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The Effect of a Threshold Proportional Reinsurance Strategy on Ruin Probabilities

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Abstract: In the context of a compound Poisson risk model, we define a threshold proportional reinsurance strategy: A retention level k_1 is applied whenever the reserves are less than a determinate threshold b, and a retention level k_2 is applied in the other case. We obtain the integro-differential equation for the Gerber-Shiu function (defined in Gerber and Shiu (1998)) in this model, which allows us to obtain the expressions for ruin probability and Laplace transforms of time of ruin for several distributions of the claim sizes. Finally, we present some numerical results.

JEL Classification: G22.

Keywords: threshold proportional reinsurance strategy, Gerber-Shiu function, ruin probability, time of ruin.

Resumen: En un modelo de Poisson compuesto, definimos una estrategia de reaseguro proporcional de umbral: Se aplica un nivel de retención k_1 siempre que las reservas sean inferiores a un determinado umbral b, y un nivel de retención k_2 en caso contrario. Obtenemos la ecuación íntegro-diferencial para la función Gerber-Shiu (definida en Gerber-Shiu (1998)) en este modelo, que nos permite obtener las expresiones de la probabilidad de ruina y de la transformada de Laplace del momento de ruina para distintas distribuciones de la cuantía individual de los siniestros. Finalmente presentamos algunos resultados numéricos.

Palabras clave: estrategia de reaseguro umbral, función Gerber-Shiu, probabilidad de ruina, momento de ruina.

1 Introduction

Studies on the effect of reinsurance strategy on solvency measures have concentrated their attention on the ultimate ruin probability. Several of them analyze the effect of reinsurance on the adjustment coefficient or Lundberg exponent (Waters (1979), Chapter 8 of Gerber (1979), Centeno (1986,2002) and Hesselager (1990)).

Many authors have considered the problem of determining the optimal level and/or type of reinsurance, where optimal is defined in terms of some stability criterion, manly the probability of ruin (Waters (1983), Goovaerts et al.(1989), Chapter 6 of Bühlmann (1996), Chapter 14 of Bowers et al. (1997), Verlaak and Beirlant (2003), Schmidli (2001, 2002), Hipp and Vogt (2003) or Taksar and Markussen (2003)). The reinsurance strategy considered can be static or dynamic. In the first case, it is assumed that the level and type of reinsurance remain constant throughout the period considered, which in many cases is infinite (Waters (1983), Centeno (1986, 2005) and Dickson and Waters (1996)). In the dynamic case, we can find papers which consider that for a fixed type of reinsurance the level of reinsurance can change continuously (Hojgaard and Taksar (1998), Schmidli (2001, 2002), Hipp and Vogt (2003) and Taksar and Markussen (2003)).In these papers, optimal stochastic control tools in continuous time are used. Dickson and Waters (2006) assume that the insurer can change the type and/or level of reinsurance at the start of each year, so they studied a discrete time stochastic control problem.

In this paper we consider a classical (compound Poisson) model for the insurer surplus, and introduce a dynamic reinsurance strategy. We assume that the insurer considers a proportional reinsurance, but the retention level is not constant and depends on the level of the surplus. We then define a threshold proportional strategy: A retention level k_1 is applied whenever the reserves are less than a determinate threshold b, and a retention level k_2 is applied in the other case. As, for the insurer, reinsurance is a tool for controling the solvency of the portfolio, it seems natural that the retention level depends on the surplus level at each moment. The threshold proportional reinsurance strategy that we propose in this paper is an easy and clear way to include this dependence.

The objective of this paper is to analyze the effect of this new strategy on the solvency measures of the insurer using the Gerber-Shiu function (defined in Gerber and Shiu (1998)), which allows us to obtain ruin probability and time of ruin.

The paper is organized as follows: In Section 2 we explain the assumptions and some preliminaries. In Section 3, we obtain the integro-differential equation for the Gerber-Shiu function in a model with a threshold reinsurance strategy. Mathematically, the process is similar to that applied by Lin and Pavlova (2006) in order to analyze dividend problems. We then analyze some special cases of the Gerber-Shiu function. In Sections 4 and 5 we obtain the expressions for the ruin probability and time of ruin if the individual claim amount is distributed as an exponential and a phase-type(2). Finally, in Section 6, some numerical results are presented.

2 Assumptions and preliminaries

In the classical risk theory model, the surplus, $R(t)$, at a given time $t \in [0, \infty)$ is defined as $R(t) = u + ct - S(t)$, with $u = R(0) \geq 0$ being the insurer's initial surplus, $S(t)$ the aggregate claims and c the rate at which the premiums are received.

$S(t)$ is modeled as a compound Poisson process

$$S(t) = \sum_{i=1}^{N(t)} Z_i,$$

where $N(t)$, the number of claims occurring until time t, follows a Poisson process with parameter λ, the amount of claims $\{Z_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables with density function $f(z)$ and $N(t)$ is independent of $\{Z_i, i \geq 1\}$.

The instantaneous premium rate, c, is proportional to the product of the mean number of claims, λ, and the mean value of the claim amount, $E[Z]$. In other words, $c = \lambda E[Z](1 + \rho)$, where ρ, called the security loading coefficient, is a positive constant, in order to fulfill the net profit condition.

