Excitation modes of vortices in submicron magnetic disks

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Classical and quantum theory of spin waves in the vortex state of a mesoscopic submicron magnetic disk have been developed with account of the finite mass density of the vortex. Oscillations of the vortex core resemble oscillations of a charged string in a potential well in the presence of the magnetic field. A conventional gyrotropic frequency appears as a gap in the spectrum of spin waves of the vortex. The mass of the vortex has been computed, and the result agrees with experimental findings. The finite vortex mass generates a high-frequency branch of spin waves. The effects of an external magnetic field and dissipation have been addressed.

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I. INTRODUCTION

Recent advances in optical and electron-beam lithography offered the possibility to fabricate arrays of micron and submicron-size magnetic structures with controlled magnetic properties. Among such structures are mesoscopic circular disks of soft ferromagnetic materials. Arrays of such disks, as well as individual disks, have been intensively studied[1–7] due to their unusual magnetic properties and potential for technological and biomedical applications.[8–10]

Micron-size circular disks exhibit a wide variety of magnetic equilibrium configurations due to geometric constraints on the spin field.[11] Their applications are based on static and dynamic properties of one of the essentially nonuniform ground states, the *vortex state*. It is characterized by the curling of the magnetization in the plane of the disk, leaving virtually no magnetic "charges." The very weak uncompensated magnetic moment of the disk sticks out of a small area confined to the vortex core (VC). The diameter of the core is comparable to the material exchange length.[3,12] The low-frequency dynamics of the vortex state is due to the *gyrotropic* mode, consisting of the spiral-like precessional motion of the VC as a whole.[13–17] It is generically distinct from conventional spin wave excitations in ferromagnets and, for that reason, has received a lot of attention from experimentalists.

Magnon dynamics in the vortex state has been investigated by a number of authors.[18–22] High-frequency spin wave modes can be excited by low-amplitude short-duration magnetic field pulses. They are classified[21] by a pair of integers (m,n), with $n = 1,2,3,\ldots$ corresponding to the number of axially symmetric nodes and $m = 0, \pm 1, \pm 2, \ldots$ counting the number of azimuthal modes. The gyrotropic mode is identified with the lowest wave number compatible with the $m = -1$ mode. Azimuthal modes can be generated by a magnetic field pulse parallel to the dot plane. They form doublets,[20,21] each doublet corresponding to the absolute value of the azimuthal number m. Radially symmetric modes can be generated by a magnetic field pulse perpendicular to the dot plane. Their magnetization dynamics is dominated by long-range dipolar interactions.[18] These excitations are localized outside the VC (for instance, at the dot edge).[18] The dependence of these modes on the thickness L and radius R of the disk, computed using the collective variable approach,[21,22] agree with experimental data.

This paper is focused on the gyrotropic mode that describes circular motion of the vortex about the center of the disk. If deformations of the VC along the axis of the disk are ignored, the gyrotropic mode can be viewed as uniform precession of the magnetic moment of the disk. Rigidity of the VC has been an underlying assumption in the existing theoretical models of spin excitations of submicron magnetic disks. While it is true that strong exchange interaction makes the vortex a well defined independent entity, the question arises whether the gyrotropic mode allows spatial dispersion similar to spin waves of a finite wavelength in ferromagnets. The aim of this paper is to study spin waves related specifically to the gyrotropic motion of the vortex. Such a wave is illustrated in Fig. 1. It must exist due to the finite elasticity of the vortex provided by the exchange interaction. Note that conceptually similar modes have been studied for vortices in liquid helium[23] (Tkachenko modes) and for Abrikosov vortices in superconductors.[24] However, the dynamics of vortices in magnetic disks is very different from the dynamics of vortex lines in helium and in superconductors. The wave motion of the vortex in a magnetic disk shown in Fig. 1 requires therefore a separate analysis that will be presented in this paper.

