NOTES
ON
MATHEMATICS

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GEI (Grau d’Empresa Internacional)
The aim of this manual is to provide GEI students with a thorough introduction to the contents of *Mathematics* course and, to this end, it will be used and further expanded upon in class. Its contents are essential for ensuring that students are in handling many of the formal and quantitative matters that underpin other subjects in this GEI.

The manual comprises three main sections. The first, *Linear Algebra on $\mathbb{R}^n$*, deals with the basic properties of vectors – i.e. directed line segments - which unlike scalar magnitudes are determined by more than one numerical value. A good grounding in this subject is fundamental for understanding the second section, devoted to *Multivariable Optimization*, in which we search for the maxima and minima (extreme points) of functions of several variables without any constraints (“classical optimization”). Finally, the third section, *Dynamic Analysis*, deals with the study of integrals of continuous functions of one real variable. This is an indispensable tool for calculating planar areas determined by functions as well as for solving differential equations, which in turn play a vital role in mathematical economics. Precisely this manual ends up studying the simplest cases of these differential equations.

Please note that each of the subjects tackled here contains various relevant applications in Economics as well as a number of interesting exercises that we will solve in class. The manual concludes with some bibliographic references and a glossary of terms and page references to help quickly find the most important concepts quoted here. Finally, this manual has been filed in the UB Digital Repository ([http://hdl.handle.net/2445/45663](http://hdl.handle.net/2445/45663)). Please note that all contents (as well as any errors that might be found) are the sole and exclusive responsibility of the author.
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1. LINEAR ALGEBRA

Numerical magnitudes that scientists usually deal with are basically of two types: *scalar* and *vectorial*. Scalar magnitudes are those determined by a sole numerical value; consider, for instance, the weight of a body, the price of a good or the rate of interest. In the other hand, vectorial magnitudes need, unlike scalars, more than one value to be explicited. In Economics vectorial magnitudes or vectors appear frequently as independent variables of functions. Consider for example a lobster fishery where output $Q$ stands for the estimated catch of lobster depending on two inputs, the stock of lobster $K$ and the harvesting effort $L$:

If this dependence between output and inputs can be expressed mathematically as: $^1$

$$Q = 2.26 \cdot K^{0.44} \cdot L^{0.48}$$

the lobster fishery can be "modeled" through the function of two variables: $^2$

$$\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$$
$$\langle K, L \rangle \mapsto Q = f(K, L) = 2.26 \cdot K^{0.44}L^{0.48}$$

being:

$$\mathbb{R}^2 = \{ \tilde{x} = (x_1, x_2) : x_1, x_2 \in \mathbb{R} \}$$

the two-dimensional set of vectors. So if we want to understand how functions of this type work we need prior to deal with vectors like these and the notion of vector space picks up the most important properties related to them. $^3$ This section discusses some of the seminal properties of vector spaces to apply them further to study metric concepts like norms, angles and distances. Finally we introduce quadratic forms we need in the next section.

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$^1$ See Sydsaeter, K & Hammond, P. J. (1995), page 390. This function is a particular case of Cobb-Douglas's function of production.

$^2$ We deal with this type of functions in the second section.

$^3$ Namely the properties of sum of vectors and product of a vector by a number.
1.1. Vector Spaces

1.1.1. \( \mathbb{R}^n \)-Vector Space, Vectors and Scalars

The concept of a \( \mathbb{R}^n \)-vector space incorporates the basic properties of the sum and the external product (the product by a number) of vectors.

**Definition:** The \( \mathbb{R}^n \)-vector space (\( \mathbb{R}^n \) for short) is given by:

1. A set of vectors: \( \mathbb{R}^n = \{ \vec{x} = (x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R} \} \)
2. A vector addition: \( \vec{x} + \vec{y} = (x_1 + y_1, \ldots, x_n + y_n) \)
3. An multiplication by scalars: \( \lambda \cdot \vec{x} = (\lambda x_1, \ldots, \lambda x_n) \), for any \( \lambda \in \mathbb{R} \).

The elements of the set of real numbers \( \mathbb{R} \) are called scalars.

From now onwards we will denote the zero-vector of the sum of \( \mathbb{R}^n \) by:

\[
\vec{0} = (0, \ldots, 0) \in \mathbb{R}^n. \quad \text{(4)}
\]

On the other hand, the opposite vector of \( \vec{x} \in \mathbb{R}^n \) will be:

\[
-\vec{x} = (-1) \cdot \vec{x}.
\]

Any \( \mathbb{R}^n \)-vector space satisfies the following basic properties:

**Properties:** In any \( \mathbb{R}^n \)-vector space we have:

a. \( \lambda \cdot \vec{0} = \vec{0} \) and \( 0 \cdot \vec{x} = \vec{0} \)

b. \( \lambda \cdot \vec{x} = \vec{0} \) implies either \( \lambda = 0 \) or \( \vec{x} = \vec{0} \).

c. \( -(\lambda \cdot \vec{x}) = (-\lambda) \cdot \vec{x} \).

d. \( \vec{x} = \vec{y} \) is equivalent to \( \vec{x} - \vec{y} = \vec{0} \).

---

\( ^4 \) This vector should not be confused with the scalar 0.
1.1.2. Linear Combination of Vectors

We begin by introducing the most basic concept related to vectors, i.e., their linear combination. For instance, the vector \((3, -1, 7)\) is a linear combination of the vectors \((1, 0, 4)\) i \((-1, 1, 1)\) since it can be written as:

\[
(3, -1, 7) = 2 \cdot (1, 0, 4) + (-1) \cdot (-1, 1, 1).
\]

Thus:

Definition: The vector \(\bar{x} \in \mathbb{R}^n\) is a linear combination of \(k\) vectors \(\bar{x}_1, \ldots, \bar{x}_k \in \mathbb{R}^n\) provided \(k\) scalars \(\lambda_1, \ldots, \lambda_k \in \mathbb{R}\) exist with the condition:

\[
\bar{x} = \lambda_1 \cdot \bar{x}_1 + \cdots + \lambda_k \cdot \bar{x}_k = \sum_{i=1}^{k} \lambda_i \cdot \bar{x}_i. \quad \text{(5)}
\]

Consider this example:

Example: Prove that the vector \((-1, 9, 4)\) is a linear combination of the vectors \((1, 2, 0)\) and \((-4, 3, 4)\). Reasons that this is not the case for the vector \((7, 3, 4)\).

SOLUTION:

In the first case we have to solve the vector equation:

\[
(-1, 9, 4) = \lambda_1 \cdot (1, 2, 0) + \lambda_2 \cdot (-4, 3, 4)
\]

that leads to the linear system of equations with the variables \(\lambda_1\) and \(\lambda_2\):

\[
\begin{align*}
\lambda_1 - 4\lambda_2 &= -1 \\
2\lambda_1 + 3\lambda_2 &= 9 \\
4\lambda_2 &= 4
\end{align*}
\]

\[
\frac{\lambda_1 - 4\lambda_2 = -1}{2\lambda_1 + 3\lambda_2 = 9} \quad \frac{\lambda_1 = \frac{3}{2} \implies \lambda_2 = 1}{\text{Solution}} \quad \text{There is linear combination.}
\]

However this is not the the case for the vector \((7, 3, 4)\) since:

\[
\begin{align*}
\lambda_1 - 4\lambda_2 &= 7 \\
2\lambda_1 + 3\lambda_2 &= 3 \\
4\lambda_2 &= 4
\end{align*}
\]

\[
\frac{\lambda_1 - 4\lambda_2 = 7}{2\lambda_1 + 3\lambda_2 = 3} \quad \text{Inconsistent system} \quad \text{There is no linear combination.}
\]

\[
\text{(5) Note that the number of vectors } k \text{ does not have to match the number } n \text{ of their components or coordinates.}
\]
1.1.3. Linear Dependence and Independence of Vectors

As we have just seen, the vector \((-1,9,4)\) can be expressed as a linear combination of \((1,2,0)\) and \((-4,3,4)\). In this case we say that:

\((-1,9,4), (1,2,0)\) and \((-4,3,4)\) are linearly dependent.

On the other hand we know this is not true for the vector \((7,3,4)\). In this case we say that:

\((7,3,4), (1,2,0)\) and \((-4,3,4)\) are linearly independent.\(^6\)

Thus by definition:

**Definition:** \(k\) vectors \(\vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^n\) are:

1. **Linearly dependent** if at least one of them, for example \(\vec{x}_i\), can be expressed as a linear combination of the others:

   \[
   \vec{x}_i = \lambda_1 \cdot \vec{x}_1 + \cdots + \lambda_{i-1} \cdot \vec{x}_{i-1} + \lambda_{i+1} \cdot \vec{x}_{i+1} + \cdots + \lambda_k \cdot \vec{x}_k = \sum_{j=1, j \neq i}^{k} \lambda_j \cdot \vec{x}_j.
   \]

2. **Linearly independent** when this is not the case.

The next theorem characterizes linear independence through the zero-linear combinations of vectors. Indeed:

**Theorem:** \(k\) vectors \(\vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^n\) are linearly independent if and only if every zero-linear combination of these \(k\) vectors has the corresponding scalars equal to zero. In other words:

\[
\lambda_1 \cdot \vec{x}_1 + \cdots + \lambda_k \cdot \vec{x}_k = \vec{0} \text{ always implies } \lambda_1 = \cdots = \lambda_k = 0.
\]

Obviously any set of vectors will be linearly dependent in case there exists a zero-linear combination among them with at least one non-zero scalar.

\(^6\) \((1,2,0)\) and \((-4,3,4)\) are also linearly independent.
Example: Prove that vectors $(7,3,4)$, $(1,2,0)$ and $(-4,3,4)$ are linearly independent while $(-1,9,4)$, $(1,2,0)$ and $(-4,3,4)$ are not.

SOLUTION:
In the first case we must solve the zero-vector equation:

$$(0,0,0) = \lambda_1 \cdot (7,3,4) + \lambda_2 \cdot (1,2,0) + \lambda_3 \cdot (-4,3,4)$$

that leads to the homogeneous system:

\[
\begin{align*}
7\lambda_1 + \lambda_2 - 4\lambda_3 &= 0 \\
3\lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \\
4\lambda_1 + 4\lambda_3 &= 0
\end{align*}
\]

Since this system is a Cramer’s system, the solution is unique and equal to the trivial zero-solution. Thus, applying the above theorem we conclude that:

\[
\begin{align*}
\lambda_1 &= 0 \\
\lambda_2 &= 0 \\
\lambda_3 &= 0
\end{align*}
\]

implies Linearly independent vectors.

In the other case, the zero-vector equation we have now to analyze is:

$$(0,0,0) = \lambda_1 \cdot (-1,9,4) + \lambda_2 \cdot (1,2,0) + \lambda_3 \cdot (-4,3,4)$$

which is equivalent to the associated homogeneous system:

\[
\begin{align*}
-\lambda_1 + \lambda_2 - 4\lambda_3 &= 0 \\
9\lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \\
4\lambda_1 + 4\lambda_3 &= 0
\end{align*}
\]

Since this system is consistent but dependent we have:

\[
\begin{align*}
-\lambda_1 + \lambda_2 - 4\lambda_3 &= 0 \\
9\lambda_1 + 2\lambda_2 + 3\lambda_3 &= 0 \\
4\lambda_1 + 4\lambda_3 &= 0
\end{align*}
\]

There is more than one solution.

Thus, the last three vectors are linearly dependent since one of the above solutions contains at least some non-zero scalars.

---

7 A Cramer’s system is a consistent and independent linear system of equations with a square associated matrix. Recall that a linear system of equations is homogeneous when all of its right-hand constant terms are zero; every homogeneous system is always consistent since it has the trivial solution equal to zero.
1.1.3.1. Relationship between Linear Independence and Rank of a Matrix

If we recall that the rank of a matrix is equal to the order of the largest associated non-zero minor this relationship states that:

**Property:** *k* vectors of \( \mathbb{R}^n \) are linearly independent if and only if the matrix formed by these vectors has a rank equal to *k*.\(^8\)

Consider the following example:

**Example:** Prove that the three vectors \((1, -1, 3, 7), (5, 2, 4, -2)\) and \((0, -6, 0, 1)\) are linearly independent while \((2, -2, 8), (5, 1, 3)\) and \((4, 2, -1)\) are not.

**SOLUTION:**

In the first case the matrix formed by the three vectors is:

\[
A = \begin{bmatrix}
1 & -1 & 3 & 7 \\
5 & 2 & 4 & -2 \\
0 & -6 & 0 & 1
\end{bmatrix}.
\]

Since we have a non-zero minor of order 3:

\[
\begin{vmatrix}
1 & -1 & 3 \\
5 & 2 & 4 \\
0 & -6 & 0
\end{vmatrix} = -66 \neq 0 \implies \text{rank } A = 3
\]

then, with the aid of the above property, we can conclude that these vectors are linearly independent. However, in the second case since:

\[
|A| = \begin{vmatrix}
2 & -2 & 8 \\
5 & 1 & 3 \\
4 & 2 & -1
\end{vmatrix} = 0 \implies \text{rank } A < 3
\]

we now have three linearly dependent vectors.

---

\(^8\) Consequently the rank of a matrix is the maximum number of linearly independent row (or column) vectors. It is clear that if the rank is less than the number of vectors, we have a set of linearly dependent vectors.
1.1.3.2. Application of Linear Independence in Economics

Example: An oil company can convert one barrel of crude oil into three different kinds of fuel according to the vector equation:

\[(x, y, z) = (1 - \theta) \cdot (2, 2, 4) + \theta \cdot (5, 0, 3)\]

where \(x\), \(y\) and \(z\) stand for the amounts of the three types of fuel obtained, and \(0 \leq \theta \leq 1\) is the percentage of lead additives added. Check if the following output vectors are possible:

\[\vec{u} = (3.5, 1, 3.5), \quad \vec{v} = \begin{pmatrix} 4/3 \\ 10/3 \end{pmatrix} \quad \text{and} \quad \vec{w} = (1, 6, 9)\]

and, if so, determine the percentage of lead additives included.

SOLUTION:

First of all, we must see if the three set of three vectors:

\[\{\vec{u}, (2, 2, 4), (5, 0, 3)\}, \quad \{\vec{v}, (2, 2, 4), (5, 0, 3)\} \quad \text{and} \quad \{\vec{w}, (2, 2, 4), (5, 0, 3)\}\]

are linearly dependent. Since:

\[
\begin{vmatrix}
3.5 & 1 & 3.5 \\
2 & 2 & 4 \\
5 & 0 & 3
\end{vmatrix} = 
\begin{vmatrix}
1 & 6 & 9 \\
2 & 2 & 4 \\
5 & 0 & 3
\end{vmatrix} = 0 \quad \text{and} \quad 
\begin{vmatrix}
4/3 & 10/3 \\
2 & 2 & 4 \\
5 & 0 & 3
\end{vmatrix} = -14/3 \neq 0
\]

we conclude that the first and the third vectors are two possible output vectors. Taking into account that:

\[(x, y, z) = (1 - \theta) \cdot (2, 2, 4) + \theta \cdot (5, 0, 3) = (2 + 3\theta, 2 - 2\theta, 4 - \theta)\]

the first case, i.e., the vector \(\vec{u}\), is the only case where the percentage \(0 \leq \theta \leq 1\) exists since:

\[\vec{u} = (3.5, 1, 3.5) = (2 + 3\theta, 2 - 2\theta, 4 - \theta) \implies \theta = 1/2\]

while \(\vec{w}\) fails:

\[\vec{w} = (1, 6, 9) = (2 + 3\theta, 2 - 2\theta, 4 - \theta) \implies \text{There is no value of } 0 \leq \theta \leq 1.\]

Thus \((3.5, 1, 3.5)\) is an output vector with \(\theta = 50\%\) of lead additives.

---

10 The vector \((x, y, z)\) is called the "output vector".
1.1.4. Spanning Set of Vectors

A third fundamental concept to consider is that of the spanning set of vectors of $\mathbb{R}^n$. For instance the three vectors:

$$(1,0,0), (0,1,0) \text{ and } (0,0,1)$$

form a spanning set of the vector space $\mathbb{R}^3$ since every vector $(a,b,c)$ can be written as a linear combination of them:

$$(a,b,c) = a \cdot (1,0,0) + b \cdot (0,1,0) + c \cdot (0,0,1).$$

**Definition:** $k$ vectors of $\mathbb{R}^n$ form a spanning set if any vector of it can be expressed as a linear combination of these $k$ vectors.