In this model, and in the more general ordinary renewal model, the claims interrocurrence times, $\{T_i\}_{i=1}^{\infty}$, are modeled as a sequence of independent and identically distributed random variables, where T_1 denotes the time until the first claim and T_i , for $i > 1$, denotes the time between the $(i-1)$-th and i-th claims. Note that in a Poisson process with parameter λ, $T_i, i \geq 1$ has an exponential distribution with mean $1/\lambda$.

The time to ruin is defined as $T = \min\{t \mid R(t) < 0\}$, with $T = \infty$ if $R(t) \geq 0$ for all $t \geq 0$. The ruin probability is

$$\psi(u) = P[T < \infty \mid R(0) = u] = E\{I(T < \infty) \mid R(0) = u\},$$

where $I(A) = 1$ if A occurs and $I(A) = 0$ otherwise.

Let us first consider first the effect of a proportional reinsurance. The ceding company (insurer) and the reinsurer agree on a cession percentage, say $(1-k)$, k being the retention level applied to each claim. Then, in one period, the expected aggregate cost assumed by the insurer is $k\lambda E[Z]$ and the expected aggregate cost assumed by the reinsurer is $(1-k)\lambda E[Z]$.

We assume that insurance and reinsurance premiums are calculated by the expected value principle with positive loading factors, $\rho_R > 0$ being the reinsurer loading factor.

The total premium income retained by the insurer, c', depends on ρ_R and k, where

$$c' = \lambda E[Z](1+\rho) - (1-k)\ (1+\rho_R)\ \lambda\ E[Z].$$

A new security loading for the insurer, ρ_N, can be defined,

$$\begin{aligned} c' &= k\lambda E[Z](1+\rho_N) = \lambda E[Z]\left((1+\rho) - (1-k)(1+\rho_R)\right) \\ &\Longrightarrow \rho_N = \rho_R - \tfrac{\rho_R - \rho}{k}, \forall k > 0. \end{aligned}$$

If $\rho = \rho_R$, the total premium paid by the policyholder c is shared between insurer and reinsurance in the same proportion k, so $c' = kc$ and $\rho_N = \rho$.

In this paper, we consider a threshold proportional reinsurance strategy, which is defined by a threshold $b \geq 0$. A retention level k_1 is applied whenever the reserves are less than b, and a retention level k_2 is applied in the other case. Then, the premium income retained is c_1 and c_2, respectively. We consider that the retention levels give new positive security loadings for the insurer , i.e. the net profit condition is always fulfilled.

Graphically,

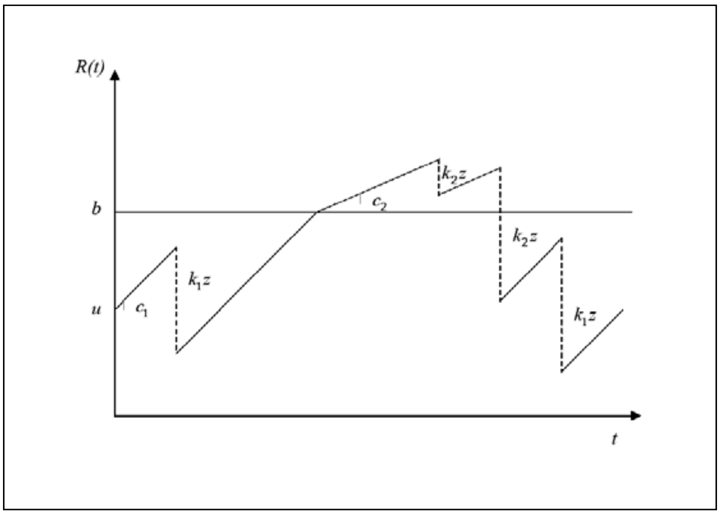

Figure 1: Threshold reinsurance strategy

Let $R^{-}(T)$ be the surplus just before ruin, and $R^{+}(T)$ the surplus at ruin if ruin occurs. Gerber and Shiu (1998, 2005) define the function

$$\phi(u) = E\left[e^{-\delta T} w\left(R^{-}(T), \left|R^{+}(T)\right|\right) I(T<\infty) \middle| R(0)=u\right], \tag{1}$$

where $\delta \geq 0$ is the discounted factor, and $w(x,y)$ is the penalty function, so that $\phi(u)$ is the expected discounted penalty payable at ruin. This function is known to satisfy a defective renewal equation (Gerber and Shiu (1998), Li and Garrido (2004), Willmot (2007)). Easy explicit formulae for $\phi(u)$ are only available for certain special cases for the claim size distribution (Landriault and Willmot (2008), Lin and Willmot (1999,2000)).

Let $\phi(u)$ with $w(x,y)=1$, then we arrive at the expression for the Laplace transform of the time of ruin $E\left[e^{-\delta T} I(T<\infty)\right]$, and if addition $\delta = 0$, then $P[T<\infty] = \psi(u)$, i.e. the ruin probability.