Most of the research on the gyrotropic motion of vortices in circularly polarized disks ignores the inertial mass of the VC. Such a mass has a dynamical origin stemming from the variation of the shape of the VC as it moves inside the disk. Meantime, experimental studies of vortex oscillations in micrometer permalloy rings[25] hinted towards a non-negligible vortex mass of order 10^{-24} kg. Despite the ring geometry in which it was measured, this value must provide a reasonable ballpark of the vortex mass in a disk because, as we shall see below, the main contribution to the mass comes from distances far from the VC. (Similar situation exists for the electromagnetic structure of the Abrikosov vortex in a superconductor.) On a theoretical side the mass of the magnetic vortex has been previously computed in a two-dimensional Heisenberg model with anisotropic exchange interaction.[26–28] In disks made of soft magnetic materials that have been experimented with, the exchange interaction is isotropic. Attempts to estimate the mass of the magnetic vortex in this case have been previously made,[29,30] based on the method proposed in Ref. 27. We will show that in this case the finite mass density of the vortex

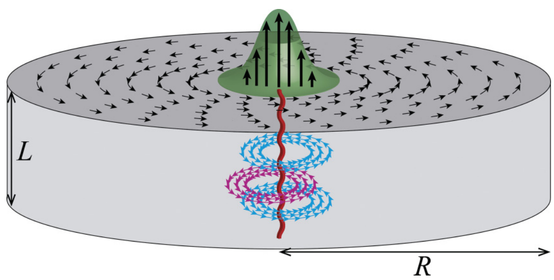

FIG. 1. (Color online) Gyroscopic spin wave in the vortex state of a mesoscopic magnetic disk.

originates from the geometrical confinement of the spin field and the magnetic dipole-dipole interactions.

In this paper, we will derive the generalized Thiele equation that describes spin waves in the vortex core of finite mass density and will obtain the spectrum of such waves. We will show that the conventional gyrotropic mode ω_G appears as a gap in the spectrum of the spin waves in the vortex, $\omega(q) = \omega_G + \alpha q^2$, when the vortex mass is neglected. From the mathematical point of view the above problem resembles, the problem of the motion of a charged string in a potential well in the presence of the magnetic field. The latter problem is a generalization of the problem of Landau levels of an electron in a two-dimensional potential well in the magnetic field. We will show that this problem has a nice exact solution for quantized oscillations of the string, thus providing the spectrum of magnons in the vortex in the quantum regime as well. Classical and quantum solutions for the spectrum of excitations of the vortex lead to the same dispersion law $\omega(q)$ in the limit of small q.

The paper is structured as follows. In Sec. II, a Lagrangian formulation of the problem is presented. Formal derivation of the massive elastic Thiele's equation that allows deformations of the vortex line is given in Sec. III. The spectrum of spin waves in the vortex core is obtained in Sec. IV. We show that a finite mass of the vortex results in the additional excitation mode that is absent in the case of zero mass. Quantum-mechanical treatment of magnons in the vortex core is developed in Sec. V. The vortex mass in a circularly polarized disk is computed in Sec. VI and is shown to be in good agreement with experimental findings. The field dependence of the vortex excitation modes and effects of dissipation are discussed in Sec. VII. Section VIII contains final conclusions and suggestions for experiment.

II. LAGRANGIAN MECHANICS OF THE VORTEX CORE

We shall describe the vortex line by the vector field $\vec{X} = (x,y)$, where $x(t,z)$ and $y(t,z)$ are coordinates of the center of the vortex core in the XY plane. Landau-Lifshitz dynamics of the fixed-length magnetization vector $\vec{M}(\Theta,\Phi) = M_s(\cos\Phi\sin\Theta, \sin\Phi\sin\Theta, \cos\Theta)$ follows from the Lagrangian[31]

$$\mathcal{L}[t;\Theta,\Phi,\dot{\Theta},\dot{\Phi},\partial_z\Theta,\partial_z\Phi] = \int dz\, d^2\vec{r}\left[\frac{M_s}{\gamma}(D_t\Phi)\cos\Theta - \mathcal{E}(\Theta,\Phi,\partial_z\Theta,\partial_z\Phi)\right], \quad (1)$$

where $\mathcal{E}(\Theta,\Phi,\partial_z\Theta,\partial_z\Phi)$ is the energy density. The dependence of the Lagrangian on the partial derivatives ∂_z of the angular coordinates comes from the elastic nature of the vortex core. It is contained in the total energy $\mathcal{E}$ that takes into account interaction between different layers of the vortex line, see below.

