_l}{2l+1} \simeq -\frac{\sqrt{2}}{6}\sqrt{(l-1)(l+2)}\frac{kQ(\eta_{rec})}{\dot{\tau}_{rec}}\frac{j_l(x_0)}{x_0}. \tag{5.93}$$

The angular power spectrum is given by

$$\begin{aligned}C_l^{EE} &\simeq \frac{1}{9\pi}(l-1)(l+2)\int\frac{dk}{k}k^3\left(\frac{kQ(\eta_{rec})}{\dot{\tau}_{rec}}\frac{j_l(x_0)}{x_0}\right)^2\\ &\simeq \frac{1}{9\pi}(l-1)(l+2)\left(\frac{\mathcal{A}_Q|\mathcal{T}_k(\eta_{rec})|}{\dot{\tau}_{rec}\eta_0}\right)^2\int\frac{dx_0}{x_0}x_0^{n_v+2}\left(\frac{j_l(x_0)}{x_0}\right)^2.\end{aligned} \tag{5.94}$$

In order to evaluate the integral we ignore the oscillating part and consider only the amplitude of the bessel function. We evaluate the integral for $x_0 \sim l$ ending with

$$\begin{aligned}C_l^{EE} &\simeq \frac{1}{9\pi}(l-1)(l+2)\left(\frac{\mathcal{A}_Q|\mathcal{T}_k(\eta_{rec})|}{\dot{\tau}_{rec}\eta_0}\right)^2 l^{n_v-2}\\ &\sim \frac{1}{9\pi}\left(\frac{\mathcal{A}_Q|\mathcal{T}_k(\eta_{rec})|}{\dot{\tau}_{rec}\eta_0}\right)^2 l^{n_v}\end{aligned}$$

$$\sim C_l^{TT} \times l^2, \tag{5.95}$$

Using $\dot{\tau}_{rec} \sim \frac{1}{\Delta\eta} \sim \frac{10}{\eta_{rec}}$

$$l(l+1)C_l^{EE} = \frac{1}{900\pi} \left(\mathcal{A}_Q |\mathcal{T}_k(\eta_{rec})|\right)^2 \left(\frac{\eta_{rec}}{\eta_0}\right)^2 l^{n_v+2}$$
$$\sim l^{n_v+2}. \tag{5.96}$$

Here, we have again the same dependence in l as for the B-mode.

In Fig. (5.3) are shown the B-mode polarization modes and temperature anisotropy power spectrum in E-A theory, for values of the parameters $c_1 = -0.019$, $c_{13} = -0.03$, $c_{14} = -\alpha = -0.0128$, calculated numerically in [NK11], and compared with tensor perturbations in GR in the case of the tensor-to-scalar ratio $r = 0.1$. The power spectra are $\mathcal{P}_V \propto k^{n_v}$ and $\mathcal{P}_T \propto k^0$.

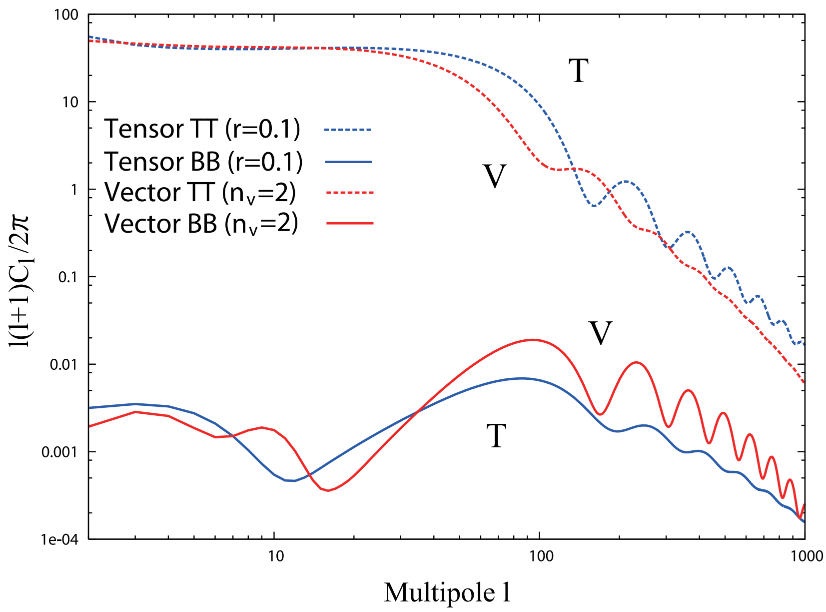


Figure 5.3: B-mode polarization modes and temperature anisotropy power spectrum in E-A theory (from [NK11]).

Tensor perturbations behave (for $l < l_{rec}$) as $l(l+1)C_l^{BB} \propto l^2$ ([PK05]). Although the amplitude of the spectrum for E-A differs from the one of GR, this difference is small for small values of the parameters, and we can compare with this last spectrum. It seems that the BB spectrum for vectors can be larger than the one coming from primordial tensor modes. The observability of this spectrum will depend on the values of the parameters.

Chapter 6

Summary and Conclusions

In this thesis we have studied cosmological perturbations in Einstein-Aether theory, where the scalar and transverse vector sectors of General Relativity are enlarged by an additional dynamical field each. We find that inflation can induce sizable perturbations in both of these new massless fields on observable scales. Our analysis also applies to the low energy limit of BPSH gravity, where the transverse vector is missing by construction [Jac10].

For the purposes of summarizing our results, we shall assume that the Aether parameters c_i $(i = 1, \ldots, 4)$ are small. This is natural, since they can be thought of as proportional to the square of the ratio of the symmetry breaking scale M to the Planck scale $c_i \sim (M/M_P)^2 \ll 1$.

To motivate the choice of the range of parameters we shall use below, let us recall that in the Einstein-Aether theory, the effective gravitational constant on small scales G_N can be different from the effective gravitational constant which appears in the Friedmann equation G_{cos}. We shall call α and c_{14} the parameters which relate these two constants to the bare Newton's constant G. They are given in Eqs. (3.9) as linear combinations of the standard c_i. In terms of α and c_{14} the effective gravitational constants are given by

$$G = \left(1 - \frac{\alpha}{2}\right) G_{\text{cos}} = \left(1 + \frac{c_{14}}{2}\right) G_N. \tag{6.1}$$

Note that for $\alpha + c_{14} = 0$ we have $G_{\text{cos}} = G_N$. The difference $G_{\text{cos}} - G_N$ is constrained by nucleosynthesis to be less than 10 %, so it seems natural to consider the range

$$|\tilde{\kappa}| \ll 1, \quad \text{where} \quad \tilde{\kappa} \equiv -\left(1 + \frac{\alpha}{c_{14}}\right). \tag{6.2}$$

This range guarantees the similarity of G_{cos} and G_N, but it is typically more restrictive than required by the nucleosynthesis bound, since $c_{14} \sim (M/M_P)^2$ is naturally small. If the parameters are such that we are outside of the range

(6.2), the effects we are investigating would be either too small to be of phenomenological interest, or too large to be compatible with observations.

The main results of the thesis are the following. First, we find that in the scalar sector, aside from the standard adiabatic mode ζ (which corresponds to the curvature of surfaces of constant matter density), there is an additional isocurvature mode which can be important for phenomenology. Geometrically, the isocurvature mode can be described as the differential e-folding number δN which separates the surfaces of constant matter density from the surfaces orthogonal to the Aether. This plays the role of a velocity potential v for the Aether with respect to matter. At the time of horizon exit during inflation, the amplitudes of δN and v are comparable to that of the standard adiabatic mode ζ

$$v \sim \delta N \sim \zeta \sim \frac{H}{M_P}\epsilon^{-1/2} \quad \text{(horizon exit)}. \tag{6.3}$$

Here H is the Hubble rate and $\epsilon \ll 1$ is the slow roll parameter during inflation, which is independent of the Aether parameters.

After horizon crossing, the curvature perturbation ζ stays constant, while the behavior of δN depends on the parameter $\tilde{\kappa}$ defined above. For $\tilde{\kappa} < 0$, the isocurvature perturbation slowly decays on large scales, while for $\tilde{\kappa} > 0$ it grows. On the other hand, the velocity perturbation is given by $v \sim (k/\dot{a})\delta N$, where k is the co-moving wave number and $\dot{a}$ is the derivative of the scale factor with respect to proper time. Hence, during inflation, when $\dot{a}$ grows, the long wavelength velocity field decays, roughly in proportion to the inverse of the scale factor. After inflation, the universe decelerates and the velocity field grows again. At the time of horizon reentry, on cosmologically relevant scales, we have

$$v \sim \delta N \sim e^{N\tilde{\kappa}}\zeta \sim e^{N\tilde{\kappa}}\, 10^{-5} \lesssim 1, \quad \text{(horizon reentry)} \tag{6.4}$$

where $N \sim 60$ is the number of e-foldings of inflation since the time when the cosmological scale first crossed the horizon. The last inequality indicates the limit of validity of the linear approximation. Note that for $\tilde{\kappa} = 0$, the isocurvature perturbation and the velocity field of the Aether are comparable to $\zeta \sim 10^{-5}$ at horizon reentry. However, with $\tilde{\kappa} \lesssim 10/N$, we can have $\delta N \lesssim 1$. If $\tilde{\kappa}$ is large enough to saturate the inequality, this still allows for mildly relativistic speeds for the Aether field $v \sim 1$ within the observable universe.

Similar results hold for the vector sector. Denoting by V the transverse component of the Aether velocity field with respect to matter, we find that on superhorizon scales

$$V \sim \left(\frac{\epsilon}{c_{14}}\right)^{1/2} v.$$

Hence, if $c_{14} < \epsilon$ (which seems quite natural if the scale of Lorentz symmetry breaking is low), the vector contribution to the velocity field will be dominant with respect to that of the longitudinal component. On the other hand, in a theory such as BPSH, the transverse component V is missing, and the scalar part v is the dominant one.

We also find that the longitudinal gauge gravitational potentials ϕ and ψ can be different even for the adiabatic mode. On superhorizon scales, we find that this effect (which can be attributed to anisotropic stress of the Aether energy momentum tensor) is of order

$$(\phi - \psi)_{\text{adiab}} \sim \phi_{\text{adiab}}\, c_{13} \sim \zeta\, c_{13} \sim 10^{-5} c_{13}, \tag{6.5}$$

where $c_{13} \sim (M/M_P)^2$ is another combination of the Aether parameters c_i, given in Eqs. (3.9). Physically, this parameter can be expressed in terms of the propagation speed of tensor modes $c_{13} = c_t^{-2} - 1$. The isocurvature mode contributes maximally to the anisotropic stress, but the potential due to the isocurvature mode is suppressed by c_{13}

$$(\phi - \psi)_{\text{isoc}} \sim \phi_{\text{isoc}} \sim c_{13}\, \delta N. \tag{6.6}$$

Since δN can be larger than ζ, the anisotropic stress can be dominated by the isocurvature mode. The anisotropic stress on observable scales is suppressed from its value at horizon crossing, due to the dynamics of the Aether on subhorizon scales. For $\tilde{\kappa} = 0$, the effect scales like k^{-2} for modes that crossed the horizon during the matter era. For modes that crossed the horizon during the radiation era, the behavior changes to k^{-1}. Current constraints on $\phi - \psi$ on cosmological scales are not very restrictive, and $|c_{13}| \lesssim 1$ seems to be allowed by observations.

The Aether manifests itself in PPN parameters through frame dependent effects, which cause anisotropies in the gravitational field of bodies which move with respect to the Aether. In this way, the velocity field generated during inflation might be detectable. It should be noted, however, that it seems difficult with present technology to observe the statistical properties of the random field from this particular type of observations. Even if the velocity field were relativistic on cosmological scales $v \sim 1$, it falls with scale as k^{-2}. In particular, the component which varies on scales of the order of 100 Mpc would then be below the virial velocity $v_{\text{vir}} \sim 10^{-3}$ of objects bound in galaxies, and it seems unlikely that we can directly sample frame dependent effects in objects which are located at distances larger than that. On the other hand, at the relatively small distances where the observation of frame dependent effects is accessible, we may still detect a large but fairly homogeneous velocity field, even one that is much larger than the virial velocity of bound objects.

Finally, we have computed the contribution of transverse vector fields V to the angular power spectrum of CMB anisotropies. We find that for

$\tilde{\kappa} = 0$, the spectrum of multipole coefficients C_ℓ^V has the same shape as that of tensor modes. The amplitude, on the other hand, is related to the spectrum C_ℓ^t for tensor modes and C_ℓ^ζ for the adiabatic scalar mode as

$$C_\ell^V \sim \frac{c_{13}^2}{c_{14}} e^{2N\tilde{\kappa}} C_\ell^t \sim \frac{\epsilon\, c_{13}^2}{c_{14}} e^{2N\tilde{\kappa}} C_\ell^\zeta. \tag{6.7}$$

This means that the vector modes in Einstein-Aether theory can easily dominate the signal from tensor modes. The analysis of polarization induced by the vector modes is therefore of phenomenological interest. Vector perturbations in Einstein-Aether theories contribute to the polarization modes of the CMB. In particular, the contribution for the B-mode is present and could be distinguishable from the contribution from tensor modes. This could provide a way to differentiate E-A theories form GR. The approach we have followed here just provides us information of the l-dependence of the spectrum until recombination. In order to obtain further information a numerical calculation should be done. The study made by [NK11], that appeared during the writing of this thesis, follows this line and suggests that the B-mode would be detectable by future CMB observations, being its spectrum bigger than the one coming from inflationary gravitational waves.

Moreover, we know that the CMB is well-fit with a primordial spectrum of *scalar* adiabatic perturbations. This imposes additional phenomenological restriction amongst the parameters c_{13} and $\tilde{\kappa}$ of Einstein-Aether theories, of the form

$$\tilde{\kappa} \lesssim \frac{1}{2N} \ln \left| \frac{c_{14}}{\epsilon\, c_{13}^2} \right|. \tag{6.8}$$

So far, we have not included the constraints which follow from the frame-dependent effects on the PPN parameters. These are summarized in Section 3.2, and take the form

$$\omega\, \alpha_1 \lesssim 10^{-7}, \quad \omega^2 \alpha_2 \lesssim 10^{-13}. \tag{6.9}$$

Here, $\omega = \max\{V, v, v_{\rm vir}\}$, is the velocity of the Aether with respect to the object whose gravitational field is being tested at post-Newtonian order, and $v_{\rm vir} \sim 10^{-3}$ is the typical virial velocity for bound objects with respect to the CMB frame. The post-Newtonian parameters α_1 and α_2 are combinations of the four Aether parameters $(\alpha, c_{14}, c_+, c_-)$. Here, following [Jac07], we have introduced $c_+ \equiv c_{13} = c_1 + c_3$ and $c_- \equiv c_1 - c_3$. Phenomenologically, it is possible to set $\alpha_1 = \alpha_2 = 0$, which determines α and c_{14} as functions of the other two parameters in the model,

$$\alpha = -c_{14} = -2\frac{c_+ c_-}{c_+ + c_-}. \tag{6.10}$$

The parameters c_+ and c_- remain rather unconstrained by observations. Stability requirements and superluminality (or Cherenkov) constraints are satisfied provided that $-1 \leqslant c_+ \leqslant 0$, $c_+/3(1+c_+) \leqslant c_- \leqslant 0$. Constraints from radiation damping in binary systems determine further constraints on the (c_+, c_-) plane, but a sizable coefficient

$$|c_{13}| \lesssim 1 \tag{6.11}$$

still seems to be allowed by all observations [Jac07]. This is important, since the gravitational effects of the Aether are suppressed by this coefficient. For instance the contribution of vectors to the CMB anisotropies is of order

$$C_\ell^V \sim c_{13}^2 V^2, \tag{6.12}$$

where $V \lesssim 1$ is the Aether velocity field. Hence, the observability of the effect depends crucially on c_{13} being sufficiently large.

This brings us to the question of fine tuning amongst the parameters of the model. In a low energy theory, one might have expected all dimensionless parameters to be of the same order,

$$c_i \sim \left(\frac{M}{M_P}\right)^2 .$$

Observability of C_ℓ^V requires an inequality of the form $c_{13}V \gtrsim 10^{-6}$, which would be natural provided that

$$\left(\frac{M}{M_P}\right)^2_{\text{obs}} \gtrsim 10^{-6} V^{-1}. \tag{6.13}$$

On the other hand, in Eq. (6.10) we have adjusted the parameters so that $\alpha_1 = \alpha_2 = 0$, but the actual restriction (6.9) is of the form $\alpha_2 \lesssim 10^{-13}\omega^{-2}$. Hence, α_2 must be well below the natural scale (6.13) by a considerable suppression factor

$$\alpha_2 \lesssim 10^{-7}\omega^{-1}\frac{V}{\omega}\left(\frac{M}{M_P}\right)^2_{\text{obs}}, \tag{6.14}$$

with $10^{-3} < \omega < 1$. In the classical theory, the parameter α_2 can always be chosen by hand to have any particular value. However, in an effective field theory (EFT) a parameter is considered to be finely tuned or technically unnatural if quantum corrections to it are larger than the desired renormalized value of the parameter. The question, therefore, is whether the very small values of $\alpha_2 \ll (M/M_P)^2$ are stable or not under quantum corrections. Withers [Wit09] has recently analyzed the Einstein-Aether theory as an EFT, with the conclusion that the parameters c_i receive only negligible logarithmic corrections. A similar result may hold in BPSH theory [BPS10b, BPS10a]. This subject is left for further study.

To conclude, the results presented here show that the preferred frame singled out by the Aether field A^μ, or by the preferred foliation of the BPSH theory, may have picked up a large random velocity field seeded by quantum fluctuations during inflation. Depending on the parameters, this may even be mildly relativistic on cosmological scales. The effects of this velocity field may be detectable in observations of frame dependent PPN effects, or in specific features in the CMB spectrum such as a sizable contribution from vector modes. These issues deserve further investigation.

Appendix A

Theory of Cosmological Perturbations

In this chapter we will review the basics of the theory of cosmological perturbations [MFB92, Muk05, KS84, Wei08, Bar80]. In order to study gravity and its modifications it is necessary to consider not just a perfectly homogeneous and isotropic universe but the fluctuations about this background. These perturbations are the origin of the structures in the universe, and so it is important to understand the behavior of these perturbations to explain the observed large scale structures (as galaxies or clusters of galaxies). It will be also necessary in order to calculate anisotropies of the Cosmic Microwave Background.

Our starting point is a background spacetime and we want to see what happens when it is perturbed. The perturbations in the metric of the spacetime must be small compared with the background metric. In the case of cosmological perturbation theory the background metric will be the Friedmann-Robertson-Walker (FRW) metric, and in our case, the flat FRW universe. So, the metric can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \tag{A.1}$$

being $\bar{g}_{\mu\nu}$ the background metric and $\delta g_{\mu\nu}$ the perturbation.