3 Integro-differential equation for the Gerber-Shiu Function

In this section, we derive the integro-differential equations satisfied by the Gerber-Shiu discounted penalty function. The discounted penalty function $\phi(u)$ behaves differently, depending on whether its initial surplus u is below or above the level b. Hence, for notational convenience, we write

$$\phi(u)=\begin{cases}\phi_1(u) & 0\le u<b\\ \phi_2(u) & u\ge b\end{cases}.$$

Theorem 1 *The discounted penalty function $\phi(u)$ satisfies the integro-differential equations*

$$\phi'(u)=\begin{cases}\phi_1'(u) & 0\le u<b\\ \phi_2'(u) & u\ge b\end{cases}, \tag{2}$$

where

$$\begin{aligned}
\phi_1'(u) &= \frac{\lambda+\delta}{c_1}\phi_1(u)-\frac{\lambda}{c_1}\int_0^{\frac{u}{k_1}}\phi_1(u-zk_1)dF(z)-\frac{\lambda}{c_1}\xi_1(u)\\
\phi_2'(u) &= \frac{\lambda+\delta}{c_2}\phi_2(u)-\frac{\lambda}{c_2}\left[\int_0^{\frac{u-b}{k_2}}\phi_2(u-zk_2)dF(z)\right.\\
&\quad\left.+\int_{\frac{u-b}{k_2}}^{\frac{u}{k_2}}\phi_1(u-zk_2)dF(z)\right]-\frac{\lambda}{c_2}\xi_2(u),
\end{aligned}$$

and

$$\begin{aligned}
\xi_1(t) &= \int_{\frac{t}{k_1}}^{\infty} w(t,zk_1-t)f(z)dz,\\
\xi_2(t) &= \int_{\frac{t}{k_2}}^{\infty} w(t,zk_2-t)f(z)dz.
\end{aligned}$$

Let $w\left(R^-\left(T\right),\left|R^+\left(T\right)\right|\right)$ be a nonnegative function of $R^-\left(T\right)>0$, the surplus immediately before ruin, and $R^+\left(T\right)>0$ the surplus at ruin.

Proof. For $0 \leq u < b$,

$$\begin{aligned}
\phi_1(u) &= \int_0^{\frac{b-u}{c_1}} e^{-\delta t}\lambda e^{-\lambda t}\left[\int_0^{\frac{u+c_1t}{k_1}} \phi(u+c_1t-zk_1)dF(z)\right. \\
&\quad \left. + \int_{\frac{u+c_1t}{k_1}}^{\infty} w(u+c_1t, zk_1-u-c_1t)dF(z)\right] dt \qquad (3) \\
&\quad + \int_{\frac{b-u}{c_1}}^{\infty} e^{-\delta t}\lambda e^{-\lambda t}\left[\int_0^{\frac{b+c_2\left(t-\frac{b-u}{c_1}\right)}{k_2}} \phi\left(b+c_2\left(t-\tfrac{b-u}{c_1}\right)-zk_2\right)dF(z)\right. \\
&\quad \left. + \int_{\frac{b+c_2\left(t-\frac{b-u}{c_1}\right)}{k_2}}^{\infty} w\left(b+c_2\left(t-\tfrac{b-u}{c_1}\right), zk_2-b-c_2\left(t-\tfrac{b-u}{c_1}\right)\right)dF(z)\right] dt \\
&= \lambda\int_0^{\frac{b-u}{c_1}} e^{-(\lambda+\delta)t}\gamma_1(u+c_1t)dt \\
&\quad +\lambda\int_{\frac{b-u}{c_1}}^{\infty} e^{-(\lambda+\delta)t}\gamma_2\left(b+c_2\left(t-\tfrac{b-u}{c_1}\right)\right)dt,
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1(t) &= \int_0^{\frac{t}{k_1}} \phi(t-zk_1)dF(z)+\xi_1(t), \\
\gamma_2(t) &= \int_0^{\frac{t}{k_2}} \phi(t-zk_2)dF(z)+\xi_2(t).
\end{aligned}$$

Now, a change of variables in (3) results in

$$\begin{aligned}
\phi_1(u) &= \tfrac{\lambda}{c_1}e^{\frac{(\lambda+\delta)u}{c_1}}\int_u^b e^{-\frac{(\lambda+\delta)t}{c_1}}\gamma_1(t)dt \qquad (4) \\
&\quad +\tfrac{\lambda}{c_2}e^{\frac{(\lambda+\delta)u}{c_1}}\int_b^{\infty} e^{-(\lambda+\delta)\left(t-\frac{(c_1-c_2)b}{c_1}\right)/c_2}\gamma_2(t)dt.
\end{aligned}$$

By differentiating (4) with respect to u we obtain

$$\phi_1'(u) = \frac{\lambda+\delta}{c_1}\phi_1(u) - \frac{\lambda}{c_1}\int_0^{\frac{u}{k_1}} \phi_1(u-zk_1)dF(z) - \frac{\lambda}{c_1}\xi_1(u).$$

Similarly, when $u \geq b$,

$$\begin{aligned}
\phi_2(u) &= \int_0^{\infty} e^{-\delta t}\lambda e^{-\lambda t}\left[\int_0^{\frac{u+c_2t}{k_2}} \phi(u+c_2t-zk_2)dF(z)\right. \\
&\quad \left. + \int_{\frac{u+c_2t}{k_2}}^{\infty} w(u+c_2t, zk_2-u-c_2t)dF(z)\right] dt \\
&= \lambda\int_0^{\infty} e^{-(\lambda+\delta)t}\gamma_2(u+c_2t)dt.
\end{aligned}$$