In this context we must take into account the following characterization:

**Property:** $k$ vectors of $\mathbb{R}^n$ form a spanning set if and only if the matrix they form has a rank equal to $n$.

**Example:** Prove that $(1,0,0)$, $(2,3,-1)$, $(5,11,-4)$ and $(-4,5,0)$ is a spanning set of $\mathbb{R}^3$.

**SOLUTION:** In this case $n = 3$ and the matrix associated with the vectors:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & -1 \\ 5 & 11 & -4 \\ -4 & 5 & 0 \end{pmatrix}$$

has a rank equal to 3 since:

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & -1 \\ 5 & 11 & -4 \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 11 & -4 \end{vmatrix} = -12 + 11 = -1 \neq 0 \implies \text{rank } A = 3 = n.$$

According to the above property this set of vectors is a spanning set of $\mathbb{R}^3$.

---

11 Note that $n$ is the number of components of each of these $k$ vectors.
1.1.5. Bases and Dimension of $\mathbb{R}^n$

Roughly speaking a basis of $\mathbb{R}^n$ is a spanning set formed by a “minimum” number of vectors in the following sense:

**Definition:** $k$ vectors form a basis of $\mathbb{R}^n$ if they form a spanning set of linearly independent vectors.

Let’s look at an example of a leading basis:

**Example:** The vectors

\[
\begin{bmatrix}
1, 0, \ldots, 0 \\
0, \ldots, 0, 1
\end{bmatrix}
\]  

form a basis of $\mathbb{R}^n$.\(^{12}\)

**SOLUTION:**

Trivially these vectors form a spanning set of $\mathbb{R}^n$ since:

\[
(x_1, x_2, \ldots, x_n) = x_1 \cdot (1,0,\ldots,0) + x_2 \cdot (0,1,0,\ldots,0) + \ldots + x_n \cdot (0,\ldots,0,1)
\]

and they are linearly independent due to the fact that the square matrix they formed, i.e. the identity matrix, is regular:\(^{13}\)

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1
\end{pmatrix}
\]

implies $\det l_n = 1 \neq 0$ implies $\operatorname{rank} l_n = n$.

From the above we can conclude that all bases of $\mathbb{R}^n$ have the same number of vectors:

**Definition:**\(^{14}\) The dimension of $\mathbb{R}^n$ is the number of vectors of its bases, i.e. $n$.

As we can see the dimension of $\mathbb{R}^n$ also matches the number of components of its vectors.

---

\(^{12}\) This is the so-called *canonical* basis of $\mathbb{R}^n$. Every basis of $\mathbb{R}^n$ provides a specific “reference system” on it.

\(^{13}\) Remember that a square matrix is regular provided that its determinant is different from 0.

\(^{14}\) It is worth noting that there are general vector spaces of infinite dimension.
1.1.5.1. Relationship between Bases and Determinant of a Square Matrix

The next characterization property is a direct result of the above properties and it enables us to check whether a set of vectors is or is not a basis of $\mathbb{R}^n$:

Property: *k* vectors of $\mathbb{R}^n$ form a basis if and only if:

a. $k = n$.

b. The square matrix formed by these vectors is regular.

Consider the following example:

**Example:** Check if the following vectors:

$\begin{pmatrix} 1,1,0,0 \\ 2,3,1,8 \\ 3,3,1,5 \\ 0,0,1,0 \end{pmatrix}$

form a basis of $\mathbb{R}^4$.

**Solution:**

As the matrix formed by them is a regular matrix:

$$
\begin{vmatrix}
1 & 1 & 0 & 0 \\
2 & 3 & 1 & 8 \\
3 & 3 & 1 & 5 \\
0 & 0 & 1 & 0
\end{vmatrix}
= (-1)^1 \cdot \begin{vmatrix} 1 & 1 & 0 & 1 \\
2 & 3 & 8 & 1 \\
3 & 3 & 1 & 2 \\
0 & 5 & 0 & 3
\end{vmatrix}
= -1 \cdot \begin{vmatrix} 1 & 1 & 0 \\
2 & 3 & 2 \\
3 & 3 & 0 \\
0 & 1 & 0
\end{vmatrix}
= -1 \cdot (-1) 
= 1

we can affirm that they form a basis of $\mathbb{R}^4$.

From now onwards we must take into account that:

**Properties:** In any $\mathbb{R}^n$-vector space we always have:

a. Any set of linearly independent vectors of $\mathbb{R}^n$ can be expanded to a basis.

b. Any spanning set of $\mathbb{R}^n$ contains at least one basis among their vectors.

Hence if we want to prove that a set of vectors form a spanning set, the second property is telling us we have to look for a basis inside it.
1.2. Scalar Product on Vector Spaces

1.2.1. Scalar Product on $\mathbb{R}^n$

We are going to introduce the algebraic notion of scalar product in order to deal with some metric concepts in $\mathbb{R}^n$ such as the norm of a vector, the angle formed by two vectors and the distance between two vectors. Among others, these concepts are geometrically related with the length of a linear segment, the angle between two straight lines and the distance between points.

**Definition:** The scalar product of vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is equal to the real number obtained through the expression:

$$\vec{x} \cdot \vec{y} = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1y_1 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i \in \mathbb{R}.$$ 

Let’s look at the following example:

**Example:** Calculate:

i. The scalar product of the two vectors $(1,0,-5)$ and $(5,-2,1)$.

ii. The value of $k \in \mathbb{R}$ so that the scalar product of $(5,-2,1)$ and $(1,-1,k)$ is 10.

**SOLUTION:** i) In this case we trivially have:

$$(1,0,-5) \cdot (5,-2,1) = 1 \cdot 5 + 0 \cdot (-2) + (-5) \cdot 1 = 5 - 5 = 0.$$ 

ii) The value of the parameter $k \in \mathbb{R}$ will be:

$$10 = (5,-2,1) \cdot (1,-1,k) = 5 \cdot 1 + (-2)(-1) + 1 \cdot k \implies k = 3.$$ 

It must be emphasized that the scalar product of two vectors is not a vector but a number or scalar, hence the name. Likewise, as we have just seen in this example, the scalar product of two non-zero vectors can nevertheless be equal to 0.

---

15 Also called either usual or standard scalar product. In general a vector space with a scalar product is an Euclidian vector space.
1.2.1.1. Application of Scalar Product in Economics

Consider the following statement:

**Example:** If the numbers \( p_1, \ldots, p_n \) denote the unit sale prices of \( n \) products \( A_1, \ldots, A_n \), prove that the total income obtained selling \( q_i \) units of \( A_i \) matches the scalar product of the two vectors:

\[
\vec{p} = (p_1, \ldots, p_n) \quad \text{and} \quad \vec{q} = (q_1, \ldots, q_n).
\]

By way of application if €5, €2 and €3 are respectively the unit sale prices of three economic goods, determine how many we need to sell if we want to obtain an income of €105 where the amount sold of the second good is half that of the first, and the amount sold of the third equals the sum of the other two.

**SOLUTION:**

It is clear that the total income \( IT \) obtained with the sale of these \( n \) products equals the sum of the partial incomes associated with each product. Thus, as the sale of \( q_i \) units of \( A_i \) provides a partial income of \( \varepsilon p_i \cdot q_i \) then the total income will be:

\[
IT = \sum_{i=1}^{n} p_i \cdot q_i = p_1 \cdot q_1 + p_2 \cdot q_2 + \cdots + p_n \cdot q_n = (p_1, \ldots, p_n) \cdot (q_1, \ldots, q_n) = \vec{p} \cdot \vec{q}.
\]

In the specific case of:

\[
\vec{p} = (p_1, p_2, p_3) = (5, 2, 3) \quad \text{and} \quad IT = \varepsilon 105
\]

we would have the linear equality:

\[
105 = IT = 5q_1 + 2q_2 + 3q_3
\]

associated with the two conditions:

\[
q_2 = 0.5 \cdot q_1 \quad \text{implies} \quad q_3 = q_1 + q_2 = \{q_2 = 0.5q_1\} = 1.5q_1.
\]

Thus:

\[
105 = 5q_1 + 2q_2 + 3q_3 = \left\{\begin{array}{l}
q_2 = 0.5q_1 \\
q_3 = 1.5q_1
\end{array}\right. = 5q_1 + 2(0.5q_1) + 3(1.5q_1) = 10.5q_1 \quad \text{Solution} \rightarrow q_1 = 10.
\]

As we can see, we must sell 10 units of \( A_1 \), 5 units of \( A_2 \) and 15 units of \( A_3 \) if we want to obtain an income of €105.
1.2.2. Norm of a Vector and Properties

We are now ready to deal with some basic metric concepts in \( \mathbb{R}^n \) through the associated scalar product.\(^{16}\) First of all we are going to introduce the notion of the norm (or the length) of a vector. As the scalar product of any vector by itself is always positive we have:

**Definition:** The norm of the vector \( \vec{x} \in \mathbb{R}^n \) is the positive square root:

\[
\| \vec{x} \| = +\sqrt{\vec{x} \cdot \vec{x}} = +\sqrt{x_1^2 + \cdots + x_n^2} = +\sqrt{\sum_{i=1}^{n} x_i^2}.
\]

In fact the norm of any vector in \( \mathbb{R}^n \) reproduces the Pythagoras theorem as we can graphically see in both two and three components:

![Vector Norm Diagram](image)

**Properties:**

a. \( \| \vec{x} \| \geq 0 \). Moreover \( \| \vec{x} \| = 0 \) is equivalent to \( \vec{x} = \mathbf{0} \)

b. \textit{“Triangular inequality”:} \( \| \vec{x} + \vec{y} \| \leq \| \vec{x} \| + \| \vec{y} \| \).

c. \textit{“Schwarz inequality”:} \( |\vec{x} \cdot \vec{y}| \leq \| \vec{x} \| \cdot \| \vec{y} \| \).

d. \textit{If} \( \vec{x} \neq \mathbf{0}, \text{ the vector } \frac{1}{\| \vec{x} \|} \cdot \vec{x} \text{ is an unit vector.}^{17} \)

---

\(^{16}\) The \( \mathbb{R}^n \)-vector space with the scalar product is also called \textit{Euclidean n-space}.

\(^{17}\) A \textit{unit vector} is a vector whose norm is equal to 1.
1.2.3. Angle between Vectors and Orthogonal Vectors

One of the main concepts in metric geometry is the angle $\alpha$ formed by two straight lines or, in our case, two vectors. Graphically:

![Diagram showing angle between two vectors]

The Schwarz inequality enables us to consider the cosine of $\alpha$ since:

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| \implies -1 \leq \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \leq 1.$$

**Definition:** Assuming that $\vec{x}, \vec{y} \neq \vec{0}$:

1. The angle $\alpha$ between the two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is such that $\cos \alpha = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$.

2. $\vec{x}, \vec{y} \in \mathbb{R}^n$ are said to be orthogonal (or perpendicular) if the cosine of the angle they form is zero. In other words: $\vec{x} \cdot \vec{y} = 0$.

Note that the cosine allows us to define the scalar product using the norm:

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos \alpha.$$

**Example:** Find the angle formed by $\vec{x} = (1,1,0)$ and $\vec{y} = (2,9,6)$.

**SOLUTION:**

As we have: $\vec{x} \cdot \vec{y} = (1,1,0) \cdot (2,9,6) = 1 \cdot 2 + 1 \cdot 9 + 0 \cdot 6 = 11$, then:

$$\cos \alpha = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} = \frac{11}{\sqrt{1^2 + 1^2 + 0^2} \sqrt{2^2 + 9^2 + 6^2}} = \frac{11}{\sqrt{2}} \implies \alpha = \frac{\pi}{4} \text{ radians.}$$

---

18 Note that the two first vectors that appear in the example on page 14 are orthogonal.
19 Thus the scalar product shows the “tendency” of two vectors to point in the same direction.
1.2.4. Distance between Vectors

Another basic metric concept is the distance between two vectors. This concept enables us to rigorously define the distance between points of \( \mathbb{R}^n \) as graphically illustrated below.

**Definition:** The distance between two vectors \( \vec{x}, \vec{y} \in \mathbb{R}^n \) is defined by the norm of their difference:

\[
d(\vec{x}, \vec{y}) = \| \vec{x} - \vec{y} \| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.
\]

Graphically:

In this case we can also talk about the distance between points \( P \) and \( Q \).

**Example:** Calculate the distance between the two vectors:

\[
\vec{x} = (3, -2, 0, 1) \quad \text{and} \quad \vec{y} = (1, -4, 0, 2).
\]

**SOLUTION:**

In this case we have:

\[
d(\vec{x}, \vec{y}) = \| \vec{x} - \vec{y} \| = \| (2, 2, 0, -1) \| = +\sqrt{2^2 + 2^2 + 0^2 + (-1)^2} = +\sqrt{9} = 3.20
\]

\[20 \text{ Consequently we can affirm that the distance between points } P = (3, -2, 0, 1) \text{ and } Q = (1, -4, 0, 2) \text{ is equal to 3.} \]
1.3. Quadratic Forms on Vector Spaces

Basically we are interested in quadratic forms due to the sufficient condition of the existence of extreme points of functions of several variables.\(^{21}\) For example, the mapping between the vector spaces \(\mathbb{R}^3\) and \(\mathbb{R}\) defined by:

\[
Q(x_1, x_2, x_3) = 3x_1^2 + 4x_1x_2 + 3x_1x_3 + 2x_2x_2 - 2x_2x_3
\]

is a quadratic form in \(\mathbb{R}^3\).\(^{22}\) Notice that if we consider the symmetric matrix defined by:\(^{23}\)

\[
A = \begin{pmatrix}
3 & 1 & 0 \\
1 & 4 & -1 \\
0 & -1 & 3
\end{pmatrix}
\]

where the entries of the main diagonal \(a_{ii}\) are the coefficients of the square variables \(x_1^2\), \(x_2^2\) and \(x_3^2\), and \(a_{ij}\) are the coefficients of the crossing-products \(x_i \cdot x_j\), then:\(^{24}\)

\[
\begin{align*}
\bar{x}^\top A \bar{x} &= (x_1, x_2, x_3) \cdot \begin{pmatrix}
3 & 1 & 0 \\
1 & 4 & -1 \\
0 & -1 & 3
\end{pmatrix} \cdot \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = (3x_1 + x_2) \\
&= x_1 (3x_1 + x_2) + x_2 (x_1 + 4x_2 - x_3) + x_3 (3x_1 - x_2 + 3x_3) = \\
&= 3x_1^2 + 4x_2^2 - 2x_2x_3 = Q(x_1, x_2, x_3).
\end{align*}
\]

Thus, in this case, we can say that the quadratic form has the matrix \(A\) as its “associated” symmetric matrix. Generally:

**Definition:** A mapping \(\mathbb{R}^n \rightarrow \mathbb{R}\) is a quadratic form provided that a symmetric matrix \(A\) of order \(n\) exists such that:

\[
Q(\bar{x}) = \bar{x}^\top A \bar{x} = (x_1, \ldots, x_n) \cdot \begin{pmatrix}
a_{11} & \cdots & a_{1n}
\vdots & \ddots & \vdots
a_{n1} & \cdots & a_{nn}
\end{pmatrix} \cdot \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix} = \sum_{i,j=1}^{n} a_{ij}x_i x_j.
\]

---

\(^{21}\) We return to this question in the next section.

\(^{22}\) Note that the sum of the exponents of the variables in each term is 2.

\(^{23}\) Recall that a matrix is symmetric when it is equal to its associated transpose matrix.

\(^{24}\) \(\bar{x}^\top\) is the “transpose” row vector of \(\bar{x}\).
1.3.1. Sign of a Quadratic Form

What is of interest to us is basically the sign of quadratic forms and in this regard we need to bear in mind the following definition:

**Definition:** The quadratic form $\mathbb{R}^n \rightarrow \mathbb{R}$ is:

1. **Positive definite** if $Q(x) > 0$, for any non-zero vector $x \in \mathbb{R}^n$.\(^{25}\)
2. **Negative definite** if $Q(x) < 0$, for any non-zero vector $x \in \mathbb{R}^n$.
3. **Positive semidefinite** if $Q(x) \geq 0$, for any vector $x \in \mathbb{R}^n$. Besides one non-zero vector $\bar{y} \in \mathbb{R}^n$ must exist such that $Q(\bar{y}) = 0$.
4. **Negative semidefinite** if $Q(x) \leq 0$, for any vector $x \in \mathbb{R}^n$. Besides one non-zero vector $\bar{y} \in \mathbb{R}^n$ must exist such that $Q(\bar{y}) = 0$.
5. **Indefinite** if it does not match any of the definitions listed above, i.e., when two vectors $\bar{x}, \bar{y} \in \mathbb{R}^n$ exist such that $Q(\bar{x}) < 0$ and $Q(\bar{y}) > 0$.