The background metric in comoving coordinates is

$$ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \tag{A.2}$$

where $a(t)$ is the scale factor. We are going to work with conformal time instead of comoving time, defined as

$$d\eta = \frac{dt}{a(t)}, \tag{A.3}$$

and the background metric looks like

$$ds^2 = a(\eta)^2 (-d\eta^2 + \delta_{ij} dx^i dx^j). \tag{A.4}$$

In the unperturbed universe we had the Friedmann equations, given by

$$\mathcal{H}^2 = \frac{8\pi G a^2}{3}\rho, \tag{A.5}$$

$$\mathcal{H}^2 - 2\frac{a''}{a} = 8\pi G a^2 p, \tag{A.6}$$

$$\mathcal{H}' = -\frac{4\pi G a^2}{3}(\rho + 3p), \tag{A.7}$$

where $\mathcal{H}$ is the conformal Hubble parameter $\mathcal{H} = aH$; and the energy continuity equation

$$\rho' = -3\mathcal{H}(\rho + p). \tag{A.8}$$

We can define the equation of state parameter as $w \equiv \frac{p}{\rho}$ and the speed of sound as $c_s^2 \equiv \frac{p'}{\rho'}$. If $w = constant$, then $c_s^2 = w$.

The perturbed metric is

$$g_{\mu\nu} = a^2 \left[-(1+2\phi)d\eta^2 + 2(B_{,i} + S_i)d\eta dx^i \right. \\ \left. +(\delta_{ij} - 2\psi\delta_{ij} + E_{,ij} + 2F_{(i,j)} + t_{ij})dx^i dx^j\right]. \tag{A.9}$$

In order to obtain the perturbed Einstein equations we need to calculate the perturbed Ricci tensor and we will need to compute the perturbed Christoffel symbols. The full set of perturbed quantities that are required to write the left-hand side of the Einstein equations and the gravitational action are collected in Appendix B.

Now we are going to focus our attention in the energy-momentum tensor.

A.0.1 Perturbations in the energy-momentum tensor

The background energy-momentum tensor is of the perfect fluid form

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} = \begin{pmatrix} \rho & \\ & p\delta_{ij} \end{pmatrix}. \tag{A.10}$$

The homogeneity and isotropy of the universe implies that the density and pressure in the background are just functions of time, $\rho = \rho(\eta)$ and $p = p(\eta)$, and the fluid is at rest, $u^i = 0$. The condition $u_\mu u^\mu = -1$ leaves us with $u^\mu = \frac{1}{a}(1, \vec{0})$.

The perturbed energy-momentum tensor is obtained substituting the perturbed variables

$$\rho \to \rho + \delta\rho, \tag{A.11}$$

$$p \to p + \delta p, \tag{A.12}$$

$$u^\mu \to u^\mu + \delta u^\mu, \tag{A.13}$$

in Eq. (A.10).

Using the constraint $u_\mu u^\mu = -1$ and $\delta u^i = \frac{1}{a} v^i$

$$u^\mu = \frac{1}{a}(1 - \phi, v^i) \to u_\mu = a(-1 - \phi, v_i + B_i), \tag{A.14}$$

where $B_i = B_{,i} + S_i$. We finally get the perturbed energy-momentum tensor

$$T^\mu_\nu = \begin{pmatrix} -(\rho + \delta\rho) & (\rho + p)(v_i + B_i) \\ -(\rho + p)v_i & (p + \delta p)\delta_{ij} \end{pmatrix}. \tag{A.15}$$

From energy-momentum conservation $\nabla_\mu T^\mu_\nu = 0$ at first order in perturbations we get

$$\delta\rho' = -3\mathcal{H}(\delta\rho + \delta p) + 3(\rho + p)\psi' - (\rho + p)v_{,ii} - \frac{1}{2}(\rho + p)E'_{,ii}, \tag{A.16}$$

completing the set of first order perturbed equations. In the next section we will see how the variables behave under gauge transformations and the way to get gauge invariant equations.

A.1 Gauge Transformations and Gauge Invariant Variables

The physical interpretation of the results of perturbation theory has to deal with the freedom in the choice of coordinates for describing the perturbations. To avoid the presence of fictitious modes or the "disappearance" of real modes it is convenient to study the gauge transformation of the perturbations. This will allow to construct the gauge-invariant variables which use will avoid the presence of unphysical modes. We could instead fix the gauge to eliminate the gauge degrees of freedom but in some cases a residual freedom can still be present, obscuring the interpretation of the results.

We have to consider a coordinate transformation

$$x^\alpha \to \tilde{x}^\alpha = x^\alpha + \xi^\alpha, \tag{A.17}$$

where ξ^α are infinitesimally small functions of space and time. The metric tensor in the coordinate system $\tilde{x}$ can be obtained using the transformation law

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \frac{\partial x^\gamma}{\partial \tilde{x}^\alpha}\frac{\partial x^\delta}{\partial \tilde{x}^\beta} g_{\gamma\delta}(x) \approx \bar{g}_{\alpha\beta} + \delta g_{\alpha\beta} - \bar{g}_{\alpha\delta}\xi^\delta_{,\beta} - \bar{g}_{\gamma\beta}\xi^\gamma_{,\alpha}. \tag{A.18}$$

In the new coordinates $\tilde{x}$ we can also split the metric into background and perturbed piece

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \bar{\tilde{g}}_{\alpha\beta}(\tilde{x}) + \delta\tilde{g}_{\alpha\beta}. \tag{A.19}$$

Comparing these two expressions and taking into account that

$$\bar{g}_{\alpha\beta}(x) \approx \bar{\tilde{g}}_{\alpha\beta}(\tilde{x}) - \bar{g}_{\alpha\beta,\gamma}\xi^{\gamma}, \tag{A.20}$$

we arrive at

$$\delta g_{\alpha\beta} \to \delta\tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} - \bar{g}_{\alpha\beta,\gamma}\xi^{\gamma} - \bar{g}_{\gamma\beta}\xi^{\gamma}_{,\alpha} - \bar{g}_{\alpha\delta}\xi^{\delta}_{,\beta}. \tag{A.21}$$

Defining $\triangle g_{\alpha\beta} = \delta\tilde{g}_{\alpha\beta} - \delta g_{\alpha\beta}$ we see that the metric components for a FRW universe transform in the following way:

$$\begin{aligned} \triangle g_{00} &= -2(aa'\xi^0 + a^2\xi^{0'}), \\ \triangle g_{0i} &= a^2(\xi^{i'} - \xi^0_{,i}), \\ \triangle g_{ij} &= a^2(\xi^i_{,j} + \xi^j_{,i}) + 2aa'\xi^0, \end{aligned} \tag{A.22}$$

using $\xi_i = \zeta_{,i} + \xi_i^{\perp}$.

We can separate them in scalar, vector and tensor components

$$\begin{aligned} &\phi \longrightarrow \phi + \mathcal{H}\xi^0 + \xi^{0'}, \\ &\psi \longrightarrow \psi - \mathcal{H}\xi^0, \\ &E \longrightarrow E + 2\zeta, \\ &B \longrightarrow B + \zeta' - \xi^0, \\ &\mathbf{F} \longrightarrow \mathbf{F} + \xi, \\ &\mathbf{S} \longrightarrow \mathbf{S} + \xi'. \end{aligned} \tag{A.23}$$

In the same way, for the case of a 4-vector the transformation will be the following

$$\delta u^{\alpha} \to \delta\tilde{u}^{\alpha} = \delta u^{\alpha} - u^{\alpha}_{,\beta}\xi^{\beta} + u^{\beta}\xi^{\alpha}_{,\beta}. \tag{A.24}$$

If we have a 4-vector of the form $u^{\alpha} = (u^0, \overrightarrow{0})$, the time and spatial components will transform

$$\begin{aligned} \triangle u^0 &= \xi^{0'}u^0 - \xi^0 u^{0'}, \\ \triangle u^i &= \xi^{i'}u^0. \end{aligned} \tag{A.25}$$

Finally for a scalar

$$\delta s \to \delta\tilde{s} = \delta s - s'\xi^0. \tag{A.26}$$

Now that we have the transformations for all types of variables we can work out ones that are invariant under gauge transformations. The simplest set of these gauge invariant variables (called Bardeen potentials) is:

$$\Psi = \psi - \mathcal{H}\left(B - \frac{1}{2}E'\right),$$

$$\Phi = \phi + \left(B' - \frac{1}{2}E''\right) + \mathcal{H}\left(B - \frac{1}{2}E'\right). \tag{A.27}$$

for the scalar sector and

$$\mathbf{Q} = \mathbf{F}' - \mathbf{S}, \tag{A.28}$$

for the vector sector. The tensor sector is already gauge invariant.

As we will see when we look for subhorizon solutions, in some cases it will be useful to write some formulas using the longitudinal (Newtonian or conformal) gauge in which $E = B = 0$. This gauge has the advantage that the gauge-invariant potentials are equal to the metric perturbations in this gauge ($\Psi = \psi_l, \Phi = \phi_l$), and there is no residual gauge freedom. Once we have all the variables in a gauge invariant form we are capable of writing the Einstein equations in terms of these variables.

A.2 Perturbed Einstein Equations

Finally, we may write the perturbed Einstein equations in terms of the gauge-invariant variables. The gauge-invariant variables for the metric components will be the Bardeen potentials and for the density, pressure and fluid velocity we have

$$\begin{aligned} \delta\rho^{GI} &= \delta\rho + \rho'\left(B - \frac{1}{2}E'\right), \\ \delta p^{GI} &= \delta p + p'\left(B - \frac{1}{2}E'\right), \\ v_i^{GI} &= v_i + \frac{1}{2}E'. \end{aligned} \tag{A.29}$$

The equations are

$$\Psi_{,ii} - 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho^{GI}, \tag{A.30}$$

$$\Psi' + \mathcal{H}\Phi = -4\pi G a^2(\rho + p)v^{GI}, \tag{A.31}$$

$$\begin{aligned} &\left[2\Psi'' + 2\mathcal{H}(\Phi' + 2\Psi') + 2(2\mathcal{H}' + \mathcal{H}^2)\Phi + (\Phi - \Psi)_{,kk}\right]\delta_{ij} \\ &\qquad + (\Psi - \Phi)_{,ij} = 8\pi G a^2 \delta p^{GI}\delta_{ij}, \end{aligned} \tag{A.32}$$

$$\text{If i}\neq\text{j}, \quad \text{then} \quad \Psi = \Phi.$$

Then the resulting equations are

$$\Phi_{,ii} - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta\rho^{GI}, \tag{A.33a}$$

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2(\rho + p)v^{GI}, \tag{A.33b}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p^{GI}. \tag{A.33c}$$

These are the scalar equations for General Relativity. In order to study the Einstein-Aether theory we will add to these equations the terms coming from the Aether sector, that we will calculate in Chapter 3.

Appendix B

Gravitational Perturbations

We will gather here the general formulas for gravitational perturbations in a Friedmann-Robertson-Walker universe. The generic perturbed metric can be written

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}.$$

The Christoffel symbols up to second order in perturbations

$$\Gamma^{\lambda}_{\mu\nu}{}^{(0)} = \frac{1}{2}\bar{g}^{\lambda\rho}\left[\bar{g}_{\mu\rho,\nu} + \bar{g}_{\nu\rho,\mu} - \bar{g}_{\mu\nu,\rho}\right], \tag{B.1}$$

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu}{}^{(1)} &= \frac{1}{2}\left[\delta g^{\lambda}_{\nu,\mu} + \delta g^{\lambda}_{\mu,\nu} - \delta g_{\mu\nu}{}^{,\lambda}\right] - \frac{1}{2}\delta g^{\lambda\rho}\left[\bar{g}_{\mu\rho,\nu} + \bar{g}_{\nu\rho,\mu} - \bar{g}_{\mu\nu,\rho}\right] \\ &= \frac{1}{2}\left[\delta g^{\lambda}_{\nu,\mu} + \delta g^{\lambda}_{\mu,\nu} - \delta g_{\mu\nu}{}^{,\lambda}\right] - \delta g^{\lambda}_{\rho}\Gamma^{\rho}_{\mu\nu}{}^{(0)},\end{aligned} \tag{B.2}$$

$$\Gamma^{\lambda}_{\mu\nu}{}^{(2)} = -\frac{1}{2}\delta g^{\lambda\rho}\left[\delta g_{\mu\rho,\nu} + \delta g_{\nu\rho,\mu} - \delta g_{\mu\nu,\rho}\right] + \delta g^{\lambda\rho}\delta g_{\xi\rho}\Gamma^{\xi}_{\mu\nu}{}^{(0)}. \tag{B.3}$$

B.1 Friedmann-Robertson-Walker

We are interested in the perturbations in a FRW universe, so in the following we are restricting ourselves to this case.

B.1.1 Metric

The perturbed metric

$$g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) = a^2\begin{pmatrix} -(1+2\phi) & B_i \\ B_i & \delta_{ij} + h_{ij}\end{pmatrix}. \tag{B.4}$$

We can split it in scalar, vector and tensor components, $B_i = B_{,i} + S_i$ and $h_{ij} = -2\psi\delta_{ij} + E_{,ij} + 2F_{(i,j)} + t_{ij}$. The inverse metric is given by

$$\begin{aligned} g^{\mu\nu} &= a^{-2}(\eta^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\rho}h^{\rho\nu}) \\ &= a^{-2}\left[\begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} - \begin{pmatrix} -2\phi & -B_i \\ -B_i & h_{ij} \end{pmatrix}\right. \\ &+ \begin{pmatrix} -4\phi^2 + B_{,i}^2 & 2(\psi-\phi)B_{,i} - B_{,j}E_{,ij} \\ 2(\psi-\phi)B_{,i} - B_{,j}E_{,ij} & 4\psi^2\delta_{ij} + E_{,ik}E_{ij} - 4\psi E_{,ij} - B_{,i}B_{,j} \end{pmatrix} \\ &+ \left.\begin{pmatrix} S_i^2 & -S_jF_{j,i} \\ -S_jF_{j,i} & 4F_{(i,k)}F_{(j,k)} - S_iS_j + t_{ik}t_{jk} \end{pmatrix}\right]. \end{aligned} \tag{B.5}$$

To calculate the gravitational action up to second order we will need to calculate the square root of the determinant of the metric up to second order. As $h_{\mu\nu} \ll$ we can expand the square root in a Taylor series

$$\begin{aligned} \sqrt{-g} &= a^4(1+\tilde{g})^{1/2} = a^4\left(1 + \frac{1}{2}\tilde{g} - \frac{1}{8}\tilde{g}^2 + \dots\right) \\ &= a^4\left(1 - \frac{1}{2}h_{00} + \frac{1}{2}h_{ii} + \frac{1}{2}h_{0i}^2 - \frac{1}{8}h_{00}^2 - \frac{1}{4}h_{00}h_{ii} + \frac{1}{4}h_{ii}h_{jj}\right. \\ &\qquad \left. - \frac{1}{8}h_{ii}^2 - \frac{1}{4}h_{ij}^2\right) \\ &= a^4\left(1 + \phi - 3\psi + \frac{1}{2}E_{,ii} - \frac{1}{2}\phi^2 + \frac{3}{2}\psi^2 + \frac{1}{4}\left(\frac{1}{2}E_{,ii}^2 - E_{,ij}^2\right)\right. \\ &\qquad \left. + \frac{1}{2}B_{,i}^2 + \frac{1}{2}\phi E_{,ii} - \frac{1}{2}\psi E_{,ii} - 3\phi\psi + \frac{1}{2}S_i^2 - F_{(i,j)}^2 - \frac{1}{4}t_{ij}^2\right), \end{aligned} \tag{B.6}$$

where

$$\tilde{g} = -h_{00} + h_{ii} - h_{00}h_{ii} + h_{0i}^2 - \frac{1}{2}h_{ij}^2 + \frac{1}{2}h_{ii}h_{jj}. \tag{B.7}$$

B.1.2 Christoffel symbols

The Christoffel symbols at zero order

$$\Gamma^{0\ (0)}_{00} = \mathcal{H}, \qquad \Gamma^{0\ (0)}_{ij} = \mathcal{H}\delta_{ij}, \qquad \Gamma^{j\ (0)}_{0i} = \mathcal{H}\delta^j_i. \tag{B.8}$$

The Christoffel symbols at first order

$$\Gamma^{0\ (1)}_{00} = \phi', \tag{B.9a}$$

$$\Gamma^{i\ (1)}_{00} = \phi_{,i} + B_i' + \mathcal{H}B_i = \phi_{,i} + B_{,i}' + \mathcal{H}B_{,i} + S_i' + \mathcal{H}S_i, \tag{B.9b}$$

$$\Gamma^{0\ (1)}_{0i} = \phi_{,i} + \mathcal{H}B_i = \phi_{,i} + \mathcal{H}B_{,i} + \mathcal{H}S_i, \tag{B.9c}$$

$$\begin{aligned}\Gamma^{0\ (1)}_{ij} =& -B_{(i,j)} + \frac{1}{2}h'_{ij} - 2\mathcal{H}\phi\delta_{ij} + \mathcal{H}h_{ij}\\ =& -\psi'\delta_{ij} - B_{,ij} + \frac{1}{2}E_{,ij} - 2\mathcal{H}(\psi+\phi)\delta_{ij} + \mathcal{H}E'_{,ij}\\ &+ F'_{(i,j)} - S_{(i,j)} + 2\mathcal{H}F_{(i,j)} + \frac{1}{2}t'_{ij} + \mathcal{H}t_{ij},\end{aligned} \tag{B.9d}$$

$$\Gamma^{j\ (1)}_{i0} = B_{[j,i]} + \frac{1}{2}h'_{ij} = -\psi'\delta_{ij} + \frac{1}{2}E'_{,ij} + S_{[j,i]} + F'_{(i,j)} + \frac{1}{2}t'_{ij}, \tag{B.9e}$$

$$\begin{aligned}\Gamma^{k\ (1)}_{ij} =& \frac{1}{2}(h_{ik,j} + h_{jk,i} - h_{ij,k}) - \mathcal{H}B_k\delta_{ij}\\ =& (\psi_{,k}\delta_{ij} - \psi_{,i}\delta_{jk} - \psi_{,j}\delta_{ik}) + \frac{1}{2}E_{,ijk} - \mathcal{H}B_{,k}\delta_{ij}\\ &+ F_{k,ij} - \mathcal{H}S_k\delta_{ij} + \frac{1}{2}(t_{ik,j} + t_{jk,i} - t_{ij,k}).\end{aligned} \tag{B.9f}$$