With a change of variable and differentiating with respect to u

$$\begin{aligned}\phi_2'(u) &= \tfrac{\lambda+\delta}{c_2}\phi_2(u) - \tfrac{\lambda}{c_2}\left[\int_0^{\frac{u-b}{k_2}} \phi_2(u-zk_2)dF(z)\right.\\ &\quad \left.+\int_{\frac{u-b}{k_2}}^{\frac{u}{k_2}} \phi_1(u-zk_2)dF(z)\right] - \tfrac{\lambda}{c_2}\xi_2(u).\end{aligned}$$

by which the proof is concluded. ■

From now on let $w(x,y)=1$. So, in (2), we have

$$\begin{cases}\begin{array}{ll}\phi_1'(u) = \frac{\lambda+\delta}{c_1}\phi_1(u) - \frac{\lambda}{c_1}\int_0^{\frac{u}{k_1}} \phi_1(u-zk_1)dF(z) \\ \qquad -\frac{\lambda}{c_1}\left[1-F\left(\frac{u}{k_1}\right)\right], & 0 \le u < b \\ \\ \phi_2'(u) = \frac{\lambda+\delta}{c_2}\phi_2(u) - \frac{\lambda}{c_2}\left[\int_0^{\frac{u-b}{k_2}} \phi_2(u-zk_2)dF(z)\right. \\ \left. +\int_{\frac{u-b}{k_2}}^{\frac{u}{k_2}} \phi_1(u-zk_2)dF(z)\right] - \frac{\lambda}{c_2}\left[1-F\left(\frac{u}{k_2}\right)\right], & u \ge b.\end{array}\end{cases} \tag{5}$$

4 Ruin probability and time of ruin with individual claim amount exponential

In this section we consider the case when the individual claim amount Z is distributed as an exponential(1).

By substituting $f(z)=e^{-z}$ in (5) and differentiating with respect to u, it is easy to obtain the ordinal differential equations,

$$\begin{array}{ll}\phi_1''(u) - \left(\frac{\lambda+\delta}{c_1} - \frac{1}{k_1}\right)\phi_1'(u) - \frac{\delta}{c_1k_1}\phi_1(u) = 0, & 0 \le u < b \\ \phi_2''(u) - \left(\frac{\lambda+\delta}{c_2} - \frac{1}{k_2}\right)\phi_2'(u) - \frac{\delta}{c_2k_2}\phi_2(u) = 0, & u \ge b.\end{array} \tag{6}$$

The corresponding characteristic equations are

$$\begin{array}{ll}r^2 - \left(\frac{\lambda+\delta}{c_1} - \frac{1}{k_1}\right) r - \frac{\delta}{c_1k_1} = 0, & 0 \le u < b \\ s^2 - \left(\frac{\lambda+\delta}{c_2} - \frac{1}{k_2}\right) s - \frac{\delta}{c_2k_2} = 0, & u \ge b,\end{array}$$

and the real roots are $r_1 < 0, r_2 \ge 0$, $s_1 < 0$ and $s_2 \ge 0$. The roots r_2 and s_2 are equal to zero if $\delta = 0$ (the ruin probability case), and positive if $\delta > 0$ (Laplace transform of the time of ruin).

Then the Laplace transform of the time of ruin $E\left[e^{-\delta T}I\left(T<\infty\right)\right]$ is

$$\phi\left(u\right)=\begin{cases}\phi_1(u)=C_1e^{r_1u}+C_2e^{r_2u}, & 0\leq\ u<b\\ \phi_2(u)=D_1e^{s_1u}+D_2e^{s_2u}, & u\geq b\end{cases} \tag{7}$$

being $C_i, D_i, i=1,2$ the coefficients of the solution of the ordinal differential equations (6). These coefficients depend on δ but not on u. From the condition $\lim\limits_{u\to\infty}\phi\left(u\right)=0$, we know that $D_2=0$, and from the continuity condition $\phi_1(b)=\phi_2(b)$ we obtain $\sum_{i=1}^{2}C_ie^{r_ib}-D_1e^{s_1b}=0$. By substituting (7) in (5), we obtain two additional conditions, $\sum_{i=1}^{2}\frac{C_i}{k_1r_i+1}=1$ and $\sum_{i=1}^{2}\frac{C_i}{k_2r_i+1}\left(1-e^{b\left(r_i+\frac{1}{k_2}\right)}\right)+\frac{D_1}{s_1k_2+1}e^{b\left(s_1+\frac{1}{k_2}\right)}=1$, which allows us to obtain the coefficients $C_i, D_i, i=1,2$. So, if we make the dependence of the coefficients on δ explicit,

