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Discounting Arduousness

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Abstract:

There is a growing literature considering deviations from standard constant discounting. In this paper we combine time-inconsistent (non-constant discounting) preferences with recursive utilities. We apply this setting to the demand side properties of what we call *arduous* goods. The rational for a non-standard discounting is that production and consumption are not separable in these kinds of goods. The necessary effort implies that individuals discount consumption of these goods in a special way: both biased preferences and dynamic recursive adjustment are present. In this way, willingness to make an effort, modeled as a discount factor, becomes endogenous.

Resum:

Hi ha una literatura creixent que considera desviacions del descompte exponencial estàndar. En aquest article combinem preferències temporalment inconsistents (descompte no constant) amb preferències recursives. Aquest formalisme l'apliquem a les propietats relatives a la demanda del que anomenem bens ardus. La justificació del descompte no estàndar proposat ve donada pel fet que la producció i el consum no són separables per a aquest tipus de bens. L'esforç implica que els individus descompten el consum d'aquests bens d'una manera especial, amb la presència de preferències esbiaixades i d'un ajustament recursiu dinàmic. D'aquesta manera, la voluntat de realitzar un esforç, caracteritzada per un factor de descompte, resulta endògena.

JEL classification: C61; D83; D99; D03

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1 Introduction

Once upon a time all goods were arduous to get. In order to drink a simple glass of water one would need to make the glass or whatever recipient, walk for a while in order to find a natural spring, etc... But as time went on, many things changed and human life became much easier. Essentially, labor division and market exchange allowed people to sell their labor in the labor market and use the resulting income to buy goods of different types. In this way consumption and production became almost completely separate activities for a wide range of goods. One could say that those goods became "easy", while only a few goods remained "arduous", or "hard". Undoubtedly this process, reinforced by technological advances, has improved the welfare of humankind. But what happens to those goods which remain arduous? Is their consumption reduced or reinforced? Does it depend upon economic variables? Does it affect human welfare? Is this a relevant economic issue on which we should spend some thought? These are some of the questions raised by a consideration of the existence of hard goods. We could think of examples of hard goods as simple consumption goods whose enjoyment requires the kind of training only attainable by practice, such as sport, listening to classic music, reading literature, going to the theater, etc... Going a step further, many other commodities might be also considered as arduous. Pure knowledge and human capital accumulation can in a way be considered as hard goods, although the result of the time and effort invested is marketable (can be sold on the market). Going even further, the formation of preferences itself might be a time- and effort-consuming activity, such as altruism formation and all kinds of interpersonal relationships -including having and raising children- as an outstanding example. The aim of this paper is to address some issues concerning the demand side of these goods, in particular the possibility that the unavoidable production effort affects the way such goods are discounted.

There is a growing literature considering deviations from standard constant discounting. In order to avoid the deficiencies of optimal growth theory due to the use of a constant discount rate, Koopmans (1960) introduced the so-called recursive utilities. Uzawa (1968) extended Koopmans' discrete-time concept of recursive utility to a continuous-time setting. The Uzawa's model was used in Nairay (1984) for the analysis of convergence and optimal properties in a consumption investment model (see also Chang (1994)). A general description of recursive (non-additive) utility functionals was given in Epstein (1987). The basic idea of recursive utilities is that the rate of time preference is not exogenous. In this way, different degrees of impatience appear (e.g. due to different experienced past consumption levels). The relevant property of time consistency of preferences in the standard case is preserved within the context of recursive utilities. Drugeon and Wigniolle (2007) recently used a particular form of these utilities in a rational explanation of addiction and satiation.

An alternative way to deviate from standard discounting was motivated by Strotz (1956), who studied the effects of choosing an exogeneous but variable rate of time preference. Strotz illustrated how for a very simple model preferences are time consistent if, and only if, time preferences are exponentials with a constant discount rate. In a discrete time framework, effects of the so-called hyperbolic (or quasi-geometric) discount functions introduced by Phelps and Pollak (1968) have recently been extensively studied. Laibson (1997) has made compelling observations about ways in which rates of time preference vary. The most relevant effect of non-constant discounting is that preferences change with time. In this sense, an agent making a decision at time t has different time preferences compared with those at the initial time t_0.

The main motivation for these approaches, especially in the first case (recursive utilities), seems to be the need to overcome limitations of the standard discount model. The second approach has recently been applied to some economic problems in which myopic behavior plays a role[1]. In this paper we apply these two main strands of literature to the consideration of hard goods.

As stated above, one might think that the main property of a hard good is that consumption and production are not separable activities, i.e. human effort cannot be substituted by the service of a capital good, and labor supply should be provided by consumers themselves. As regards this property, it turns out that hard good production tends to be affected very little by technological progress. These are the supply side properties of hard goods, which nevertheless might affect the demand side. In particular, in this paper we focus on a possible deviation from standard discounting in intertemporal decision making due to the non-separability of consumption and production activities, which makes effort an inevitable ingredient of these types of goods. To some extent, Becker's (1965) analysis of the allocation of time is somehow related to the issue we are tackling, as long as it considers how individuals (or households) allocate time - a limited, personal and non-transferable resource - to a consumption production joint activity. We take a different approach. Given that those commodities are not only time- but also effort-consuming, we consider this effort as interfering in the discounting process. In this way, we model an endogenous discount factor, interpreted as the willingness to make an effort to consume arduous good. On the one hand we consider

[1]Thaler and Bernartzi (2004) and Imrohoglu et al (2003) analyze the role of social security as a saving commitment device for myopic individuals with time inconsistent preferences of this kind. Fehr et al (2008) combine rational and hyperbolic consumers in order to isolate this commitment effect, while Pestieau and Possen (2006) introduce both myopic and prodigal individuals. Finally Diamond and Koszegi (2003) discuss the self control problems that arise when both the retirement decision and the saving decision are exogenous.

the case in which the agent has presently biased preferences with respect to the hard good. As long as effort is always previous to the benefits in hard good consumption, agents discount future benefits associated to effort more heavily, and the agent ends up under-consuming compared with the case where benefits are instantaneous[2]. On the other hand one might think that these kinds of commodities are affected by a special learning process, since the more they are consumed the more valued they might be, following a kind of learning-by-consuming process. This leads us to the literature on recursive utility, in which the discount factor depends on the vector of past consumption.

Summarizing, we consider a model where the agent has to decide between consumption in a standard (easy) good and a hard good. The main features are:

- Future utilities due to consumption in the easy good are discounted in the standard way, by using a constant instantaneous discount rate.
- Future utility due to consumption in the arduous good is discounted taking into account effects of non-constant discounting, and effects due to previous experience in consumption of the hard good, together with a dependence on the quantity of stock accumulated.

The paper is organized as follows. In Section 2 the consideration of a single hard good is addressed, while in Section 3 the interaction between a hard and an easy good is considered. The particular case in which the agent uses standard constant, but different discount rates for the easy and hard

[2]See O'Donoghue and Rabin (1999) and Brocas and Carrillo (2001). Both papers analyze (in a simplified setting) the self control problem when preferences are presently biased and agents face activities or projects characterized either by immediate cost and delayed reward, or the opposite. The later considers the role of competition between agents and of complementarity of projects.

goods is discussed in Section 4. This simplified situation illustrates the main features of the general model, and allows us to solve an example explicitly. Section 5 analyzes the case where the planning horizon (the time devoted by the agent to consumption of the hard good) is not an exogenous variable, but a decision (control) variable. Finally, Section 6 contains the main conclusions of the paper.