The spatial dependence of angular coordinates (Θ,Φ) for the vortex state is given by $\Theta = \Theta(t;\vec{r},z) = \Theta(\vec{r}-\vec{X}(t,z),t)$ and $\Phi = \Phi(t;\vec{r},z) = \Phi(\vec{r}-\vec{X}(t,z),t)$. We only consider long-wave solutions that do not deform the vortex core in any z cross section of the disk. This means that the angular coordinates depend on t and z via the coordinates of the vortex core $\vec{X}(t,z)$. The covariant derivative with respect to time $D_t\Phi$ along the vortex core is given by

$$D_t\Phi = \nabla_{\dot{\vec{X}}(t,z)}\Phi = -\dot{\vec{X}}(t,z)\cdot\nabla_{\vec{r}}\Phi[\vec{r}-\vec{X}(t,z)], \quad (2)$$

where "dot" denotes partial derivative with respect to t.

Taking all these considerations into account, the above Lagrangian becomes

$$\mathcal{L}[t;\vec{X},\dot{\vec{X}},\partial_z\vec{X}] = \int dz\tilde{\mathcal{L}}[t,z;\vec{X},\dot{\vec{X}},\partial_z\vec{X}] \quad (3)$$

with the Lagrangian density being

$$\tilde{\mathcal{L}}[t,z;\vec{X},\dot{\vec{X}},\partial_z\vec{X}] = \int d^2\vec{r}\left[\frac{M_s}{\gamma}(-\dot{\vec{X}}(t,z)\cdot\nabla_{\vec{r}}\Phi)\cos\Theta - \mathcal{E}(\Theta,\Phi,\partial_z\Theta,\partial_z\Phi)\right]. \quad (4)$$

Thus the generalized momentum densities are given by

$$\vec{\Pi}_t[t,z;\vec{X},\dot{\vec{X}},\partial_z\vec{X}] \equiv \frac{\delta\tilde{\mathcal{L}}}{\delta[\dot{\vec{X}}(t,z)]} = -\frac{M_s}{\gamma}\int d^2\vec{r}\,(\nabla_{\vec{r}}\Phi)\cos\Theta, \quad (5)$$

$$\vec{\Pi}_z[t,z;\vec{X},\dot{\vec{X}},\partial_z\vec{X}] \equiv \frac{\delta\tilde{\mathcal{L}}}{\delta[\partial_z\vec{X}(t,z)]} = -\frac{\delta\omega(\vec{X},\partial_z\vec{X})}{\delta[\partial_z\vec{X}(t,z)]}, \quad (6)$$

with $\omega(\vec{X},\partial_z\vec{X}) = \int d^2\vec{r}\,\mathcal{E}(\Theta,\Phi,\partial_z\Theta,\partial_z\Phi)$ being the linear energy density. The dynamics of the vortex core is governed by the Euler-Lagrange equation,

$$D_t\vec{\Pi}_t + \partial_z\vec{\Pi}_z - \frac{\delta\tilde{\mathcal{L}}}{\delta\vec{X}(t,z)} = 0. \quad (7)$$

Notice that

$$\begin{aligned} D_t\Lambda(t,z;\ \vec{r},\vec{v}) &= \dot{\xi}(t,z)(\Lambda) = \frac{\partial}{\partial t}\Lambda[\vec{r}-\vec{X}(t,z),\dot{\vec{X}}(t,z)] \\ &= -\dot{\vec{X}}(t,z)\cdot\nabla_{\vec{r}}\Lambda[\vec{r}-\vec{X}(t,z),\dot{\vec{X}}(t,z)] \\ &\quad + \ddot{\vec{X}}(t,z)\cdot\nabla_{\vec{v}}\Lambda[\vec{r}-\vec{X}(t,z),\dot{\vec{X}}(t,z)] \end{aligned} \quad (8)$$

means covariant derivative along the curve that is tangent to the vortex core, $\xi(t,z)$. All terms involving $\dot{\vec{X}}(t,z)$ and $\ddot{\vec{X}}(t,z)$ in the Euler-Lagrange equation come from $D_t\vec{\Pi}_t$, which is given by

$$D_t\vec{\Pi}_t = -\frac{M_s}{\gamma}\int d^2\vec{r}\; D_t\,(\cos\Theta\nabla_{\vec{r}}\Phi)$$
$$= -\frac{M_s}{\gamma}\int d^2\vec{r}[\nabla_{\vec{r}}(-\dot{X}_j\partial_j\Phi + \ddot{X}_j\tilde{\partial}_j\Phi)\cos\Theta$$
$$+\nabla_{\vec{r}}\Phi(-\dot{X}_j\partial_j\cos\Theta + \ddot{X}_j\tilde{\partial}_j\cos\Theta)]$$
$$= \hat{e}_i M_{ij}\ddot{X}_j - \hat{e}_i K_{ij}\dot{X}_j, \tag{9}$$