$$C_1\left(\delta\right)=\frac{a_{2,1}a_{1,1}\left((k_2s_1+1)r_2(k_2-k_1)-a_{1,2}k_2(r_2-s_1)e^{\frac{a_{2,2}}{k_2}b}\right)}{(k_2s_1+1)(r_1-r_2)(k_1-k_2)-k_2\left(a_{1,1}a_{2,2}(s_1-r_1)e^{\frac{a_{2,1}}{k_2}b}-a_{1,2}a_{2,1}(s_1-r_2)e^{\frac{a_{2,2}}{k_2}b}\right)},$$

$$C_2\left(\delta\right)=a_{1,2}-\frac{a_{1,2}}{a_{1,1}}C_1\left(\delta\right),$$

$$D_1\left(\delta\right)=a_{1,2}e^{(r_2-s_1)b}+\left(e^{(r_1-s_1)b}-\frac{a_{1,2}}{a_{1,1}}e^{(r_2-s_1)b}\right)C_1\left(\delta\right),$$

where $a_{i,j}=\left(k_ir_j+1\right), i,j=1,2.$

To obtain the *ruin probability*, $\phi(u)=E\left[I\left(T<\infty\right)\right]=\psi(u)$, let $\delta=0$ in (7), then,

$$\psi\left(u\right)=\begin{cases}\psi_1(u)=1-\left(1+\rho_1\right)C_1\left(0\right)+C_1\left(0\right)e^{-\frac{\rho_1}{k_1(1+\rho_1)}u}, & 0\leq u<b\\ \\ \psi_2(u)=e^{\frac{\rho_2}{k_2(1+\rho_2)}b}\left(1-\left((1+\rho_1)\right.\right. & \\ \qquad\left.\left.-e^{-\frac{\rho_1}{k_1(1+\rho_1)}b}\right)C_1\left(0\right)\right)e^{-\frac{\rho_2}{k_2(1+\rho_2)}u}, & u\geq b,\end{cases} \tag{8}$$

where

$$C_1\left(0\right)=\frac{h}{h\left(1+\rho_1\right)+\left(k_1-k_2\right)\rho_1\left(1+\rho_1\right)e^{-\frac{b}{k_2}}+\left(k_2\rho_1-h\right)e^{-\frac{\rho_1}{k_1(1+\rho_1)}b}},$$

with $h=\left(k_1+\rho_1\left(k_1-k_2\right)\right)\rho_2$.

From (7) and (8), if the moments of time of ruin exist, it is easy to obtain them from Laplace transform of time of ruin,

$$E\left[T^n I\left(T<\infty\right)\right]=(-1)^n \left.\frac{\partial^n \phi\left(u\right)}{\partial \delta^n}\right|_{\delta=0}.$$

For example, the expected time of ruin if ruin occurs is given by

$$E\left[T \mid T<\infty\right]=-\frac{\left.\frac{\partial \phi(u)}{\partial \delta}\right|_{\delta=0}}{\psi(u)}.$$

Then for $0\leq u<b$,

$$E\left[T \mid T<\infty\right]=-\frac{\left.\frac{\partial C_1(\delta)}{\partial \delta}\right|_{\delta=0} e^{-\frac{\rho_1}{k_1(1+\rho_1)}u}-\frac{C_1(0)ue^{-\frac{\rho_1}{k_1(1+\rho_1)}u}}{\lambda k_1\rho_1(1+\rho_1)}+\left.\frac{\partial C_2(\delta)}{\partial \delta}\right|_{\delta=0}+\frac{C_2(0)u}{\lambda k_1\rho_1}}{1-(1+\rho_1)C_1(0)+C_1(0)e^{-\frac{\rho_1}{k_1(1+\rho_1)}u}},$$

and for $u\geq b$,

$$E\left[T \mid T<\infty\right]=-\frac{\left.\frac{\partial D_1(\delta)}{\partial \delta}\right|_{\delta=0}}{D_1\left(0\right)}+\frac{1}{\lambda k_2\rho_2\left(1+\rho_2\right)}u. \tag{9}$$

We can observe that for $u\geq b$ the expression obtained for $E\left[T \mid T<\infty\right]$ is a first degree polynomial on u and, in addition, if $k_2=1$, the slope of expression (9) coincides with the slope in a model without reinsurance (see Gerber (1979), p. 138).

5 Ruin probability and time of ruin with individual claim amount Phase-type(2)

In this section we consider the case when the individual claim amount Z follows a phase-type(2) distribution (all linear combinations and convolutions of two exponential distributions (with not necessarily equal means) are included). In Dickson and Hipp (2000) it is shown that these distributions have a density satisfying the following second order differential equation:

$$f(z)+A_1f'(z)+A_2f''(z)=0 \qquad \text{for} \quad z>0 \tag{10}$$

where

$$A_2>0. \tag{11}$$

Dickson and Hipp (2000) in equation (2.1) show that,

$$1 - A_1 f(0) - A_2 f'(0) = 0. \tag{12}$$

This relationship is useful for simplifying some expressions that appear in this section.