2 A basic model for hard goods

2.1 Preliminaries

Let us first recall the basic description of utility functions within the context of recursive utilities and time-inconsistent preferences. We work in a continuous time setting.

According to Uzawa (1968), a utility function U_0 is said to be recursive if

$$U_0 = \int_0^T e^{-\int_0^s \rho(c(\tau))\,d\tau} u(c(s))\,ds + e^{-\int_0^T \rho(c(\tau))\,d\tau} F(x(T),T)\ . \qquad (1)$$

For an axiomatization of this functional form in a discrete time setting, see Epstein (1983). In particular, an agent at time t will maximize

$$U_t = \int_t^T e^{-\int_t^s \rho(c(\tau))\,d\tau} u(c(s))\,ds + e^{-\int_t^T \rho(c(\tau))\,d\tau} F(x(T),T)\ . \qquad (2)$$

Note that the discount factor, given by

$$D(s,t) = e^{-\int_t^s \rho(c(\tau))\,d\tau}\ ,$$

differs from the standard one, since the instantaneous discount rate of time preference is not constant, but depends on past consumption.

Remark 1 *Most papers on recursive utilities are addressed to the study of economic growth, where models are usually constructed within an infinite horizon setting ($T = \infty$). Therefore, no final function appears. In our model* *for hard goods, where we work in a finite horizon setting, the introduction of a final function plays a crucial role. Since recursive utilities are time-consistent (the maximization problem associated with (1) is a standard optimal control problem), only the expression of U_0 (with $T = \infty$ and no final function) is usually found in the literature.*

Secondly, concerning time-inconsistent preferences, following the ideas first introduced in Strotz (1956), Phelps and Pollak (1968) proposed the so-called quasi-hyperbolic (or quasi-geometric) discounting. If $\delta \in (0, 1)$ is the standard geometric discount factor, the utility function for the agent at time t (the so-called t-agent in the literature of non-constant discounting) is defined as

$$U_t = u_t + \beta(\delta u_{t+1} + \delta^2 u_{t+2} + \delta^3 u_{t+3} + \cdots) \, ,$$

where $0 < \beta \leq 1$, and u_k, $k = t, t+1, \ldots,$ denotes the utility in period k. Therefore, the agent at time t applies not only the geometric discount factor δ, but also a "future discount factor" $\beta > 0$ to all future periods. Clearly, if $\beta \neq 1$, preferences for the agent in different periods of time will change. In fact, Laibson (1997) argues that β would be substantially less than one on an annual basis; perhaps between one-half and two-thirds. Both discount factors, δ and β, are exogenously given. In a recent paper, Young (2007) derives a 'generalized Euler equation' for an agent with multi-period deviations from geometric discounting. A dynamic programming equation and numerical analysis for solving the general problem can be found in Fujii and Karp (2008).

In a continuous time setting, Barro (1999) and Karp (2007) extended the standard assumption of constant discount rate of time preference to the case of a non-constant instantaneous discount rate $r(s)$, which is assumed to be non-increasing. The discount factor at time t used to evaluate a payoff at time $t + \tau$, for $\tau \geq 0$, is $\theta(\tau) = \exp\left(-\int_0^\tau r(s)\, ds\right)$. Then, the objective of the

t-agent is to maximize

$$U_t = \int_t^T e^{-\int_0^{s-t} r(\tau)\, d\tau} u(c(s))\, ds + e^{-\int_0^{T-t} r(\tau)\, d\tau} F(x(T), T)\ . \qquad (3)$$

In our model for hard goods, we put together elements of the utility functions (2) and (3) in a precise way.

2.2 The model with one hard good

As a starting point, in this section we derive the optimal consumption level of an arduous good with no other alternative choice. Although the main interest of hard goods arises when the agent has to allocate his effort between easy and hard goods, we begin by presenting this simpler problem for the sake of clarity. In order to stress that effort comes before enjoyment, we consider the extreme case in which the agent invests resources throughout the decision period in order to accumulate an arduous good that will only be enjoyed in the final time. This extreme assumption can be relaxed without changing the essential features of the model.

Let $x(t) = (x_1(t), \ldots, x_n(t))$ be the set of stock variables describing the economic system (accumulated arduous good, for instance), and denote by $a(t) = (a_1(t), \ldots, a_m(t))$ the decision variables describing the effort (consumption in arduous good) expended in completing different tasks. In general, the dynamical evolution of $x(t)$ will depend on the effort made $a(t)$, and also perhaps on the state of the system. Therefore, $\dot{x}_i(s) = f_i(x(s), a(s))$, $i = 1, \ldots, n$, given $x(0) = x_0$. For simplicity, in the following we restrict our attention, without loss of generality, to the case of just one arduous good whose effort is described by one decision variable $a(t)$, and such that the utility due to consumption depends on one state variable $x(t)$.

If the payoff due to consumption in the hard good is only achieved at the end of the (finite) planning horizon T, at $t = 0$ the agent will look to maximize $U_0 = D(T, 0)F(x(T), T)$, where $D(T, 0)$ denotes the discount factor. If, in addition, the consumer's time preferences have a bias to the present and also depend on the previous consumption levels, we can define the discount factor as

$$D(T,0) = e^{-\int_0^T \rho(a(\tau),\tau)\,d\tau} \; .$$

The above discount factor is the natural generalization of the discount factors of recursive utilities and time-inconsistent preferences.

In general, at each moment t, the t-agent will seek to maximize

$$U_t = D(T,t)F(x(T),T) \; , \tag{4}$$

where $x(t)$ evolves according to the differential equation

$$\dot{x}(s) = f(x(s),a(s)) \; , \qquad x(t) = x_t \, , \tag{5}$$

with $x(0) = x_0$. The discount factor of the t-agent is given by

$$D(T,t) = e^{-\int_t^T \rho(a(\tau),\tau-t)\,d\tau} \; . \tag{6}$$

Therefore, the t-agent values the consumption of the hard good at time T at an instantaneous discount rate $\rho(a(T),T-t)$. Note that we avoid considering that a produces disutility. Instead we consider that the level of a affects the willingness to make an effort captured by the recursive part of the discount factor.

For an explanation of the above discount factor, note that (6) is the continuous time limit of the discrete discount factor

$$D(T,t) = e^{-\sum_{\tau=t}^{T-1} \rho(a(\tau),\tau+1-t)} \; .$$

When discounting the future, the agent takes into account two factors: the distance to the future and his previous consumption level experienced at the previous moment. Since there is substantial evidence that agents are impatient about choices in the short term but are more patient when choosing between long-term alternatives, then according to the non-constant discounting literature, we should assume that $\dfrac{\partial \rho(a,s)}{\partial s} \leq 0$. However, since a represents an effort, and the utility is obtained at the end of the planning horizon, it seems natural to assume the opposite, i.e. $\dfrac{\partial \rho(a,s)}{\partial s} \geq 0$. This reflects the fact that agents will value more highly the consumption in the arduous good (effort) the closer they are to enjoying the benefits of the accumulated effort. The dependence of $\rho(a,s)$ in a is not clear in general. For instance, it seems natural to assume that, if the level of previous effort is high, a greater effort produces tiresome and hence reduces the willingness to make an effort (increases the discount factor). In this case, $\dfrac{\partial \rho(a,s)}{\partial a} \geq 0$. On the other hand, this tiring effect dissapears if the previous effort is very low or null, and in this case it may happen that $\dfrac{\partial \rho(a,s)}{\partial a} \leq 0$.