The Christoffel symbols at second order

$$\begin{aligned}\Gamma^{0\ (2)}_{00} =& -2\phi\phi' + \phi_{,i}B_i + B_iB'_i + \mathcal{H}B_i^2\\ =& -2\phi\phi' + \phi_{,i}B_{,i} + B_{,i}B'_{,i} + \mathcal{H}B_{,i}^2 + S_iS'_i + \mathcal{H}S_i^2,\end{aligned} \tag{B.10a}$$

$$\begin{aligned}\Gamma^{0\ (2)}_{0i} =& -2\phi\phi_{,i} + B_jB_{[j,i]} + \frac{1}{2}B_jh'_{ij} - 2\mathcal{H}\phi B_i\\ =& -2\phi\phi_{,i} - \psi'B_{,i} + \frac{1}{2}B_{,j}E'_{,ij} - 2\mathcal{H}\phi B_{,i} + S_jF'_{(i,j)} + S_jS_{[j,i]},\end{aligned} \tag{B.10b}$$

$$\begin{aligned}\Gamma^{i\ (2)}_{00} =& -\phi'B_i - \phi_{,j}h_{ij} - B'_jh_{ij} - \mathcal{H}B_jh_{ij}\\ =& -\phi'B_{,i} + 2\psi\phi_{,i} - \phi_{,l}E_{,il} + 2\psi B'_{,i} - E_{,il}B'_{,l} + 2\mathcal{H}\psi B_{,i}\\ &- \mathcal{H}E_{,il}B_{,l} - 2F_{(i,l)}S'_l - 2\mathcal{H}F_{(i,l)}S_l,\end{aligned} \tag{B.10c}$$

$$\begin{aligned}\Gamma^{0\ (2)}_{ij} =& 2\phi B_{(i,j)} - \phi h'_{ij} + \frac{1}{2}B_k(h_{ik,j} + h_{jk,i} - h_{ij,k}) + 4\mathcal{H}\phi^2\delta_{ij}\\ &- 2\mathcal{H}\phi h_{ij} - \mathcal{H}B_k^2\delta_{ij}\\ =& 2\phi B_{,ij} + 2\phi\psi'\delta_{ij} - \phi E'_{,ij} + \frac{1}{2}B_{,l}E_{,ijl} + \psi_{,l}B_{,l}\delta_{ij} - \psi_{,i}B_{,j}\\ &- \psi_{,j}B_{,i} - 2\mathcal{H}\phi E_{,ij} + 4\mathcal{H}(\phi^2 + \phi\psi)\delta_{ij} - \mathcal{H}B_{,l}^2\delta_{ij}\\ &+ S_lF_{l,ij} - \mathcal{H}S_l^2\delta_{ij},\end{aligned} \tag{B.10d}$$

$$\begin{aligned}\Gamma^{j\ (2)}_{0i} =& -\phi_{,i}B_j - h_{jk}B_{[k,i]} - \frac{1}{2}h_{jk}h'_{ik} - \mathcal{H}B_iB_j\\ =& -\phi_{,i}B_{,j} - 2\psi\psi'\delta_{ij} - \frac{1}{2}E'_{,il}E_{,jl} + \psi E'_{,ij} + \psi'E_{,ij} - \mathcal{H}B_{,i}B_{,j}\\ &- 2F'_{(i,l)}F_{(j,l)} - 2F_{(j,l)}S_{[l,i]} - \mathcal{H}S_iS_j - \frac{1}{2}t'_{il}t_{jl},\end{aligned} \tag{B.10e}$$

$$\begin{aligned}\Gamma^{k\ (2)}_{ij} =& B_kB_{(i,j)} - \frac{1}{2}B_kh'_{ij} - \frac{1}{2}h_{kl}(h_{il,j} + h_{jl,i} - h_{ij,l}) + 2\mathcal{H}\phi B_k\delta_{ij}\\ &+ \mathcal{H}B_lh_{kl}\delta_{ij} - \mathcal{H}B_kh_{ij}\end{aligned}$$

$$
\begin{aligned}
&= B_{,ij}B_{,k} + \psi' B_{,k}\delta_{ij} - \frac{1}{2}B_{,k}E'_{,ij} - \frac{1}{2}E_{,ijl}E_{,kl} \\
&\quad + 2\psi(\psi_{,k}\delta_{ij} - \psi_{,i}\delta_{jk} - \psi_{,j}\delta_{ik}) + \psi E_{,ijk} - \psi_{,l}E_{,kl}\delta_{ij} \\
&\quad + 2\psi_{(,i}E_{,j)k} + 2\mathcal{H}\phi B_{,k}\delta_{ij} + \mathcal{H}B_{,l}E_{,kl}\delta_{ij} - \mathcal{H}B_{,k}E_{,ij} \\
&\quad + S_k S_{(i,j)} - S_k F'_{(i,j)} - 2F_{(k,l)}F_{l,ij} + 2\mathcal{H}(S_l F_{(k,l)}\delta_{ij} - S_k F_{(i,j)}) \\
&\quad - \frac{1}{2}t_{kl}(t_{il,j} + t_{jl,i} - t_{ij,l}). \qquad \text{(B.10f)}
\end{aligned}
$$

B.1.3 Ricci tensor and scalar

The Ricci tensor at zero order is

$$R^{(0)}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\rho\alpha}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\rho}_{\mu\alpha}, \qquad \text{(B.11)}$$

$$
\begin{aligned}
R^{(0)}_{00} &= -\Gamma^{i}_{0i,0} + \Gamma^{0}_{00}\Gamma^{i}_{0i} - \Gamma^{i}_{0j}\Gamma^{j}_{0i} = -3\mathcal{H}', && \text{(B.12a)} \\
R^{(0)}_{0i} &= 0 && \text{(B.12b)} \\
R^{(0)}_{ij} &= \Gamma^{0}_{ij,0} + \Gamma^{0}_{00}\Gamma^{0}_{ij} + \Gamma^{0}_{ij}\Gamma^{k}_{0k} - \Gamma^{k}_{0i}\Gamma^{0}_{kj} - \Gamma^{k}_{0j}\Gamma^{0}_{ki} = (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij}. && \text{(B.12c)}
\end{aligned}
$$

And the Ricci scalar

$$R^{(0)} = a^{-2}\eta^{\mu\nu}R^{(0)}_{\mu\nu} = 6(\mathcal{H}' + \mathcal{H}^2)a^{-2}. \qquad \text{(B.13)}$$

At order one

$$
\begin{aligned}
R^{(1)}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha}{}^{(1)} - \Gamma^{\alpha}_{\mu\alpha,\nu}{}^{(1)} + \Gamma^{\alpha}_{\rho\alpha}{}^{(1)}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\alpha}_{\rho\alpha}\Gamma^{\rho}_{\mu\nu}{}^{(1)} \\
-\Gamma^{\alpha}_{\rho\nu}{}^{(1)}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\rho}_{\mu\alpha}{}^{(1)}, \qquad \text{(B.14)}
\end{aligned}
$$

$$
\begin{aligned}
R^{(1)}_{00} &= \Gamma^{i}_{00,i}{}^{(1)} - \Gamma^{i}_{0i,0}{}^{(1)} + \Gamma^{0}_{00}\Gamma^{i}_{0i}{}^{(1)} + \Gamma^{i}_{0i}\Gamma^{0}_{00}{}^{(1)} - 2\Gamma^{i}_{0j}\Gamma^{j}_{0i}{}^{(1)} \\
&= \phi_{,ii} + B'_{i,i} - \frac{1}{2}h''_{ii} + 3\mathcal{H}\phi' + \mathcal{H}B_{i,i} - \frac{1}{2}\mathcal{H}h'_{ii} \\
&= \phi_{,ii} + B'_{,ii} + 3\psi'' - \frac{1}{2}E''_{,ii} + 3\mathcal{H}(\phi' + \psi') + \mathcal{H}B_{,ii} - \frac{1}{2}\mathcal{H}E'_{,ii}, && \text{(B.15a)} \\
R^{(1)}_{0i} &= \Gamma^{0}_{0i,0}{}^{(1)} + \Gamma^{j}_{0i,j}{}^{(1)} - \Gamma^{0}_{00,i}{}^{(1)} - \Gamma^{j}_{0j,i}{}^{(1)} + \Gamma^{j}_{0j}\Gamma^{0}_{0i}{}^{(1)} + \Gamma^{j}_{0i}\Gamma^{k}_{jk}{}^{(1)} \\
&\quad - \Gamma^{j}_{0k}\Gamma^{k}_{ij}{}^{(1)} - \Gamma^{0}_{ij}\Gamma^{j}_{00}{}^{(1)} \\
&= B_{[j,i]j} + h'_{j[i,j]} + 2\mathcal{H}\phi_{,i} + (\mathcal{H}' + 2\mathcal{H}^2)B_i \\
&= 2\psi'_{,i} + 2\mathcal{H}\phi_{,i} + (\mathcal{H}' + 2\mathcal{H}^2)B_{,i} + \frac{1}{2}(F' - S)_{i,jj} + (\mathcal{H}' + 2\mathcal{H}^2)S_i, && \text{(B.15b)}
\end{aligned}
$$

$$\begin{aligned}
R_{ij}^{(1)} &= \Gamma^{0\ (1)}_{ij,0} + \Gamma^{k\ (1)}_{ij,k} - \Gamma^{0\ (1)}_{0i,j} - \Gamma^{k\ (1)}_{ik,j} + \left(\Gamma^0_{00} + \Gamma^k_{0k}\right)\Gamma^{0\ (1)}_{ij} \\
&\quad + \Gamma^0_{ij}\left(\Gamma^{0\ (1)}_{00} + \Gamma^{k\ (1)}_{0k}\right) - \Gamma^0_{ik}\Gamma^{k\ (1)}_{0j} - \Gamma^0_{jk}\Gamma^{k\ (1)}_{0i} - \Gamma^k_{0i}\Gamma^{0\ (1)}_{kj} \\
&\quad - \Gamma^k_{0j}\Gamma^{0\ (1)}_{ik} \\
&= -\phi_{,ij} - B'_{(i,j)} + \frac{1}{2}h''_{ij} + \frac{1}{2}(h_{ik,jk} + h_{jk,ik} - h_{ij,kk} - h_{kk,ij}) \\
&\quad - \mathcal{H}\phi'\delta_{ij} + \mathcal{H}h'_{ij} + \frac{1}{2}\mathcal{H}h'_{kk}\delta_{ij} - 2\mathcal{H}B_{(i,j)} - \mathcal{H}B_{k,k}\delta_{ij} \\
&\quad - 2(\mathcal{H}' + 2\mathcal{H}^2)\phi\delta_{ij} + (\mathcal{H}' + 2\mathcal{H}^2)h_{ij} \\
&= (\psi_{,kk} - \psi'')\delta_{ij} + \psi_{,ij} - \left(\phi + B' - \frac{1}{2}E''\right)_{,ij} - 2\mathcal{H}B_{,ij} + \mathcal{H}E'_{,ij} \\
&\quad + (2\mathcal{H}^2 + \mathcal{H}')(E_{,ij} - 2(\phi + \psi)\delta_{ij}) - \mathcal{H}(\phi' + 5\psi')\delta_{ij} \\
&\quad - \mathcal{H}\left(B_{,kk} - \frac{1}{2}E'_{,kk}\right)\delta_{ij} \\
&\quad + F''_{(i,j)} - S'_{(i,j)} + 2\mathcal{H}(F'_{(i,j)} - S_{(i,j)}) + (2\mathcal{H}' + 4\mathcal{H}^2)F_{(i,j)} \\
&\quad + \frac{1}{2}(t''_{ij} - t_{ij,kk}) + \mathcal{H}t'_{ij} + (\mathcal{H}' + 2\mathcal{H}^2)t_{ij}.
\end{aligned} \tag{B.15c}$$

$$\begin{aligned}
R^{(1)} &= a^{-2}\left(\eta^{\mu\nu}R^{(1)}_{\mu\nu} - h^{\mu\nu}R^{(0)}_{\mu\nu}\right) \\
&= a^{-2}\left[-2\phi_{,ii} - 2B'_{i,i} + h''_{ii} + h_{ik,ik} - h_{ii,kk} - 6\mathcal{H}\phi' - 6\mathcal{H}B_{i,i}\right. \\
&\qquad \left. + 3\mathcal{H}h'_{ii} - 12(\mathcal{H}' + \mathcal{H}^2)\phi\right] \\
&= a^{-2}\left[-2\phi_{,ii} - 2B'_{,ii} - 6\psi'' + E''_{,ii} - 2\psi_{,ii} + 6\psi_{,ii}\right. \\
&\qquad \left. - 6\mathcal{H}\phi' - 6\mathcal{H}B_{,ii} - 18\mathcal{H}\psi' + 3\mathcal{H}E'_{,ii} - 12(\mathcal{H}' + \mathcal{H}^2)\phi\right] \\
&= a^{-2}\left[-2\phi_{,ii} - 2B'_{,ii} - 6\psi'' + E''_{,ii} + 4\psi_{,ii} - 6\mathcal{H}\phi' - 6\mathcal{H}B_{,ii}\right. \\
&\qquad \left. - 18\mathcal{H}\psi' + 3\mathcal{H}E'_{,ii} - 12(\mathcal{H}' + \mathcal{H}^2)\phi\right].
\end{aligned} \tag{B.16}$$

At second order

$$\begin{aligned}
R^{(2)}_{\mu\nu} &= \Gamma^{\alpha\ (2)}_{\mu\nu,\alpha} - \Gamma^{\alpha\ (2)}_{\mu\alpha,\nu} + \Gamma^{\alpha\ (1)}_{\rho\alpha}\Gamma^{\rho\ (1)}_{\mu\nu} - \Gamma^{\alpha\ (1)}_{\rho\nu}\Gamma^{\rho\ (1)}_{\mu\alpha} \\
&\quad + \Gamma^{\rho\ (2)}_{\mu\nu}\Gamma^{\alpha}_{\rho\alpha} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\alpha\ (2)}_{\rho\alpha} - \Gamma^{\rho\ (2)}_{\mu\alpha}\Gamma^{\alpha}_{\rho\nu} - \Gamma^{\rho}_{\mu\alpha}\Gamma^{\alpha\ (2)}_{\rho\nu}
\end{aligned} \tag{B.17}$$

We will write separately the scalar, vector and tensor sectors; first the R_{00} component

$$\begin{aligned}
R^{(2)}_{00} &= \Gamma^{i\ (2)}_{00,i} - \Gamma^{i\ (2)}_{0i,0} + \Gamma^{i\ (1)}_{00}\left(\Gamma^{j\ (1)}_{ij} - \Gamma^{0\ (1)}_{0i}\right) + \Gamma^{0\ (1)}_{00}\Gamma^{i\ (1)}_{0i} - \Gamma^{j\ (1)}_{0i}\Gamma^{i\ (1)}_{0j} \\
&\quad + \Gamma^{0\ (2)}_{00}\Gamma^{i}_{0i} + \Gamma^{0}_{00}\Gamma^{i\ (2)}_{0i} - 2\Gamma^{i\ (2)}_{0j}\Gamma^{j}_{0i} \\
&= -\phi^2_{,i} - \phi'B_{i,i} - \phi_{,ij}h_{ij} - \phi_{,j}h_{ij,i} + \frac{1}{2}(\phi_i h_{jj,i} + \phi' h'_{ii})
\end{aligned}$$

$$
\begin{aligned}
& - B_{[i,j]}B_{[j,i]} + \frac{1}{2}B_i' h_{jj,i} + B_{[j,i]}' h_{ij} + B_{[j,i]} h_{ij}' + \frac{1}{4}h_{ij}' + \frac{1}{2}h_{ij}'' h_{ij} \\
& - 6\mathcal{H}\phi'\phi + \mathcal{H}\phi_{,i}B_i + 3\mathcal{H}B_i' B_i + (2\mathcal{H}^2 + \mathcal{H}')B_i^2 + \mathcal{H}B_{[j,i]}h_{ij} \\
& - \mathcal{H}B_{j,i}h_{ij} - \mathcal{H}B_j h_{ij,i} + \frac{1}{2}\mathcal{H}B_i h_{jj,i} + \frac{1}{2}\mathcal{H}h_{ij}' h_{ij}, \qquad \text{(B.18a)}
\end{aligned}
$$

$$
\begin{aligned}
R_{00}^{(2)s} = & \, 3\psi'^2 + 6\psi''\psi - \phi_{,i}^2 - 3\psi'\phi' - \phi_{,i}\psi_{,i} + 2\phi_{,ii}\psi - \phi' B_{,ii} + \frac{1}{4}E_{,ij}'^2 \\
& - \psi' E_{,ii}' - \psi'' E_{,ii} - \psi E_{,ii}'' + \frac{1}{2}(\phi' E_{,ii}' - \phi_{,i}E_{,ijj} - 2\phi_{,ij}E_{,ij}) \\
& + \frac{1}{2}B_{,i}' E_{,ijj} - 3\psi_{,i}B_{,i}' + 6\mathcal{H}(\psi\psi' - \phi\phi') + \mathcal{H}\phi_{,i}B_{,i} - \mathcal{H}\psi_{,i}B_{,i} \\
& + 2\mathcal{H}\psi B_{,ii} + \frac{1}{2}\mathcal{H}E_{,ij}' E_{,ij} - \mathcal{H}\psi E_{,ii}' - \mathcal{H}\psi' E_{,ii} - \frac{1}{2}\mathcal{H}B_{,i}E_{,ijj} \\
& - \mathcal{H}B_{,ij}E_{,ij} + 3\mathcal{H}B_{,i}' B_{,i} + (\mathcal{H}' + 2\mathcal{H}^2)B_{,i}^2, \qquad \text{(B.18b)}
\end{aligned}
$$

$$
\begin{aligned}
R_{00}^{(2)v} = & - S_{[i,j]}S_{[j,i]} + F_{(i,j)}'^2 + 2F_{(i,j)}'' F_{(i,j)} + 2S_{[j,i]}' F_{(i,j)} + 2S_{[j,i]}F_{(i,j)}' \\
& + \mathcal{H}F_{i,j}' F_{i,j} + 3\mathcal{H}S_i' S_i + \mathcal{H}' S_i^2 + 2\mathcal{H}^2 S_i^2 + 2\mathcal{H}S_{[j,i]}F_{(i,j)} \\
& - 2\mathcal{H}S_{j,i}F_{(i,j)} - 2\mathcal{H}S_j F_{(i,j)i}, \qquad \text{(B.18c)}
\end{aligned}
$$

$$
R_{00}^{(2)t} = \frac{1}{2}t_{ij}'' t_{ij} + \frac{1}{4}t_{ij}'^2 + \frac{1}{2}\mathcal{H}t_{ij}' t_{ij}. \qquad \text{(B.18d)}
$$