For $0 \leq u < b$, differentiating (5) we obtain,

$$\begin{aligned}\phi_1''(u) &= \frac{\lambda+\delta}{c_1}\phi_1'(u) - \frac{\lambda}{c_1}\left(f(0)\frac{1}{k_1}\phi_1(u)\right. \\ &\left.+\frac{1}{k_1}\int_0^{\frac{u}{k_1}}\phi_1(u-zk_1)f'(z)\,dz\right) + \frac{\lambda}{k_1c_1}f\left(\frac{u}{k_1}\right).\end{aligned} \tag{13}$$

Now, from (10), we substitute $f'(z) = -\frac{A_2}{A_1}f''(z) - \frac{f(z)}{A_1}$ in (13), and knowing from (5) that

$$\frac{\lambda}{c_1}\int_0^{\frac{u}{k_1}}\phi_1(u-zk_1)f(z)dz = \frac{\lambda+\delta}{c_1}\phi_1(u) - \phi_1'(u) - \frac{\lambda}{c_1}\left(1 - F\left(\frac{u}{k_1}\right)\right),$$

we obtain,

$$\begin{aligned}\phi_1''(u) &= \left(\tfrac{\lambda+\delta}{c_1} - \tfrac{1}{k_1A_1}\right)\phi_1'(u) - \left(\tfrac{\lambda+\delta}{c_1k_1A_1} - \tfrac{\lambda}{k_1c_1}f(0)\right)\phi_1(u) \\ &+\tfrac{\lambda}{c_1k_1A_1}\left(F\left(\tfrac{u}{k_1}\right) - 1\right) + \tfrac{\lambda A_2}{c_1k_1A_1}\left(\int_0^{\frac{u}{k_1}}\phi_1(u-zk_1)f''(z)\,dz\right) \\ &+\tfrac{\lambda}{c_1k_1}f\left(\tfrac{u}{k_1}\right).\end{aligned} \tag{14}$$

Differentiating (14) and knowing that $f'''(z) = -\frac{A_1}{A_2}f''(z) - \frac{f'(z)}{A_2}$,

$$\begin{aligned}\phi_1'''(u) &= \left(\tfrac{\lambda+\delta}{c_1} - \tfrac{A_1}{k_1A_2}\right)\phi_1''(u) \\ &+\left(-\tfrac{1}{k_1^2A_2} + \tfrac{A_1(\lambda+\delta)}{c_1k_1A_2} - \tfrac{\lambda}{k_1c_1}f(0)\right)\phi_1'(u) \\ &+\tfrac{\lambda}{k_1^2c_1}\left(\tfrac{\delta}{\lambda A_2} + \left(\tfrac{1-A_1f(0)-A_2f'(0)}{A_2}\right)\right)\phi_1(u) \\ &+\tfrac{A_1\lambda}{A_2c_1^2k_1}f\left(\tfrac{u}{k_1}\right) + \tfrac{\lambda}{A_2c_1^2k_1}F\left(\tfrac{u}{k_1}\right) + \tfrac{\lambda}{c_1^2k_1}f'\left(\tfrac{u}{k_1}\right) - \tfrac{\lambda}{k_1^2c_1A_2}.\end{aligned} \tag{15}$$

From (12) and (10), (15) is

$$\begin{aligned}\phi_1'''(u) &= \left(\tfrac{\lambda+\delta}{c_1} - \tfrac{A_1}{k_1A_2}\right)\phi_1''(u) + \left(\tfrac{A_1(\lambda+\delta)}{c_1k_1A_2} - \tfrac{1}{k_1^2A_2} - \tfrac{\lambda}{k_1c_1}f(0)\right)\phi_1'(u) \\ &+\tfrac{\delta}{A_2k_1^2c_1}\phi_1(u).\end{aligned} \tag{16}$$

Following a similar process, for $u \geq b$, we obtain

$$\begin{aligned}\phi_2'''(u) &= \left(\tfrac{\lambda+\delta}{c_2} - \tfrac{A_1}{k_2 A_2}\right)\phi_2''(u) + \left(\tfrac{A_1(\lambda+\delta)}{c_2 k_2 A_2} - \tfrac{1}{k_2^2 A_2} - \tfrac{\lambda}{k_2 c_2} f(0)\right)\phi_2'(u) \\ &\quad + \tfrac{\delta}{A_2 k_2^2 c_2}\phi_2(u). \end{aligned} \tag{17}$$

The characteristic equations of (16) and (17) are

$$\begin{aligned} c_1 k_1 r^3 - \left((\lambda+\delta)k_1 - c_1\tfrac{A_1}{A_2}\right) r^2 + \left(\lambda f(0) + \tfrac{c_1}{k_1 A_2} - \tfrac{A_1(\lambda+\delta)}{A_2}\right) r - \tfrac{\delta}{A_2 k_1} &= 0, \\ c_2 k_2 r^3 - \left((\lambda+\delta)k_2 - c_2\tfrac{A_1}{A_2}\right) r^2 + \left(\lambda f(0) + \tfrac{c_2}{k_2 A_2} - \tfrac{A_1(\lambda+\delta)}{A_2}\right) r - \tfrac{\delta}{A_2 k_2} &= 0. \end{aligned}$$

Let us assume, in order to simplify the expressions, that $r_i, s_i, i = 1,2,3$ are real and distinct. Then

$$\phi(u) = \begin{cases} \phi_1(u) = \sum\limits_{i=1}^{3} F_i e^{r_i u}, & 0 \leq u < b \\ \phi_2(u) = \sum\limits_{i=1}^{3} G_i e^{s_i u}, & u \geq b. \end{cases} \tag{18}$$

Note that the coefficients $F_i, G_i,\ i = 1,2,3$ are functions of b. In order to obtain these coefficients, 6 equations are needed. The first equation is obtained from the condition $\lim_{u \longrightarrow \infty} \phi(u) = 0$. The second equation can be obtained considering that $\phi(u)$ must be continuous, note that $\phi_1(b) = \phi_2(b)$. The other 4 equations are obtained by substituting (18) in (5).