**Example:** Prove that the quadratic form:

$$Q(x,y,z) = x^2 + y^2 + z^2 - 2xy$$

is positive semidefinite.

**SOLUTION:**

This is clear because this quadratic form is always positive:

$$Q(x,y,z) = x^2 + y^2 + z^2 - 2xy = (x-y)^2 + z^2 \geq 0$$

and the image of the non-zero vector $(1,1,0)$ is 0:

$$Q(1,1,0) = (1-1)^2 + 0^2 = 0.$$  

Determining the sign of a quadratic form using the above definition is not an easy task. Thus we are going to introduce a relatively simple method for doing so in the easiest cases.

\(^{25}\) Note that the corresponding image of the zero-vector by any quadratic form is always equal to 0.
1.3.2. Classification of Quadratic Forms by means of Principal Minors

First of all we need to recall the definition of principal minor of a matrix. Indeed, the principal minors of a square matrix of the type:

\[ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \]

are, precisely, the minors with the main diagonal formed by some of the n elements:

\[ a_{11}, \ldots, a_{nn} \]

of the main diagonal of matrix \( A \).

Let’s examine the following example:

**Example:** Calculate the principal minors of:

\[ A = \begin{pmatrix} -5 & 3 & 1 \\ 3 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} \]

**SOLUTION:**

First we have the three first-order principal minors \( M_{p1} \) that match the three entries of the main diagonal:

\[ M_{p1} = \{-5; -2; -3\}. \]

The three second-order principal minors \( M_{p2} \) are:

\[ M_{p2} = \begin{vmatrix} -5 & 3 \\ 3 & -2 \end{vmatrix}, \begin{vmatrix} 3 & -2 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} -5 & 1 \\ 1 & -3 \end{vmatrix} = \{1; 6; 14\}. \]

As matrix \( A \) is of the third order, the sole third-order principal minor \( M_{p3} \) is the associated determinant:

\[ M_{p3} = |A| = \begin{vmatrix} -5 & 3 & 1 \\ 3 & -2 & 0 \\ 1 & 0 & -3 \end{vmatrix} = \{-1\}. \]
1.3.2.1. Relationship between the Sign of Quadratic Forms and Principal Minors

The theorem we enunciate below characterizes the sign of a quadratic form through the numerical sign of the principal minors associated with the related matrix.

**Theorem:** A quadratic form with associated symmetric matrix $A$ is:

- **Positive definite if and only if all the principal minors of $A$ are strictly positive.**
- **Negative definite if and only if all the principal minors of even order of $A$ are strictly positive and those of odd order are strictly negative.**
- **Positive semidefinite if and only if all the principal minors of $A$ are positive being always** $\det A = 0$.
- **Negative semidefinite if and only if all the principal minors of even order of $A$ are positive and those of odd order are negative or zero being always** $\det A = 0$.\(^{26}\)
- **Indefinite otherwise.**

**Example:** Calculate both the analytic expression and the sign of the quadratic form with associated symmetric matrix:

$$\begin{pmatrix} -5 & 3 & 1 \\ 3 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}.$$ 

**SOLUTION:**
According to the definition of quadratic form, the analytic expression will be:

$$Q(x, y, z) = -5x^2 - 2y^2 - 3z^2 + 6xy + 2xz.$$ 

Since:

- $Mp_1 = \{-5; -2; -3\}$, $Mp_2 = \{1; 6; 14\}$ and $Mp_3 = \{-1\}$

according to the previous theorem we can conclude that this quadratic form is negative definite.

\(^{26}\) Note that a quadratic form can not be semidefinite if the determinant of the associated matrix is not equal to 0.
1.3.2.2. Application of the Sign of Quadratic Forms in Economics

Example: A company dedicated to develop three types of wine has annual revenues given by the function:

\[ \Pi(x, y, z) = x^2 + y^2 + 2z^2 - 2xy - 2\sqrt{2}xz \]

where \( x, y \) and \( z \) denote respectively the amount of hectoliters of each type of wine elaborated. Under these assumptions prove that: (i) The company may have losses. (ii) This is not the case if the number of hectoliters elaborated of the first type is half of the third.

SOLUTION:

i) First of all observe that the function of revenues is a quadratic form. Since the associated symmetric matrix is:

\[
A = \begin{pmatrix}
1 & -1 & -\sqrt{2} \\
-1 & 1 & 0 \\
-\sqrt{2} & 0 & 2
\end{pmatrix}
\]

and the principal minors are:

\( M_{p_1} = \{1; 1; 2\} \), \( M_{p_2} = \{0; 2; 0\} \) and \( M_{p_3} = \{-2\} \)

the function of revenues is indefinite. According to the definition of indefinite forms, we might conclude that this company may have losses. In fact this is the case if the production of the three types of wine is the same:

\[
(x = y = z = n) \implies \Pi(n, n, n) = 2n^2 - (2\sqrt{2})n^2 < 0.
\]

ii) In this case the function of revenues would be the “restricted” quadratic form:

\[
x = 0.5z - \frac{zx}{2} \implies \Pi(x, y, 2x) = (9 - 4\sqrt{2})x^2 + y^2 - 2xy.
\]

Since the associated matrix is:

\[
A = \begin{pmatrix}
9 - 4\sqrt{2} & -1 \\
-1 & 1
\end{pmatrix},
\]

and the principal minors are strictly positive:

\( M_{p_1} = \{9 - 4\sqrt{2}; 1\} \) and \( M_{p_2} = |A| = \{8 - 4\sqrt{2}\} \)

this quadratic form is definite positive, which means that the company doesn’t have losses.
1.4. Exercises

1. Find the values of the parameter $a \in \mathbb{R}$ so that the vector $(-2, -1, 5, 0)$ is a linear combination of $(2, 4, 7, 6)$ and $(a, 2, -1, a)$.

2. Find the values of $a \in \mathbb{R}$ so that the vectors $(a, 0, -3), (2, -a, 5)$ and $(0, 1, a)$ form a basis of $\mathbb{R}^3$.

3. Prove that $(-1, 0, 4, 3), (6, 5, 0, 3)$ and $(0, -2, 1, 0)$ are linearly independent and find another vector which when taken together form a basis of $\mathbb{R}^4$.

4. Prove that regardless of the value that $m \neq 0$ takes, the set of four vectors $(-m, 3, 5), (0, 11, -9), (m, 4, -6)$ and $(0, 1, m)$ is a spanning set of $\mathbb{R}^3$ and determine a basis among them depending on $m \neq 0$.

5. Given the vectors $\mathbf{u} = (2, 1, 1)$ and $\mathbf{v} = (3, -2, 2)$ determine:
   
   i. Their scalar product.
   
   ii. The associated norms.
   
   iii. The associated unit vectors.
   
   iv. The angle that they form.
   
   v. The distance between them.

6. Prove that two non-zero orthogonal vectors are always linearly independent and determine one basis of $\mathbb{R}^3$ formed by pairwise orthogonal vectors including the vector $(1, 2, 3)$.

7. Find the sign of the following quadratic forms:
   
   i. $Q(x, y, z) = 2x^2 + 3y^2 + 2z^2 + 2xz$.
   
   ii. $Q(x, y, z) = x^2 + 2y^2 + z^2 - 2xy + 2yz$.
   
   iii. $Q(x, y, z) = -3x^2 - 4y^2 - 3z^2 + 4\sqrt{2}xy + 4xz$.
   
   iv. $Q(x, y, z) = -x^2 - 14y^2 - 2z^2 + 4xz$ restricted to the equation $x + y + z = 0$.

---

27 These bases are called orthogonal.

28 In this case we have to obtain one “restricted” quadratic form depending on two variables.
SOLUTIONS:

1. \( a = 2 \).

2. \( a \neq -1 \).

3. For example \((0,0,0,1)\).

4. For example \((-m,3,5), (0,11,-9)\) and \((m,4,-6)\).

5. 
   i. \( \vec{u} \cdot \vec{v} = 6 \).
   
   ii. \( \|\vec{u}\| = \sqrt{6} \) and \( \|\vec{v}\| = \sqrt{17} \).

   iii. \( \vec{u}_0 = \left( \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right) \) and \( \vec{v}_0 = \left( \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{2}{\sqrt{17}} \right) \).

   iv. \( \alpha = 0.935 \) radians.

   v. \( d(\vec{u},\vec{v}) = \sqrt{11} \).

6. An orthogonal basis including the vector \((1,2,3)\) would be \((1,2,3), (-3,0,1)\) and \((1,-5,3)\).

7. 
   i. Positive definite.
   
   ii. Positive semidefinite.
   
   iii. Indefinite.
   
   iv. Negative definite
2. MULTIVARIABLE OPTIMIZATION

As an introduction to this subject consider the function of two variables:

\[ C(q_1, q_2) = 3q_1q_2 + 1.5q_1^2 + 2q_2^2 \]

as the total cost of a firm that produces two articles, A and B, in quantities \( q_1 \) and \( q_2 \). According to this, suppose that A and B are sold in a market generating a total income:

\[ I(q_1, q_2) = p_1q_1 + p_2q_2 \]

where \( p_1 \) and \( p_2 \) stand for the sale prices of A and B. An interesting problem in Economics appears when we want to know the production of A and B that maximizes benefits:

\[ B(q_1, q_2) = I(q_1, q_2) - C(q_1, q_2) \]

For example if sale prices are:

\[ p_1 = €42 \quad \text{and} \quad p_2 = €51 \]

benefits adopt the form:

\[ B(q_1, q_2) = (42q_1 + 51q_2) - (3q_1q_2 + 1.5q_1^2 + 2q_2^2) = -1.5q_1^2 - 2q_2^2 - 3q_1q_2 + 42q_1 + 51q_2. \]

In this case it can be proved that \( q_1 = 5 \) units of A and \( q_2 = 9 \) units of B maximize benefits with value \( B(5, 9) = €334.5 \). On the other hand if:

\[ p_1 = 66 - 3q_1 \quad \text{and} \quad p_2 = 72 - q_2 \]

are the demand functions\(^{29}\) of A and B benefits will be equal to:

\[ B(q_1, q_2) = (p_1q_1 + p_2q_2) - C(q_1, q_2) = -4.5q_1^2 - 3q_2^2 - 3q_1q_2 + 66q_1 + 72q_2. \]

Now \( q_1 = 4 \) units of A and \( q_2 = 10 \) of B maximize benefits with value \( B(4, 10) = €492.\(^{30}\)

In order to obtain these figures we must know the basic properties of functions of several variables related to domains and level curves, as well as the procedure of finding their associated extreme points in case they exist. This section is concerned on this issue dealing with some of the most important analytical concepts such as partial derivatives and Hessian matrices of functions.

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\(^{29}\) Demand function are functions relating prices and quantities demanded at these prices.

\(^{30}\) We will solve these exemples in class.
2.1. Function of Several Variables

2.1.1. Function of Several Variables and Domain

The basic structure in which we are going to move is the vector space $\mathbb{R}^n$ associated with the scalar product defined above, i.e. the Euclidean vector space $\mathbb{R}^n$. Formally:

**Definition:** A function of $n$ variables (function for short) is a mapping $A \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ that assigns a real number $z \in \mathbb{R}$ to each $n$-vector $\vec{x} = (x_1, \ldots, x_n)$ in $A \subseteq \mathbb{R}^n$. In other words:

$$z = f(\vec{x}) = f(x_1, \ldots, x_n), \text{ for any } \vec{x} \in A \subseteq \mathbb{R}^n.$$

By definition, the set $A \subseteq \mathbb{R}^n$ formed by the points of $\mathbb{R}^n$ that support image is the domain of the function.

Note that if $n = 1$ a function of several variables is simply a function of one real variable.\(^{31}\)

Let’s see an example:

**Example:** Determine the domain of the function:

$$z = f(x, y) = +\sqrt{x^2 + y^2} - 1.$$

**SOLUTION:**

It is easy to see this is a function of two variables ($n = 1$) since it is defined on $\mathbb{R}^2$ and takes real positive values:

$$+\sqrt{x^2 + y^2} - 1 \in \mathbb{R}_+.$$

Thus the associated domain will be:

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}.$$

\(^{31}\) From now onwards we assume the main properties of the functions of one variable related to continuity and derivability are known.

\(^{32}\) $\mathbb{R}_+$ denotes the set of real positive numbers.
2.1.1.1. Example of Graphical Representation of Domains

From a practical point of view we only will graph domains of functions of two variables. This is the first task that we must learn about this type of functions. Let’s now see an easy example:

**Example:** Determine graphically the domain of the above function:

\[ f(x, y) = +\sqrt{x^2 + y^2 - 1}. \]

**SOLUTION:** We already know that it is a function with domain:

\[ A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\} \]

Since:

\[ x^2 + y^2 = 1 \]

represents the circumference of radius 1 centered at point \((0, 0)\), the associated domain will be either the circle that it contains or the outer part of the plane this circumference determines. In either case, the circumference will form part of the domain since the inequality is not strict. Taking into account that \((0, 0)\) does not belong to the domain \(A\) we conclude that the second alternative is the correct one. Graphically:

---

33 We recommend to review the graphic representation of lines, parabolas and hyperbolas.
34 Recall that any equation of the form \((x-a)^2 + (y-b)^2 = r^2\) represents the circumference of radius \(r > 0\) centered at point \((a, b)\).
2.1.2. Level Curves of a Function of Two Variables

We know that any function of a single variable can be represented by a “curve” in the plane. However, in the case of two variables, the “surfaces” that functions of this type generate in space are very difficult to draw. In practice what happens is that this problem is simplified by considering only the curves that join those points of the domain that have the same value by the function. It should be stressed that these curves provide a lot of information when the drawing of functions is not a straightforward task. Formally:

Definition: The level curves of $A \subset \mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ are the plane-curves defined by the equation:

$$f(x, y) = k \in \mathbb{R}$$

and are denoted by $c_k$.\(^{35}\)

Example: Represent graphically the level curves of $z = f(x, y) = xy$.

SOLUTION:
The level curves $c_k$ are the plane curves provided by the equation:

$$xy = k.$$  

Note that these curves are in fact hyperbolas having the two reference axes as asymptotes. For any $k > 0$ the hyperbola appears in the first and third quadrants, whereas for $k < 0$ it does in the second and fourth. Graphically:

\(^{35}\) In terms of economics, the level curves associated with production functions are called *isoquants*, while those associated with utility functions are called *indifference curves*.  

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29
2.2. Partial Derivatives of Functions

2.2.1. Partial Derivatives of a Function of Several Variables

Similarly to the way in which we define the derivative of a function of a single variable, the same procedure can be adopted with functions of several variables. This gives rise to the concept of the partial derivative. Formally:

**Definition:** The function \( f : A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) has derivative with respect to the variable \( x_i \) at point \( \bar{a} = (a_1, \ldots, a_n) \in A \subset \mathbb{R}^n \) in case of the existence of the limit:

\[
\lim_{t \to 0} \frac{f(a_1, \ldots, a_i + t, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{t} = \frac{\partial f(\bar{a})}{\partial x_i} \in \mathbb{R}.
\]

Thus, given a function of \( n \) variables and any point from its domain, we have in principle \( n \) partial derivatives at this point. Obviously their existence will depend on the existence of the corresponding limits.

**Example:** Calculate the partial derivatives at point \((1,1)\) of the function:

\[ z = f(x,y) = \frac{y}{x}. \]

**SOLUTION:** In this case \( \bar{a} = (1,1) \) and hence we have:

\[
\frac{\partial f(1,1)}{\partial x} = \lim_{t \to 0} \frac{f(1+t,1) - f(1,1)}{t} = \lim_{t \to 0} \frac{1}{t(1+t)} = -\lim_{t \to 0} \frac{-t}{(1+t)} = -1.
\]

and:

\[
\frac{\partial f(1,1)}{\partial y} = \lim_{t \to 0} \frac{f(1,1+t) - f(1,1)}{t} = \lim_{t \to 0} \frac{(1+t)/1 - 1}{t} = \lim_{t \to 0} \frac{t}{t} = 1.
\]

---

\(^{36}\) Take care of this symbolism of partial derivatives.