and then the R_{ij} component

$$
\begin{aligned}
R_{ij}^{(2)} = & \, \Gamma_{ij,0}^{0\,(2)} + \Gamma_{ij,k}^{k\,(2)} - \Gamma_{i0,j}^{0\,(2)} - \Gamma_{ik,j}^{k\,(2)} + \Gamma_{ij}^{0\,(1)}\left(\Gamma_{00}^{0\,(1)} + \Gamma_{0k}^{k\,(1)}\right) \\
& + \Gamma_{ij}^{k\,(1)}\left(\Gamma_{0k}^{0\,(1)} + \Gamma_{lk}^{l\,(1)}\right) - \Gamma_{0i}^{0\,(1)}\Gamma_{0j}^{0\,(1)} - \Gamma_{jk}^{0\,(1)}\Gamma_{0i}^{k\,(1)} \\
& - \Gamma_{ik}^{0\,(1)}\Gamma_{0j}^{k\,(1)} - \Gamma_{ik}^{l\,(1)}\Gamma_{jl}^{k\,(1)} + \Gamma_{ij}^{0\,(2)}\left(\Gamma_{00}^{0} + \Gamma_{0k}^{k}\right) \\
& + \Gamma_{ij}^{0}\left(\Gamma_{00}^{0\,(2)} + \Gamma_{0k}^{k\,(2)}\right) - \Gamma_{i0}^{k}\Gamma_{kj}^{0\,(2)} - \Gamma_{0j}^{k\,(2)}\Gamma_{ik}^{0} - \Gamma_{j0}^{k}\Gamma_{ki}^{0\,(2)} \\
& - \Gamma_{0i}^{k\,(2)}\Gamma_{jk}^{0} \\
= & \, \phi_{,i}\phi_{,j} + 2\phi\phi_{,ij} + \phi' B_{(i,j)} + 2\phi B_{(i,j)}' - \frac{1}{2}\phi' h_{ij}' - \phi h_{ij}'' \\
& + \frac{1}{2}\phi_{,k}(h_{ik,j} + h_{jk,i} - h_{ij,k}) + B_{[k,i]}B_{(j,k)} + B_{[k,j]}B_{(i,k)} \\
& - B_{k,i}B_{k,j} + B_{k,k}B_{(i,j)} - B_k B_{k,ij} + B_k B_{(i,j)k} \\
& + \frac{1}{2}B_k'(h_{ik,j} + h_{jk,i} - h_{ij,k}) + \frac{1}{2}B_k(h_{ik,j}' + h_{jk,i}') \\
& - \frac{1}{2}B_{k,k}h_{ij}' - \frac{1}{2}B_{(i,j)}h_{kk}' + \frac{1}{2}B_{i,k}h_{jk}' - \frac{1}{2}B_{[k,j]}h_{ik}' \\
& - B_k h_{ij,k}' + \frac{1}{2}h_{kl}(h_{kl,ij} + h_{ij,kl} - h_{jl,ik} - h_{ik,jl}) \\
& + \frac{1}{4}(h_{il,k}h_{kl,j} + h_{kl,i}h_{kl,j} - h_{ik,l}h_{kl,j})
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}(h_{il,k}h_{jl,k}-h_{ik,l}h_{jl,k})-\frac{1}{2}h_{kl,k}(h_{il,j}+h_{jl,i}-h_{ij,l})\\
&+\frac{1}{4}h_{ll,k}(h_{ik,j}+h_{jk,i}-h_{ij,k})+\frac{1}{4}h'_{ij}h'_{kk}-\frac{1}{2}h'_{ik}h'_{jk}\\
&+4\mathcal{H}\phi'\phi\delta_{ij}+\mathcal{H}\phi_{,k}B_k\delta_{ij}+2\mathcal{H}\phi B_{k,k}\delta_{ij}+4\mathcal{H}\phi B_{(i,j)}\\
&-\mathcal{H}\phi' h_{ij}-2\mathcal{H}\phi h'_{ij}-\mathcal{H}\phi h'_{kk}\delta_{ij}-\mathcal{H}B'_kB_k\delta_{ij}+\mathcal{H}B_lh_{kl,k}\delta_{ij}\\
&+\mathcal{H}B_k(h_{ik,j}+h_{jk,i}-h_{ij,k})-\frac{1}{2}\mathcal{H}B_kh_{ll,k}\delta_{ij}+\mathcal{H}B_{l,k}h_{kl}\delta_{ij}\\
&-\mathcal{H}B_{[k,l]}h_{lk}\delta_{ij}-\mathcal{H}B_{k,k}h_{ij}+\frac{1}{2}\mathcal{H}(h'_{kk}h_{ij}-h'_{kl}h_{kl}\delta_{ij})\\
&+(2\mathcal{H}^2+\mathcal{H}')(4\phi^2\delta_{ij}-2\phi h_{ij}-B_k^2\delta_{ij}),
\end{aligned}
\tag{B.19a}
$$

$$
\begin{aligned}
R^{(2)s}_{ij}=&\Big(\psi'^2+2\psi\psi_{,kk}+\psi_{,k}^2+\psi'\phi'+2\phi\psi''+\phi_{,k}\psi_{,k}+B_{,kk}\psi'\\
&\quad+B'_{,k}\psi_{,k}+2B_{,k}\psi'_{,k}-\frac{1}{2}\psi'E'_{,kk}-\frac{1}{2}\psi_{,k}E_{,kll}-\psi_{,kl}E_{,kl}\Big)\delta_{ij}\\
&+3\psi_{,i}\psi_{,j}+2\psi\psi_{,ij}-2\phi_{,i}\psi_{,j}+\phi_{,i}\phi_{,j}+2\phi\phi_{,ij}+2\psi'B_{,ij}\\
&-2B'_{,i}\psi_{,j}-2B_{,i}\psi'_{,j}+\phi'B_{,ij}+2\phi B'_{,ij}\\
&+\frac{1}{4}E'_{,ij}E'_{,kk}-\frac{1}{2}E'_{,ik}E'_{,jk}+\frac{1}{4}E_{,ilk}E_{,jlk}-\frac{1}{4}E_{,ijk}E_{,llk}\\
&+\frac{1}{2}\big(\phi_{,k}E_{,ijk}+B'_{,k}E_{,ijk}-B_{,kk}E'_{,ij}+B_{,ik}E'_{,jk}-B_{,ij}E'_{,kk}\\
&-\phi'E'_{,ij}+\psi'E'_{,ij}-\psi_{,k}E_{,ijk}\big)+2\psi_{,ik}E_{,jk}-\psi_{,ij}E_{,kk}-\phi E''_{,ij}\\
&+B_{,kk}B_{,ij}-B_{,ik}B_{,jk}+\Big(10\mathcal{H}\phi\psi'+2\mathcal{H}\phi'\psi+4\mathcal{H}\phi'\phi\\
&+\mathcal{H}\phi_{,k}B_{,k}+2\mathcal{H}\phi B_{,kk}+3\mathcal{H}\psi_{,k}B_{,k}-\frac{1}{2}\mathcal{H}E'_{,kl}E_{,kl}+\mathcal{H}\psi'E_{,kk}\\
&-\mathcal{H}\phi E'_{,kk}+\frac{1}{2}\mathcal{H}B_{,k}E_{,kll}+\mathcal{H}B_{,kl}E_{,kl}-\mathcal{H}B'_{,k}B_{,k}\Big)\delta_{ij}\\
&-\mathcal{H}\phi'E'_{,ij}-2\mathcal{H}\phi E'_{,ij}-3\mathcal{H}\psi E_{,ij}+4\mathcal{H}\phi B_{,ij}-4\mathcal{H}\psi_{,i}B_{,j}\\
&+\frac{1}{2}\mathcal{H}E'_{,kk}E_{,ij}+\mathcal{H}B_{,k}E_{,ijk}-\mathcal{H}B_{,kk}E_{,ij}\\
&+(\mathcal{H}'+2\mathcal{H}^2)\left(4\phi^2-B_{,k}^2+4\phi\psi\right)\delta_{ij}-2(\mathcal{H}'+2\mathcal{H}^2)\phi E_{,ij},
\end{aligned}
\tag{B.19b}
$$

$$
\begin{aligned}
R^{(2)v}_{ij}=&S_kS_{(i,j)k}-S_{,k}S_{k,ij}-\frac{1}{2}\left(S_{k,i}S_{k,j}+S_{i,k}S_{j,k}\right)-2F'_{(i,k)}F'_{(j,k)}\\
&+\frac{1}{2}\left(F_{l,ik}F_{l,jk}-F_{l,kk}F_{l,ij}+F_{l,ik}F_{(j,k)l}-F_{k,il}F_{(j,l)k}\right)\\
&+S'_kF_{k,ij}+F'_{k,ij}S_k+V_{i,k}F'_{(j,k)}\\
&-\mathcal{H}\left(S'_kS_k+2F'_{k,l}F_{k,l}-F_{k,ll}S_k-F_{(k,l)}S_{k,l}\right)\delta_{ij}+2\mathcal{H}F_{k,ij}S_k\\
&-(\mathcal{H}'+2\mathcal{H}^2)S_k^2\delta_{ij},
\end{aligned}
\tag{B.19c}
$$

$$\begin{aligned}R_{ij}^{(2)t} =& \frac{1}{2}t_{kl}\left(t_{kl,ij}+t_{ij,kl}-t_{jl,ik}-t_{ik,jl}\right)+\frac{1}{4}t_{kl,i}t_{kl,j}-\frac{1}{2}t'_{ik}t'_{jk}\\ &-\frac{1}{2}\mathcal{H}t'_{lk}t_{lk}\delta_{ij}.\end{aligned} \tag{B.19d}$$

The R_{0i} does not appear in the calculation of the Ricci scalar at second order, so we do not include it here. The second order Ricci scalar is

$$\begin{aligned}R^{(2)} &= a^{-2}\left(\eta^{\mu\nu}R^{(2)}_{\mu\nu}-h^{\mu\nu}R^{(1)}_{\mu\nu}+h^{\mu}{}_{\rho}h^{\rho\nu}R^{(0)}_{\mu\nu}\right)\\ &= a^{-2}\Big(\eta^{00}R^{(2)}_{00}+\eta^{ij}R^{(2)}_{ij}-h^{00}R^{(1)}_{00}-h^{ij}R^{(1)}_{ij}-2h^{0i}R^{(1)}_{0i}\\ &\qquad+\left(h^{0}{}_{0}h^{00}+h^{0}{}_{i}h^{i0}\right)R^{(0)}_{00}+\left(h^{i}{}_{k}h^{kj}+h^{i}{}_{0}h^{0j}\right)R^{(0)}_{ij}\Big)\\ &= a^{-2}\Big[2\phi_{,i}^2+4\phi\phi_{,ii}+2\phi' B_{i,i}+4\phi B'_{i,i}+2\phi_{,ij}h_{ij}+2\phi_{,k}h_{lk,l}\\ &\qquad-\phi_k h_{ll,k}-2\phi h''_{ii}-\phi' h'_{ii}+\frac{1}{2}B_{i,j}B_{j,i}-\frac{3}{2}B^2_{i,j}+B_{i,i}B_{j,j}\\ &\qquad+2B_jB_{i,ij}-2B_iB_{i,jj}+\frac{1}{4}h'^2_{ii}-\frac{3}{4}h'^2_{ij}-h''_{ij}h_{ij}\\ &\qquad+h_{jk}h_{jk,ii}+h_{jk}h_{ii,jk}-h_{jk}h_{ik,ij}-h_{ij}h_{ik,jk}+\frac{3}{4}h^2_{ij,k}\\ &\qquad-\frac{1}{4}h_{ii,j}h_{kk,j}+\frac{1}{4}(h_{ik,j}h_{jk,i}-h_{ij,k}h_{jk,i}-2h_{ij,k}h_{ik,j})\\ &\qquad+h_{ij,i}h_{kk,j}-h_{ij,i}h_{kj,k}+B'_jh_{ij,i}-B'_jh_{ii,j}+B'_{i,j}h_{ij}\\ &\qquad-B_{j,j}h'_{ii}-2B_{[j,i]}h'_{ij}+2B_jh'_{ij,i}-2B_jh'_{ii,j}+B_{(i,j)}h'_{ij}\\ &\qquad+24\mathcal{H}\phi'\phi+6\mathcal{H}\phi_{,i}B_{,i}+12\mathcal{H}\phi B_{i,i}-6\mathcal{H}\phi h'_{ii}-6\mathcal{H}B'_iB_i\\ &\qquad-3\mathcal{H}B_jh_{ii,j}+6\mathcal{H}B_jh_{ij,i}+6\mathcal{H}B_{j,i}h_{ij}-3\mathcal{H}h'_{ij}h_{ij}\\ &\qquad+24(\mathcal{H}'+\mathcal{H}^2)\phi^2-6(\mathcal{H}'+\mathcal{H}^2)B_i^2\Big]\end{aligned} \tag{B.20}$$

B.1.4 Lagrangian density

Finally, with all these pieces we can calculate the Lagrangian density of GR up to second order. As before, we will split it in scalar, vector and tensor components, and we will use integration by parts to obtain the more compact form.

$$\mathcal{L}^{(2)} = \frac{1}{2}M_P^2\left(\sqrt{-g}^{(1)}R^{(1)}+\sqrt{-g}^{(0)}R^{(2)}+\sqrt{-g}^{(2)}R^{(0)}\right), \tag{B.21}$$

$$\begin{aligned}\mathcal{L}_s^{(2)} = \frac{1}{2}M_P^2a^2\Big[&2\phi_{,i}^2+2\phi\phi_{,ii}+2\phi' B_{,ii}+2\phi B'_{,ii}+6\phi\psi''+6\phi'\psi'\\ &-\phi E''_{,ii}-\phi' E'_{,ii}+2\phi_{,ii}\psi+2\phi_{,i}\psi_{,i}+4\phi\psi_{,ii}\end{aligned}$$

$$
\begin{aligned}
&+2\phi_{,ij}E_{,ij} - \phi_{,ii}E_{,jj} + \phi_{,i}E_{,ijj} + B_{,ii}B_{,jj} - B_{,ij}^2 \\
&+4B'_{,i}\psi_{,i} + 4B'_{,ii}\psi + 8B_{,i}\psi'_{,i} + 4B_{,ii}\psi' + B'_{,ij}E_{,ij} \\
&-B'_{,ii}E_{,jj} + B_{,ij}E'_{,ij} - B_{,ii}E'_{,jj} + 6\psi''\psi + 6\psi_{,i}^2 + 4\psi_{,ii}\psi \\
&-\psi''E_{,ii} - \psi E''_{,ii} - 2\psi_{,j}E_{,iij} - 2\psi_{,ij}E_{,ij} + \frac{1}{4}E'^2_{,ii} \\
&-\frac{3}{4}E'^2_{,ij} + \frac{1}{2}E''_{,ii}E_{,jj} - E''_{,ij}E_{,ij} + \frac{1}{4}E^2_{,ijk} - \frac{1}{4}E_{,iij}E_{,kkj} \\
&+18\mathcal{H}\phi'\phi + 6\mathcal{H}\left(\phi_{,i}B_{,i} + 6\phi B_{,ii}\right) + 18\mathcal{H}\left(\phi\psi' + \phi'\psi\right) \\
&-3\mathcal{H}\left(\phi E'_{,ii} + \phi' E_{,ii}\right) - 6\mathcal{H}B'_{,i}B_{,i} + 6\mathcal{H}\left(B_{,i}\psi_{,i} + B_{,ii}\psi\right) \\
&+3\mathcal{H}B_{,i}E_{,ijj} + 6\mathcal{H}B_{,ij}E_{,ij} - 3\mathcal{H}B_{,ii}E_{,jj} + 18\mathcal{H}\psi'\psi \\
&-3\mathcal{H}\left(\psi' E_{,ii} + \psi E'_{,ii}\right) + \frac{3}{2}\mathcal{H}E'_{,ii}E_{,jj} - 3\mathcal{H}E'_{,ij}E_{,ij} \\
&+9\frac{a''}{a}\phi^2 + 18\frac{a''}{a}\phi\psi - 3\frac{a''}{a}\phi E_{,ii} - 3\frac{a''}{a}B^2_{,i} + 9\frac{a''}{a}\psi^2 \\
&\left.-3\frac{a''}{a}\psi E_{,ii} + \frac{3}{4}\frac{a''}{a}E^2_{,ii} - \frac{3}{2}\frac{a''}{a}E_{,ij}E_{,ij}\right] \\
\overset{i.b.p.}{=}\ & \frac{1}{2}M_P^2a^2\left[4\phi\psi_{,ii} - 4B_{,ii}\psi' + 2\psi_{,i}^2 - 6\psi'^2 + 2\psi'E'_{,ii} - 4\mathcal{H}\phi B_{,ii}\right. \\
&-12\mathcal{H}\phi\psi' + 2\mathcal{H}\phi E'_{,ii} - 9\mathcal{H}^2\phi^2 - 18\mathcal{H}^2\phi\psi + 3\mathcal{H}^2\phi E_{,ii} \\
&\left.+3\mathcal{H}^2B^2_{,i} + (2\mathcal{H}' + \mathcal{H}^2)\left(3\psi^2 - \psi E_{,ii} - \frac{1}{4}E^2_{,ij}\right)\right],
\end{aligned}
\tag{B.22a}
$$

$$
\begin{aligned}
\mathcal{L}_v^{(2)} =\ & \frac{1}{2}M_P^2a^2\left[\frac{1}{2}S_{i,j}S_{j,i} - \frac{3}{2}S_{i,j}^2 - 2S_iS_{i,jj} + S'_iF_{i,jj} + 2S_iF'_{i,jj}\right. \\
&+2S_{i,j}F'_{(i,j)} - 2S'_{i,j}F_{(i,j)} - 3F'^2_{(i,j)} - 4F''_{(i,j)}F_{(i,j)} \\
&+F^2_{i,jk} - F_{i,jj}F_{i,kk} + F_{i,jk}F_{j,ik} - \frac{1}{2}F_{i,jk}F_{k,ij} \\
&-\frac{1}{2}F_{j,ik}F_{k,ij} + 6\mathcal{H}\left(S_iF_{i,jj} + S_{i,j}F_{i,j} + S_{i,j}F_{j,i} - S'_iS_i\right. \\
&\left.\left.-F'_{i,j}F_{i,j} - F'_{i,j}F_{j,i}\right) - 3\frac{a''}{a}\left(F^2_{i,j} + S^2_i + F_{i,j}F_{j,i}\right)\right] \\
\overset{i.b.p.}{=}\ & \frac{1}{2}M_P^2a^2\left[\frac{1}{2}(F'_{i,j} - S_{i,j})^2 + 3\mathcal{H}^2S_i^2 - (2\mathcal{H}' + \mathcal{H}^2)F^2_{i,j}\right],
\end{aligned}
\tag{B.22b}
$$

$$
\begin{aligned}
\mathcal{L}_t^{(2)} =\ & \frac{1}{2}M_P^2a^2\left[-\frac{3}{4}t'^2_{ij} - t''_{ij}t_{ij} + t_{ij}t_{ij,kk} - \frac{3}{4}t^2_{ij,k} - \frac{1}{2}t_{ij,k}t_{ik,j}\right. \\
&\left.-3\mathcal{H}t'_{ij}t_{ij} - \frac{3}{2}\frac{a''}{a}t^2_{ij}\right] \\
\overset{i.b.p.}{=}\ & \frac{1}{8}M_P^2a^2\left[t'^2_{ij} - t^2_{ij,k} - 2(2\mathcal{H}' + \mathcal{H}^2)t^2_{ij}\right].
\end{aligned}
\tag{B.22c}
$$