The formalism above describes the situation of an agent whose decisions about the optimal amount of arduous good at each instant depend on the distance to final enjoyment and the past level of effort. If the dominant effect of past effort is a lower willingness to make an effort, the time path of a decreases and an undesirable result may arise in the form of laziness or hard good consumption trap.

An alternative would be to consider that rather than the flow it is the stock of accumulated arduous good that affects the willingness to make an effort, by means of a kind of learning-by-doing or consuming process. In fact, in many of those arduous activities such as reading, learning, or playing sports, the more time one devotes to them the less tiresome and more enjoyable they become. In this case it would be natural to assume that accumulated knowledge of the good has a positive effect on their valuation. For instance, if $\dot{x}(t) = a(t)$, $x(t)$ represents the accumulated stock throughout the period $[0,t]$, and we can assume that the discount factor in equation (4) takes the form

$$D(T,t) = e^{-\int_t^T \rho(x(\tau),\tau - t)\, d\tau} \ .$$

By assuming that $\dfrac{\partial \rho(x,s)}{\partial x} \leq 0$, we might obtain a positive time path of arduous good consumption.

In this paper we consider both possibilities, so that the discount factor depends on both the flow and the stock. In this way, an effort can give rise to a short-run negative effect - tiredness - and a long-run positive effect - knowledge. The discount factor becomes

$$D(T,t) = e^{-\int_t^T \rho(x(\tau),a(\tau),\tau-t)\,d\tau}\ , \tag{7}$$

where the instantaneous discount rate $\rho(x,a,s)$ satisfies the conditions $\dfrac{\partial \rho}{\partial x} \leq 0$ and $\dfrac{\partial \rho}{\partial s} \geq 0$.

Problem (4),(7) is not a standard optimal control problem. It is clear that if the agent is naive (according to the literature of non-constant discounting), he will solve a series of optimal control problems (one for each t), and the (time-inconsistent) solution will be obtained by patching together the "optimal" solutions obtained by each t-agent. If we look for a time-consistent solution (the agent is sophisticated), things are more complicated. In the following subsection we derive a dynamic programming equation satisfied by the equilibria describing the time consistent solution of a sophisticated agent to the problem.

2.3 A dynamic programming equation

In order to obtain a dynamic programming equation (DPE) in continuous time, let $V^S(x_t,t)$ be the value function of the sophisticated t-agent with initial condition $x(t) = x_t$. It is clear that, for the T-agent, $V^S(x_T,T) = F(x_T,T)$. In general, the t-agent, knowing the reaction of the s-agents, $s > t$, to his decision $a(t)$ at time t, chooses $a^*(t)$ as the maximizer of the utility function (4) with initial value $x(t) = x_t$. Let $a^*(s)$, $s \in [t,T]$ be the equilibrium rule obtained in this way, and $x^*(t)$ the solution to $\dot{x}^*(s) =$

$f(x^*(s), a^*(s))$, $x^*(t) = x_t$. Then the value function of the sophisticated t-agent is given by

$$V^S(x_t, t) = e^{-\int_t^T \rho(x^*(\tau), a^*(\tau), \tau - t)\, d\tau} V^S(x^*(T), T) \ . \tag{8}$$

Next we derive a DPE satisfied by the value function $V^S(x_t, t)$. We adopt an approach similar in spirit to that in Barro (1999) or Ekeland and Lazrak (2008) for non-constant discounting, where it is implicitly assumed that the t-agent can precommit his future behavior during an infinitesimal period of time $[t, t+\epsilon]$, and then the limit $\epsilon \to 0$ is taken. We also assume that both the decision rule and the function $f(x, a)$ are continuous, and the value function is continuously differentiable in all its arguments.

Note that, for $t \in [0, T]$, the sophisticated t-agent, knowing his future behavior as a function of his present actions, chooses his consumption level at time t as the maximizer of his present value, so we can informally write

$$V^S(x_t, T) = \max_{\{a(t)\}} \{D(T, t) F(x(T), T)\} \ . \tag{9}$$

Since the value function for the $(t+\epsilon)$-agent can be written as

$$V^S(x_{t+\epsilon}, t+\epsilon) = e^{-\int_{t+\epsilon}^T \rho(x^*(\tau), a^*(\tau), \tau - t - \epsilon)\, d\tau} V^S(x_T, T) \ , \tag{10}$$

where $x(t+\epsilon) = x_{t+\epsilon}$ is obtained from (5) with $x(t) = x_t$, solving $V^S(x_T, T)$ in (8) and (10) we obtain

$$e^{\int_{t+\epsilon}^T \rho(x^*(\tau), a^*(\tau), \tau - t - \epsilon)\, d\tau} V^S(x_{t+\epsilon}, t+\epsilon) = e^{\int_t^T \rho(x^*(\tau), a^*(\tau), \tau - t)\, d\tau} V^S(x_t, t) \ . \tag{11}$$

If the decision rule is a continuous function in t, then

$$e^{\int_{t+\epsilon}^T \rho(x^*(\tau), a^*(\tau), \tau - t - \epsilon)\, d\tau} V^S(x_{t+\epsilon}, t+\epsilon) = e^{\int_t^T \rho(x^*(\tau), a^*(\tau), \tau - t)\, d\tau} V^S(x_t, t) +$$
$$+ \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[e^{\int_{t+\epsilon}^T \rho(x^*(\tau), a^*(\tau), \tau - t - \epsilon)\, d\tau} \cdot V^S(x_{t+\epsilon}, t+\epsilon) \right] + o(\epsilon) \ .$$

Since

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} V^S(x_{t+\epsilon}, t+\epsilon) = \frac{\partial V^S(x_t, t)}{\partial t} + \frac{\partial V^S(x_t, t)}{\partial x} \cdot f(x_t, a^*(t))$$

and

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} e^{\int_{t+\epsilon}^{T} \rho(x^*(\tau),a^*(\tau),\tau-t-\epsilon)\,d\tau} =$$

$$= e^{\int_{t}^{T} \rho(x^*(\tau),a^*(\tau),\tau-t)\,d\tau} \cdot \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[\int_{t+\epsilon}^{T} \rho(x^*(\tau),a^*(\tau),\tau-t-\epsilon)\,d\tau\right]$$

where

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[\int_{t+\epsilon}^{T} \rho(x^*(\tau),a^*(\tau),\tau-t-\epsilon)\,d\tau\right] =$$

$$= -\rho(x_t^*, a^*(t), 0) - \int_0^{T-t} \frac{\partial\rho(x^*(s+t),a^*(s+t),s)}{\partial s}\,ds\ ,$$

from (9) and (11) we obtain the dynamic programming equation

$$-\frac{\partial V^S}{\partial t} = \max_{\{a(t)\}} \left[\frac{\partial V^S}{\partial x}\cdot f - \left(\rho(x(t),a(t),0) + \int_0^{T-t} \frac{\partial\rho(x(s+t),a(s+t),s)}{\partial s}\,ds\right) V^S\right]\ . \tag{12}$$

Hence, we have proved:

Proposition 1 *If the decision rule and the function $f(x,a)$ are continuous, and the value function $V^S(x,t)$ for the problem (4) with discount factor (7) is of class C^1, it satisfies the DPE given by Equation (12) with boundary condition $V^S(x,T) = F(x,T)$.*

Clearly, as resources in this setting have no alternative other than hard good consumption, the agent will decide to consume it as long as it gives a positive marginal benefit. Condition (12) shows all the marginal effects involved. With respect to the standard HJB equation, a new term appears which groups together the effect of all changes in the time preference rate. First, provided that the agent has consumed some a, the passage of time alters the discount factor, either in a positive or in a negative direction. Second, there is an increase in the time preference rate due to the direct effect of time on the biased part of the time preference rate. Overall, the net effect can be either positive or negative.