To obtain the ruin probability, let $\delta = 0$ in (16) and (17), then $\phi(u) = E\left[I\left(T < \infty\right)\right] = \psi(u)$. The six equations in order to obtain the coefficients are the same as in the Laplace transform of time of ruin, taking into account that $r_3 = s_3 = 0$.

As an example, we analyze the particular case Erlang$(2,\beta)$, i.e. $f(z) = \beta^2 z e^{-\beta z}$. Erlang$(2,\beta)$ is a phase-type(2) distribution with $A_1 = \dfrac{2}{\beta}$ and $A_2 = \dfrac{1}{\beta^2}$ (Dickson and Drekic (2004)). Then, the characteristics equations are

$$\begin{aligned} r^3 + \left(\tfrac{2\beta}{k_1} - \tfrac{\lambda+\delta}{c_1}\right) r^2 + \left(\tfrac{\beta^2}{k_1^2} - \tfrac{2\beta(\lambda+\delta)}{c_1 k_1}\right) r - \tfrac{\delta\beta^2}{c_1 k_1^2} &= 0, \quad 0 \leq u < b \\ s^3 + \left(\tfrac{2\beta}{k_2} - \tfrac{\lambda+\delta}{c_2}\right) s^2 + \left(\tfrac{\beta^2}{k_2^2} - \tfrac{2\beta(\lambda+\delta)}{c_2 k_2}\right) s - \tfrac{\delta\beta^2}{c_2 k_2^2} &= 0, \qquad u \geq b. \end{aligned}$$

It is easy to demonstrate that two of the roots are negative ($r_{i,} s_i < 0, i = 1,2$) and that $r_{3,} s_3 > 0$ if $\delta > 0$ or $r_{3,} s_3 = 0$ if $\delta = 0$. The system of equations that we need to find the coefficients is

$G_3 = 0,$
$\sum_{i=1}^{3} F_i e^{r_i b} - \sum_{i=1}^{2} G_i e^{s_i b} = 0,$
$\sum_{i=1}^{3} \frac{F_i}{r_i k_1+\beta} = \frac{1}{\beta},$
$\sum_{i=1}^{3} \frac{F_i}{(r_i k_1+\beta)^2} = \frac{1}{\beta^2},$
$\sum_{i=1}^{3} \frac{F_i}{k_2 r_i + \beta}\left(1 - e^{b\left(r_i+\frac{\beta}{k_2}\right)}\right) + \sum_{i=1}^{2} \frac{G_i e^{b\left(s_i+\frac{\beta}{k_2}\right)}}{k_2 s_i + \beta} = \frac{1}{\beta},$
$\sum_{i=1}^{3} \frac{F_i\left(e^{b\left(r_i+\frac{\beta}{k_2}\right)}\left(b(k_2 r_i+\beta) - k_2\right) + k_2\right)}{(k_2 r_i + \beta)^2} - \sum_{i=1}^{2} \frac{G_i e^{b\left(s_i+\frac{\beta}{k_2}\right)}\left(b(k_2 s_i+\beta) - k_2\right)}{(k_2 s_i + \beta)^2} = \frac{k_2}{\beta^2}.$

To obtain the ruin probability, the six equations to find the coefficients are also the previous ones, taking into account that $\delta = r_3 = s_3 = 0$.

6 Numerical examples

In this section we show some numerical results, computed with *Mathematica 6.0*, for the ruin probability in a model modified with a threshold reinsurance strategy, with $Z \sim$ Exponential(1), $\lambda = 1$, $\rho = 0.2$ and $\rho_R = 0.3$. These results are compared with those obtained in a a model with a proportional reinsurance strategy.

First, for $u = 5$ and $b = 10$, in Figure 2, we represent the different combinations of k_1 and k_2 that give the same ruin probability. For this example the minimal ruin probability is 0.326325 for $k_1 = 0.68733$ and $k_2 = 0.626034$.

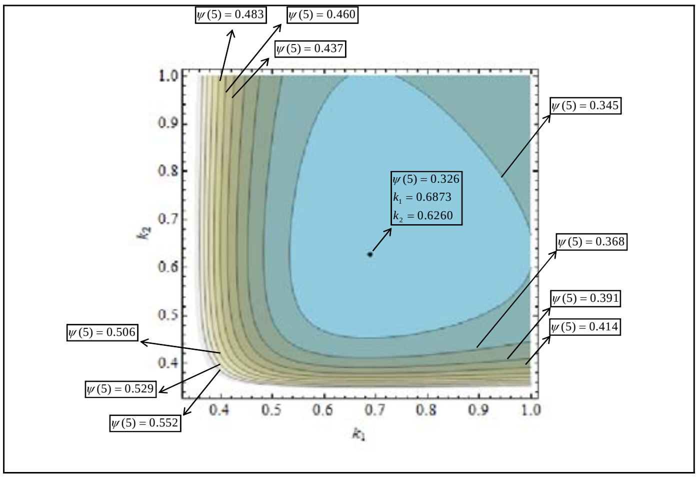

Figure 2: Combinations of k_1 and k_2 to obtain the same $\psi(5)$.