\(^{37}\) Note that the partial derivatives of a given function at a point do not necessarily have to match each other.
2.2.1.1. Geometrical Interpretation of Partial Derivatives

Geometrically, the partial derivative with respect to $x$ of the function $z = f(x, y)$ at point $(a, b) \in A$ is the slope of the tangent $r$ on the plane $p$:

In other words:

$$\frac{\partial f(a, b)}{\partial x} = \tan \alpha.$$ 

The partial derivative with respect to $y$ of the function $z = f(x, y)$ at point $(a, b) \in A$ is the slope of the tangent $r$ on the plane $p$:

In other words:

$$\frac{\partial f(a, b)}{\partial y} = \tan \alpha.$$
2.2.1.2. Calculation of Partial Derivatives

From the above definition it is easy to see that the calculus of partial derivatives can be reduced to that of the derivatives in a single variable. This is due to the fact that the partial derivative with respect to any specific variable is obtained by “deriving” the function (as we already know) respect to this variable being constant the remaining variables. Let’s look at an example:

**Example:** Calculate the partial derivatives of:

i. \( z = f(x, y) = 4x^2 - 7y^3 + 2x^2y - 75y + 8 \).

ii. \( z = f(x, y) = \frac{1}{2} \ln \left( \frac{x-y}{x+y} \right) \).

iii. \( z = f(x, y) = x^y \).

**SOLUTION:**

i) In this case the partial derivative with respect to \( x \) is:

\[
\frac{\partial z}{\partial x} = \{ y \; \text{constant} \} = 8x - 0 + 4x \cdot y - 0 + 0 = 8x + 4xy
\]

and that corresponding to \( y \) will be:

\[
\frac{\partial z}{\partial y} = \{ x \; \text{constant} \} = 0 - 21y^2 + 2x^2 \cdot 1 - 75 \cdot 1 + 0 = -21y^2 + 2x^2 - 75.
\]

ii) Now the two partial derivatives will be:

\[
\frac{\partial z}{\partial x} = \{ y \; \text{constant} \} = \frac{1}{2} \left( x + y \right) \left( \frac{1 \cdot (x + y) - (x - y) \cdot 1}{(x + y)^2} \right) = \frac{y}{x^2 - y^2}
\]

and:

\[
\frac{\partial z}{\partial y} = \{ x \; \text{constant} \} = \frac{1}{2} \left( x + y \right) \left( \frac{-1 \cdot (x + y) - (x - y) \cdot 1}{(x + y)^2} \right) = \frac{-x}{x^2 - y^2}.
\]

iii) In this case we have:

\[
\frac{\partial z}{\partial x} = \{ y \; \text{constant} \} = y \cdot x^{y-1} \; \text{and} \; \frac{\partial z}{\partial y} = \{ x \; \text{constant} \} = x^y \cdot \ln x.
\]
2.2.2. Gradient of a Function

We are ready to introduce the seminal concept of the vector gradient of a function of several variables at a point. Formally:

**Definition:** The gradient vector (gradient for short) of $A \subset \mathbb{R}^n \xrightarrow{f} \mathbb{R}$ at the point $\bar{a} \in A$ is the vector formed by all the partial derivatives of the function at this point. Namely:

$$\nabla f(\bar{a}) = \left( \frac{\partial f(\bar{a})}{\partial x_1}, \ldots, \frac{\partial f(\bar{a})}{\partial x_n} \right).$$

We have to stress that the gradient associated with a function of $n$ variables is a vector with $n$ components. Hence, if any of the partial derivatives appearing in the definition do not exist, neither does the gradient.

**Example:** Calculate, in the case that it does exist, the gradient at point $(1,0)$ of:

i. $z = f(x,y) = \frac{y}{x}$

ii. $z = f(x,y) = \sqrt{xy}$

**SOLUTION:**

i) In this case the gradient does exist and it is:

$$\nabla f(1,0) = \left( \frac{\partial f(1,0)}{\partial x}, \frac{\partial f(1,0)}{\partial y} \right) = \left( -\frac{1}{x^2}, \frac{1}{y} \right) = \left( -\frac{1}{1^2}, \frac{1}{1} \right) = (0,1).$$

ii) Here, the gradient does not exist since:

$$\nabla f(1,0) = \left( \frac{\partial f(1,0)}{\partial x}, \frac{\partial f(1,0)}{\partial y} \right) = \left( \frac{1}{2\sqrt{xy}}, \frac{1}{2\sqrt{xy}} \right) = \left( \frac{1}{2\sqrt{1}}, \frac{1}{2\sqrt{1}} \right) = \left( \frac{1}{2}, \frac{1}{2} \right).$$
2.2.3. Second-Order Partial Derivatives of a Function

A fundamental concept in the field we are studying is that of the Hessian matrix of a function of several variables associated to a point. This matrix is formed by the second-order partial derivatives of that function evaluated at the point. Hence, we have to start by introducing the concept of the second-order partial derivative:

**Definition:** The second-order partial derivative of $A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ at point $\bar{a} \in A$ with respect to the variables $(x_i, x_j)$ is the partial derivative with respect to the second variable $x_j$ at point $\bar{a} \in A$ of the partial derivative function with respect to the first variable $x_i$:

$$\frac{\partial^2 f(\bar{a})}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial f(\bar{a})}{\partial x_i} \right)$$

When $x_i = x_j$ we put $\frac{\partial^2 f(\bar{a})}{\partial x^2}$.

**Example:** Evaluate at point $(1,1)$ the second-order partial derivatives of the function:

$$f(x, y) = x \cdot \ln y$$

**SOLUTION:** Since the general gradient is:

$${\nabla} f(x, y) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( \ln y, \frac{x}{y} \right)$$

we deduce that:

$$\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} (\ln y) = \{y \text{ constant}\} = 0 \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (\ln y) = \frac{1}{y} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (x / y) = \frac{1}{y} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} (x / y) = \frac{x}{y^2}
\end{align*}$$

$$\begin{align*}
\left. \frac{\partial^2 f}{\partial x^2} \right|_{(x, y) = (1, 1)} &= 0 \\
\left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(x, y) = (1, 1)} &= 1 \\
\left. \frac{\partial^2 f}{\partial y \partial x} \right|_{(x, y) = (1, 1)} &= 1 \\
\left. \frac{\partial^2 f}{\partial y^2} \right|_{(x, y) = (1, 1)} &= -1
\end{align*}$$
2.2.3.1. Hessian Matrix of a Function

As we shall see, the concept of the Hessian matrix is of great significance in the context of
the optimization of functions of several variables.\(^{38}\)

**Definition:** *The Hessian matrix of the function* \( f : \mathbb{R}^n \to \mathbb{R} \) *at point* \( \bar{a} \in A \) *is the square matrix defined by:*

\[
H_f(\bar{a}) = \begin{bmatrix}
\frac{\partial^2 f(\bar{a})}{\partial x_1^2} & \frac{\partial^2 f(\bar{a})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\bar{a})}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f(\bar{a})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\bar{a})}{\partial x_2^2} & \cdots & \frac{\partial^2 f(\bar{a})}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f(\bar{a})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\bar{a})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\bar{a})}{\partial x_n^2}
\end{bmatrix}.
\]

**Example:** *Evaluate the Hessian matrix of* \( f(x,y) = x \cdot \ln y \) *at point* \((1,1)\).

**SOLUTION:**

Applying the results obtained from the previous page, we see that the Hessian matrix of
the function at \((1,1)\) is:

\[
H_f(1,1) = \begin{bmatrix}
\frac{\partial^2 f(1,1)}{\partial x^2} & \frac{\partial^2 f(1,1)}{\partial x \partial y} \\
\frac{\partial^2 f(1,1)}{\partial y \partial x} & \frac{\partial^2 f(1,1)}{\partial y^2}
\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.
\]

However, if the point changes, so does the Hessian matrix. For example, the Hessian matrix
at point \((0,2)\) will be:

\[
H_f(0,2) = \begin{bmatrix}
\frac{\partial^2 f(0,2)}{\partial x^2} & \frac{\partial^2 f(0,2)}{\partial x \partial y} \\
\frac{\partial^2 f(0,2)}{\partial y \partial x} & \frac{\partial^2 f(0,2)}{\partial y^2}
\end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.
\]

\(^{38}\) In fact this concept plays a similar role to that of the second derivative of a function of a single
variable.
2.2.4. Applications of Partial Derivatives

2.2.4.1. Application in Economics I: Marginalism

Let \( A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be an economic function\(^{39}\) having partial derivative with respect to \( x_i \) at point \( \bar{a} \in A \), i.e:

\[
\frac{\partial f(\bar{a})}{\partial x_i} = \lim_{t \to 0} \frac{f(a_1, \ldots, a_i + t, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{t} \in \mathbb{R}.
\]

If we consider \( t = 1 \) in this equality we would obtain the approximation:

\[
f(a_1, \ldots, a_i + 1, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n) \approx \frac{\partial f(\bar{a})}{\partial x_i}.
\]

Thus, this partial derivative approximately measures the change in the function value \( f(a_1, \ldots, a_n) \) caused by an increase of 1 unit in the component \( a_i \in \mathbb{R} \).

Consider this application in terms of economics:

**Example:** Let:

\[
\Pi(x, y) = -2x^2 - 2.5y^2 + 1100x + 1300y - 70,000
\]

be a function of the benefit in euros associated with the production and sale of two articles \( A \) and \( B \). If currently \( x = 250 \) and \( y = 220 \) units are produced, calculate the approximate change in the benefit if the production of \( B \) is increased by one unit.

**SOLUTION:**

In order to estimate the change in the benefit we need to consider the partial derivative of the function with respect to the variable \( y \) at point \( (250, 220) \). Since:

\[
\frac{\partial \Pi}{\partial y} = -5y + 1300 \quad \text{at } x = 250 \text{ and } y = 220
\]

we deduce that the benefit increases by approximately €200:

\[
\Delta \Pi(250, 220) = \Pi(250, 221) - \Pi(250, 220) \approx \frac{\partial \Pi(250, 220)}{\partial y} = 200.40
\]

---

\(^{39}\) For instance, a function of benefits, costs, utility, etc.

\(^{40}\) This approximation works quite well since the real value of this increase is equal to €197.5.
2.2.4.2. Application of in Economics II: Partial Elasticity

If $A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ admits partial derivative with respect to $x_i$ at point $\bar{a} = (a_1, \ldots, a_n) \in A$ such that $f(\bar{a}) \neq 0$, we can define the partial elasticity with respect to the variable $x_i$ at point $\bar{a} \in A$ as the number:

$$E_{x_i} f(\bar{a}) = \frac{a_i}{f(\bar{a})} \frac{\partial f(\bar{a})}{\partial x_i}.$$

Since:

$$\frac{\partial f(\bar{a})}{\partial x_i} = \frac{f(a_1, \ldots, a_i + \Delta a_i, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{\Delta a_i} = \{\text{Definition}\} = \frac{\Delta f(\bar{a})}{\Delta a_i}$$

taking into account the previous definitions we can affirm that the value of the partial elasticity $E_{x_i} f(\bar{a})$ is approximately equal to the percentage change in the function value $f(a_1, \ldots, a_n)$ caused by a 1% increase in the component $a_i \in \mathbb{R}$:

$$\frac{\Delta a_i}{a_i} = 1\% \implies \frac{\Delta f(\bar{a})}{f(\bar{a})} = \frac{a_i}{f(\bar{a})} \left( \frac{\Delta f(\bar{a})}{\Delta a_i} \right) \approx E_{x_i} f(\bar{a})\%.$$

Moreover if this component increases (or decreases) at a $r\%$ rate, the value $f(a_1, \ldots, a_n)$ varies approximately by:

$$(E_{x_i} f(\bar{a}) \cdot r)\%.$$

**Example:** Estimate the increase in benefits in the above example if the production of A increases by 2% from the present level of 250 units.

**SOLUTION:** Since:

$$E_x \Pi = \frac{x}{\Pi(x,y)} \cdot \frac{\partial \Pi}{\partial x} = \frac{-4x^2 + 1100x}{\Pi(x,y)} \bigg|_{x=250 \text{ and } y=220} \rightarrow E_x \Pi(250,220) = 0.102041$$

we deduce that the percentage of change in benefits when the production of A increases by $r = 2\%$ is approximately:

$$E_x \Pi(250,220) \cdot r\% = 0.102041 \cdot 2\% = 0.204082\% \approx 0.2\%.$$

---

41 The increase or decrease of the function depends on the sign of the corresponding partial elasticity.

42 In fact the real percentage of change is equal to 0.183%.
2.2.4.3. Application of the Gradient: Maximal Increase

Among the main properties of the gradient of a function at a point we should mention that, as a vector, it always points in the direction of maximal increase of the function from this point.\(^{43}\) Consequently if we want to maximize functions as fast as possible we are forced to follow the direction of gradient vectors; conversely, if we wish to minimize we have to reverse their direction.\(^{44}\)

Let’s look at an example:

**Example:** Given the function:

\[ z = f(x,y) = 3x^2y - y^3 + x^2 \]

i. Calculate the gradient at points \((1,0)\) and \((1/3, -1/3)\).

ii. If we want to minimize this function as fast as possible from point \((1,0)\), what would be the direction to follow? Can we say from point \((1/3, -1/3)\)?

**SOLUTION:**

i) The associated gradients of the function at these points are:

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = \left( 6xy + 2x, 3x^2 - 3y^2 \right)
\]

\[
\begin{align*}
\nabla f(1,0) &= (2,3) \\
\n\nabla f \left( \frac{1}{3}, -\frac{1}{3} \right) &= (0,0)
\end{align*}
\]

ii) Thus if we want to minimize the function value as fast as possible from \((1,0)\), we have to follow the direction of the reverse of this gradient, i.e. the direction of the vector:

\[
\vec{u} = -\nabla f(1,0) = (-2, -3).
\]

Now, due to the fact that:

\[
\nabla f(1/3, -1/3) = (0,0)
\]

we cannot say anything about the increase (or decrease) of the function from \((0,0)\).

---

\(^{43}\) This property deals with differentiable functions. In fact all the functions we study here are differentiable. See Sydsaeter, K.; Hammond, P.J. (1995) for a definition of this concept.

\(^{44}\) In the case the gradient is equal to the zero-vector we cannot say anything in principle. We will discuss this issue in the next topic.
2.3. Classical Optimization

2.3.1. Global and Local Extreme Points of a Function

As we shall see below, the definition of extreme points of a function of several variables is similar to that corresponding to a single variable:

**Definition:** The function $A \subset \mathbb{R}^n \rightarrow \mathbb{R}$ at point $\tilde{a} \in A$ has:

1. A **global maximum** if $f(\tilde{x}) \leq f(\tilde{a})$, for any point $\tilde{x} \in A$.
2. A **global minimum** if $f(\tilde{a}) \leq f(\tilde{x})$, for any point $\tilde{x} \in A$.
3. A **local maximum** if there is an open ball centered at $\tilde{a} \in A$, included in the domain $A \subset \mathbb{R}^n$, such that for any point $\tilde{x} \in A$ of this open ball we have:

   $$ f(\tilde{x}) \leq f(\tilde{a}). $$

4. A **local minimum** if there is an open ball centered at $\tilde{a} \in A$, included in the domain $A \subset \mathbb{R}^n$, such that for any point $\tilde{x} \in A$ of this open ball we have:

   $$ f(\tilde{a}) \leq f(\tilde{x}). $$

In fact not every global extreme point is local and vice versa. For example the piecewise defined function:

$$ f(x,y) = \begin{cases} x^2 + y^2, & 0 \leq x^2 + y^2 \leq 1 \\ 2 - x^2 - y^2, & x^2 + y^2 > 1 \end{cases} $$

has a local minimum at $(0,0)$ that is not global. Graphically:

---

45 Also called *optima*.
46 By definition, an **open ball** of radius $r > 0$ centered at a point $\tilde{a} \in \mathbb{R}^n$ is the set of points of $\mathbb{R}^n$ placed at a distance of $\tilde{a} \in \mathbb{R}^n$ smaller than $r$. For instance, open balls in the plane are disks without the corresponding circumference.
2.3.1.1. Extreme Values of a Function

The extreme value of a function of several variables is the value reached by the function at an extreme point. Formally:

Definition: Given a function \( A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) and an extreme point \( \bar{a} = (a_1, \ldots, a_n) \in A \), the real value defined by:

\[
z_0 = f(\bar{a}) = f(a_1, \ldots, a_n) \in \mathbb{R}
\]

is an extreme value of the function.