If the instantaneous discount rate $\rho(x, a, t)$ is additively separable, $\rho(x, a, t) = \rho_1(x, a) + \rho_2(t)$, the DPE (12) simplifies to

$$\rho_2(T-t)V^S - \frac{\partial V^S}{\partial t} = \max_{\{a(t)\}} \left[\frac{\partial V^S}{\partial x} \cdot f - \rho_1(x(t), a(t))V^S \right] .$$

If $V(x, t)$ is the solution in case $\rho_2(s) = 0$, then $V^S = \exp\left(-\int_0^{T-t} \rho_2(s)\, ds\right) V$ and the equilibrium trajectory is independent of $\rho_2(s)$. Therefore, the equilibrium rule $a^*(x, t)$ coincides with the optimal control for the problem with $\rho_2(s) = 0$. In fact, it becomes clear from the formulation of the problem that the equilibrium decision rule and the associated state trajectory are independent of the instantaneous discount rate $\rho_2()s$ for both naive and sophisticated agents.

3 A more general model for easy and hard goods

Let us now assume that there is an easy good competing with the hard good, so that the agent must decide how to allocate his resources between the two goods over the time. In the description of the utility function for the hard good, we follow the model presented in the previous section, which reflects one of the main characteristics of these kinds of goods: the benefits of consumption in these good are achieved only at the end of a time horizon T, after a period of time $[0, T]$ of continuous effort. Nevertheless, resources can now also be devoted to other consumption activities, which give an immediate reward and have no long-term effects on utility. Given that those activities involve no effort, we assume that future utilities due to consumption of easy goods are discounted in a standard way. If $c(s)$ represents the instantaneous consumption at time s of the easy good, which provides the agent an instantaneous utility $u(c(s))$, the objective of the agent at time $t = 0$ will be to maximize

$$U_0 = \int_0^T e^{-rs} u(c(s))\, ds + D(T,0) F(x(T),T) \ ,$$

where $D(T,0)$ is the discount factor of the hard good given by (7), and r is the instantaneous discount rate applied on the utility obtained by consumption in the easy good. In general, at each moment $t \in [0,T]$, the t-agent aims to maximize

$$U_t = \int_t^T e^{-r(s-t)} u(c(s))\, ds + D(T,t) F(x(T),T) \ , \tag{13}$$

where

$$\dot{x}(s) = f(x(s), a(s), c(s)) \ , \qquad x(t) = x_t \tag{14}$$

with $x(0) = x_0$.

There are different ways of modeling the resources constraint in (14). One of them could be to assume that $\dfrac{\partial f}{\partial a} > 0$ and $\dfrac{\partial f}{\partial c} < 0$. We could take, for instance, $\dot{x} = f(x,a) - c$, with f an increasing function in a.

In this paper we follow an alternative approach. At time t, the agent earns a wage $w(t)$. The hard good has a cost in effort and time, the latter being reflected in the fact that it reduces the working time. A natural consequence of this is that the actual wage becomes $(1-a)w$. We also assume that all the salary is spent on consumption in the easy good (there is no saving), so $(1-a)w = c$. The need to make an effort is not directly captured by a disutility but by the special shape of discounting. Since $\dot{x} = f(x,a,c)$, we can eliminate the dependence on the consumption of the easy good in the model: $\dot{x} = f(x,a,(1-a)w) = \bar{f}(x,a,t)$, and $u(c) = u((1-w)a) = \bar{u}(a,t)$. Hence we have to solve the following problem:

$$\max \left\{ U_t = \int_t^T e^{-r(s-t)} \bar{u}(a,s)\, ds + D(T,t) F(x(T),T) \right\} \ , \tag{15}$$

$$\dot{x}(s) = \bar{f}(x(s), a(s), s) \ , \qquad x(t) = x_t \ , \quad x(0) = 0 \ . \tag{16}$$

Once again, we define the value function $V^S(x_t,t)$ of the sophisticated t-agent with initial condition $x(t)=x_t$ as

$$V^S(x_t,t)=\int_t^T e^{-r(s-t)}\bar{u}(a^*(s),s)\,ds+e^{-\int_t^T \rho(x^*(\tau),a^*(\tau),\tau-t)\,d\tau}F(x^*(T),T)\ , \tag{17}$$

where $a^*(s)$, $s\in[t,T]$, is the equilibrium rule and $\dot{x}^*(s)=\bar{f}(x^*(s),a^*(s),s)$ with $x(t)=x_t$. Clearly, $V^S(x_T,T)=F(x_T,T)$. We assume that $V^S(x_t,t)$ is of class C^1, and the decision rule and $\bar{f}(x,a,s)$ are continuous. As in the previous section, we can informally write

$$V^S(x_t,t)=\max_{\{a(t)\}}\left\{\int_t^T e^{-r(s-t)}\bar{u}(a(s),s)\,ds+D(T,t)F(x(T),T)\right\}\ . \tag{18}$$

By solving $F(x^*(T),T)$ in the right hand term in (17) we obtain

$$F(x^*(T),T)=e^{\int_t^T \rho(x^*(\tau),a^*(\tau),\tau-t)\,d\tau}\left[V^S(x_t,t)-\int_t^T e^{-r(s-t)}\bar{u}(a^*(s),s)\,ds\right]\ . \tag{19}$$

In a similar way, since

$$V^S(x_{t+\epsilon},t+\epsilon)=\left\{\int_{t+\epsilon}^T e^{-r(s-t-\epsilon)}\bar{u}(c^*(s),s)\,ds+D(T,t+\epsilon)F(x^*(T),T)\right\}\ ,$$

we can solve $F(x^*(T),T)$ as

$$F(x^*(T),T)=$$

$$=e^{\int_{t+\epsilon}^T \rho(x^*(\tau),a^*(\tau),\tau-t-\epsilon)\,d\tau}\left[V^S(x_{t+\epsilon},t+\epsilon)-e^{r\epsilon}\int_{t+\epsilon}^T e^{-r(s-t)}\bar{u}(a^*(s),s)\,ds\right]=$$

$$=e^{\int_t^T \rho(x^*(\tau),a^*(\tau),\tau-t)\,d\tau}\left[V^S(x_t,t)-\int_t^T e^{-r(s-t)}\bar{u}(a^*(s),s)\,ds\right]+$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left[e^{\int_{t+\epsilon}^T \rho(x^*(\tau),a^*(\tau),\tau-t-\epsilon)\,d\tau}\left(V^S(x_{t+\epsilon},t+\epsilon)-e^{r\epsilon}\int_{t+\epsilon}^T e^{-r(s-t)}\bar{u}(a^*(s),s)\,ds\right)\right]+o(\epsilon)\ .$$

Therefore, using (19) we get

$$0=\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\left[e^{\int_{t+\epsilon}^T \rho(x^*(\tau),a^*(\tau),\tau-t-\epsilon)\,d\tau}\left(V^S(x_{t+\epsilon},t+\epsilon)-e^{r\epsilon}\int_{t+\epsilon}^T e^{-r(s-t)}\bar{u}(a^*(s),s)\,ds\right)\right]\ .$$