B.1.5 Einstein tensor

The Einstein tensor at order zero

$$G^{(0)}_{\mu\nu} = R^{(0)}_{\mu\nu} - \frac{1}{2}a^2\eta_{\mu\nu}R^{(0)}, \tag{B.23}$$

$$G^{(0)}_{00} = 3\mathcal{H}^2, \qquad G^{(0)}_{0i} = 0, \qquad G^{(0)}_{ij} = -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij}. \tag{B.24}$$

And finally at first order

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{a^2}{2}\left(\eta_{\mu\nu}R^{(1)} + h_{\mu\nu}R^{(0)}\right), \tag{B.25}$$

$$\begin{aligned}
G^{(1)}_{00} &= R^{(1)}_{00} - \frac{a^2}{2}\left(h_{00}R + \eta_{00}R^{(1)}\right) \\
&= 2\psi_{,ii} - \mathcal{H}\left(6\psi' + 2B_{,ii} - E'_{,ii}\right), \qquad \text{(B.26a)} \\
G^{(1)}_{0i} &= R^{(1)}_{0i} - \frac{a^2}{2}h_{0i}R \\
&= 2\psi'_{,i} + 2\mathcal{H}\phi_{,i} - (\mathcal{H}^2 + 2\mathcal{H}')B_{,i} + \frac{1}{2}(F'_{i,jj} - S_{i,jj}) \\
&\quad - (\mathcal{H}^2 + 2\mathcal{H}')S_i, \qquad \text{(B.26b)} \\
G^{(1)}_{ij} &= R^{(1)}_{ij} - \frac{a^2}{2}\left(h_{ij}R + \eta_{ij}R^{(1)}\right) \\
&= \left[2\psi'' + \left(\phi - \psi + B' - \frac{1}{2}E''\right)_{,kk} + 2\mathcal{H}\left(\phi' + 2\psi'\right)\right. \\
&\quad \left. + 2\mathcal{H}\left(B_{,kk} - \frac{1}{2}E'_{,kk}\right) + 2(2\mathcal{H}' + \mathcal{H}^2)(\phi + \psi)\right]\delta_{ij} \\
&\quad + \left(\psi - \phi - B' + \frac{1}{2}E'' - 2\mathcal{H}\left(B - \frac{1}{2}E'\right) - (2\mathcal{H}' + \mathcal{H}^2)E\right)_{,ij} \\
&\quad + (F' - S)'_{(i,j)} + 2\mathcal{H}(F' - S)_{(i,j)} - 2(2\mathcal{H}' + \mathcal{H}^2)F_{(i,j)} \\
&\quad + \frac{1}{2}(t''_{ij} - t_{ij,kk}) + \mathcal{H}t'_{ij} - (2\mathcal{H}' + \mathcal{H}^2)t_{ij}. \qquad \text{(B.26c)}
\end{aligned}$$

Appendix C

General Equations of Motion in the Longitudinal Gauge

We will quote here the general equations of motion for the Einstein-Aether theory for a generic content of matter, considering that it can be cast in a perfect fluid form. Once we have the general equations we will write them in the longitudinal gauge. These equations will be then used to calculate the equations of motion in the particular cases of the inflationary universe and the radiation and matter epochs.

C.1 Matter Lagrangian and the Energy-momentum Tensor

The matter lagrangian can be written in the form

$$S_m = \int d^4x\sqrt{-g}L_m = \int d^4x\sqrt{-g}p(X,\varphi),$$

which stress-energy tensor can be expressed in the perfect fluid form identifying

$$p = p(X,\varphi) \quad \left(\text{with } X = -\frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\right), \tag{C.1}$$

$$\rho = 2Xp_{,X} - p \quad \text{and } u_\mu = \frac{\varphi_{,\mu}}{\sqrt{2X}}. \tag{C.2}$$

The energy-momentum tensor at zero order is

$$T^0_{\ 0} = -\rho, \quad T^0_{\ i} = 0, \quad T^i_{\ j} = p\delta^i_j, \tag{C.3}$$

and the continuity equation

$$\nabla T^\mu_{\ 0} = 0 \Rightarrow \rho' = -3\mathcal{H}(\rho + p). \tag{C.4}$$

The perturbed stress-energy tensor expressed in terms of $\delta\varphi$ is

$$T^{0\ (m)}_{0} = -\delta\rho = \left[3\mathcal{H}(\rho+p)\frac{\delta\varphi}{\varphi'} - \frac{\rho+p}{c_s^2}\left(-\phi + \left(\frac{\delta\varphi}{\varphi'}\right)' + \mathcal{H}\frac{\delta\varphi}{\varphi'}\right)\right], \tag{C.5a}$$

$$T^{0\ (m)}_{i} = -(\rho+p)\left(\frac{\delta\varphi}{\varphi'}\right)_{,i}, \tag{C.5b}$$

$$T^{i\ (m)}_{0} = (\rho+p)\left(\frac{\delta\varphi}{\varphi'} + B\right)_{,i}, \tag{C.5c}$$

$$T^{i\ (m)}_{j} = \delta p\delta_{ij} = \left[(\rho+p)\phi + 4\mathcal{H}(\rho+p)\frac{\delta\varphi}{\varphi'} + \left[(\rho+p)\frac{\delta\varphi}{\varphi'}\right]'\right], \tag{C.5d}$$

$$\delta\rho = -3\mathcal{H}(\rho+p)\frac{\delta\varphi}{\varphi'} + \frac{\rho+p}{c_s^2}\left(-\phi + \left(\frac{\delta\varphi}{\varphi'}\right)' + \mathcal{H}\frac{\delta\varphi}{\varphi'}\right), \tag{C.6}$$

where $c_s^2 = \frac{p_X}{\rho_{,X}} = \frac{\rho+p}{2X\rho_{,X}}$,

$$\delta p = -(\rho+p)\phi + 4\mathcal{H}(\rho+p)\frac{\delta\varphi}{\varphi'} + \left[(\rho+p)\frac{\delta\varphi}{\varphi'}\right]', \tag{C.7}$$

and from $\nabla_\mu T^\mu_0 = 0$

$$\delta\rho' = -3\mathcal{H}(\delta\rho + \delta p) + 3(\rho+p)\psi' + (\rho+p)\left(\frac{\delta\varphi}{\varphi'} + B - \frac{1}{2}E'\right)_{,ii}. \tag{C.8}$$

The background equations are given by 3.13.
The perturbed equations are

$$\begin{aligned} 2\psi_{,ii} - 6\mathcal{H}\psi' - 2\mathcal{H}\left(B_{,ii} + \frac{1}{2}E'_{,ii}\right) &= 8\pi G a^2(\delta\rho + 2\rho\phi) \\ &- c_{14}\left(\phi + C' + B' + \mathcal{H}(B+C)\right)_{,ii} + \alpha\mathcal{H}\left(C_{,ii} + \frac{1}{2}E'_{,ii} - 3\psi'\right), \end{aligned} \tag{C.9a}$$

$$\begin{aligned} 2(\psi' + \mathcal{H}\phi)_{,i} + \left(\mathcal{H}^2 - 2\frac{a''}{a}\right)B_{,i} &= 8\pi G a^2\left((\rho+p)\left(\frac{\delta\varphi}{\varphi'}\right)_{,i} + pB_{,i}\right) \\ &- c_{14}\left(\phi' + C'' + B''\right)_{,i} - c_{14}\mathcal{H}\left(\phi + 2(C'+B')\right)_{,i} - c_{14}\frac{a''}{a}(C+B)_{,i} \\ &+ \alpha\left(\left(\frac{a''}{a} - 2\mathcal{H}^2\right)C_{,i} - \frac{3}{2}\mathcal{H}^2 B_{,i}\right), \end{aligned} \tag{C.9b}$$

$$
\begin{aligned}
&\left[2\psi'' + (\phi-\psi)_{,kk} + 2\mathcal{H}(\phi' + 2\psi') + \left(4\frac{a''}{a} - 2\mathcal{H}^2\right)(\phi+\psi)\right]\delta_{ij} \\
&+ (\psi-\phi)_{,ij} + \left[B'_{,kk} - \frac{1}{2}E''_{,kk} + 2\mathcal{H}\left(B_{,kk} - \frac{1}{2}E'_{,kk}\right)\right]\delta_{ij} \\
&- \left(B'_{,ij} - \frac{1}{2}E''_{,ij}\right) - 2\mathcal{H}\left(B_{,ij} - \frac{1}{2}E'_{,ij}\right) + \left(\mathcal{H}^2 - \frac{a''}{a}\right)E_{,ij} \\
&= \alpha\left[\psi'' + \mathcal{H}(2\psi' + \phi') + \left(2\frac{a''}{a} - \mathcal{H}^2\right)(\phi+\psi)\right]\delta_{ij} \\
&- c_2\left(C'_{,kk} + \frac{1}{2}E''_{,kk} + 2\mathcal{H}\left(C_{,kk} + \frac{1}{2}E'_{,kk}\right)\right)\delta_{ij} \\
&+ \frac{\alpha}{2}\left(\mathcal{H}^2 - 2\frac{a''}{a}\right)E_{,ij} - c_{13}\left(C'_{,ij} + \frac{1}{2}E''_{,ij} + 2\mathcal{H}\left(C_{,ij} + \frac{1}{2}E'_{,ij}\right)\right) \\
&+ 8\pi G a^2\big((\delta p - 2p\psi)\delta_{ij} + pE_{,ij}\big). \qquad \text{(C.9c)}
\end{aligned}
$$

The vector equations

$$
\nabla_\mu J^\mu{}_\nu + c_4 A^\mu \nabla_\mu A^\sigma \nabla_\nu A_\sigma = \lambda A_\nu \qquad \text{(C.10)}
$$

For $\nu = 0$

$$
\begin{aligned}
&c_1\nabla_\mu\nabla^\mu A_0 + c_2\nabla_0\nabla_\mu A^\mu + c_3\nabla_\mu\nabla_0 A^\mu + c_4 A^\mu\nabla_\mu A^\sigma\nabla_0 A_\sigma = \lambda A_0 \\
&\Rightarrow 0 + O(\epsilon^2) \qquad \text{(C.11a)}
\end{aligned}
$$

For $\nu = i$

$$
\begin{aligned}
&c_1\nabla_\mu\nabla^\mu A_i + c_2\nabla_i\nabla_\mu A^\mu + c_3\nabla_\mu\nabla_i A^\mu + c_4 A^\mu\nabla_\mu A^\sigma\nabla_i A_\sigma = \lambda A_i \\
&\beta\left(C_{,kk} + \frac{1}{2}E'_{,kk}\right) - \alpha(\psi' + \mathcal{H}\phi) + \alpha\left(\frac{a''}{a} - 2\mathcal{H}^2\right)(C+B) \\
&- c_{14}\left(\phi' + C'' + B'' + \mathcal{H}\phi + 2\mathcal{H}(C' + B') + \frac{a''}{a}(C+B)\right) = 0. \qquad \text{(C.11b)}
\end{aligned}
$$

C.2 Longitudinal Gauge

We are going to fix the gauge now, and decided to use the longitudinal gauge (B=E=0). Writing the equations for mixed indices and in this gauge

$$
\begin{aligned}
&2k^2\psi + 3(2-\alpha)\mathcal{H}\psi' + c_{14}k^2\phi + c_{14}k^2C' + (c_{14}-\alpha)\mathcal{H}k^2C \\
&\quad + \frac{2-\alpha}{c_s^2}\left((3c_s^2 - 1)\mathcal{H}^2 + \mathcal{H}'\right)\phi \\
&\qquad = \frac{8\pi G a^2(\rho+p)}{c_s^2}\left((3c_s^2-1)\mathcal{H}\frac{\delta\varphi}{\varphi'} - \left(\frac{\delta\varphi}{\varphi'}\right)'\right), \qquad \text{(C.12a)}
\end{aligned}
$$

$$2\psi' + c_{14}\phi' + (2+c_{14})\mathcal{H}\phi + c_{14}\left(C'' + 2\mathcal{H}C' + \frac{a''}{a}C\right) - \alpha\left(\frac{a''}{a} - 2\mathcal{H}^2\right)C = 8\pi G a^2(\rho+p)\frac{\delta\varphi}{\varphi'}, \tag{C.12b}$$

$$3(2-\alpha)\left(\psi'' + \mathcal{H}(\phi' + 2\psi') + (\mathcal{H}' + 2\mathcal{H}^2)\phi\right) - 2k^2(\phi-\psi) - \alpha k^2\left(C' + 2\mathcal{H}C\right) = 24\pi G a^2(\rho+p)\left(4\mathcal{H}\frac{\delta\varphi}{\varphi'} + \frac{(\rho+p)'}{(\rho+p)}\frac{\delta\varphi}{\varphi'} + \frac{\delta\varphi'}{\varphi'}\right), \tag{C.12c}$$

$$\psi - \phi = -c_{13}\left(C' + 2\mathcal{H}C\right). \tag{C.12d}$$

The vector equation

$$c_{14}\left(C'' + 2\mathcal{H}C' + \frac{a''}{a}C\right) + \alpha\left(2\mathcal{H}^2 - \frac{a''}{a}\right)C + \beta k^2 C + \alpha(\psi' + \mathcal{H}\phi) + c_{14}(\phi' + \mathcal{H}\phi) = 0. \tag{C.13}$$

Simplifications

Subtracting Eq. (C.13) to Eq. (C.12b)

$$(2-\alpha)(\psi' + \mathcal{H}\phi) - \beta k^2 C = 8\pi G a^2(\rho+p)\frac{\delta\varphi}{\varphi'}, \tag{C.14}$$

and using Eq. (C.12d) to eliminate ϕ

$$(2-\alpha)\left(\psi' + \mathcal{H}\psi + c_{13}\mathcal{H}(C' + 2\mathcal{H}C)\right) - \beta k^2 C = 8\pi G a^2(\rho+p)\frac{\delta\varphi}{\varphi'}. \tag{C.15}$$

Rewriting Eq. (C.12b) using Eq. (C.12d) to eliminate ϕ and Eq. (C.15) to eliminate $\frac{\delta\varphi}{\varphi'}$ and simplifying we obtain one equation for ψ and C

$$C'' + \left(2 + \frac{c_{13}(c_{14}+\alpha)}{c_{14}(1+c_{13}))}\right)\mathcal{H}C' + \left(\frac{c_{14} - \alpha + 2c_{13}c_{14}}{c_{14}(1+c_{13})}\mathcal{H}' + \frac{(c_{14}+\alpha)(1+2c_{13})}{c_{14}(1+c_{13})}\mathcal{H}^2\right)C + \frac{\beta}{c_{14}(1+c_{13})}k^2 C + \frac{c_{14}+\alpha}{c_{14}(1+c_{13})}(\psi' + \mathcal{H}\psi) = 0. \tag{C.16}$$

We can get this equation also substituting Eq. (C.12d) in the vector equation.

Now we need to get a second equation for ψ and C. Adding Eqs. (C.12a) and (C.12c) and using Eq. (C.12d) to eliminate ϕ

$$
\begin{aligned}
&(2-\alpha)\psi'' + 6(2-\alpha)\mathcal{H}\psi' + (2-\alpha)\left(\frac{c_s^2+1}{c_s^2}\mathcal{H}' + \frac{5c_s^2-1}{c_s^2}\mathcal{H}^2\right)\psi \\
&+ (2-\alpha)c_{13}\mathcal{H}C'' + (2-\alpha)c_{13}\left(\frac{c_s^2+1}{c_s^2}\mathcal{H}' + \frac{7c_s^2-1}{c_s^2}\mathcal{H}^2\right)C' \\
&+(2-\alpha)c_{13}\left(2\frac{2c_s^2+1}{c_s^2}\mathcal{H}' + 2\frac{5c_s^2-1}{c_s^2}\mathcal{H}^2\right)\mathcal{H}C + \left(c_{14}(1+c_{13})-\beta\right)k^2C' \\
&+ \left(c_{14}(1+2c_{13})-\alpha-2\beta\right)k^2\mathcal{H}C + (2+c_{14})k^2\psi \\
&= 8\pi G a^2(\rho+p)\left(\frac{7c_s^2-1}{c_s^2}\mathcal{H}\frac{\delta\varphi}{\varphi'} + \frac{(\rho+p)'}{(\rho+p)}\frac{\delta\varphi}{\varphi'} + \frac{c_s^2-1}{c_s^2}\frac{\delta\varphi'}{\varphi'}\right), \qquad \text{(C.17)}
\end{aligned}
$$

substituting Eq. (C.15) and $\dfrac{(\rho+p)'}{\rho+p} = \dfrac{2\mathcal{H}\mathcal{H}'-\mathcal{H}''}{\mathcal{H}^2-\mathcal{H}'} - 2\mathcal{H}$, and simplifying we end up with the second equation for ψ and C

$$
\begin{aligned}
&\psi'' + \left(2\mathcal{H} - \frac{2\mathcal{H}\mathcal{H}'-\mathcal{H}''}{\mathcal{H}^2-\mathcal{H}'}\right)\psi' + \left(2\mathcal{H}' - \frac{2\mathcal{H}\mathcal{H}'-\mathcal{H}''}{\mathcal{H}^2-\mathcal{H}'}\mathcal{H}\right)\psi \\
&\quad + c_{13}\mathcal{H}C'' + c_{13}\left(2\mathcal{H}' + 2\mathcal{H}^2 - \mathcal{H}\frac{2\mathcal{H}\mathcal{H}'-\mathcal{H}''}{\mathcal{H}^2-\mathcal{H}'}\right)C' \\
&\quad + 2c_{13}\left(3\mathcal{H}' - \mathcal{H}\frac{2\mathcal{H}\mathcal{H}'-\mathcal{H}''}{\mathcal{H}^2-\mathcal{H}'}\right)\mathcal{H}C + \frac{2+c_{14}}{2-\alpha}c_s^2k^2\psi \\
&\quad + \frac{c_{14}(1+c_{13})c_s^2-\beta}{2-\alpha}k^2C' \\
&\quad + \left(\frac{(c_{14}(1+2c_{13})+2c_{13})c_s^2-\beta}{2-\alpha}\mathcal{H} + \frac{\beta}{2-\alpha}\frac{2\mathcal{H}\mathcal{H}'-\mathcal{H}''}{\mathcal{H}^2-\mathcal{H}'}\right)k^2C = 0. \qquad \text{(C.18)}
\end{aligned}
$$

Appendix D

Scalar Equations during Inflation

In this appendix we discuss the scalar sector in the longitudinal gauge in the presence of an inflaton field.