Let’s see the following example:

Example: Prove that the function:

\[
z = f(x, y) = +\sqrt{4 - x^2 - 4y^2}
\]

has a global maximum at \((0,0)\) with maximum value equal to 2, and a global minimum at \((0,1)\) with minimum value equal to 0.

SOLUTION:

Since:

\[
0 \leq f(x, y) = +\sqrt{4 - x^2 - 4y^2} = +\sqrt{4 - (x^2 + 4y^2)} \leq +\sqrt{4 - 0} = +\sqrt{4} = 2
\]

and:

\[
f(0,1) = +\sqrt{4 - 0^2 - 4 \cdot 1^2} = +\sqrt{0} = 0
\]

and:

\[
f(0,0) = +\sqrt{4 - 0^2 - 4 \cdot 0^2} = +\sqrt{4} = 2
\]

we conclude that \((0,1)\) is a global minimum and \((0,0)\) is a global maximum of the function with associated extreme values respectively equal to 0 and 2.

---

\(^{47}\) This extreme point can be either a global or a local extreme point.
2.3.2. Necessary First-Order Condition of Existence of Extreme Points

In what follows we assume that the functions have 1st and 2nd order continuous partial derivatives. The necessary first-order condition for a function to have a local extreme point is that the associated gradient at this point, in the case that it does exist, is precisely the zero-vector:

Theorem: A necessary condition for \( \bar{a} \in A \) to be a local extreme point of \( A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) is that:

\[
\nabla f(\bar{a}) = \left( \frac{\partial f(\bar{a})}{\partial x_1}, \ldots, \frac{\partial f(\bar{a})}{\partial x_n} \right) = \left( 0, \ldots, 0 \right).
\]

Consider this example:

Example: Check that the global maximum \((0,0)\) of:

\[
f(x,y) = +\sqrt{4-x^2-4y^2}
\]

satisfies the necessary first-order condition.

SOLUTION:

This is straightforward since:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{-2x}{2\sqrt{4-x^2-4y^2}} = \frac{-x}{\sqrt{4-x^2-4y^2}} \\
\frac{\partial f}{\partial y} &= \frac{-8y}{2\sqrt{4-x^2-4y^2}} = \frac{-4y}{\sqrt{4-x^2-4y^2}}
\end{align*}
\]

\[
\rightarrow \nabla f(0,0) = \left( \frac{\partial f(0,0)}{\partial x}, \frac{\partial f(0,0)}{\partial y} \right) = (0,0).
\]

It is worth noting that the global minimum \((0,1)\) does not satisfy this condition because the gradient is not defined at this point as we can easily check. It should be emphasized that this first-order condition is satisfied only when the extreme points have a defined gradient.

---

48 We say “first-order” because the gradient is involved.

49 This condition is essential and all the functions studied here include it.
2.3.2.1. Stationary and Saddle Points of a Function

Thus all those points with associated zero-gradient play a key role in the process of finding extreme points. So consider the following definition:

Definition: \( \tilde{a} \in A \) is a stationary point of \( A \subset \mathbb{R}^n \xrightarrow{f} \mathbb{R} \) provided that it has an associated zero-gradient vector.

Example: Calculate the stationary points of \( f(x,y,z) = 2x^2 + y^2 - z^2 + x + 5 \).

SOLUTION:

The sole stationary point of this function is \( \left( -\frac{1}{4}, 0, 0 \right) \) since:

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left( 4x + 1, 2y, -2z \right) \xrightarrow{\text{Solution}} x = -\frac{1}{4}, y = z = 0.
\]

Yet a stationary point does not always have to be an extreme point. A stationary point that is not an extreme point is known as a saddle point:

Definition: \( \tilde{a} \in A \) is a saddle point of \( A \subset \mathbb{R}^n \xrightarrow{f} \mathbb{R} \) provided that it is a stationary point that is not an extreme point.\(^{50}\)

Graphically the point \((a,b)\) is a saddle point of \( z = f(x,y) \):

\(^{50}\) The stationary point of the above example, as we shall see below, is a saddle point.
2.3.3. *Necessary Second-Order Condition of Existence of Extreme Points*

This condition involves the Hessian matrices of the function under study. We must stress here that every Hessian matrix can be viewed as the symmetric matrix associated to a certain quadratic form:

**Theorem:** Let \( A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. Then:

a. If it has a local minimum at \( \bar{a} \in A \), the Hessian matrix \( Hf(\bar{a}) \) is either positive definite or semidefinite.

b. If it has a local maximum at \( \bar{a} \in A \), the Hessian matrix \( Hf(\bar{a}) \) is either negative definite or semidefinite.

**Example:** Check that the maximum \((0,0)\) of the function \( f(x,y) = +\sqrt{4 - x^2 - 4y^2} \) satisfies the associated necessary second-order condition.

**SOLUTION:** Since:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{-x}{\sqrt{4 - x^2 - 4y^2}} \\
\frac{\partial f}{\partial y} &= \frac{-4y}{\sqrt{4 - x^2 - 4y^2}}
\end{align*}
\]

implies

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= \frac{-4 + 4y^2}{(4 - x^2 - 4y^2)^{3/2}} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{-4xy}{(4 - x^2 - 4y^2)^{3/2}} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{-16 + 4x^2}{(4 - x^2 - 4y^2)^{3/2}}
\end{align*}
\]

the maximum \((0,0)\) satisfies the required condition since the Hessian matrix:

\[
Hf(0,0) = \begin{pmatrix}
\frac{\partial^2 f(0,0)}{\partial x^2} & \frac{\partial^2 f(0,0)}{\partial x \partial y} \\
\frac{\partial^2 f(0,0)}{\partial y \partial x} & \frac{\partial^2 f(0,0)}{\partial y^2}
\end{pmatrix} = \begin{pmatrix}
-0.5 & 0 \\
0 & -2
\end{pmatrix}
\]

is negative definite.

---

\(^{51}\) This theorem and the sequel justify the study of quadratic forms.
2.3.3.1. The Necessary Second-Order Condition and Saddle Points

Note that the necessary second-order condition introduced below affirms that the Hessian matrix associated with any extreme point cannot be indefinite. Hence we have:

**Property:** Let \( A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. If the Hessian matrix \( Hf(\bar{a}) \) at a stationary point \( \bar{a} \in A \) is indefinite then \( \bar{a} \in A \) is a saddle point.

Let’s look at an example:

**Example:** Prove that the function:

\[
f(x, y, z) = 2x^2 + y^2 - z^2 + x + 5
\]

has a saddle point at \( \left( -\frac{1}{4}, 0, 0 \right) \).

**SOLUTION:**

We already know that this point is a stationary point of the above function.\(^{52}\) Therefore since:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x + 1 \\
\frac{\partial f}{\partial y} &= 2y \\
\frac{\partial f}{\partial z} &= -2z
\end{align*}
\]

implies \( Hf(x, y, z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\
\frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2 \end{pmatrix}
\]

we deduce that the Hessian matrix at this point:

\[
Hf\left(-\frac{1}{4}, 0, 0\right) = \begin{pmatrix} 4 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & -2 \end{pmatrix}
\]

is indefinite and consequently that \( \left( -\frac{1}{4}, 0, 0 \right) \) is a saddle point of this function.

\(^{52}\) See page 42.
2.3.4. **Sufficient Second-Order Condition of Existence of Extreme Points**

This condition is also related to the Hessian matrices associated with the function under study and shows us when stationary points are local extreme points.

**Theorem:** Let \( A \subset \mathbb{R}^n \rightarrow \mathbb{R} \) be a function. If \( \bar{a} \in A \) is a stationary point then:

a. If the Hessian matrix \( Hf(\bar{a}) \) is positive definite, \( \bar{a} \in A \) is a local minimum.

b. If the Hessian matrix \( Hf(\bar{a}) \) is negative definite, \( \bar{a} \in A \) is a local maximum.

Consider this example:

**Example:** Prove that the stationary point \((0,0)\) is a local maximum of:

\[
f(x,y) = +\sqrt{4-x^2-4y^2}.
\]

**SOLUTION:**

Since the Hessian matrix at the stationary point \((0,0)\):

\[
Hf(0,0) = \begin{pmatrix}
\frac{\partial^2 f(0,0)}{\partial x^2} & \frac{\partial^2 f(0,0)}{\partial x \partial y} \\
\frac{\partial^2 f(0,0)}{\partial y \partial x} & \frac{\partial^2 f(0,0)}{\partial y^2}
\end{pmatrix} = \begin{pmatrix}
-0.5 & 0 \\
0 & -2
\end{pmatrix}
\]

is negative definite, we have that the stationary point \((0,0)\) satisfies the sufficient second-order condition. Accordingly, this point is a local maximum of this function.

Remember that we cannot apply the above analytical procedure in order to obtain extreme points in those situations in which there is neither gradient nor Hessian matrix well defined. In these cases we have to study individually case by case.

---

53 That is why we refer to it as a “second-order” condition.
54 See page 43.
Let’s look at a complete example:

**Example:** Find the extreme points and the associated extreme values of:

\[ f(x,y) = 8xy + \frac{1}{x} + \frac{1}{y}. \]

**SOLUTION:**

Note that the domain of this function is:

\[ A = \{(x,y) \in \mathbb{R}^2 : x, y \neq 0\}. \]

Finding their extreme points means solving the following system of non-linear equations:

\[
\begin{align*}
0 &= \frac{\partial f}{\partial x} = 8y - \frac{1}{x^2} \\
0 &= \frac{\partial f}{\partial y} = 8x - \frac{1}{y^2}
\end{align*}
\]

Equivalent

\[
\begin{align*}
8x^2y &= 1 \\
8xy^2 &= 1
\end{align*}
\]

Since:

\[
\begin{align*}
8x^2y &= 1 \\
8xy^2 &= 1
\end{align*}
\]

implies

\[
0 = 1 - 1 = 8x^2y - 8xy^2 = 8xy(x - y) \quad \text{as } x, y \neq 0
\]

\[
x = y
\]

then:

\[
y = x \implies 1 = 8x^3 \implies x = y = \frac{1}{2}.
\]

Thus, this function has a stationary point at \((0.5,0.5)\). Now since the associated Hessian matrix at this point:

\[
Hf = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix} = \begin{pmatrix}
\frac{2}{x^3} & 8 \\
8 & \frac{2}{y^3}
\end{pmatrix}
\]

implies

\[
Hf \begin{pmatrix}
1 \\
2
\end{pmatrix} = \begin{pmatrix}
\frac{2}{0.5^3} & 8 \\
8 & \frac{2}{0.5^3}
\end{pmatrix} = \begin{pmatrix}
16 & 8 \\
8 & 16
\end{pmatrix}
\]

is positive definite we conclude that \((0.5,0.5)\) is a local minimum. Thus, the extreme (minimum) value of the function will be:

\[
z_0 = f(0.5,0.5) = 8 \cdot 0.5 \cdot 0.5 + \frac{1}{0.5} + \frac{1}{0.5} = 6.
\]
2.3.4.1. Application of Extreme Points in Econometrics

Consider now this example of application of the “linear regression” method commonly used in Econometrics:

Example: It is known that a certain phenomenon under study follows a linear law of type

\[ Y = f(X) = a \cdot X + b, \]

where \( X \) and \( Y \) represent accurate economic indicators and \( a, b > 0 \) are parameters. On the other hand suppose we have just experimentally obtained four values of \( Y \) corresponding to four values of \( X \) according to the table:

<table>
<thead>
<tr>
<th>( X_i )</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y_i )</td>
<td>3.1</td>
<td>3.4</td>
<td>4.2</td>
<td>4.5</td>
</tr>
</tbody>
</table>

Under these conditions find the values of parameters \( a, b > 0 \) such that the above linear law describes as best as possible the phenomenon.

SOLUTION: The linear regression method supposes to minimize the “loss” function:

\[
L(a, b) = \sum_{i=1}^{4} (Y_i - f(X_i))^2 = \sum_{i=1}^{4} (Y_i - (a \cdot X_i + b))^2
\]

\[
= (3.1 - f(1))^2 + (3.2 - f(2))^2 + (4.2 - f(4))^2 + (4.5 - f(6))^2
\]

\[
= (3.1 - (a \cdot 1 + b))^2 + (3.2 - (a \cdot 2 + b))^2 + (4.2 - (a \cdot 4 + b))^2 + (4.5 - (a \cdot 6 + b))^2
\]

\[
= 57a^2 + 4b^2 + 26ab - 107.4a - 30.4b + 59.06
\]

Since:

\[
0 = -\frac{\partial L}{\partial a} = 114a + 26b - 107.4
\]

\[
0 = -\frac{\partial L}{\partial b} = 8b + 26a - 30.4
\]

\[
\begin{align*}
0 = 114a + 26b - 107.4 &\quad\rightarrow\quad a = \frac{86}{295} = 0.291 \\
0 = 8b + 26a - 30.4 &\quad\rightarrow\quad b = \frac{1683}{590} = 2.852
\end{align*}
\]

and the general Hessian matrix:

\[
HL = \begin{pmatrix}
114 & 26 \\
26 & 8
\end{pmatrix}
\]

is positive definite we conclude that the function:

\[ Y = 0.291 \cdot X + 2.852 \]

is the best choice in order to linearly describe the phenomenon.
2.3.5. *Constrained Optimization: the Direct-Case Method*

This method of optimization may be applied when a system of linear equations with the independent variables as unknowns acts as a “constrain” on the function. Basically what is required here is to introduce the solution of this system into the function and then optimizing the resulting “auxiliary” function. Let’s see an example:

**Example: Find the extreme points of** \( f(x,y,z) = 2xy - 3xz + yz \) **constrained to the equation:**

\[ 3x - 2y + z = 1. \]

**SOLUTION:** Since the “solution” in terms of the variable \( z \) of this linear equation is:

\[ z = 1 - 3x + 2y \]

the direct-case method basically means to introduce this dependence into the function and optimizing the new function obtained:

\[ g(x,y) = f(x,y,1-3x+2y) = 9x^2 + 2y^2 - 7xy - 3x + y. \]

Since:

\[
\begin{align*}
0 &= \frac{\partial g}{\partial x} = 18x - 7y - 3 \\
0 &= \frac{\partial g}{\partial y} = 4y - 7x + 1 \\
\end{align*}
\]

and the Hessian matrix at this stationary point:

\[
Hg = \begin{pmatrix}
18 & -7 \\
-7 & 4
\end{pmatrix}
\]

implies

\[
Hg \left( \frac{5}{23}, \frac{12}{92} \right) = \begin{pmatrix}
18 & -7 \\
-7 & 4
\end{pmatrix}
\]

is positive definite, point \( \left( \frac{5}{23}, \frac{12}{92} \right) \) is a minimum of \( g(x,y) \). Due to the fact that:

\[
z = 1 - 3x + 2y \left|_{x=5/23 \text{ and } y=12/92} \right. \to z = \frac{14}{23}
\]

the direct-case method enables us to affirm that point:

\[
(x,y,z) = \left( \frac{5}{23}, \frac{12}{92}, \frac{14}{23} \right)
\]

is a local minimum of \( f(x,y,z) \) constrained to \( 3x - 2y + z = 1 \).
2.3.5.1. Application in Economics: Marshall’s Model of Consumption

Let’s see now an interesting application of constrained optimization in terms of economics:

Example: Let:

\[ U(x,y) = 0.05xy \]

be the consumption utility associated with the purchase of \( x \) units of commodity A and \( y \) units of commodity B. If €5 and €8 are their corresponding unit sale prices determine the amount of A and B that maximize the utility of spending €400 in both products.