From the above equation, and after several calculations, we obtain

$$\frac{\partial V^S}{\partial t}+\frac{\partial V^S}{\partial x}\cdot\bar{f}+\bar{u}(a^*(t),t))-\left(\rho(x(t),a^*(t),0)+\int_0^{T-t}\frac{\partial\rho(x^*(s+t),a^*(s+t),s)}{\partial s}\,ds\right)\cdot$$

$$\cdot \left(V^S \int_t^T e^{-r(s-t)} \bar{u}(a^*(s,s))\, ds \right) - r \int_t^T e^{-r(s-t)} \bar{u}(a^*(s), s)\, ds = 0 \, .$$

Finally, by solving equation (16) for the equilibrium rule, $a^*(s)s = a^*_s(x_t, s)$, and from (18) then we easily obtain:

Proposition 2 *Let $V^S(x,t)$ be the value function of the sophisticated t-agent with initial condition $x(t) = x$. If the decision rule and the function $\bar{f}(x,a,s)$ are continuous, and $V^S(x,t)$ is of class C^1, it satisfies the DPE*

$$K - \frac{\partial V^S}{\partial t} = \max_{\{a(t)\}} \left[\bar{u}(a(t),t) + \frac{\partial V^S}{\partial x} \cdot \bar{f}(x(t),a(t),t) - \rho(x(t),a(t),0)V^S + \right.$$

$$\left. + \left(\rho(x(t),a(t),0) - r\right) \int_t^T e^{-r(s-t)} \bar{u}(a^*(s),s)\, ds \right] \, , \tag{20}$$

$$V^S(x_T, T) = F(x_T, T) \, , \tag{21}$$

where $K = K(x_t, t)$ is given by

$$K = \int_0^{T-t} \frac{\partial \rho(x^*(s+t), a^*_{s+t}(x_t,s), s)}{\partial s}\, ds \left(V^S - \int_t^T e^{-r(s-t)} \bar{u}(a^*_s(x_t,s),s)\, ds \right) \, . \tag{22}$$

The solution $V^S(x,t)$ to equations (20-22) is a Markov Perfect equilibrium (MPE), and if a^* is the maximum in Equation (20), then the associated control trajectory $a^*(x,t)$, $t \in [0,T]$, is the equilibrium rule.

Remark 2 *Note that the derivation above can be reproduced step by step for the general problem obtained when we do not impose the budget constraint $(1 - a(t))w(t) = c(t)$. The resulting DPE is very similar to (20-22), with just some notational changes.*

In particular, when $\rho(x,a,s) = \rho_1(x,a) + \rho_2(s)$, Equation (20) simplifies to

$$\tilde{K} + \rho_2(T-t)V^S - \frac{\partial V^S}{\partial t} = \tag{23}$$

$$= \max_{\{a\}} \left[\bar{u}(a,t) + \frac{\partial V^S}{\partial x} \cdot \bar{f} - \rho_1(x,a) \left(V^S - \int_t^T e^{-r(s-t)} \bar{u}(a^*(s),s)\, ds \right) \right] \, ,$$

where $\tilde{K} = \tilde{K}(x_t, s)$ is given by

$$\tilde{K} = (r - \rho_2(T-t)) \int_t^T e^{-r(s-t)} \bar{u}(a_s^*(x_t, s), s)\, ds\ . \tag{24}$$

The way we model the discount factor, interpreted as the willingness to make an effort, gives interesting insights which cannot be reproduced by using a standard intertemporal utility function with a constant instantaneous discount rate of time preference. First, due to the existence of a biased time preference, if the distance to the final time T is large, the t-agent consumes a low amount of hard good. Only when t approaches to T will the t-agent begin to value the final utility more. This effect is reinforced by the possibility of getting utility only from the easy goods. In other words, the agent underconsumes and could even end up in a sort of unwished laziness trap, consuming no hard good at all. Secondly, the fact that discounting depends endogenously on previous consumption decision enriches the time path of $a(t)$. For example, if the flow of effort applied has a negative short run effect on the willingness to make an effort -showing tiresome-, this negative effect on hard good consumption prolongs the time spent in a laziness trap. Nevertheless, if the accumulated stock of hard good has a positive long run effect -reflecting an increase in valuation due to learning by consuming process-, if effort is positive it will produce a gradual correction of the bias. As we will show in the next section, these effects of undervaluing consumption of hard good in the long-run are already present if the hard good is discounted at a constant but greater instantaneous discount rate ρ compared with that for the easy good, i.e., $\rho > r \geq 0$.

4 Constant but different discount rates for hard and easy goods. An illustration

Let us assume that inter-temporal utilities are non-recursive, i.e., the discount factor does not depend on previous consumption and the state variable. Let us also assume that there is no temporal bias in the instantaneous discount rate of time preference for the arduous good, i.e., $\rho(x, c, \tau - t) = \rho$, where ρ is a constant. We consider $\rho > r \geq 0$. This is a very interesting case which illustrates some of the main characteristics concerning consumption in arduous goods: when t approaches to T, the valuation of the hard good increases, compared with the valuation of previous t-agents. In addition, the relative simplicity of this simplified model allows us to solve explicitly a particular example showing this property. Note that the DPE in Proposition 2 is a very complicated functional equation with non-local terms. If ρ is constant, the DPE above simplifies to

$$\tilde{K} + \rho V^S - \frac{\partial V^S}{\partial t} = \max_{\{a\}} \left[\bar{u}(a, t) + \frac{\partial V^S}{\partial x} \cdot \bar{f} \right] \ , \tag{25}$$

with

$$\tilde{K} = (r - \rho) \int_t^T e^{-r(s-t)} \bar{u}(a_s^*(x_t, s), s) \, ds \ . \tag{26}$$

When $\rho = r$ we recover the standard Hamilton-Jacobi-Bellman equation. But if $\rho \neq r$, the new term $\tilde{K}$ recovers the effects due to the difference of instantaneous discount rates of time preference. This term reflects the fact that naive agents will be time-inconsistent, in general.

Within this context, let us illustrate how equations (25)-(26) can be used in order to find a time-consistent solution for our model of easy and hard goods. In order to find a solution in closed-form, we take $r = 0$ (the agent does not discount the utility due to consumption in the easy good) and logarithmic utilities for both goods, with w a constant. We have to solve

$$\max_{a,c} \int_t^T \ln c \, ds + e^{-\rho(T-t)} \ln x(T) \ , \quad \text{for} \quad t \in [0, T] \ , \tag{27}$$

$$\dot{x} = a \ , \quad x(t) = x_t \ , \tag{28}$$

$$c = (1-a)w \ . \tag{29}$$

For this simple problem, let us derive and compare the solutions obtained for an agent who has the possibility of precommiting his future behaviour at $t=0$ (precommitment solution) with those for (time-inconsistent) naive and (time-consistent) sophisticated agents without precommitment.