In the longitudinal gauge, there are five fields in the scalar sector: ϕ, ψ, C, $\delta\lambda$ and the inflaton perturbation $\delta\varphi$. Therefore, we need five independent equations to uniquely determine their values. Contracting Eq. (3.7) with A^β and using the constraint $A^\beta A_\beta = -1$ yields an equation that expresses the Lagrange multiplier in terms of the Aether and the metric,

$$\delta\lambda = \frac{1}{a^2}\Bigg[- 6(\alpha - c_2)\mathcal{H}^2\phi + 6c_2\frac{a''}{a}\phi + 3c_2\mathcal{H}\phi' + +c_3k^2\phi \\ - 3(2\beta - c_2)\mathcal{H}\psi' + 3c_2\psi'' - (\beta + c_1)\mathcal{H}k^2C + (\beta - c_1)k^2C'\Bigg], \quad \text{(D.1)}$$

which shows explicitly how the Lagrange multiplier can be expressed in terms of the remaining fields. The time component of the linearized Aether field Eq. (3.7) is identically satisfied. The linearized spatial components give

$$C'' + 2\mathcal{H}C' + \left[\frac{\alpha}{c_{14}}\left(2\mathcal{H}^2 - \frac{a''}{a}\right) + \frac{a''}{a}\right]C \\ + \frac{\beta}{c_{14}}k^2C + \left(1 + \frac{\alpha}{c_{14}}\right)\mathcal{H}\phi + \phi' + \frac{\alpha}{c_{14}}\psi' = 0, \quad \text{(D.2)}$$

which combined with the ${}^0{}_i$ Einstein equation results in

$$\mathcal{H}\phi + \psi' - \frac{\beta}{2-\alpha}k^2C = 4\pi G\frac{2}{2-\alpha}\varphi'\delta\varphi. \quad \text{(D.3)}$$

Eq. (D.3) expresses $\delta\varphi$ in terms of the remaining scalars, and allows us to eliminate $\delta\varphi$ from our system of equations. On large scales, this equation has the same form it would have in the absence of the Aether, with the difference that the effective Newton's constant has the renormalized value implied by Eqs. (3.13). The part of the ${}^i{}_j$ Einstein equations which is not proportional to $\delta^i{}_j$ is

$$\phi = \psi + c_{13}(C' + 2\mathcal{H}C), \tag{D.4}$$

which immediately reveals that the Einstein-Aether is a source of anisotropic stress in the scalar sector. This equation allows us to express ϕ in terms of ψ and C, and thus eliminate yet another variable from the equations. Note that scalar fields and perfect fluids cannot source anisotropic stress, which is why a value of $\psi - \phi$ different from zero is sometimes attributed to modified gravity. Finally, the sum of the ${}^0{}_0$ and the ${}^i{}_j$ Einstein equations proportional to $\delta^i{}_j$ is

$$\begin{aligned}\psi'' + 5\mathcal{H}\psi' + \mathcal{H}\phi' + 2\left(\frac{a''}{a} + \mathcal{H}^2\right)\phi + \frac{c_{14}-1}{2-\alpha}k^2\phi + \frac{3}{2-\alpha}k^2\psi \\ + \frac{c_{14}-c_2}{2-\alpha}k^2C' + \frac{c_{14}-\alpha-2c_2}{2-\alpha}\mathcal{H}k^2C = \frac{8\pi G}{2-\alpha}3\mathcal{H}(1-w)\varphi'\delta\varphi,\end{aligned} \tag{D.5}$$

where we have used Eq. (D.1) to eliminate $\delta\lambda$, and that during power-law expansion the equation of state parameter w is constant. Eqs. (D.1), (D.2), (D.3), (D.4) and (D.5) form a set of five differential equations for the five unknowns. We can use the constraints (D.3) and (D.4) to eliminate ϕ and $\delta\varphi$ from Eqs. (D.2) and (D.5), arriving at

$$\begin{aligned}C'' + \left(2 + \frac{c_{13}(c_{14}+\alpha)}{c_{14}(1+c_{13})}\right)\mathcal{H}C' + \left(\frac{c_{14}-\alpha+2c_{13}c_{14}}{c_{14}(1+c_{13})}\frac{a''}{a} + \frac{2\alpha}{c_{14}}\mathcal{H}^2\right)C \\ + \frac{\beta}{c_{14}(1+c_{13})}k^2C + \frac{c_{14}+\alpha}{c_{14}(1+c_{13})}(\psi' + \mathcal{H}\psi) = 0,\end{aligned} \tag{D.6}$$

and

$$\begin{aligned}&\psi'' + 3(1+w)\mathcal{H}\psi' + 2\left(\frac{a''}{a} - 2\mathcal{H}^2\right)\psi + 3(1+w)\mathcal{H}^2\psi \\ &+ c_{13}\mathcal{H}C'' + 2c_{13}\left(\frac{a''}{a} - \mathcal{H}^2\right)C' + 3(1+w)c_{13}\mathcal{H}^2C' \\ &+ c_{13}\left(6\frac{a''}{a} - 10\mathcal{H}^2\right)\mathcal{H}C + 6(1+w)c_{13}\mathcal{H}^3C \\ &+ \frac{2+c_{14}}{2-\alpha}k^2\psi + \frac{c_{14}(1+c_{13})-\beta}{2-\alpha}k^2C' \\ &+ \frac{c_{14}(1+2c_{13})+4\beta-\alpha-3(1+w)\beta}{2-\alpha}\mathcal{H}k^2C = 0.\end{aligned} \tag{D.7}$$

Because this is a system of two second-order linear differential equations, we need to specify four independent initial conditions, so there must exist four linearly independent solutions. This is also what we expect by simply counting matter fields. In the limit of weak gravitational couplings, we may neglect metric perturbations, so we just have one degree of freedom in the inflaton perturbations and one degree of freedom in the Aether field perturbations, for a total of four initial conditions to determine uniquely the evolution of the system. As we deviate from the limit of weak coupling, neglecting metric perturbations ceases to be a good approximation, but the number of degrees of freedom in the theory remains unchanged.

D.1 Short wavelength Solutions

In the short wavelength regime, $k|\eta| \gg 1$, the solutions of the equations of motion (D.6) and (D.7) behave approximately like in flat space. The notion of an approximate solution can be formalized by introducing $k\eta$ as an expansion parameter. In the limit $k|\eta| \gg 1$ the solution of Eqs. (D.6) and (D.7) can be cast in the form

$$\psi = \widetilde{\psi}(k\eta)\,\exp(-ic_s k\eta), \quad C = \frac{1}{k}\widetilde{C}(k\eta)\,\psi, \tag{D.8}$$

where $\widetilde{\psi}$ and $\widetilde{C}$ are functions whose power series expansion starts at a finite positive power of $k\eta$, and c_s is a "sound speed" to be determined. Substituting the ansatz (D.8) into Eqs. (D.6) and (D.7), and keeping the leading powers of $k\eta$ yields a set of algebraic equations with two positive frequency and two negative frequency solutions, for a total of four solutions, as expected. At leading order, $\widetilde{\psi}$ remains unconstrained and can be taken to be constant. The positive frequency solutions are given by

$$(c_s)_a = c_a, \qquad \widetilde{C}_a = i\frac{\alpha - 2}{\beta}c_a, \tag{D.9a}$$

$$(c_s)_\varphi = 1, \qquad \widetilde{C}_\varphi = i\frac{c_{14} + \alpha}{\beta - c_{14}(1 + c_{13})}, \tag{D.9b}$$

where c_a is the sound speed of Eq. (4.24).

These two modes correspond to the two independent short wavelength solutions (4.26a) and (4.26b) that we found in Subsection 4.1.2. To see that this is the case, we may use the expression of δN and ζ_a in the longitudinal gauge

$$\delta N = \frac{\mathcal{H}\delta\varphi}{\varphi'} + \mathcal{H}C, \tag{D.10a}$$

$$\zeta_a = \psi - \mathcal{H}C. \tag{D.10b}$$

In the first equation, $\delta\varphi$ should be expressed in terms of ψ and C through the relation

$$\delta\varphi = \frac{M_P^2}{\varphi'}\left[(2-\alpha)\left(\psi' + \mathcal{H}\psi + c_{13}\mathcal{H}C' + 2c_{13}\mathcal{H}^2 C\right) - \beta k^2 C\right], \tag{D.11}$$

which follows from (D.3) and (D.4). By comparison with (4.26a) and (4.26b) we also obtain the overall normalization factor $\widetilde{\psi}$. For the first mode, we have

$$\psi_a \to \frac{Z_a^{1/2}}{a} \frac{e^{-ic_a k\eta}}{\sqrt{2c_a k}}, \quad C_a \to \frac{1}{k}\widetilde{C}_a \, \psi_a, \tag{D.12}$$

where Z_a is given in Eq. (4.23), and $\widetilde{C}_a$ in Eq. (D.9a). For the second mode, we have

$$\psi_\varphi \to \frac{Z_\varphi^{1/2}}{a} \frac{e^{-ik\eta}}{\sqrt{2k}}, \quad C_\varphi \to \frac{1}{k}\widetilde{C}_\varphi \, \psi_\varphi, \tag{D.13}$$

where $\widetilde{C}_\varphi$ is given in Eq. (D.9b) and

$$Z_\varphi^{1/2} \equiv i \frac{\varphi'}{k\, M_P^2} \left(2 + \frac{c_{14}(\beta + \alpha(1+c_{13}))}{\beta - c_{14}(1+c_{13})}\right)^{-1}. \tag{D.14}$$

Substitution of (D.12) into (D.10) reproduces Eq. (4.26b), while substitution of (D.13) into (D.10), together with the background Eqs. (3.13), reproduces Eq. (4.26a). The vacuum is thus characterized by the two independent solutions of Eqs. (D.6) and (D.7) that approach (D.12) and (D.13) in the limit $k|\eta| \to \infty$.

D.2 Long wavelength Solutions

In the limit of long wavelengths, $k|\eta| \ll 1$, we may neglect terms proportional to k^2 in Eqs. (D.6) and (D.7). In this limit, the power-law ansatz

$$\psi = (-\eta)^t, \quad C = \mathcal{C} \cdot (-\eta) \cdot \psi \tag{D.15}$$

reduces the two coupled differential equations (D.6) and (D.7) to an algebraic system for the two constants t and $\mathcal{C}$,

$$\begin{aligned} \Bigg[t^2 &+ \frac{c_{14}(1+c_{13})(5+3w) + 2c_{13}(c_{14}+\alpha)}{c_{14}(1+c_{13})(1+3w)} t \\ &+ \frac{2(c_{14}+\alpha)(3(1+w) + c_{13}(5+3w))}{c_{14}(1+c_{13})(1+3w)^2} \Bigg] \mathcal{C} \\ &= \frac{c_{14}+\alpha}{c_{14}(1+c_{13})} \left(t + \frac{2}{1+3w}\right), \end{aligned} \tag{D.16a}$$

$$t(5 + t + 3w + 3wt)(1 + 3w - 2c_{13}\,\mathcal{C}) = 0. \tag{D.16b}$$

Because Eqs. (D.16) are linear in $\mathcal{C}$, they may be reduced to a single quartic equation for t, with four different solutions, as it should be.

D.2.1 Adiabatic modes ($\delta N = 0$)

Two solutions of the coupled equations (D.16) follow directly from Eq. (D.16b),

$$t_1 = 0, \qquad \mathcal{C}_1 = \frac{1+3w}{3(1+w)+c_{13}(5+3w)}, \tag{D.17a}$$

$$t_2 = -\frac{5+3w}{1+3w}, \qquad \mathcal{C}_2 = -\frac{1+3w}{2}. \tag{D.17b}$$

The corresponding perturbations are the two "adiabatic" modes that always exist at long wavelengths, regardless of the matter content of the universe [Wei03]. Along these two modes, the (spatial) curvature perturbation on comoving slices[1],

$$\zeta \equiv \psi + \frac{2}{3}\frac{\mathcal{H}\phi + \psi'}{\mathcal{H}(1+w)}, \tag{D.18}$$

and the difference of the two metric potentials (which is proportional to the anisotropic stress) are given by

$$\zeta_1 = \frac{(5+3w)(1+c_{13})}{3(1+w)+c_{13}(5+3w)}\psi_1, \quad \phi_1 - \psi_1 = -\frac{c_{13}}{1+c_{13}}\zeta_1, \tag{D.19a}$$

$$\zeta_2 = 0, \qquad \phi_2 - \psi_2 = 0. \tag{D.19b}$$

It can be readily checked that for these modes $\delta N = 0$, so that matter is at rest in the Aether frame. Though these adiabatic modes have the properties described in [Wei03], they do not share the properties postulated in [Wei04b, Wei04a, Wei08]. In particular, for the first adiabatic mode, the anisotropic stress is non-zero. The form of the two adiabatic modes for an arbitrary expansion history and matter content is derived in Appendix F.

D.2.2 Isocurvature modes ($\zeta = 0, \delta N \neq 0$)

The two remaining solutions of Eqs. (D.16) require $\mathcal{C} = (1+3w)/2c_{13}$, which gives

$$\psi = -c_{13}\mathcal{H}C \propto (-\eta)^{t_\pm}. \tag{D.20}$$

From (D.16a), the exponents are given by

$$2t_\pm \quad = \quad -\left(\frac{5+3w}{1+3w}\right) \pm \sqrt{\left(\frac{5+3w}{1+3w}\right)^2 + 4\kappa}, \tag{D.21}$$

[1] Recall from equation (4.8) that we mean comoving with respect to all forms of matter, excluding the Aether. The ${}^0{}_i$ Einstein equation (D.3) however reveals that the contribution of the Aether to the total velocity perturbation is negligible on large scales. Hence, on large scales, hypersurfaces comoving with matter and comoving with matter plus Aether are actually the same.

where κ is given by Eq. (4.30). It is straightforward to check that for these modes we have

$$\zeta_\pm = 0, \tag{D.22a}$$

$$\phi_\pm - \psi_\pm = \left(\frac{1+3w}{2}\right) t_{(\mp)}\, \psi_\pm, \tag{D.22b}$$

$$\delta N_\pm = -\left(\frac{1+c_{13}}{c_{13}}\right) \psi_\pm \;\propto (-\eta)^{t_{(\pm)}}. \tag{D.22c}$$

These are two isocurvature modes, in the sense that the curvature perturbation on comoving slices ζ vanishes for any value of w.

From Eq. (D.21) it is straightforward to check that, for any value of w, one of the two modes is a decaying one. Whether the second mode is growing or decaying depends on the sign of κ, which is in turn determined by the sign of $1 + (\alpha/c_{14})$. For $\kappa < 0$ the second solution is also a decaying one, but for $\kappa > 0$ there is a growing mode. In the special case $\kappa = 0$, there is a constant non-decaying long wavelength solution.

The existence of a growing non-adiabatic isocurvature mode in Einstein-Aether theories for $(\alpha/c_{14}) < -1$ can have important phenomenological consequences, as we discuss in the main text.

Appendix E

Scalar Equations for Radiation and Matter

We have found the solutions during inflation in the previous chapter, both in the short and long wavelength limits. Now, we are going to write the equations (again in the longitudinal gauge) and full solutions during the epochs of radiation and matter domination. The analysis of these results is done in Section 4.1.5.

E.1 Radiation

In a radiation-dominated universe, $a \propto \eta$

$$\mathcal{H} = \frac{1}{\eta};\ \mathcal{H}' = \frac{-1}{\eta^2};\ \mathcal{H}'' = \frac{2}{\eta^3};\ (\mathcal{H}' = -\mathcal{H}^2).$$

From Eqs. (C.18, C.17) we end up with a pair of differential equations for ψ and C

$$\begin{aligned}\psi'' + 4\mathcal{H}\psi' + \frac{2+c_{14}}{2-\alpha}c_s^2 k^2\psi + c_{13}\mathcal{H}C'' + 2c_{13}\mathcal{H}^2 C' + 2c_{13}\mathcal{H}\mathcal{H}'C \\ + \frac{c_{14}(1+c_{13})c_s^2 - \beta}{2-\alpha}k^2 C' + \frac{(c_{14}(1+2c_{13}) + 2c_{13})c_s^2 - 3\beta}{2-\alpha}\mathcal{H}k^2 C = 0,\end{aligned} \tag{E.1}$$

and

$$\begin{aligned}C'' + \left(2 + \frac{c_{13}(c_{14}+\alpha)}{c_{14}(1+c_{13})}\right)\mathcal{H}C' + 2\frac{\alpha}{c_{14}}\mathcal{H}^2 C + \frac{\beta}{c_{14}(1+c_{13})}k^2 C \\ + \frac{c_{14}+\alpha}{c_{14}(1+c_{13})}(\psi' + \mathcal{H}\psi) = 0.\end{aligned} \tag{E.2}$$

In Section 4.1.5 we calculate the solutions of these equations in the short wavelength limit and do the matching with the long wavelength solutions in order to get the normalization. The complete normalization coefficients for the adiabatic mode are

$$A_1 = \frac{1}{3(1+c_{13})\sqrt{\pi(2-\alpha)}} \left(\frac{\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}}{A_{den}} \right.$$
$$\left. - \frac{\frac{(2-\alpha)(2+3c_{13})}{\beta}\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}}{A_{den}} \right), \quad \text{(E.3)}$$

$$A_2 = \frac{1}{3(1+c_{13})\sqrt{\pi(2-\alpha)}} \frac{\cos\frac{1}{\sqrt{3}} - \frac{\sqrt{3}(\alpha+c_{14})(2+3c_{13})}{c_{14}(1+c_{13})-3\beta}\sin\frac{1}{\sqrt{3}}}{A_{den}}, \quad \text{(E.4)}$$

and for the isocurvature mode

$$A_1 = -\frac{1}{(1+c_{13})\sqrt{\pi(2-\alpha)}} \left(\frac{\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}}{A_{den}} \right.$$
$$\left. - \frac{\frac{(2-\alpha)c_{13}}{\beta}\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}}{A_{den}} \right), \quad \text{(E.5)}$$

$$A_2 = \frac{1}{(1+c_{13})\sqrt{\pi(2-\alpha)}} \frac{\frac{\sqrt{3}(\alpha+c_{14})c_{13}}{c_{14}(1+c_{13})-3\beta}\sin\frac{1}{\sqrt{3}} - \cos\frac{1}{\sqrt{3}}}{A_{den}}. \quad \text{(E.6)}$$

where

$$A_{den} = \frac{\sqrt{3}(\alpha+c_{14})}{c_{14}(1+c_{13})-3\beta}\sin\frac{1}{\sqrt{3}}\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}$$
$$- \cos\frac{1}{\sqrt{3}}\sqrt{\frac{(2+c_{14})(2-\alpha)}{c_{14}\beta(1+c_{13})}}\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}. \quad \text{(E.7)}$$