SOLUTION:

Since the purchase “constraint” is the linear equation:

\[ 5x + 8y = 400 \]

we can solve the problem:

\[
\begin{align*}
\text{maximize} & \quad U(x,y) = 0.05xy \\
\text{subject to} & \quad 5x + 8y = 400 \\
& \quad x, y \geq 0
\end{align*}
\]

applying the direct-case method. Since:

\[ y = 50 - 0.625x \quad \implies \quad u = u(x) = U(x, 50 - 0.625x) = 2.5x - 0.03125x^2. \]

Since:

\[ \frac{du}{dx} = 2.5 - 0.0625x \quad \implies \quad x = 40 \quad \text{and} \quad \frac{d^2u}{du^2} = -0.0625 < 0 \]

we deduce that \( x = 40 \) maximizes \( u(x) \). Due to the fact that:

\[ y = 50 - 0.625x \quad \implies \quad y = 25 \]

we can affirm that 40 units of A and 25 units of B maximize the utility of the consumption of €400 being the optimal utility:

\[ U(40,25) = u(40) = 50. \]

55 This type of problems in Economics are called programs.
2.4. Exercises

1. Determine both the domain and the level curves of \( f(x,y) = \sqrt{x^2 + y} \).

2. Prove that \( z = 0.5 \ln (x^2 + xy - y^2) \) satisfies the equality: \( x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 1 \).

3. Calculate the Hessian matrix of \( f(x,y,z) = x^2y - yz^2 \) at point \((1,1,1)\).

4. Study the existence of extreme points of:
   i. \( f(x,y) = x^2 + y^2 - 6xy - 39x + 18y + 20 \).
   ii. \( f(x,y) = 2x + 3y - x^2 - 2y^2 + xy \).
   iii. \( f(x,y) = x^2 + y^2 - 2\ln x - 18\ln y \).
   iv. \( f(x,y) = xy - x^3 - y^3 \).
   v. \( f(x,y,z) = x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 12yz \).

5. An employer who manufactures two products A and B knows that, at a price of €25, he can sell 175 units of A and that for every euro he lowers this price the sales increase by 5 units. If the price of B is always €30 and the function:
   \[ C = \frac{x^2}{5} + \frac{y^2}{3} + 1,925 \]
   is the cost function, where \( x \) and \( y \) represent the output of A and B, determine how many units of these products he should manufacture in order to maximize benefits.

6. A company manufactures fabrics of 103 meters long and 1 meter wide that are dyed green, yellow and red. If the cost of production is:
   \[ C(x,y,z) = 0.03x^2 + 0.05y^2 + 0.11z^2 + 40.05 \]
   where \( x, y \) and \( z \) are the respective number of square meters of each color depicted, determine the amount of green, yellow and red of each fabric if this company wants to minimize the cost of production.\(^{56}\)

---

\(^{56}\) This exercise needs to be solved using the direct-case method.
SOLUTIONS:

1.

\[ H_f(1,1,1) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & -2 \end{pmatrix}. \]

3.

4.

i. \( \left( \frac{15}{16}, -\frac{99}{16} \right) \) is a saddle point.

ii. \( \left( \frac{11}{7}, \frac{8}{7} \right) \) is a local maximum.

iii. \((1,3)\) is a local minimum.

iv. \((0,0)\) is a saddle point and \( \left( \frac{1}{3}, \frac{1}{3} \right) \) is a local maximum.

v. \((0,0,0)\) is a saddle point.

5. 75 units of A and 45 units of B. The maximum value of benefits is €1,000.

6. 55 m² green, 33 m² yellow and 42 m² red. The minimum cost is €210.
This section is basically related with differential equations in which integrals play a very important role. As illustrative case consider the radioactive dating-method using the 14-carbon isotope $^{14}\text{C}$. We need to know that this method is based on the fact that the ratio of $^{14}\text{C}$ in every living organism remains constant throughout its life and, as it is no replaced, it decreases after death. From an experimentally point of view we can assume that at a moment $t > 0$ after the death $(t = 0)$, the rate of decay of $^{14}\text{C}$ is proportional to the amount $^{14}\text{C}(t)$ present at $t > 0$:

$$\frac{d^{14}\text{C}(t)}{dt} = -1.24486 \cdot 10^{-4} \cdot ^{14}\text{C}(t).$$

Observe that in this equation the rate of decay at $t > 0$ is estimated by the derivative of the function $^{14}\text{C}(t)$. So this equation is a special case of differential equation in which the unknown variable is precisely $^{14}\text{C}(t)$. Under these assumptions we want to know how much 14-carbon isotope $^{14}\text{C}$ is present into an organism at a moment $t > 0$ after its death, i.e. the formal expression of $^{14}\text{C}(t)$. Taking integrals on this equation we can solve it obtaining as a general solution:

$$^{14}\text{C}(t) = ^{14}\text{C}_0 \cdot e^{-1.24486 \cdot 10^{-4} \cdot t}$$

in which $^{14}\text{C}_0$ is the amount of $^{14}\text{C}$ at the moment of death ($^{14}\text{C}_0 = ^{14}\text{C}(0)$). Graphically:

This section is devoted to study how to calculate integrals, taking a look at some of their applications in calculating areas and Economics, and applying them to solve some of the most basic first-order differential equations.

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57 So this rate acts like a “speed” of decay of the 14-carbon isotope.

58 This differential equation will be solved later in class.
3.1. Indefinite Integrals

3.1.1. Primitive of a Function of One Real Variable

From an “informal” point of view we can say that integrating a function of a real variable is the opposite of calculating its derivative.\(^{59}\) Hence by definition:

**Definition:** A primitive of a function of one real variable \(A \subset \mathbb{R} \rightarrow \mathbb{R}\) is another function \(F(x)\) such that its derivative is precisely \(f(x)\):

\[
F'(x) = f(x).
\]

**Example:** Prove that:

\[
F(x) = \frac{x^3}{3} + 1
\]

is a primitive of \(f(x) = x^2\).

**SOLUTION:** Evidently since \(F(x) = \frac{x^3}{3} + 1\) \(\implies\) \(F'(x) = x^2 = f(x)\).

We should emphasize that a primitive of a function is not unique. Indeed:

**Property:** If \(F_1(x)\) and \(F_2(x)\) are two primitives of a same function, a constant \(C \in \mathbb{R}\) always exists such that:

\[
F_2(x) = F_1(x) + C.
\]

Note that in the above case all the primitives of \(f(x) = x^2\) would be the functions:

\[
F(x) = \frac{x^3}{3} + C, \text{ with } C \in \mathbb{R}.
\]

\(^{59}\) That is why the result of the process of calculating the integral of a function is sometimes called an antiderivative.
3.1.2. Indefinite Integral of a Function and Properties

The last property allows us to define the concept of the indefinite integral:

**Definition:** The indefinite integral (integral for short) of \( A \subset \mathbb{R} \rightarrow \mathbb{R} \) is, \(^{60}\)

\[
\int f(x) \, dx = F(x) + C
\]

in which \( F(x) \) is a primitive of function \( f(x) \) and \( C \in \mathbb{R} \) is a constant.

**Example:** Find the integral of \( f(x) = \frac{1}{x} \) and the associated primitive passing through \((1,2)\).

**SOLUTION:**

Since \( f(x) = \ln x \) is a primitive of \( f(x) \) we have as indefinite integral:

\[
\int f(x) \, dx = \int \frac{1}{x} \, dx = \ln x + C.
\]

Now since:

\[
y = \ln x + C \quad \overset{(x,y)=(1,2)}{\rightarrow} \quad 2 = \ln 1 + C = C
\]

the primitive passing through \((1,2)\) is \( y = \ln x + 2 \). Graphically:

![Graph of indefinite integral](image)

So from a geometrical point of view any indefinite integral represents a family of curves in the plane. Each of these curves is precisely a primitive.

---

\(^{60}\) This symbolism is related to the concept of the definite integral which we will study later.
3.1.2.1. Immediate Integrals

In order to calculate indefinite integrals it is essential to take into account the following list of *immediate integrals*, i.e. those that can be directly resolved from the proper definition. Indeed:

a. \[ \int a \cdot dx = ax + C, \text{ for any constant } a \in \mathbb{R} \]
b. \[ \int x^a dx = \frac{x^{a+1}}{a+1} + C, \text{ where } a \neq -1 \]
c. \[ \int x^{-1} dx = \int \frac{1}{x} dx = \ln x + C \]
d. \[ \int a^x dx = \frac{a^x}{\ln a} + C, \text{ where } a > 0 \]
e. \[ \int \sin x dx = -\cos x + C \]
f. \[ \int \cos x dx = \sin x + C \]
g. \[ \int \frac{1}{\cos^2 x} dx = \int (1 + \tan^2 x) dx = \tan x + C \]
h. \[ \int \frac{1}{1 + x^2} dx = \arctan x + C \]

Unfortunately the process of integration does not support anything similar to the chain rule of the corresponding process of differentiation and this means that not all functions have an indefinite integral in the above sense. For instance, the following two integrals:

\[ \int \left( \frac{\sin x}{x} \right) dx \text{ and } \int e^{-x^2} dx \]

cannot be properly expressed in terms of suitable functions. Consequently we have to integrate each different type of integral using an appropriate method; in fact the main purpose here is to study the easiest cases.\(^{61}\)

---

\(^{61}\) We have chosen two specific cases: logarithmic integration and integration by substitution.
3.1.2.2. Properties of Indefinite Integrals: Logarithm Integration

To solve indefinite integrals we have first to consider the following properties along with the immediate integrals:

Properties:

a. \[ \int f'(x) \, dx = f(x) + C. \]

b. \[ \int (f(x) + g(x)) \, dx = \int f(x) \, dx + \int g(x) \, dx. \]

c. \[ \int \lambda \cdot f(x) \, dx = \lambda \cdot \int f(x) \, dx, \text{ for any constant } \lambda \in \mathbb{R}. \]

d. Logarithm integration: \[ \int \frac{f'(x)}{f(x)} \, dx = \ln f(x) + C. \]

Example: Calculate:

i. \[ \int (x^4 - 7x^3 + 5) \, dx \]

ii. \[ \int \left( \frac{\sqrt{x} - 2}{x^2} \right) \, dx \]

iii. \[ \int \frac{x \, dx}{1 + x^2} \]

iv. \[ \int \tan x \, dx \]

SOLUTION:

i) Applying the above properties we have:

\[ \int (x^4 - 7x^3 + 5) \, dx = \int x^4 \, dx - 7 \int x^3 \, dx + 5 \int dx = \frac{x^5}{5} - 7 \left( \frac{x^4}{4} \right) + 5x + C. \]

ii) Now applying the list of immediate integrals:

\[ \int \left( \frac{\sqrt{x} - 2}{x^2} \right) \, dx = \int \left( \frac{1}{x^{1/2}} - 2x^{-2} \right) \, dx = \frac{x^{1/2}}{1/2 + 1} - 2 \frac{x^{-2+1}}{-2+1} + C = \frac{2x\sqrt{x}}{3} + \frac{2}{x} + C. \]

iii) Thanks to logarithm integration we deduce that:

\[ \int \frac{x \, dx}{1 + x^2} = \frac{1}{2} \int \frac{2x \, dx}{1 + x^2} = \left\{ (1 + x^2) = 2x \right\} = \frac{\ln(1 + x^2)}{2} + C \]

iv) In the same way as before we have now:

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{-\sin x}{\cos x} \, dx = \left\{ (\cos x)' = -\sin x \right\} = -\ln \cos x + C. \]

---

62 Roughly speaking this property can be seen as a specific integration method.
3.1.2.3. Integration by Substitution: Differential of a Function

Remember that the differential of a differentiable function $A \subset \mathbb{R} \rightarrow \mathbb{R}$ is nothing but the expression:

$$df(x) = f'(x) \cdot dx \overset{\text{Equivalent}}{=} \frac{df(x)}{dx} = f'(x).$$

The method of integration by substitution is based on the so-called “change of variable”. In simple words introducing a change of variable means substituting the variable $x$ with a function $\phi(t)$ depending on a new variable $t$:\footnote{Formally speaking this new function must be bijective and differentiable.}

$$x = \phi(t) \overset{\text{Differential}}{=} dx = \phi'(t)dt \overset{64}{\text{.}}$$

The following example shows us the basic guidelines of how this method works:

**Example:** Calculate:

$$\int (x\sqrt{x-1})dx$$

*using the change of variable* $x-1 = t$.

**SOLUTION:**

In this case the function $\phi(t)$ would be:

$$x = \phi(t) = t + 1 \quad dx = \phi'(t)dt = (t + 1)'dt = 1\cdot dt = dt$$

Thus:

$$\int x\sqrt{x-1} \cdot dx = \int x = 1 \cdot (t + 1)\sqrt{t}dt = \int (t + 1)^{1/2}dt = \int (t^{3/2} + t^{1/2})dt =$$

$$= \frac{t^{3/2}}{3/2 + 1} + \frac{t^{1/2}}{1/2 + 1} + C = \{t = x - 1\} = \frac{2(x - 1)^{5/2}}{5} + \frac{2(x - 1)^{3/2}}{3} + C.$$
Let’s find the following integrals applying the substitution method:

**Example:** Calculate the following integrals:

i. \( \int (1-x)^7 \, dx \)  
ii. \( \int \frac{dx}{(x-3)^2} \)  
iii. \( \int \frac{x^3}{x-1} \, dx \)  
iv. \( \int e^{x^2} \cdot x^2 \, dx \)  
v. \( \int \sqrt{1-3x} \cdot dx \)  
vi. \( \int \frac{0.5dx}{\sqrt{x+5}} \)

**SOLUTION:**

i) In this case we can introduce the change \( 1-x = t \):

\[
\int (1-x)^7 \, dx = \left\{ 1-x = t \implies x = \phi(t) = 1-t \right\} = \int t^7 \cdot (-dt) = -\int t^7 \, dt = -\frac{t^8}{8} + C = \{ t = 1-x \} = -\frac{(1-x)^8}{8} + C.
\]

ii) Now introducing the change \( x - 3 = t \) we have:

\[
\int \frac{dx}{(x-3)^2} = \left\{ x - 3 = t \implies x = 3 + t \right\} = \int \frac{dt}{t^2} = \int t^{-2} \, dt = \frac{t^{-1}}{-1} + C = \{ t = x - 3 \} = -\frac{1}{x-3} + C.
\]

iii) Since \( \frac{x^3}{x-1} = (x^2 + x + 1) + \frac{1}{x-1} \) then:

\[
\int \frac{x^3 \, dx}{x-1} = \int \left( x^2 + x + 1 + \frac{1}{x-1} \right) \, dx = \frac{x^3}{3} + \frac{x^2}{2} + x + \ln(x-1) + C.
\]

iv) Applying the change \( x^3 = t \) we have:

\[
\int e^{x^2} \cdot x^2 \, dx = \left\{ x^3 = t \implies 3x^2 \, dx = dt \right\} = \int e^t \cdot (\frac{dt}{3x^2}) = \frac{1}{3} \int e^t \, dt = \frac{1}{3} e^t + C = \frac{e^{x^2}}{3} + C.
\]

v) With the change \( 1-3x = t \) we have:

\[
\int \sqrt{1-3x} \, dx = \left\{ \frac{1-3x}{-3dx} = dt \right\} = -\frac{1}{3} \int t^{\frac{1}{2}} \, dt = -\frac{1}{3} \left( \frac{t^{\frac{3}{2}}}{(1/3)+1} \right) + C = -\frac{(1-3x)^{\frac{3}{2}}}{4} + C.
\]

vi) As in the former case and considering the change \( x + 5 = t \) we deduce:

\[
\int \frac{0.5dx}{\sqrt{x+5}} = \left\{ x + 5 = t \implies dx = dt \right\} = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \int t^{-\frac{1}{2}} \, dt = \frac{1}{2} \left( \frac{t^{\frac{1}{2}+1}}{0.5+1} \right) + C = \sqrt{t} + C = \sqrt{x+5} + C.
\]
3.2. Definite Integrals

3.2.1. Definite Integral of a Function of One Real Variable and Properties

Let \([a, b] \subset \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function with the positive restriction:

\[ f(x) \geq 0 , \text{ for any } a \leq x \leq b . \]

In this case the \textit{definite integral} of \(f(x)\) between the points \(a\) and \(b\) symbolized by:

\[ \int_{a}^{b} f(x) \, dx \]

measures the area \(A\) delimited by the \(x\)-axis, the function \(f(x)\) and the two vertical lines \(x = a\) and \(x = b\). Graphically:

\[ A = \int_{a}^{b} f(x) \, dx . \]

Consequently:

\[ A = \int_{a}^{b} f(x) \, dx . \]

It is worth noting that the positive restriction is not an obstacle when calculating areas, as we will have the chance to see right now.