4.1 Precommitment solution

In this case, the agent at time $t=0$ maximizes his discounted utility function according to his present preferences. Future t-agents ($t>0$) cannot modify the decisions, although from their perspective they are not optimal. The solution is obtained by solving the standard optimal control problem associated to the 0-agent. The Hamiltonian function is

$$H = \ln c + p\, a = \ln\left[w(1-a)\right] + p\, a \ . \tag{30}$$

By applying Pontryagin's Maximum Principle (PMP) to the Hamiltonian function (30) with the transversality condition $p(T) = \dfrac{\partial(e^{-\rho T}\ln x(T))}{\partial x(T)} = \dfrac{e^{-\rho T}}{x(T)}$ we obtain

$$a^P(t) = \frac{e^{-\rho T} - x_0}{e^{-\rho T} + T} \ , \quad x^P(t) = x_0 + \frac{e^{-\rho T} - x_0}{e^{-\rho T} + T}\, t \ . \tag{31}$$

Hence, the effort devoted to the hard good remains constant, the state variable is a linear function in t and the consumption rule is linear in x^P,

$$a^P(t) = \frac{e^{-\rho T}}{e^{-\rho T} + T - t} - \frac{1}{e^{-\rho T} + T - t}\, x^P(t) \ . \tag{32}$$

4.2 Solution for a naive agent

If the agent is naive, the t-agent will solve Problem (27-29) as a standard optimal control problem with initial condition $x(t) = x_t$. By applying the

PMP to the Hamiltonian function (30) with the transversality condition $p(T) = \dfrac{\partial(e^{-\rho(T-t)}\ln x(T))}{\partial x(T)} = \dfrac{e^{-\rho(T-t)}}{x(T)}$, the optimal solution is given by

$$a(s) = \frac{e^{-\rho(T-t)} - x_s}{e^{-\rho(T-t)} + T - s} \ , \quad \forall s \in [t, T] \ .$$

However, this solution will not be optimal for future s-agents, $s > t$; since preferences change over the time, the decision rule will change continuously, and the solution is not time-consistent. The optimal solution from the perspective of the t-agent will be satisfied only for $s = t$. Therefore

$$a^N(t) = \frac{e^{-\rho(T-t)}}{e^{-\rho(T-t)} + T - t} - \frac{1}{e^{-\rho(T-t)} + T - t} x^N(t) \ . \tag{33}$$

Note that the solution for the problem of a naive agent does not coincide with the precommitment solution, unless $\rho = 0$.

4.3 Solution for a sophisticated agent

We apply Proposition 2 which, for our problem, particularizes to equations (25-26). The right hand term in equation (25) becomes

$$\max_{\{a\}} \left[u(w(1-a)) + \frac{\partial V^S}{\partial x} \cdot f \right] = \ln w - \ln V_x^S + V_x^S - 1 \ ,$$

where we have used that

$$\frac{\partial \left[\ln\left(w(1-a)\right) + V_x^S a \right]}{\partial a} = 0 \Rightarrow a = 1 - \frac{1}{V_x^S} \ .$$

From the precommitment and naive solutions, we conjecture that the consumption rule for the arduous good is linear in the state variable, i.e., $a = A(t) + B(t)x$. Since $V_x^S = 1/(1-a) = 1/(1 - A(t) - B(t)x)$, then

$$V(x,t) = -\frac{1}{B(t)} \ln\ (1 - A(t) - B(t)x) + C(t) \ .$$

From the final condition

$$V(x,T) = -\frac{1}{B(T)} \ln\left(1 - A(T) - B(T)x\right) + C(T) = \ln x(T)$$

we get $A(T) = 1$, $B(T) = -1$ and $C(T) = 0$.

Then, equation (25) becomes

$$\tilde{K} + \rho\left(-\frac{1}{B(t)}\ln\,(1 - A(t) - B(t)x) + C(t)\right) -$$

$$-\frac{B'(t)}{B^2(t)}\ln\,(1 - A(t) - B(t)x) - \frac{A'(t) + B'(t)x}{B(t)(1 - A(t) - B(t)x)} - C'(t) =$$

$$= \ln w + \ln\,(1 - A(t) - B(t)x) + \frac{1}{1 - A(t) - B(t)x} - 1\;, \tag{34}$$

where

$$\tilde{K} = -\rho\int_t^T [\ln w + \ln\,(1 - A(s) - B(s)x(s))]\;ds =$$

$$= -\rho(T - t)\ln w - \rho\int_t^T \ln\,(1 - A(s) - B(s)x(s))\;ds\;.$$

Since $\dot{x}(s) = a(s) = A(s) + B(s)x(s)$ with the initial condition $x(t) = x_t$, then

$$x(s) = e^{\int_t^s B(\tau)\,d\tau}\left[x_t + \int_t^s e^{-\int_t^\tau B(z)\,dz}A(\tau)\,d\tau\right]$$

and we can rewrite $\tilde{K}$ as

$$\tilde{K} = -\rho(T - t)\ln w - \rho\int_t^T \ln\,(1 - \Lambda(s,t) - \Gamma(s,t)x_t)\;ds\;, \tag{35}$$

where

$$\Lambda(s,t) = A(s) + B(s)e^{\int_t^s B(\tau)\,d\tau}\int_t^s e^{-\int_t^\tau B(z)\,dz}A(\tau)\,d\tau\;, \tag{36}$$

$$\Gamma(s,t) = B(s)e^{\int_t^s B(\tau)\,d\tau}\;. \tag{37}$$

Note that $\Lambda(t,t) = A(t)$ and $\Gamma(t,t) = B(t)$.

Lemma 1 *The function $\tilde{K}$ given by (35-37) can be expressed as*

$$\tilde{K} = -\rho(T - t)ln\,w + \rho\alpha(t) - \rho(T - t)ln\,(1 - A(t) - B(t))\;,$$

where

$$\alpha(t) = \int_t^T (T - s)\left(\frac{B'(s)}{B(s)} + B(s)\right)\;ds\;.$$

Proof: We define

$$G(x,t) = \int_t^T \ln\,(1 - \Lambda(s,t) - \Gamma(s,t)x_t)\;ds\;.$$

Then

$$\frac{\partial G}{\partial x} = -\int_t^T \frac{\Gamma(s,t)}{1-\Lambda(s,t)-\Gamma(s,t)x}\,ds$$

and

$$\frac{\partial G}{\partial t} = -\ln(1 - A(t) - B(t)x) - \int_t^T \frac{\Lambda_t(s,t)+\Gamma_t(s,t)x}{1-\Lambda(s,t)-\Gamma(s,t)x}\,ds\;.$$

Since

$$-\int_t^T \frac{\Lambda_t(s,t)+\Gamma_t(s,t)x}{1-\Lambda(s,t)-\Gamma(s,t)x}\,ds =$$

$$= -\int_t^T \frac{-A(t)B(s)e^{\int_t^s B(\tau)\,d\tau} - B(t)B(s)e^{\int_t^s B(\tau)\,d\tau}x}{1-\Lambda(s,t)-\Gamma(s,t)x}\,ds =$$

$$= (A(t)+B(t)x)\int_t^T \frac{B(s)e^{\int_t^s B(\tau)\,d\tau}}{1-\Lambda(s,t)-\Gamma(s,t)x}\,ds =$$

$$= (A(t)+B(t)x)\int_t^T \frac{\Gamma(s,t)}{1-\Lambda(s,t)-\Gamma(s,t)x}\,ds$$

then we find that $G(x,t)$ satisfies the first order partial differential equation

$$\frac{\partial G}{\partial t} = -\ln(1-A(t)-B(t)x) - (A(t)+B(t)x)\frac{\partial G}{\partial x}\;. \tag{38}$$