In Table 1 we present the results for the combinations of k_1 and k_2 that give the minimal ruin probability for different values of u (with $b = 10$), obtained with an algorithm for numerical minimization with constraints included in *Mathematica 6.0.* In addition, we include in the last column the expected time of ruin for each level of the initial surplus.

u	$\psi_{\min}^{k_1 \neq k_2}(u)$	k_1	k_2	$E[T \mid T < \infty]$
0	0.828764	1	0.631577	5.43174
2	0.583198	0.808025	0.627857	19.0077
4	0.396654	0.702299	0.6262	39.5646
6	0.26833	0.67847	0.625949	59.6679
8	0.181283	0.669768	0.625874	79.6959
10	0.122386	0.668654	0.625866	99.6881
12	0.0826157	0.668654	0.625815	119.688
14	0.0557689	0.668654	0.625787	139.688
16	0.0376463	0.668653	0.625768	159.688
18	0.0254127	0.668653	0.625756	179.688
20	0.0171546	0.668653	0.625747	199.688
40	0.000337037	0.668653	0.625712	399.688
100	2.55×10^{-9}	0.668653	0.625696	999.687
1000	4.03×10^{-86}	0.668653	0.625687	9999.69

Table 1: Minimal ruin probability with threshold reinsurance strategy

Now we are going to compare the threshold proportional reinsurance strategy with a proportional reinsurance strategy with a fixed retention level that doesn't depend on the level of reserves (this strategy can be obtained as a particular case of the threshold proportional reinsurance strategy for $k_1 = k_2 = k$).

Let $k_1 = k_2 = k$ in the results obtained in Section 4. In this case, it is easy to obtain explicit expressions for the value of k that minimizes the ruin probability, which is denoted by k_{op}, and for the minimal ruin probability, denoted by $\psi_{\min}^{k_{op}}(u)$,

$$k_{op} = \frac{(\rho_R - \rho)\left(\rho + 2u + \rho_R(2u-1) + \sqrt{(\rho - \rho_R)^2 + 4(1+\rho_R)u^2}\right)}{2(1+\rho_R)(\rho + \rho_R(u-1))},$$

$$\psi_{\min}^{k_{op}}(u) = \frac{e^{\frac{(2+\rho_R)u - \sqrt{(\rho-\rho_R)^2 + 4(1+\rho_R)u^2}}{\rho - \rho_R}}\left(\rho + 2u + \rho_R(2u-1) + \sqrt{(\rho-\rho_R)^2 + 4(1+\rho_R)u^2}\right)}{(1+\rho_R)\left(\rho_R - \rho + 2u + \sqrt{(\rho-\rho_R)^2 + 4(1+\rho_R)u^2}\right)}.$$

In Figure 3, we can observe the difference between $\psi_{\min}^{k_{op}}(u)$ and $\psi_{\min}^{k_1 \neq k_2}(u)$ for different values of u, i.e., the difference between minimal ruin probability with proportional reinsurance with a fixed retention level k, and the minimal ruin probability with a threshold proportional reinsurance strategy.

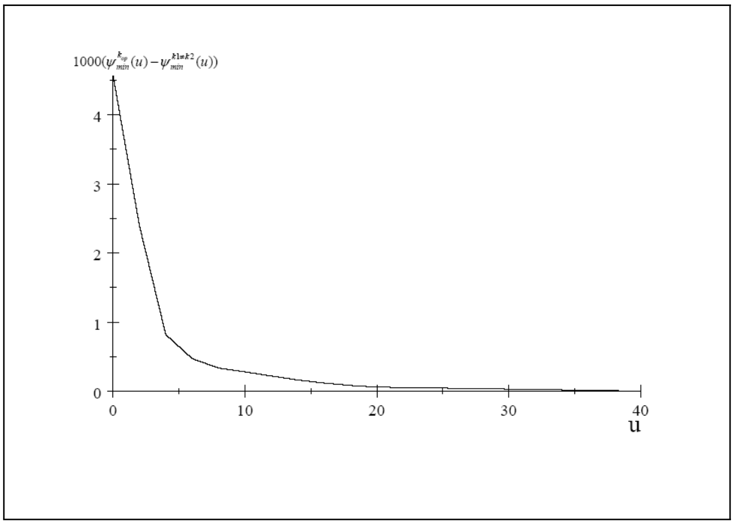

Figure 3: $1000(\psi_{\min}^{k_{op}}(u) - \psi_{\min}^{k_1 \neq k_2}(u))$ for different values of the initial surplus u.

We can observe that the difference is important for small u, and that this difference decreases with u. So, for small values of the initial surplus, the threshold proportional reinsurance strategy allows us to obtain better results in terms of ruin probability than a proportional reinsurance with a fixed retention level k.

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