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3.2.1.1. Properties of Definite Integrals

Properties:

a. \[ \int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \] and \[ \int_{a}^{b} \lambda \cdot f(x) \, dx = \lambda \cdot \int_{a}^{b} f(x) \, dx. \]

b. \[ \int_{a}^{a} f(x) \, dx = 0. \]

c. \[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

d. \[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx, \text{ for any } a \leq c \leq b. \]

e. For all \( a \leq x \leq b, \) \[ \begin{cases} f(x) \leq 0 \\ f(x) \geq 0 \end{cases} \] implies \[ \begin{cases} \int_{a}^{b} f(x) \, dx \leq 0 \\ \int_{a}^{b} f(x) \, dx \geq 0 \end{cases}. \]

From these properties we may deduce that if a continuous function \([a,b] \subseteq \mathbb{R} \xrightarrow{f} \mathbb{R}\) is negative over \([a,b],\) the value of the area \( A \) determined as before by the \( x \)-axis, the function and the lines \( x = a \) and \( x = b \): 

\[ A = \left| \int_{a}^{b} f(x) \, dx \right| = -\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx. \]

\[ \text{If we wish to use the definite integral to work areas out we ought always consider as a matter of fact its absolute value.} \]
3.2.2. Calculating Definite Integrals: Barrow’s Rule

Barrow’s rule allows us to calculate the definite integral of a continuous function provided it has primitives. Indeed:

**Theorem (Barrow’s rule):** Let $F(x)$ be a primitive of $[a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$. In this case we have:

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a) = \left[ F(x) \right]_{a}^{b}.$$ 

**Example:** Calculate the definite integral:

$$\int_{-\pi/2}^{\pi/2} \left( \frac{\cos x}{2 - \sin x} \right) \, dx.$$

**SOLUTION:**

Since the indefinite integral of this function is:

$$\int \left( \frac{\cos x}{2 - \sin x} \right) \, dx = -\int \left( \frac{-\cos x}{2 - \sin x} \right) \, dx = -\ln(2 - \sin x) + C,$$

we deduce from Barrow’s rule that:

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x \, dx}{2 - \sin x} = \left[ -\ln(2 - \sin x) \right]_{-\pi/2}^{\pi/2} = \left[ -\ln \left( 2 - \sin \left( -\frac{\pi}{2} \right) \right) \right] - \left[ -\ln \left( 2 - \sin \left( \frac{\pi}{2} \right) \right) \right] = \ln 3.$$

Due to the fact that this function is positive between $-\pi/2$ and $\pi/2$ this definite integral matches the area $A$ determined by the function as illustrated below:

---

67 Unfortunately Barrow’s rule cannot be applied on a function without primitives.

68 The second equality is a symbolism.
3.2.3. Applications of the Definite Integral

3.2.3.1. Metric Applications: Calculating Areas

a. We want to calculate the area $A$ determined as:

\[
A = A_1 + A_2 = -\int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = -\int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx.
\]

In this case: $A = A_1 + A_2 = -\int_{-1}^{0} (x-1)^3 + 1 \, dx + \int_{0}^{1} (x-1)^3 + 1 \, dx$.

Example: Find the area $A$ determined by the function $f(x) = (x-1)^3 + 1$, the lines $x = -1$, $x = 1$ and the $x$–axis.

SOLUTION: Since:

\[
0 = f(x) = (x-1)^3 + 1 \quad \text{implies} \quad x = 0
\]

we deduce that the Area $A$ is equal to:

\[
A = \int_{-1}^{0} ((x-1)^3 + 1) \, dx + \int_{0}^{1} ((x-1)^3 + 1) \, dx.
\]

Since:

\[
\int((x-1)^3 + 1) \, dx = \frac{1}{4}(x-1)^4 + x + C
\]

we conclude that:

\[
A = \left[ \frac{(x-1)^4}{4} + x \right]_{-1}^{0} + \left[ \frac{(x-1)^4}{4} + x \right]_{0}^{1} = \frac{11}{4} + \frac{3}{4} = \frac{14}{4} = 3.5.
\]

As a matter of precaution we will deal with the absolute value of these definite integrals.

\[69\]
b. We want to calculate now areas $A$ enclosed between two functions:

In this case we have: $A = \left| \int_{a}^{b} (f(x) - g(x)) \, dx \right|$.\(^{70}\)

**Example:** Calculate the area $A$ enclosed between the parabola $y = x^2 + 1$ and the straight line $x + y = 3$.

**SOLUTION:**

These two functions cut off at the points $(-2, 5)$ and $(1, 2)$. Indeed:

$$x^2 + 1 = y = 3 - x \quad \text{implies} \quad x^2 + x - 2 = 0 \quad \text{Solution} \quad x = \begin{cases} -2 \\ 1 \end{cases}$$

Therefore the area $A$ required will be equal to:

$$A = \left| \int_{-2}^{1} ((x^2 + 1) - (3 - x)) \, dx \right|.$$ 

Thus:

$$A = \left| \int_{-2}^{1} (x^2 + x - 2) \, dx \right| = \left| \left[ \frac{x^3}{3} + \frac{x^2}{2} - 2x \right]_{-2}^{1} \right| = \left| \left( \frac{1^3}{3} + \frac{1^2}{2} - 2 \cdot 1 \right) - \left( \frac{(-2)^3}{3} + \frac{(-2)^2}{2} - 2(-2) \right) \right| = \left| \frac{9}{2} - \frac{9}{2} \right| = \frac{9}{2} \div \frac{9}{2}.$$

\(^{70}\) We will always consider the absolute value of the definite integral of the difference between the two functions.
c. We want to calculate the area $A$ determined by two consecutive functions:

In this case the area will be: $A = A_1 + A_2 = \int_c^d f(x)\,dx + \int_c^d g(x)\,dx$.

**Example:** Calculate the area $A$ determined by the x-axis and the function:

$$f(x) = \begin{cases} 
1 + x, & x < 0 \\
1 - x, & x \leq 0 \\
\cos x, & 0 \leq x 
\end{cases}$$

between points $-1$ and $\frac{\pi}{2}$.

**SOLUTION:** Since this function is positive between $-1$ and $\frac{\pi}{2}$ and changes its definition at point 0 the area $A$ will be:

$$A = \int_{-1}^{\frac{\pi}{2}} f(x)\,dx = \int_{-1}^{0} \left(\frac{1+x}{1-x}\right)\,dx + \int_{0}^{\frac{\pi}{2}} \cos x\,dx.$$

Bearing in mind that:

$$\int \left(\frac{1+x}{1-x}\right)\,dx = \int \left(-1 + \frac{2}{1-x}\right)\,dx = -x - 2\ln(1-x) + C$$

and

$$\int \cos x\,dx = \sin x + C$$

we conclude that:

$$A = \int_{-1}^{0} \left(\frac{1+x}{1-x}\right)\,dx + \int_{0}^{\frac{\pi}{2}} \cos x\,dx = \left[-x - 2\ln(1-x)\right]_{-1}^{0} + \left[\sin x\right]_{0}^{\frac{\pi}{2}} =$$

$$= \left(-0 - 2\ln(1-0) - (-(-1) - 2\ln(1-(-1)))\right) + \left(\sin \frac{\pi}{2} - \sin 0\right) =$$

$$= (-1 + 2\ln2) + (1 - 0) = 2\ln2.$$

71 Note that this function is strictly positive between points $-1$ and $\frac{\pi}{2}$.
3.2.3.2. Application in Economics: Marginalism

The first application concerns with *marginal* economic functions such as the marginal cost, the marginal profit, etc. Consider this example:

**Example:** Consider that the marginal cost of the production of an article $A$ of consumption is given by the function:

$$CMa(q) = 2q - 10$$

in which the variable $q$ stands for the number of output units. If the unit sales price of $A$ is €520 and that all the output is sold:

i. **Determine the function of cost of $A$ if the total profit of 10 units amounts to €5100.**

ii. **Calculate the output of $A$ that maximizes the profit.**

**SOLUTION:**

i) Recalling that marginal cost $CMa(q)$ is the derivative of the function of total cost $C(q)$ we have:

$$\frac{dC(q)}{dq} = CMa(q) \implies C(q) = \int CMa(q) dq = \int (2q - 10) dq = q^2 - 10q + C.$$

Since the income function is:

$$I(q) = 520 \cdot q$$

the profit function will be:

$$B(q) = I(q) - C(q) = 520 \cdot q - (q^2 - 10q + C) = -q^2 + 530q - C.$$

Taking into account that $B(10) = €5100$ we deduce that:

$$5100 = B(10) = -10^2 + 530 \cdot 10 - C \implies C = €100.$$

Thus the function of total cost is:

$$C(q) = q^2 - 10q + 100.$$

ii) Bearing in mind that:

$$0 = \frac{dB}{dq} = -2q + 530 \implies q = 265 \quad \text{and} \quad \frac{d^2B}{dq^2} = -2 < 0$$

the maximum benefit is obtained by producing and selling 265 units of $A$ with a maximum value of benefits $B(265) = €70,125$. 
3.2.3.3. Application in Economics: the Consumer and Producer Surplus

Another interesting application involves finding both the consumer and the producer surplus of a good.

Example: If the demand \( D \) and supply \( S \) of a product \( P \) are given by:

\[
p_S = 2q + 1 \quad \text{and} \quad p_D = 25 - q^2
\]

i. Determine the equilibrium price of \( P \) as well as the quantity demanded at that price.\(^{72}\)

ii. Find the consumer and the producer surplus of \( P \) associated with this equilibrium price.

SOLUTION:

i) We know that equilibrium prices are given by the equality \( p_O = p_S \). Thus this equilibrium quantity \( q_0 \) demanded of \( P \) will be:

\[
p_D = p_S \implies q^2 + 2q - 24 = 0 \quad \text{Solution} \quad q_0 = 4 \text{ units.}^{73}
\]

Hence the equilibrium price of \( P \) is:

\[
p_0 = p_D(4) = p_S(4) = \varepsilon 9.
\]

ii) In general the consumer surplus \( \varepsilon_D \) and the producer surplus \( \varepsilon_S \) of \( P \) associated with both the equilibrium price \( p_0 \) and the equilibrium quantity \( q_0 \) are given by the equalities:

\[
\varepsilon_D = \left( \int_0^{q_0} p_D dq \right) - p_0q_0 \quad \text{and} \quad \varepsilon_S = p_0q_0 - \int_0^{q_0} p_S dq.
\]

Thus in our case we have:

\[
\varepsilon_D = \left( \int_0^{q_0} p_D dq \right) - p_0q_0 = 4 \left( 25 - q^2 \right) dq - 9 \cdot 4 = \left( 25q - \frac{q^3}{3} \right)_0^4 - 36 = \varepsilon 42.6
\]

and:

\[
\varepsilon_S = p_0q_0 - \int_0^{q_0} p_S dq = 9 \cdot 4 - \int_0^{q_0} (2q + 1) dq = 36 - \left( q^2 + q \right)_0^4 = \varepsilon 16.
\]

\(^{72}\) Equilibrium price appears when demand and supply matches. So the quantity demanded at that price is called the “equilibrium quantity”.

\(^{73}\) Obviously we do not consider the negative solution of this equation.
3.2.3.4. Application in Economics: Accumulating a Variable

The third of these applications deals with the stock levels of a good and it specifically consists in calculating the time this product takes to run out under an increasing demand.

Example: It is known that the demand for a rare mineral is of the form:

\[ D(t) = 2 \cdot 10^6 e^{0.04t} \text{ tonnes} \]

where \( t \geq 0 \) denotes the passing years since the start of this demand. If the reserves at that time amounted to 20,000 million tonnes estimate how long it will take for the mineral to run out.

SOLUTION:

Assuming that the global demand for this mineral is continuous in time,\(^7^4\) the “total amount” \( Q(t_0) \) demanded from the initial instant \( t = 0 \) until year \( t_0 > 0 \) will be given by the definite integral:

\[ Q(t_0) = \int_0^{t_0} D(t) \, dt. \]

Taking into account that the initial reserves amounted to 20,000 millions tonnes we need to calculate the value of \( t_0 > 0 \) in which the total amount is exactly:

\[ Q(t_0) = 20,000 \cdot 10^6. \]

Therefore:

\[ 20,000 \cdot 10^6 = Q(t_0) = \int_0^{t_0} D(t) \, dt = \int_0^{t_0} 2 \cdot 10^6 \cdot e^{0.04t} \, dt = 2 \cdot 10^6 \left( \frac{e^{0.04t_0}}{0.04} \right)_0^{t_0} = 2 \cdot 10^6 \left( e^{0.04t_0} - 1 \right). \]

Consequently:

\[ 10,000 = \frac{e^{0.04t_0} - 1}{0.04} \implies t_0 = \frac{\ln 401}{0.04} = 149.85. \]

Thus the mineral will take almost 150 years to run out.

\(^7^4\) This assumption is required in order to solve this situation by definite integrals.

67
3.3. First-Order Differential Equations

Given that this section is intended to serve as just an introduction to this topic, we will only study the first-order differential equations and among these the simplest. We begin by giving the following definition:

**Definition:** A first-order differential equation is an equation that depends on the variable \( x \), the variable dependent \( y = f(x) \) (being \( f(x) \) an unknown differentiable function) and the derivative of \( y \). Any first-order differential equation can be expressed:

1. In implicit form: \( F(x, y, y') = 0 \).
2. In explicit form: \( y' = f(x, y) \).
3. In continuous form: \( p(x, y)\cdot dx + q(x, y)\cdot dy = 0 \), where \( y' = \frac{dy}{dx} \).

**Example:** Determine the three forms of the following first-order differential equations:

i. \( \ln y' + 2y = 0 \).

ii. \( y' = y + x + 1 \).

iii. \( x \cdot dx + y \cdot dy = 0 \).

**SOLUTION:**

i) In this case we have a differential equation expressed in implicit form. The explicit and continuous forms would be:

\[
y' = e^{2y} \quad \text{and} \quad dx - e^{2y}dy = 0.
\]

ii) This differential equation is expressed in explicit form since the derivative \( y' \) is isolated. The implicit and continuous forms will be:

\[
y' - y - x - 1 = 0 \quad \text{and} \quad dx - \left( \frac{1}{y + x + 1} \right)dy = 0.
\]

iii) Now this differential equation is in continuous form. The implicit and explicit associated forms would be:

\[
y \cdot y' + x = 0 \quad \text{and} \quad y' = -\frac{x}{y}.
\]
3.3.1. Solution of a First-Order Differential Equation

The concept of the solution of a differential equation is crucial. In fact the aim here is to find the solutions of the most basic first-order differential equations and integrals will be essential.

**Definition:** A solution of a first-order differential equation is any differentiable function that satisfies it. There are basically two types of solutions:

1. *General solution:* A solution depending on one real parameter.
2. *Particular solution:* Any solution obtained from the general solution when numerical values are given to its parameter.

**Example:** Prove that function \( y = C \cdot e^{-x} + x - 1 \), where \( C \in \mathbb{R} \), is the general solution of the equation \( y' + y = x \). Is the line \( y - x + 1 = 0 \) a particular solution? Reason the answer.

**SOLUTION:** Since we have a first-order differential equation depending on one parameter, we only need to check that this function satisfies the differential equation:

\[
y = C \cdot e^{-x} + x - 1 \quad \text{implies} \quad y' + y = (-C \cdot e^{-x} + 1) + (C \cdot e^{-x} + x - 1) = x.
\]

Note that if we put \( C = 0 \) in the general solution we can conclude that the line \( y = x - 1 \) is one of their particular solutions. The integral curves are graphically:

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75 Geometrically speaking these solutions are also called integral curves. As we are going to see the general solution is the family formed by their integral curves.
3.3.2. Integrating First-Order Differential Equations

3.3.2.1. Separated Differential Equations

This type of differential equations represents the easiest case.

**Definition:** Separated differential equations are first-order differential equations that take the continuous form:

\[ p(x) \cdot dx + q(y) \cdot dy = 0 \]

where \( p(x) \) and \( q(y) \) are functions of one real variable.

Separated differential equations can be solved directly as long as both functions \( p(x) \) and \( q(y) \) accept indefinite integral. Let’s look at the following example:

**Example:** Find the general solution of \( xdx + ydy = 0 \), as well as the particular solution (i.e. the integral curve) passing through the point \((1,1)\).

**SOLUTION:**

As we can see we have a separated differential equation with functions:

\( p(x) = x \) and \( q(y) = y \).