We conjecture that

$$G(x,t) = \alpha(t) + \gamma(t)\ln(1-A(t)-B(t)x_t)\;. \tag{39}$$

Then equation (38) becomes

$$\alpha'(t) + \gamma'(t)\ln(1-A(t)-B(t)x) - \gamma(t)\frac{A'(t)+B'(t)x}{1-A(t)-B(t)x} =$$

$$= -\ln(1-A(t)-B(t)x) + (A(t)+B(t)x)\frac{B(t)}{1-A(t)-B(t)x}\gamma(t)$$

which can be written as

$$\alpha'(t) + \gamma(t)\frac{B'(t)}{B(t)} + \gamma(t)B(t) + (\gamma'(t)+1)\;\ln(1-A(t)-B(t)x) -$$

$$-\frac{\gamma(t)}{1-A(t)-B(t)x}\left(A'(t)+\frac{B'(t)}{B(t)}\left(1-A(t)\right)+A(t)B(t)+B(t)(1-A(t))\right)=0\ .$$

Since the equation above must be satisfied for every pair (x,t), necessarily

$$\alpha'(t)+\gamma(t)\frac{B'(t)}{B(t)}+\gamma(t)B(t)=0\ , \tag{40}$$

$$\gamma(t)\left(A'(t)+\frac{B'(t)}{B(t)}(1-A(t))+A(t)B(t)+B(t)(1-A(t))\right)=0\ ,\ \text{and} \tag{41}$$

$$\gamma'(t)+1=0\ . \tag{42}$$

From the definition of $G(x,t)$ it is clear that $G(x,T)=0$, i.e.,

$$0=\alpha(T)+\gamma(T)\ln\left(1-A(T)-B(T)x\right)=\alpha(T)+\gamma(T)\ln x$$

so $\alpha(T)=\gamma(T)=0$. From (42) we obtain $\gamma(t)=T-t$. Since $\gamma\neq 0$, equation (41) becomes a consistency condition to be satisfied by functions $A(t),B(t)$, concretely

$$A'(t)+\frac{B'(t)}{B(t)}-\frac{A(t)B'(t)}{B(t)}+B(t)=0\ . \tag{43}$$

Then, the conjecture (39) is verified and, by integrating equation (40), the result follows. □

Using Lemma 1, equation (34) becomes

$$-\rho(T-t)\ln w-\rho\alpha(t)-\rho(T-t)\ln\left(1-A(t)-B(t)x\right)-\frac{\rho}{B(t)}\ln\left(1-A(t)-B(t)x\right)+$$

$$+\rho C(t)-\frac{B'(t)}{B^2(t)}\ln\left(1-A(t)-B(t)x\right)-\frac{A'(t)}{B(t)}\frac{1}{1-A(t)-B(t)x}-$$

$$-\frac{B'(t)}{B(t)}\left(-\frac{1}{B(t)}+\frac{1-A(t)}{B(t)}\frac{1}{1-A(t)-B(t)x}\right)-C'(t)=$$

$$=\ln w+\ln\left(1-A(t)-B(t)x\right)+\frac{1}{1-A(t)-B(t)x}-1\ .$$

Since the above equation has to be satisfied for every x, the equations

$$-\rho(T-t)-\frac{\rho}{B(t)}-\frac{B'(t)}{B^2(t)}=1\ ,\quad B(T)=-1\ , \tag{44}$$

$$-\rho(T-t)\ln w - \rho\,\alpha(t) + \rho\,C(t) + \frac{B'(t)}{B^2(t)} - C'(t) = \ln w - 1\ , \quad C(T) = 0 \quad (45)$$

and (43) must be satisfied. It is easy to check that the solution to (44) and (43) with the final condition $A(T) = 1$ is given by

$$A(t) = \frac{e^{-\rho(T-t)}}{e^{-\rho(T-t)} + T - t}\ , \quad B(t) = -\frac{1}{e^{-\rho(T-t)} + T - t} \quad (46)$$

and $C(t)$ is the solution to (45). Therefore, the equilibrium decision rule coincides both for naive and sophisticated agents. This property is no longer satisfied for more general utility functions.

4.4 Comparison of the solutions

Let us briefly compare the precommitment and naive/sophisticated solutions. We assume that $x_0 = 0$.

With respect to the precommitment solution, note that the effort devoted to consumption in the arduous good remains constant, and is a decreasing function in ρ and T, as might be expected. The final value of the state variable $x^P(T)$ and the final function $\ln x^P(T)$ are also decreasing functions in ρ, and consumption in the easy good is increasing in the discount rate ρ.

With respect to the naive and sophisticated solutions, at time $t = 0$, $a^P(0) = a^N(0) = a^S(0)$, but $\dot{a}^N(0) = \dot{a}^S(0) > 0$. This reflects the fact that if the 0-agent takes into account that his time preferences will change in the future, he values the final function more, recognizing in this way the higher valuation of the effort devoted to consumption in the hard good when the time t approaches to the final time T in which the benefits of the effort are obtained. More precisely, from (33) and (28) we obtain

$$\dot{a}^N(t) = \frac{\rho e^{-\rho(T-t)}}{e^{-\rho(T-t)} + T - t}(1 - a^N(t))\ . \quad (47)$$

Since $a^N(t) \geq 0$, then $a^N(t)$ is non decreasing for every $t \in [0, T]$. Note that the first term in the right hand term in (47), $\frac{\rho e^{-\rho(T-t)}}{e^{-\rho(T-t)} + T - t}$, is an increasing function in t, but when $a^N(t)$ increases, the second term $1-a^N(t)$ contributes by reducing the value of $\dot{a}^N(t)$. Let us illustrate numerically the time evolution of $a^N(t)$. The solution to equation (47) is given by

$$a^N(t) = 1 - \frac{T}{e^{-\rho(T-t)} + T - t} e^{\int_0^t \frac{ds}{e^{-\rho(T-s)}+T-s}} .$$

For $T = 3$, the values of $a^N(t)$, $t = 0, 1, 2, 3$, for $\rho = 0.05$ and $\rho = 0.2$ are given in Table 1.

$T=3$	$t=0$	$t=1$	$t=2$	$t=3$
$\rho=0.05$	0.2229	0.2331	0.2478	0.2732
$\rho=0.2$	0.1516	0.1874	0.2404	0.3357

Table 1: Consumption in the arduous good in the case $T = 3$.

For $T = 5$, the values of $a^N(t)$, $t = 0, 1, 2, 3, 4, 5$, for $\rho = 0.05$ and $\rho = 0.2$ are given in Table 2.

$T=5$	$t=0$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$\rho=0.05$	0.1348	0.1413	0.1496	0.1607	0.1768	0.2046
$\rho=0.2$	0.0685	0.0840	0.1067	0.1412	0.1973	0.2980

Table 2: Consumption in the arduous good in the case $T = 5$.

Tables 1 and 2 illustrate how the effort in the arduous good increases when t approaches to T in the sophisticated and naive solutions. For the precommitment solutions the effort is constant and coincides with those for $t = 0$.

With regard to the value of the state variable, $x^P(T)$ and $x^N(T) = x^S(T)$ are not higher than 1, but $x^N(T) \geq x^P(T)$ (the inequality is strict if $\rho > 0$). Note that, if $x^N(t)$ is near to 1, the effort goes to 0. The agent is actually minimizing the loss of final utility due to underconsumption in the hard good.

5 Terminal time as a decision variable

In Propositions 1 and 2, we implicitly assume that the terminal time T is fixed. If the terminal time is a decision variable of the agent, the DPE in the previous propositions remain true, but a new transversality condition has to be imposed reflecting the "optimal" decision T^*.