In the limit $|\alpha + c_{14}| \ll |c_{14}|$

$$A_1 \sim \frac{1}{(1+c_{13})\sqrt{\pi(2-\alpha)}}\sec\frac{1}{\sqrt{3}}\left(-c_{13} \right.$$
$$\left. + \frac{\beta}{2-\alpha}\sqrt{\frac{c_{14}(1+c_{13})(2-\alpha)}{\beta(2+c_{14})}}\cot\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} \right), \quad \text{(E.8)}$$

$$A_2 \sim \frac{1}{2-\alpha}\sqrt{\frac{c_{14}\beta}{\pi(1+c_{13})(2+c_{14})}}\csc\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}. \quad \text{(E.9)}$$

E.2 Matter

In a matter dominated universe, $a \propto \eta^2$

$$\mathcal{H} = \frac{2}{\eta};\ \mathcal{H}' = \frac{-2}{\eta^2};\ \mathcal{H}'' = \frac{4}{\eta^3};\ (\mathcal{H}' = -\frac{1}{2}\mathcal{H}^2).$$

From Eqs. (C.18, C.17) we end up with a pair of differential equations for ψ and C.

$$\begin{aligned}\psi'' + 3\mathcal{H}\psi' + \frac{2+c_{14}}{2-\alpha}c_s^2k^2\psi + c_{13}\mathcal{H}C'' + 2c_{13}\mathcal{H}^2C' + 2c_{13}\mathcal{H}\mathcal{H}'C \\ + \frac{c_{14}(1+c_{13})c_s^2 - \beta}{2-\alpha}k^2C' + \frac{(c_{14}(1+2c_{13}) + 2c_{13})c_s^2 - \beta}{2-\alpha}\mathcal{H}k^2C = 0,\end{aligned} \tag{E.10}$$

and

$$\begin{aligned}C'' + \left(2 + \frac{c_{13}(c_{14}+\alpha)}{c_{14}(1+c_{13})}\right)\mathcal{H}C' + \left(\frac{(c_{14}+\alpha)(1+2c_{13})}{2c_{14}(1+c_{13})} + \frac{\alpha}{c_{14}}\right)\mathcal{H}^2C \\ + \frac{\beta}{c_{14}(1+c_{13})}k^2C + \frac{c_{14}+\alpha}{c_{14}(1+c_{13})}(\psi' + \mathcal{H}\psi) = 0.\end{aligned} \tag{E.11}$$

In the short wavelength limit the equations reduce to

$$C'' + \frac{\beta}{c_{14}(1+c_{13})}k^2C + \frac{\alpha + c_{14}}{c_{14}(1+c_{13})}\psi' = 0, \tag{E.12}$$

$$\psi'' - \frac{\beta}{2-\alpha}k^2C' = 0. \tag{E.13}$$

From the second equation we get

$$\psi' = A + \frac{\beta}{2-\alpha}k^2C, \tag{E.14}$$

and plugging this into the first equation we get a second order differential equation for C,

$$C'' + \frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}k^2C + \frac{\alpha + c_{14}}{c_{14}(1+c_{13})}A = 0, \tag{E.15}$$

with solution

$$\begin{aligned}C = -\frac{(\alpha + c_{14})(2-\alpha)}{\beta(2+c_{14})}\frac{A}{k^2} + B\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta \\ + D\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta.\end{aligned} \tag{E.16}$$

Now, integrating Eq. (4.67) we get the solution for ψ

$$\begin{aligned}\psi = F + \frac{2-\alpha}{2+c_{14}}A\eta \\ + \sqrt{\frac{\beta c_{14}(1+c_{13})}{(2+c_{14})(2-\alpha)}}k\left(B\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta\right. \\ \left.-D\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta\right).\end{aligned} \tag{E.17}$$

The four constants (A, B, D, F) will be obtained through the matching with the long wavelength solutions, both for the adiabatic

$$\tilde{\psi}_{ad} = \frac{3+5c_{13}}{5(1+c_{13})}\zeta_0, \tag{E.18}$$

$$\tilde{C}_{ad} = -\frac{1}{5(1+c_{13})}\eta\,\zeta_0, \tag{E.19}$$

and isocurvature cases

$$\tilde{\psi}_{iso} = -\frac{c_{13}}{1+c_{13}}\left(\frac{\eta}{\eta_{eq}}\right)^{t_+}\delta\mathrm{N}_0, \tag{E.20}$$

$$\tilde{C}_{iso} = \frac{1}{2(1+c_{13})}\left(\frac{\eta}{\eta_{eq}}\right)^{t_+}\eta\,\delta\mathrm{N}_0, \tag{E.21}$$

where

$$\zeta_0 \simeq \delta\mathrm{N}_0 \simeq \frac{H}{2\pi}\sqrt{\frac{16\pi G}{3(2-\alpha)}}, \tag{E.22}$$

and

$$t_+ \simeq -\frac{6}{5}\left(1+\frac{\alpha}{c_{14}}\right). \tag{E.23}$$

The final solutions for the adiabatic mode

$$\begin{aligned}C_{ad} = -\frac{(\alpha+c_{14})(2-\alpha)}{\beta(2+c_{14})}\frac{A_{ad}}{k^2} + B_{ad}\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta \\ +D_{ad}\sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}}k\eta,\end{aligned} \tag{E.24}$$

and

$$\psi_{ad} = F_{ad} + \frac{2-\alpha}{2+c_{14}} A_{ad}\eta + \sqrt{\frac{\beta c_{14}(1+c_{13})}{(2+c_{14})(2-\alpha)}} k \left(B_{ad} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta - D_{ad} \cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta \right), \tag{E.25}$$

where

$$A_{ad} = \frac{4\beta k}{5(2-\alpha)(1+c_{13})\sqrt{3\pi(2-\alpha)}}, \tag{E.26}$$

$$B_{ad} = \frac{4}{5(1+c_{13})\sqrt{3\pi(2-\alpha)}} \frac{1}{k} \times \left(\sqrt{\frac{c_{14}(1+c_{13})(2-\alpha)}{\beta(2+c_{14})}} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} - \frac{2-\alpha}{2+c_{14}} \cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} \right), \tag{E.27}$$

$$D_{ad} = \frac{-4}{5(1+c_{13})\sqrt{3\pi(2-\alpha)}} \frac{1}{k} \times \left(\sqrt{\frac{c_{14}(1+c_{13})(2-\alpha)}{\beta(2+c_{14})}} \cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} + \frac{2-\alpha}{2+c_{14}} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} \right), \tag{E.28}$$

$$F_{ad} = \frac{4\left(6-\beta+10c_{13}+2(1+2c_{13})c_{14}\right)}{5(1+c_{13})\left(2+c_{14}\right)\sqrt{3\pi(2-\alpha)}}. \tag{E.29}$$

For the isocurvature mode (considering $|c_{14}+\alpha \ll c_{14}|$ and thus $t_+ \sim 0$)

$$C_{iso} = -\frac{(\alpha+c_{14})(2-\alpha)}{\beta(2+c_{14})} \frac{A_{iso}}{k^2} + B_{iso} \cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta + D_{iso} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta, \tag{E.30}$$

and

$$\psi_{iso} = F_{iso} + \frac{2-\alpha}{2+c_{14}} A_{iso}\eta + \sqrt{\frac{\beta c_{14}(1+c_{13})}{(2+c_{14})(2-\alpha)}} k \left(B_{iso} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta - D_{iso}\cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} k\eta \right), \tag{E.31}$$

where

$$A_{iso} = \frac{-2k\beta}{(2-\alpha)(1+c_{13})\sqrt{3\pi(2-\alpha)}}, \tag{E.32}$$

$$B_{iso} = \frac{-2}{(1+c_{13})\sqrt{3\pi(2-\alpha)}} \frac{1}{k} \times \left(\sqrt{\frac{c_{14}(1+c_{13})(2-\alpha)}{\beta(2+c_{14})}} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} - \frac{2-\alpha}{2+c_{14}} \cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} \right), \tag{E.33}$$

$$D_{iso} = \frac{2}{(1+c_{13})\sqrt{3\pi(2-\alpha)}} \frac{1}{k} \times \left(\sqrt{\frac{c_{14}(1+c_{13})(2-\alpha)}{\beta(2+c_{14})}} \cos\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} + \frac{2-\alpha}{2+c_{14}} \sin\sqrt{\frac{\beta(2+c_{14})}{c_{14}(1+c_{13})(2-\alpha)}} \right), \tag{E.34}$$

$$F_{iso} = \frac{2\left(\beta - 4c_{13} - (-1+c_{13})c_{14}\right)}{(1+c_{13})\left(2+c_{14}\right)\sqrt{3\pi(2-\alpha)}}. \tag{E.35}$$

Appendix F

Long Wavelength Adiabatic and Isocurvature Modes

The properties of the two long wavelength adiabatic and isocurvature modes for arbitrary expansion history and fairly general matter content can be also obtained by following a procedure outlined by Weinberg in [Wei03].

F.1 Adiabatic Modes

Consider the gauge transformations generated by

$$\eta \to \eta + \epsilon(\eta) \quad \text{and} \quad x^i \to x^i + \omega\, x^i, \tag{F.1}$$

where ω is a constant. Using the transformation properties of the metric one finds that these transformations preserve the structure of longitudinal gauge. In particular, they induce the following transformations on the metric and Aether perturbations,

$$\phi \to \phi - \epsilon' - \mathcal{H}\epsilon, \quad \psi \to \psi + \omega + \mathcal{H}\epsilon, \quad C \to C + \epsilon. \tag{F.2}$$

Because the equations of motion are invariant under gauge transformations, the difference of two sets of perturbations that differ by a gauge transformation is a solution of the linearized equations,

$$\phi = -\epsilon' - \mathcal{H}\epsilon, \quad \psi = \omega + \mathcal{H}\epsilon, \quad C = \epsilon. \tag{F.3}$$

The corresponding values of the remaining perturbation variables can be also determined by their transformation properties under (F.1). For instance, for any scalar perturbation $\delta\varphi$ or any velocity perturbation $\delta u_i \equiv \partial_i \delta u$ the solutions have

$$\delta\varphi = -\epsilon\varphi', \quad \delta u = a\,\epsilon. \tag{F.4}$$

Of course, these space-independent solutions are just gauge modes, physically equivalent to no perturbation at all. But they can be extended to actual space-dependent perturbations if the linearized ${}^0{}_i$ and ${}^i{}_k$ Einstein equations are satisfied for these putative solutions. The ${}^0{}_i$ equation is automatically satisfied for the ansatz (F.3) and (F.4). On the other hand, in the presence of the Aether the ${}^i{}_j$ Einstein equation (D.4) imposes the constraint

$$\epsilon' + 2\mathcal{H}\epsilon + \frac{1}{1+c_{13}}\omega = 0, \tag{F.5}$$

where we have assumed that the remaining matter does not contribute to the scalar anisotropic stress. The general solution of Eq. (F.5) is the superposition of two solutions, with

$$\epsilon_1 = -\frac{1}{a^2}\frac{\omega}{1+c_{13}}\int^\eta d\tilde{\eta}\, a^2(\tilde{\eta}), \qquad \omega_1 = \omega, \tag{F.6a}$$

$$\epsilon_2 = \frac{C_0}{a^2}, \qquad \omega_2 = 0, \tag{F.6b}$$

where C_0 is an integration constant. The first solution yields the non-decaying mode, which in the "gravity" sector reads

$$\phi_1 = \frac{\omega}{1+c_{13}}\left(1 - \frac{\mathcal{H}}{a^2}\int d\tilde{\eta}\, a^2(\tilde{\eta})\right), \tag{F.7a}$$

$$\psi_1 = \omega\left(1 - \frac{\mathcal{H}}{a^2}\frac{1}{1+c_{13}}\int d\tilde{\eta}\, a^2(\tilde{\eta})\right), \tag{F.7b}$$

$$C_1 = -\frac{\omega}{1+c_{13}}\frac{1}{a^2}\int d\tilde{\eta}\, a^2(\tilde{\eta}). \tag{F.7c}$$

This reduces to the adiabatic mode (D.17a) for a constant equation of state. For this mode the curvature perturbation is constant, $\zeta = \omega$, and the anisotropic stress is non-zero (if $c_{13} \neq 0$). The second solution in Eq. (F.6a) corresponds to a decaying mode, which, for a constant equation of state, agrees with the adiabatic mode in Eq. (D.17b),

$$\phi_2 = C_0\frac{\mathcal{H}}{a^2}, \quad \psi_2 = C_0\frac{\mathcal{H}}{a^2}, \quad C = \frac{C_0}{a^2}. \tag{F.8}$$

For this second adiabatic mode, the curvature perturbation vanishes, $\zeta = 0$, and so does the anisotropic stress.

F.2 Isocurvature Modes

An extension of the previous method also unveils the two isocurvature modes, under the assumption that the Aether does not couple to matter. Consider the ansatz

$$\phi = c_{13}(C' + \mathcal{H}C), \quad \psi = -c_{13}\mathcal{H}C, \tag{F.9}$$

which arises from the gauge transformation (F.1) with $\omega = 0$ and $\epsilon = -c_{13}C$. Acting on any velocity u_μ and any scalar φ (not necessarily the inflaton), the same gauge transformation leads to the matter perturbations

$$\delta\varphi = c_{13}\varphi' C, \quad \delta u = -c_{13} a C. \tag{F.10}$$

Since by assumption the Aether does not couple to matter, and for the same reasons as in the adiabatic case, we expect Eqs. (F.9) and (F.10) then to be a solution of the matter equations of motion, no matter what the Aether perturbation C actually is. Of course, for arbitrary values of C, we cannot expect the ansatz to satisfy Einstein's equations, since the Aether does couple to gravity. Inspection of the latter however reveals that the ${}^0{}_0$, ${}^0{}_i$ and diagonal ${}^i{}_j$ equations only contain spatial gradients of the Aether field, which can be neglected in the long wavelength limit. The only equation in which the Aether perturbation is not negligible at long wavelengths is (D.4), which is actually satisfied by the ansatz (F.9). Hence, it only remains to find out what the Aether perturbation C is. Substituting Eq. (F.9) into the Aether field equation (D.2) results in a differential equation for the yet undetermined Aether perturbation,

$$C'' + 2\mathcal{H}C' + \left[\left(1 + \frac{\alpha}{c_{14}}\right)\mathcal{H}^2 + \left(1 - \frac{\alpha}{c_{14}}\right)\mathcal{H}'\right] C = 0. \tag{F.11}$$

This equation has two independent solutions, which when plugged into (F.9) and (F.10) give the two independent isocurvature modes, for which $\zeta = \omega = 0$. None of these solutions can be adiabatic, as the adiabatic mode has $\epsilon = C$, while along these solutions $\epsilon = -c_{13}C$ (recall that $c_{13} = -1$ is a singular case). A measure of the non-adiabaticity of these modes is the difference in the e-folding number between surfaces comoving with Aether, and those comoving with matter, which equals

$$\delta N = (1 + c_{13})\mathcal{H}C = -\frac{1 + c_{13}}{c_{13}}\psi, \tag{F.12}$$

and thus differs from zero if $c_{13} \neq -1$. Since along these solutions all matter components (aside from the Aether) share the same velocity, the two modes describe a matter-Aether isocurvature perturbation, which is the only kind of isocurvature perturbation that can be generated if the Aether does not couple to matter. For a constant equation of state, these two isocurvature modes reproduce those found in Subsection D.2.2.

Note that this method of generating solutions would break down if the anisotropic stress of matter on large scales were not negligible, as would happen for instance if the matter sector contained a second Aether field.

Bibliography

[AAHD+06] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, *Causality, Analyticity and an IR Obstruction to UV Completion*, JHEP **10** (2006) 014, [hep-th/0602178]. 17, 18

[AHCLM04] N. Arkani-Hamed, H.-C. Cheng, M. A. Luty and S. Mukohyama, *Ghost Condensation and a Consistent Infrared Modification of Gravity*, JHEP **0405** (2004) 074, [hep-th/0312099]. ii, 4

[AHGS03] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, *Effective Field Theory for Massive Gravitons and Gravity in Theory Space*, Ann. Phys. **305** (2003) 96–118, [hep-th/0210184]. ii, 4

[APDM99] C. Armendariz-Picon, T. Damour and V. Mukhanov, *k-Inflation*, Phys. Lett. B **458** (1999) 209–218, [hep-th/9904075]. ii, 4

[APDT09] C. Armendariz-Picon and A. Diez-Tejedor, *Aether Unleashed*, JCAP **0912** (2009) 018, [arXiv:0904.0809]. iii, 5

[APL03] C. Armendariz-Picon and E. A. Lim, *Vacuum Choices and the Predictions of Inflation*, JCAP **0312** (2003) 006, [hep-th/0303103]. 31

[APMS00] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, *A Dynamical Solution to the Problem of a Small Cosmological Constant and Late-time Cosmic Acceleration*, Phys. Rev. Lett. **85** (2000) 4438–4441, [astro-ph/0004134]. ii, 4

[APMS01] C. Armendariz-Picon, V. Mukhanov and P. J. Steinhardt, *Essentials of k-essence*, Phys. Rev. D **63** (2001) 103510, [astro-ph/0006373]. ii, 4

[Bar80] J. Bardeen, *Gauge-invariant cosmological perturbations*, Phys. Rev. D (1980). 89

[BCD96] J. F. Bell, F. Camilo and T. Damour, *A Tighter Test of Local Lorentz Invariance using PSR J2317+1439*, Astrophys. J. **464** (1996) 857, [astro-ph/9512100]. 15

[BCNP07] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, *Spontaneous Lorentz Breaking and Massive Gravity*, Phys. Rev. Lett. **99** (2007) 131101, [hep-th/0703264]. iii, 5

[BCNP08] Z. Berezhiani, D. Comelli, F. Nesti and L. Pilo, *Exact Spherically Symmetric Solutions in Massive Gravity*, JHEP **0807** (2008) 130, [arXiv:0803.1687]. iii, 5

[BCNP09] D. Blas, D. Comelli, F. Nesti and L. Pilo, *Lorentz Breaking Massive Gravity in Curved Space*, Phys. Rev. D **80** (2009) 044025, [arXiv:0905.1699]. ii, iii, 4, 5

[BDG07] D. Blas, C. Deffayet and J. Garriga, *Bigravity and Lorentz-violating massive gravity*, Phys. Rev. D **76** (2007) 104036, [arXiv:0705.1982]. iii, 5