Taking integrals in the continuous form of this differential equation we have:

\[ xdx + ydy = 0 \implies \int xdx + \int ydy = C \implies \frac{x^2}{2} + \frac{y^2}{2} = C. \]

Thus the general solution of this differential equation is the family of the circumferences centered at the point \((0,0)\). So the integral curve passing through point \((1,1)\) will be the circumference centered at \((0,0)\) with radius \( r = \sqrt{2} \):

\[ (x,y) = (1,1) \implies C = \frac{y^2}{2} + \frac{x^2}{2} = \frac{1^2}{2} + \frac{1^2}{2} = 1 \text{ integral curve } x^2 + y^2 = 2. \]

\[ ^{76} \text{The parameter } C \text{ is the primitive of } 0 \text{ and should be considered a “dummy” variable, namely a variable that takes any real value; in this case all the positive real values.} \]
3.3.2.2. Separable Differential Equations

**Definition:** Separable differential equations are those first-order differential equations that take the continuous form:

\[ p_1(x) \cdot p_2(y) \cdot dx + q_1(x) \cdot q_2(y) \cdot dy = 0 \]

where \( p_1(x), p_2(y), q_1(x) \) and \( q_2(y) \) are functions of one real variable.

Observe that these equations can be transformed into separated differential equations if we divide the whole equation by the product \( q_1(x) \cdot p_2(y) \). Indeed:

\[
\frac{p_1(x)p_2(y)dx + q_1(x)q_2(y)dy}{q_1(x)p_2(y)} = \frac{p_1(x)}{q_1(x)}dx + \frac{q_2(y)}{p_2(y)}dy = 0.
\]

Consider the following example:

**Example:** Determine the general solution of \( ydx + xdy = 0 \) as well as the associated integral curves.

**SOLUTION:**

We have a separable differential equation where:

\[ p_1(x) = 1, \quad p_2(y) = y, \quad q_1(x) = x \quad \text{and} \quad q_2(y) = 1. \]

If we divide by \( q_1(x) \cdot p_2(y) = xy \) we obtain the separated differential equation:

\[
\left(\frac{1}{x}\right)dx + \left(\frac{1}{y}\right)dy = 0.
\]

Hence:

\[
\int \frac{dx}{x} + \int \frac{dy}{y} = C \implies \ln x + \ln y = C \implies xy = e^C \implies xy = C. \quad \text{77}
\]

Note that in this case the associated integral curves (particular solutions) are all the hyperbolas with the two coordinated axes as asymptotes.

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77 In this context we can substitute \( e^C \) by \( C \) since \( C \) is a dummy variable as pointed out in the previous note.
3.3.2.3. First-Order Linear Differential Equations

This is the latest type of first-order differential equations that we will deal.

**Definition:** First-order linear differential equations take the form:

\[ y' + p(x) \cdot y = q(x) \]

where coefficients \( p(x) \) and \( q(x) \) are functions of one real variable.\(^78\) Moreover:

1. If \( q(x) = 0 \), \( y' + p(x) \cdot y = 0 \) is the reduced first-order linear differential equation.\(^79\)
2. If \( p(x) = p \) and \( q(x) = q \) are constants, \( y' + p \cdot y = q \) is the first-order linear differential equation with constant coefficients.

It is worth noting that the last two cases are of separated differential equations.\(^80\) Let’s look at the following examples:

**Example:** Find the general solution of (i) \( y' - 3y = 7 \), (ii) \( y' - \frac{2y}{x} = 0 \) and (iii) \( x^2 \cdot y' - y = 5 \).

**SOLUTION:** i) The general solution is:

\[
\int \frac{dy}{3y + 7} = \int dx \implies \frac{1}{3} \ln(3y + 7) = x + C \implies y = \frac{e^{3(x+C)} - 7}{3}.
\]

ii) In this case of reduced linear differential equation we have:

\[
\frac{dy}{dx} = \frac{2y}{x} \implies \ln y = \int \frac{dy}{y} = 2 \int \frac{dx}{x} = 2 \ln x + C \implies y = e^{2\ln x} \cdot e^C = C \cdot x^2.
\]

iii) In this case we have the separable differential equation:

\[
x^2 \cdot y' - y = 5 \implies \frac{1}{x^2} \cdot y' = \frac{5}{x^2} \implies y' = \frac{1}{x^2} (y + 5) \implies dy = \frac{1}{x^2} (y + 5) dx
\]

with general solution:

\[
\ln(y + 5) = \int \frac{dy}{y + 5} = \int \frac{dx}{x^2} = -\frac{1}{x} + C \implies y = e^{-\left(\frac{1}{x}\right)} - 5.
\]

\(^78\) This differentiable equation is called linear since no function acts upon either \( y' \) or \( y \).

\(^79\) Thus the first-order linear differentiable equation with \( q(x) \neq 0 \) is called complete.

\(^80\) Note that the same situation holds when coefficients \( p(x) \) and \( q(x) \) are proportional. The third case of the following exemple illustrates this.
If we want to solve the complete first-order linear differential equation case next example gives us the guidelines to be followed:

**Example:** Find the general solution of the differential equation:

\[ y' - 2 \left( \frac{y}{x} \right) = x. \]

**SOLUTION:** First of all observe that this first-order linear differential equation cannot be reduced to a separable differential equation. To solve the complete case we will consider the change of variable given by:

\[ y = u \cdot v \]

where:

- \( u = u(x) \) is a particular solution of the associated reduced differential equation.
- \( v = v(x) \) is another function of real variable to determine.

Since we just have seen that the general solution of the reduced case is:

\[ y = e^c \cdot x^2, \text{ alternatively } y = C \cdot x^2 \]

we may choose as a function \( u = u(x) \):

\[ u = x^2 \]

as a particular solution of the reduced case. \(^81\) Hence we have:

\[ u = x^2 \implies y = u \cdot v = x^2 \cdot v \]

and introducing \( y = x^2 \cdot v \) in the complete case we deduce:

\[
\begin{align*}
  y = x^2 \cdot v \implies x &= y' - 2 \left( \frac{y}{x} \right) = (x^2 \cdot v)' - 2 \left( \frac{x^2 \cdot v}{x} \right) = \\
  &= (2x \cdot v + x^2 \cdot v') - 2x \cdot v = x^2 \cdot v' \\
\end{align*}
\]

Therefore:

\[
  v = \int v' \cdot dx = \int \frac{dx}{x} = \ln x + C
\]

and the general solution of the complete first-order differential equation will be:

\[ y = u \cdot v = x^2 \cdot (\ln x + C). \]

\(^{81}\) We have put either \( C = 0 \) in \( y = e^c \cdot x^2 \) or \( C = 1 \) in \( y = C \cdot x^2 \).
3.3.3. Applications in Economics

3.3.3.1. Instant Compound Interest

The following mathematical model formalizes a financial scenario in which monetary interest becomes capital in every instant of time.

Example: Suppose that the increase of a certain amount of money in each arbitrary time interval \([t, t + \Delta t]\) is proportional to both the amount \(C(t)\) and the duration \(\Delta t\) in years of that interval, i.e. \(\Delta C(t) = i \cdot C(t) \cdot \Delta t\), where \(\Delta C(t) = C(t + \Delta t) - C(t)\) and \(i > 0\) is the rate of instantly compound interest. Prove that the capital \(C(t)\) increases exponentially with the passing of time. If \(i = 3\%\), how long must we wait for any initial capital \(C(0)\) to double?

SOLUTION: If we suppose \(C(t)\) to be differentiable we have the first-order differential equation with respect to the temporary variable \(t\):

\[
\frac{dC(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta C(t)}{\Delta t} = \lim_{\Delta t \to 0} (i \cdot C(t)) = i \cdot C(t) \implies \frac{dC(t)}{dt} = i \cdot C(t).
\]

This differential equation has as general solution:

\[
C(t) = C(0) \cdot e^{it},
\]

where \(C(0)\) is the initial monetary amount at \(t = 0\). Graphically:

If \(i = 3\%\) the time \(t_0 > 0\) we must to wait until \(C(t_0) = 2 \cdot C(0)\) is:

\[
2 \cdot C(0) = C(t_0) = C(0) e^{0.03 \cdot t_0} \implies 2 = e^{0.03 \cdot t_0} \implies t_0 = \frac{\ln 2}{0.03} = 23.1 \text{ years.}
\]

82 We will resolve this differential equation in class.
3.3.3.2. Temporary Evolution of Prices: The Walras’s Linear Model

In this paragraph we will analyze this model of price evolution using a specific example:

**Example:** Suppose that at moment \( t > 0 \) the rate of growth of a price \( p(t) \) of a certain good A is of the form:

\[
\frac{dp(t)}{dt} = 0.5 \cdot (D(t) - S(t))
\]

being \( D(t) = 21 - 2p(t) \) the demand and \( S(t) = 10p(t) - 3 \) the supply of A at \( t \geq 0 \). In this case determine the temporary evolution of the price \( p(t) \).

**SOLUTION:**

Observe that the formal expression of \( p(t) \) satisfies the separable differential equation:

\[
\frac{dp(t)}{dt} = 0.5 \cdot (D(t) - S(t)) = 12 - 6p(t)
\]

with general solution:

\[
p(t) = 2 + (p(0) - 2) \cdot e^{-6t} \quad \text{83}
\]

We can see that \( p(t) \) is stable over time in the sense that:

\[
\lim_{t \to +\infty} p(t) = 2 + (p(0) - 2)e^{-\infty} = 2 + (p(0) - 2) \cdot 0 = 2.
\]

It is worth noting that this limit value is precisely the equilibrium price \( p_e \) of A:

\[
D = S \implies 21 - 2 \cdot p_e = 10 \cdot p_e - 3 \implies p_e = 2.
\]

Graphically:

---

83 We will solve this differential equation in class. Note that \( p(0) \) is the initial price of A.
3.3.3.3. Application on Demography: the Logistic Growth

Example: Suppose that at time \( t \geq 0 \) the number \( N = N(t) \) in thousands of individuals of a certain population of 160,000 people aware of a rumor is proportional to both its amount and that of the unawared individuals according to the differential equation:

\[
N' = \frac{dN}{dt} = 0.02 \cdot N \cdot (160 - N).
\]

Determine the population that will be aware of the rumor within half a year \(( t = 0.5 \) if it was initially \(( t = 0 \) of 20,000 individuals. What will happen eventually?

SOLUTION: Dividing this differential equation by \( N^2 \) we obtain a linear differential equation depending on the variable \( Y = Y(t) = 1/N(t) \):

\[
Y = \frac{1}{N} \implies Y' = -\frac{1}{N^2} \cdot N' = -\left(0.02 \cdot \left(\frac{160}{N} - 1\right)\right) = -3.2 \cdot Y + 0.02.
\]

This first-order linear differential equation related to variable \( Y = Y(t) \) helps us to obtain the particular solution of the initial one satisfying the condition \( N(0) = 20 \):

\[
N(t) = \frac{160}{1 + 7 \cdot e^{-3.2t}}.
\]

Thus the population aware of the rumor within half a year will be approximately of:

\[
N(0.5) = \frac{160}{1 + 7 \cdot e^{-3.2 \cdot 0.5}} = 66.3 \text{ thousand individuals}
\]

Eventually the entire population will be aware of the rumor:

\[
\lim_{t \to \infty} N(t) = \frac{160}{1 + 7 \cdot e^{-\infty}} = \left\{e^{-\infty} = 0\right\} = 160.
\]

Graphically:\(^{84}\)

\(^{84}\) This curve is known as logistic curve.
3.4. Exercises

1. Calculate the following indefinite integrals:
   
   i. \( \int \frac{x^2 - 7}{\sqrt{x}} \, dx \)   
   ii. \( \int \frac{dx}{1-2x} \)   
   iii. \( \int \frac{5x \, dx}{e^{x^2}} \)   
   iv. \( \int \frac{2 \, dx}{x \ln x} \)   
   v. \( \int \frac{(x^2 + 2x) \, dx}{x^3 + 3x^2 - 12} \)   
   vi. \( \int \frac{x^2 - x}{x + 1} \, dx \)   
   vii. \( \int \frac{dx}{(x-3)^2} \)   
   viii. \( \int \frac{e^{2x} \, dx}{1 + e^{2x}} \)   
   ix. \( \int \frac{dx}{\sqrt{3x + 5}} \)   
   x. \( \int \frac{1}{x^2} \, dx \)

2. Calculate the following definite integrals:
   
   i. \( \int_1^4 \frac{1 + \sqrt{x}}{x^2} \, dx \)   
   ii. \( \int_0^1 \frac{\ln x}{e^{2x}} \, dx \)   
   iii. \( \int_0^{\ln 2} (1 + e^{2x}) \, dx \)   
   iv. \( \int_0^{0.25} 0.5 \, dx \)

3. Calculate the area \( A \) determined by the function \( f(x) = x \cdot e^{-x^2} \) and the \( x \)-axis between the two lines \( x = -1 \) and \( x = 1 \).

4. Find the area \( A \) closed by \( y = \frac{1}{2+x} \) and the line \( 12y + x = 5 \).

5. Calculate the area \( A \) determined by the function:
   
   \( f(x) = \frac{3}{\sqrt{1+x}} \)
   
   the line \( y = 1.5x + 3 \) and the \( x \)-axis between lines \( x = -1 \) and \( x = 3 \).\textsuperscript{85}

6. The marginal cost function in euros associated with a manufacturer of electric cars is \( C_{Ma}(q) = 0.8q + 4 \). If this manufacturer produces 50 units, calculate how much it would cost to double the production.

7. Find the general solution of the following differential equations:
   
   i. \( x \, dx = (1 - x^2) \, dy \)   
   ii. \( x \, dy + (1 + y)^2 \, dx = 0 \)   
   iii. \( ex \, dx = 2ye^{-x^2} \, dy \)   
   iv. \( y' = e^{-x} \)   
   v. \( y = \ln y' \)   
   vi. \( xy' + y^2 = 0 \)   
   vii. \( 2y' - 3y = 12 \)   
   viii. \( y' + 3x^2y = 0 \)   
   ix. \( y'+2xy = 4e^{2x} \)

8. Determine the cost \( C(q) \) of a good if its marginal cost is proportional to \( C(q) \) with a constant of proportionality \( k = 32 \cdot 10^{-2} \) knowing that producing one unit costs €7.389. What is the fixed cost of production?

\textsuperscript{85} Bear in mind that the function and the line cut off at point \((0,3)\).
SOLUTIONS:

1. 
   i. \( \frac{2x^{5/2}}{5} - 14x^{1/2} + C \)  
   vi. \( \frac{x^2}{2} - 2x + 2\ln(x + 1) + C \)
   
   ii. \( -\frac{1}{2}\ln(1 - 2x) + C \)  
   vii. \( -\frac{1}{4(x - 3)^2} + C \)
   
   iii. \( -\frac{5e^{-x^2}}{2} + C \)  
   viii. \( \frac{1}{2}\ln(1 + e^{2x}) + C \)
   
   iv. \( 2\ln(\ln x) + C \)  
   ix. \( \frac{2}{3}\sqrt{3x + 5} + C \)
   
   v. \( \frac{1}{3}(x^3 + 3x^2 - 12) + C \)  
   x. \( -\frac{2^{1/x}}{\ln 2} + C \)

2. 
   i. \( \frac{7}{4} \)
   
   ii. \( \frac{3}{4} \)
   
   iii. \( \ln 2 + \frac{3}{2} \approx 2.19 \)
   
   iv. \( 1 - \frac{\sqrt{3}}{2} \approx 0.134 \)

3. \( A = 1 - e^{-1} \approx 0.632 \)

4. \( A = \frac{7}{24} + \ln\left(\frac{3}{4}\right) = 0.004 \)

5. \( A = \frac{33}{4} \)

6. €3200

7. 
   i. \( y = C - 0.5\ln(1 - x^2) \)  
   ii. \( y = \left(\frac{1}{\ln x + C}\right) - 1 \)  
   iii. \( y^2 = \frac{e^{1+x^2}}{2} + C \)
   
   iv. \( y = \ln\left(\frac{1}{e^{-x} + C}\right) \)  
   v. \( y = \ln\left(\frac{1}{C - x}\right) \)  
   vi. \( y = \frac{1}{\ln x + C} \)
   
   vii. \( y = \left(\frac{1}{3}\right)e^{1.5x + C} - 4 \)  
   viii. \( y = e^{x + C} \)  
   ix. \( y = e^{2x} + C \cdot e^{-2x} \)

8. \( C(q) = e^{0.32q+1.68} \) and the fixed cost is €5.365.
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