Since preferences are changing due to the differences in the discount rates, under no commitment the terminal time T is decided by the final agent (the T-agent in this case). Naive agents will solve the free terminal time problem by assuming that, at each moment t, they decide the value of T; that is, the decision-maker at time t will solve a standard optimal control problem where the terminal time is fixed in such a way that it is optimal from the viewpoint of the t-agent. Therefore, in general, not only the consumption decisions of naive agents will be time-inconsistent, but also the terminal times. If we look for a time-consistent decision rule under no commitment (the agent is sophisticated), we must analyze the game theoretic framework in which the final time is decided by the final agent, who is the one to decide whether or not to stop. We can adapt the proof in Marín-Solano and Navas (2009) in the case of non-constant discounting. For our problem with easy and hard goods, since T^* is optimum from the perspective of the T^*-agent, if the optimal decision for the $(T^* + \epsilon)$-agent, $\epsilon > 0$, is to stop, then from the optimality of T^* we get $F(x(T^*), T^*) \geq V^S_{T^*+\epsilon}(x(T^*), T^*)$, where $V^S_{T^*+\epsilon}$ denotes the value function of the T^*-agent if the terminal time is $T^* + \epsilon$. In this case,

$$F(x(T^*), T^*) \geq \int_{T^*}^{T^*+\epsilon} e^{-r(s-T^*)} u(c, s) ds + D(T^*+\epsilon, T^*) F(x(T^*+\epsilon), T^*+\epsilon) =$$

$$= F(x(T^*), T^*) + \epsilon \left[u(c, s) - \rho(x, a, 0) F + \left(\frac{\partial F}{\partial x} \cdot \dot{x} + \frac{\partial F}{\partial T} \right) \right] \Bigg|_{T^*} + o(\epsilon) \ . \tag{48}$$

Otherwise, if the $(T^* + \epsilon)$-agent prefers to continue until a new terminal time $T' \geq T^* + \epsilon$ (which will be optimum for the T'-agent), the optimality of T^* for the T^*-agent implies that $F(x(T^*), T^*) \geq V^S_{T'}(x(T^*), T^*)$, where $V^S_{T'}(x(T^* + \epsilon), T^* + \epsilon) \geq F(x(T^* + \epsilon), T^* + \epsilon)$. Therefore,

$$
\begin{aligned}
&F(x(T^*), T^*) \geq V^S_{T'}(x(T^*), T^*) = \\
&= u(c(T^*), T^*)\epsilon + D(T^* + \epsilon, T^*) V^S_{T'}(x(T^* + \epsilon), T^* + \epsilon) + o(\epsilon) \geq \\
&\geq u(c(T^*), T^*)\epsilon + D(T^* + \epsilon, T^*) F(x(T^* + \epsilon), T^* + \epsilon) + o(\epsilon) = \\
&= F(x(T^*), T^*) + \epsilon \left[u(c, s) - \rho(x, a, 0)F + \left(\frac{\partial F}{\partial x} \cdot \dot{x} + \frac{\partial F}{\partial T} \right) \right] \Bigg|_{T^*} + o(\epsilon) \ .
\end{aligned}
\tag{49}
$$

Taking the limit $\epsilon \to 0^+$ in (48) and (49) we obtain

$$
\left[u(c, s) - \rho(x, a, 0)F + \frac{\partial F}{\partial x} \cdot f + \frac{\partial F}{\partial T} \right] \Bigg|_{T^*_+} \leq 0 \ . \tag{50}
$$

Next, a $(T^* - \epsilon)$-agent, $\epsilon > 0$, will decide to continue until T^* only if

$$
F(x(T^* - \epsilon), T^* - \epsilon) \leq \int_{T^* - \epsilon}^{T^*} e^{-r(T^* - s)} u(c, s) ds + D(T^* + \epsilon, T^*) F(x(T^*), T^*)
$$

and in the limit $\epsilon \to 0^+$ we obtain

$$
\left[u(c, s) - \rho(x, a, 0)F + \frac{\partial F}{\partial x} \cdot f + \frac{\partial F}{\partial T} \right] \Bigg|_{T^*_-} \geq 0 \ . \tag{51}
$$

Since we are assuming that $u(c)$ and $F(x, T)$ are of class C^1, from (50) and (51) we obtain:

Proposition 3 *In Problem (15)-(16) with T free, if the agent is sophisticated and there is no commitment in the terminal time, then the following condition is satisfied in the terminal time T^*:*

$$
\left[u(c, s) + \frac{\partial F}{\partial x} \cdot f \right] \Bigg|_{T^*} = \left[\rho(x, a, 0)F - \frac{\partial F}{\partial T} \right] \Bigg|_{T^*} \ .
$$

In the above analysis we have implicitly assumed that the terminal state $x(T)$ is free.

In the example analyzed in the previous section, if T is free, then $T^* = \infty$ unless $\rho < 0$, which makes no economic sense.

6 Concluding remarks

In this paper we model the consumption decision of what we call arduous goods, which are basically consumption goods in which not only time but also effort is unavoidable, due to the non-separability of consumption and production tasks.

In particular, we derive a general formulation of the decision problem where the willingness to make an effort -to accumulate arduous goods- is modeled as a changing discount factor combining two different strands of literature. First, the time preference is modeled to be non-constant with time - the agent has biased preferences with respect to these goods. This implies that the solution path for consumption obtained by using standard optimal control theory is time-inconsistent. Secondly, the willingness to make and effort is considered to be partially endogenous, in the spirit of recursive utilities. Specifically, the rate of time preference depends on past hard good consumption levels, both the flow - effort - and the stock - accumulated arduous good.

First we analyze the case in which the agents can only spend their resources on the hard good. As a result of their biased preferences, the stronger the bias and the further the agent from the final period, the higher is the possibility of entering an arduous goods consumption -or a laziness- trap. Additionally, the recursive part of the discount factor will interact with this, producing inertia in these effects. For example, if the flow of effort applied has a negative short run effect on the willingness to make an effort -showing tiresome-, this negative effect on hard good consumption prolongs the time spent in a laziness trap. Nevertheless, if the accumulated stock of hard good has a positive long run effect -reflecting an increase in valuation due to learning by consuming process-, provided that effort is positive it will produce a gradual correction of the bias.

Secondly we introduce the possibility of consuming an easy good, competing with the arduous good. The latter is discounted as above, while the former is discounted in a standard way. In general, the possibility of consuming an easy good will reduce the amount of arduous good consumed, and the possibility of an arduous good trap is reinforced.

Thirdly, given the difficulties in obtaining closed form solutions to this problem, we derive a simplified illustration in which both goods are discounted using standard discount factors, though at different constant instantaneous discount rates. Despite its simplicity, the differential constant-time preference suffices to obtain time-inconsistent results and hence differences between the precommitment solution and the naive and sophisticated -the latter two being equal for the particular utility functions employed in the example. Even though the discount factor does not depend on time, the differential value of the rate of time preference plays the same role.

Fourthly, the consideration of a free-terminal time for the arduous good consumption shows that the trade off the agent faces in accumulating hard good for an additional period is also affected by the willingness to make an effort, and hence distorted by this.

Finally, and given that we only focus on the demand side of the problem, it is worth recalling that the arduous goods analyzed in this paper are strictly consumption goods. Nevertheless, given that the main property of these goods is the non-separability of consumption and production activities, it becomes relevant to consider the supply side of the problem. We consider it partially in the above model, since we take into account that the time spent on arduous good cannot be directed to earning income in order to consume easy goods. However, in this context there might be other interactions between non-market and market production that would be worth approaching. A natural extension of the above analysis would be to consider the case in which the time and effort devoted to hard good consumption has an impact (potentially positive) on the level of human capital acquired by the agent.

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