[Bek04] J. D. Bekenstein, *Relativistic gravitation theory for the MOND paradigm*, Phys. Rev. D **70** (2004) 083509, erratum-ibid. D71 (2005) 069901, [astro-ph/0403694]. iii, 5

[BM09] D. Bessada and O. D. Miranda, *CMB anisotropies induced by tensor modes in Massive Gravity*, JCAP **0908** (2009) 033, [arXiv:0908.1360]. iii, 4, 5

[BMV08] E. Babichev, V. Mukhanov and A. Vikman, *k-Essence, superluminal propagation, causality and emergent geometry*, JHEP **02** (2008) 101, [arXiv:0708.0561]. 18

[BP00] R. Bousso and J. Polchinski, *Quantization of Four-form Fluxes and Dynamical Neutralization of the Cosmological Constant*, JHEP **0006** (2000) 006, [hep-th/0004134]. ii, 4

[BPS09] D. Blas, O. Pujolàs and S. Sibiryakov, *On the Extra Mode and Inconsistency of Hořava Gravity*, JHEP **10** (2009) 029, [arXiv:0906.3046]. iii, iv, 5, 6, 18

[BPS10a] D. Blas, O. Pujolàs and S. Sibiryakov, *A healthy extension of Hořava gravity*, Phys. Rev. Lett. **104** (2010) 181302, [arXiv:0909.3525]. iv, 5, 18, 19, 20, 24, 87

[BPS10b] D. Blas, O. Pujolàs and S. Sibiryakov, *Comment on 'Strong coupling in extended Hořava-Lifshitz gravity'*, Phys. Lett. B **688** (2010) 350–355, [arXiv:0912.0550]. iv, 5, 18, 24, 87

[Bru07] J.-P. Bruneton, *On causality and superluminal behavior in classical field theories. Applications to k-essence theories and MOND-like theories of gravity*, Phys. Rev. D **75** (2007) 085013, [gr-qc/0607055]. 18

[BT10] R. Bean and M. Tangmatitham, *Current constraints on the cosmic growth history*, Phys. Rev. D **81** (2010) 083534, [arXiv:1002.4197]. 36

[Cap02] S. Capozziello, *Curvature Quintessence*, Int. J. Mod. Phys. D **11** (2002) 483–492, [gr-qc/0201033]. ii, 4

[CDTT04] S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, *Is Cosmic Speed-Up Due to New Gravitational Physics?*, Phys. Rev. D **70** (2004) 043528, [astro-ph/0306438]. ii, 4

[CJM04] J. M. Cline, S. Jeon and G. D. Moore, *The phantom menaced: constraints on low-energy effective ghosts*, Phys. Rev. D **70** (2004) 043543, [hep-ph/0311312]. 55

[CL04] S. M. Carroll and E. A. Lim, *Lorentz-Violating Vector Fields Slow the Universe Down*, Phys. Rev. D **70** (2004) 123525, [hep-th/0407149]. 14

[CM10] A. H. Chamseddine and V. Mukhanov, *Higgs for Graviton: Simple and Elegant Solution*, JHEP **08** (2010) 011, [arXiv:1002.3877]. ii, 4

[DCC+09] S. F. Daniel, R. R. Caldwell, A. Cooray, P. Serra and A. Melchiorri, *A Multi-Parameter Investigation of Gravitational Slip*, Phys. Rev. D **80** (2009) 023532, [arXiv:0901.0919]. 36

[DEF94] T. Damour and G. Esposito-Farese, *Testing for Preferred-Frame Effects in Gravity with Artificial Earth Satellites*, Phys. Rev. D **49** (1994) 1693–1706, [gr-qc/9311034]. 38

[DFST09] S. Dubovsky, R. Flauger, A. Starobinsky and I. Tkachev, *Signatures of a Graviton Mass in the Cosmic Microwave Background*, Phys. Rev. D **81** (2009) 023523, [arXiv:0907.1658]. iii, 4, 5

[DGP00] G. Dvali, G. Gabadadze and M. Porrati, *4D Gravity on a Brane in 5D Minkowski Space*, Phys. Lett. B **485** (2000) 208–214, [hep-th/0005016]. ii, 4

[DH03] S. Dodelson and L. Hui, *A Horizon Ratio Bound for Inflationary Fluctuations*, Phys. Rev. Lett. **91** (2003) 131301, [astro-ph/0305113]. 52

[DK02] T. Damour and I. I. Kogan, *Effective Lagrangians and Universality Classes of Nonlinear Bigravity*, Phys. Rev. D **66** (2002) 104024, [hep-th/0206042]. iii, 5

[DPRW65] R. H. Dicke, P. J. E. Peebles, P. G. Roll and D. T. Wilkinson, *Cosmic Black-Body Radiation.*, Astrophysical Journal **142** (1965) 414. i, 3

[DTT05] S. L. Dubovsky, P. G. Tinyakov and I. I. Tkachev, *Massive graviton as a testable cold dark matter candidate*, Phys. Rev. Lett. **94** (2005) 181102, [hep-th/0411158]. iii, 4, 5

[Dub04] S. L. Dubovsky, *Phases of massive gravity*, JHEP **0410** (2004) 076, [hep-th/0409124]. ii, iii, 4, 5

[Dur01] R. Durrer, *The theory of CMB anisotropies*, Journal of Physical Studies **5** (2001) 177, [astro-ph/0109522v1]. 57, 65

[EJ04] C. Eling and T. Jacobson, *Static post-Newtonian equivalence of GR and gravity with a dynamical preferred frame*, Phys. Rev. D **69** (2004) 064005, [gr-qc/0310044]. 15, 36

[EMS05] J. W. Elliott, G. D. Moore and H. Stoica, *Constraining the New Aether: Gravitational Cherenkov Radiation*, JHEP **08** (2005) 066, [hep-ph/0505211]. 17

[GJW05] M. L. Graesser, A. Jenkins and M. B. Wise, *Spontaneous Lorentz Violation and the Long-Range Gravitational Preferred-Frame Effect*, Phys. Lett. B **613** (2005) 5–10, [hep-th/0501223]. 38

[GM99] J. Garriga and V. F. Mukhanov, *Perturbations in k-inflation*, Phys. Lett. B **458** (1999) 219–225, [hep-th/9904176]. ii, 4

[GMSM10] T. Giannantonio, M. Martinelli, A. Silvestri and A. Melchiorri, *New constraints on parametrised modified gravity from correlations of the CMB with large scale structure*, JCAP **1004** (2010) 030, [arXiv:0909.2045]. 36

[GMST98] J. Garriga, X. Montes, M. Sasaki and T. Tanaka, *Canonical Quantization of Cosmological Perturbations in the One Bubble Open Universe*, Nucl. Phys. B **513** (1998) 343–374, [astro-ph/9706229]. 27

[GR80] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, New York Academic Press, New York, U.S.A., 1980. 69

[Hoř09a] P. Hořava, *Quantum Gravity at a Lifshitz Point*, Phys. Rev. D **79** (2009) 084008, [arXiv:0901.3775]. iii, 5, 18

[Hoř09b] P. Hořava, *Spectral Dimension of the Universe in Quantum Gravity at a Lifshitz Point*, Phys. Rev. Lett. **102** (2009) 161301, [arXiv:0902.3657]. iii, 5, 18

[HW97] W. Hu and M. White, *CMB Anisotropies: Total Angular Momentum Method*, Phys. Rev. D **56** (1997) 596–615, [astro-ph/9702170]. 69, 77, 78

[Jac93] R. Jackiw, *(Constrained) Quantization Without Tears*, hep-th/9306075 (1993), [hep-th/9306075]. 27

[Jac07] T. Jacobson, *Einstein-aether gravity: a status report*, PoS **QG-PH** (2007) 020, [arXiv:0801.1547]. 13, 14, 15, 86, 87

[Jac10] T. Jacobson, *Extended Hořava gravity and Einstein-aether theory*, Phys. Rev. D **81** (2010) 101502, [arXiv:1001.4823]. iv, 6, 18, 20, 24, 83

[JM01] T. Jacobson and D. Mattingly, *Gravity with a dynamical preferred frame*, Phys. Rev. D **64** (2001) 024028, [gr-qc/0007031]. iii, 5, 11

[JM04] T. Jacobson and D. Mattingly, *Einstein-Aether Waves*, Phys. Rev. D **70** (2004) 024003, [gr-qc/0402005]. 31, 51, 54

[KDN+09] E. Komatsu *et al.*, *Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation*, Astrophys. J. Suppl. **180** (2009) 330–376, [arXiv:0803.0547]. i, 4, 16, 75

[KM06] T. Koivisto and D. F. Mota, *Dark Energy Anisotropic Stress and Large Scale Structure Formation*, Phys. Rev. D **73** (2006) 083502, [astro-ph/0512135]. 36

[KS84] H. Kodama and M. Sasaki, *Cosmological Perturbation Theory*, Progress of Theoretical Physics Supplement **78** (1984) 1. 89

[KUY10] T. Kobayashi, Y. Urakawa and M. Yamaguchi, *Cosmological perturbations in a healthy extension of Hořava gravity*, JCAP **04** (2010) 025, [arXiv:1002.3101]. iv, v, 6, 7, 25, 39, 45, 46, 47, 48, 49

[Lim05] E. A. Lim, *Can We See Lorentz-Violating Vector Fields in the CMB?*, Phys. Rev. D **71** (2005) 063504, [astro-ph/0407437]. iv, 6, 18, 53, 54

[LL09] D. H. Lyth and A. Liddle, *The primordial density perturbation: cosmology, inflation and the origin of structure*, Cambridge University Press, Cambridge, UK, 2009, 497 pp. 57

[LM85] F. Lucchin and S. Matarrese, *Power Law Inflation*, Phys. Rev. D **32** (1985) 1316. 15

[LM02] I. Low and A. V. Manohar, *Spontaneously Broken Spacetime Symmetries and Goldstone's Theorem*, Phys. Rev. Lett. **88** (2002) 101602, [hep-th/0110285]. iii, 5

[LMB08] B. Li, D. F. Mota and J. D. Barrow, *Detecting a Lorentz-Violating Field in Cosmology*, Phys. Rev. D **77** (2008) 024032, [arXiv:0709.4581]. iv, 6, 53

[MFB92] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, *Theory of cosmological perturbations*, Physics Reports **215** (1992) 203. 89

[Mil83] M. Milgrom, *A Modification of the Newtonian dynamics as a possible alternative to the hidden mass hypothesis*, Astrophys. J. **270** (1983) 365–370. iii, 5

[Muk04] V. Mukhanov, *CMB-slow, or How to Estimate Cosmological Parameters by Hand*, International Journal of Theoretical Physics **43** (2004) 623–668, [astro-ph/0303072v1]. 57

[Muk05] V. Mukhanov, *Physical foundations of cosmology*, Cambrige University Press, Cambrige, UK, 2005, 421 pp. 57, 60, 63, 65, 66, 69, 72, 73, 74, 89

[NK11] M. Nakashima and T. Kobayashi, *The Music of the Aetherwave - B-mode Polarization in Einstein-Aether Theory*, arXiv:1103.2197 **astro-ph.CO** (2011), [arXiv:1103.2197]. iv, ix, 6, 78, 79, 81, 86

[Nor87] K. Nordtvedt, *Probing gravity to the second post-Newtonian order and to one part in 10 to the 7th using the spin axis of the sun*, Astrophys. J. **320** (1987) 871–874. 16

[PAG+99] S. Perlmutter *et al.*, *Measurements of Omega and Lambda from 42 High-Redshift Supernovae*, The Astrophysical Journal **517** (1999) 565, [astro-ph/9812133v1]. i, 3

[PK05] J. R. Pritchard and M. Kamionkowski, *Cosmic Microwave Background Fluctuations from Gravitational Waves: An Analytic Approach*, Annals Phys. **318** (2005) 2–36, [astro-ph/0412581]. 81

[PR88] P. J. E. Peebles and B. Ratra, *Cosmology with a Time Variable Cosmological Constant*, Astrophys. J. **325** (1988) L17. ii, 4

[PW65] A. A. Penzias and R. W. Wilson, *A Measurement of Excess Antenna Temperature at 4080 Mc/s.*, Astrophysical Journal **142** (1965) 419. i, 3

[RFC+98] A. G. Riess *et al.*, *Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant*, The Astronomical Journal **116** (1998) 1009, [astro-ph/9805201v1]. i, 3

[Rub07] V. Rubakov, *Lorentz-violating graviton masses: getting around ghosts, low strong coupling scale and VDVZ discontinuity*, hep-th/0407104 (2007), [hep-th/0407104]. iii, 4, 5

[SBK+92] G. F. Smoot *et al.*, *Structure in the COBE differential microwave radiometer first-year maps*, Astrophysical Journal **396** (1992) L1. i, 4

[Sta85] A. A. Starobinsky, *Cosmic Background Anisotropy Induced by Isotropic Flat- Spectrum Gravitational-Wave Perturbations*, Sov. Astron. Lett. **11** (1985) 133. 73

[SZ96] U. Seljak and M. Zaldarriaga, *A Line-of-Sight Integration Approach to Cosmic Microwave Background Anisotropies*, Astrophysical Journal v.469 **469** (1996) 437, [astro-ph/9603033]. 57

[TWL93] M. S. Turner, M. White and J. E. Lidsey, *Tensor Perturbations in Inflationary Models as a Probe of Cosmology*, Phys. Rev. D **48** (1993) 4613–4622, [astro-ph/9306029]. 73, 74

[Wei03] S. Weinberg, *Adiabatic Modes in Cosmology*, Phys. Rev. D **67** (2003) 123504, [astro-ph/0302326]. 33, 115, 123

[Wei04a] S. Weinberg, *Can Non-Adiabatic Perturbations Arise After Single-Field Inflation?*, Phys. Rev. D **70** (2004) 043541, [astro-ph/0401313]. 33, 115

[Wei04b] S. Weinberg, *Must Cosmological Perturbations Remain Non-Adiabatic After Multi-Field Inflation?*, Phys. Rev. D **70** (2004) 083522, [astro-ph/0405397]. 33, 115

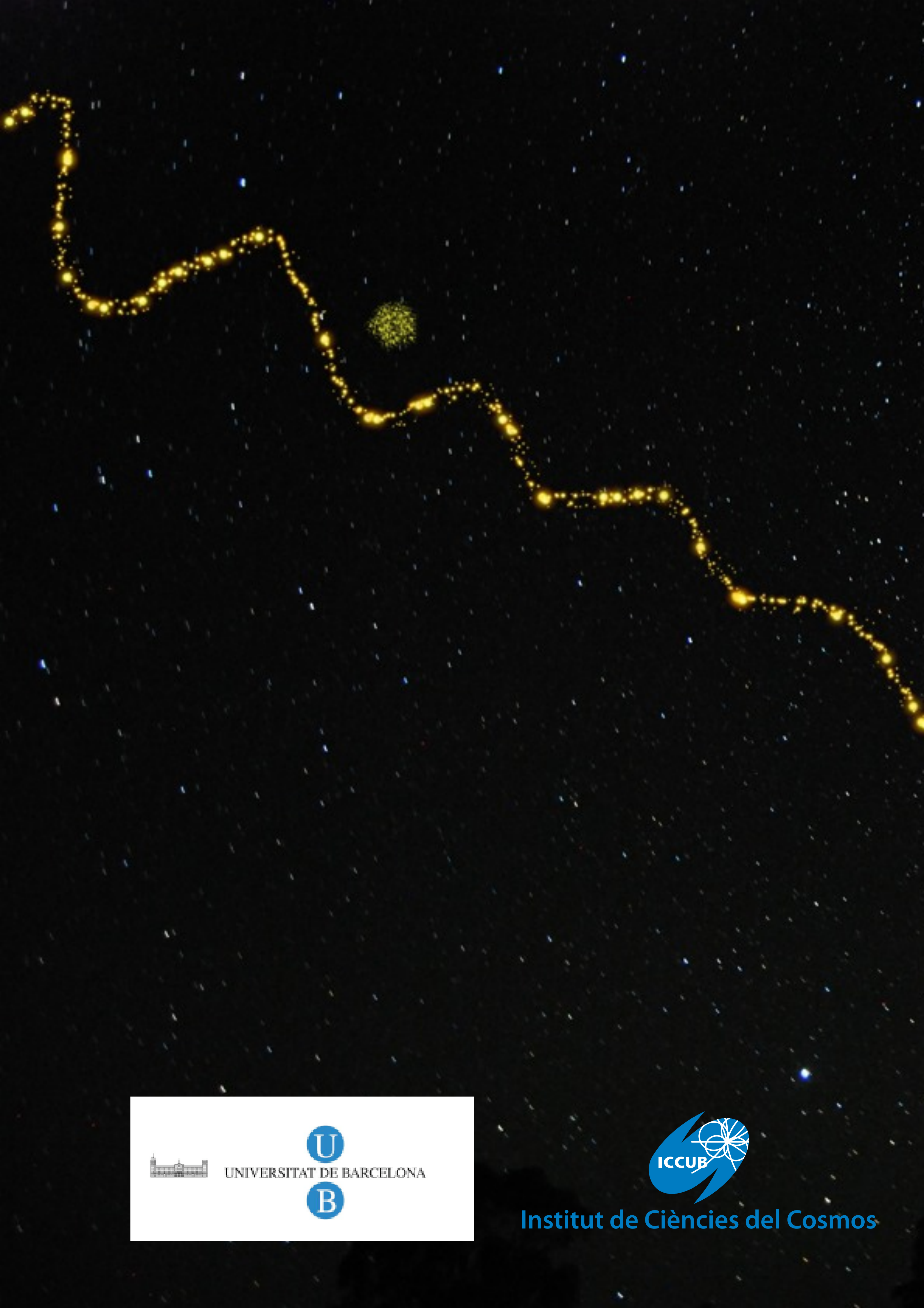
[Wei08] S. Weinberg, *Cosmology*, Oxford University Press, Oxford, UK, 2008, 593 pp. 33, 57, 89, 115

[Wil93] C. M. Will, *Theory and Experiment in Gravitational Physics*, Cambrige University Press, Cambrige, UK, 1993, 342 pp. ii, 4, 15

[Wit09] B. Withers, *Einstein-aether as a quantum effective field theory*, Class. Quant. Grav. **26** (2009) 225009, [arXiv:0905.2446]. 87

[ZFZ08] J. A. Zuntz, P. G. Ferreira and T. G. Zlosnik, *Constraining Lorentz violation with cosmology*, Phys. Rev. Lett. **101** (2008) 261102, [arXiv:0808.1824]. iv, 6

[ZZB+10] J. Zuntz, T. G. Zlosnik, F. Bourliot, P. G. Ferreira and G. D. Starkman, *Vector field models of modified gravity and the dark sector*, Phys. Rev. D **81** (2010) 104015, [arXiv:1002.0849]. iv, 6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