where

$$M_{ij} = -\frac{M_s}{\gamma}\int d^2\vec{r}\;[(\partial_i\tilde{\partial}_j\Phi)\cos\Theta + (\partial_i\Phi)\tilde{\partial}_j\cos\Theta],$$
$$K_{ij} = -\frac{M_s}{\gamma}\int d^2\vec{r}\;[(\partial_i\partial_j\Phi)\cos\Theta + (\partial_i\Phi)\partial_j\cos\Theta], \tag{10}$$

and $\partial_j \equiv \nabla_{r_j}$, $\tilde{\partial}_j = \nabla_{v_j}$.

We want $\hat{e}_i K_{ij}\dot{X}_j$ to be of the form $\vec{\rho}_G \times \dot{\vec{X}}$ that results in the identity $\epsilon_{ijk}\rho_{G,j} = K_{ik}$. From this, we obtain $\rho_{G,j} = -\frac{1}{2}\epsilon_{ikj}K_{ik}$, which translates into the vector form as

$$\vec{\rho}_G = \frac{M_s}{2\gamma}\int d^2\vec{r}\epsilon_{ikj}[(\partial_i\partial_k\Phi)\cos\Theta + (\partial_i\Phi)(\partial_k\cos\Theta)]\hat{e}_j$$
$$= \frac{M_s}{2\gamma}\int d^2\vec{r}\;[(\nabla_{\vec{r}}\times\nabla_{\vec{r}}\Phi)\cos\Theta + \nabla_{\vec{r}}\Phi\times\nabla_{\vec{r}}\cos\Theta]. \tag{11}$$

To compute the mass tensor and the gyrovector, we have to find the solutions (Θ,Φ) of the Landau-Lifshitz equation in the low-dynamics regime that is characterized by the condition $|\dot{\vec{X}}| \ll 1$. In this regime, solutions can be expanded as a perturbative series on the differential speed, $|\dot{\vec{X}}|$, of the vortex core

$$\Theta(t,z;\vec{r}) = \Theta^{(0)}(z;\vec{r}) + \Theta^{(1)}(t,z;\vec{r}) + \cdots,$$
$$\Phi(t,z;\vec{r}) = \Phi^{(0)}(z;\vec{r}) + \Phi^{(1)}(t,z;\vec{r}) + \cdots. \tag{12}$$

Notice that the zeroth order is time independent, otherwise the gyrovector would depend on time.

The approach that neglects deformation of the vortex core in any z cross section of the disk is correct only for weak deviations of the centerline of the vortex core from the straight line along the Z axis. We now proceed to the study of the Landau-Lifshitz equation for the set of variables (Θ,Φ) in such weak bending regime. It can be obtained by applying the variational principle to the Lagrangian density

$$\tilde{\mathcal{L}}[t,z;\Theta,\Phi,\dot{\Theta},\dot{\Phi},\partial_z\Theta,\partial_z\Phi]$$
$$= \int d^2\vec{r}\left[\frac{M_s}{\gamma}(D_t\Phi)\cos\Theta - \mathcal{E}(\Theta,\Phi,\nabla_{\vec{r}}\Theta,\nabla_{\vec{r}}\Phi,\partial_z\Theta,\partial_z\Phi)\right]. \tag{13}$$

Notice that

$$\mathcal{E}(\Theta,\Phi,\nabla_{\vec{r}}\Theta,\nabla_{\vec{r}}\Phi,\partial_z\Theta,\partial_z\Phi) = \mathcal{E}_{XY}(\Theta,\Phi,\nabla_{\vec{r}}\Theta,\nabla_{\vec{r}}\Phi)$$
$$+\mathcal{E}_{\rm el}(\Theta,\Phi,\partial_z\Theta,\partial_z\Phi) \tag{14}$$

with $\mathcal{E}_{XY}(\Theta,\Phi,\nabla_{\vec{r}}\Theta,\nabla_{\vec{r}}\Phi)$ being the sum of the exchange, anisotropy, and dipolar energy responsible for the formation of the vortex, and

$$\mathcal{E}_{\rm el}(\Theta,\Phi,\partial_z\Theta,\partial_z\Phi) = A_{\rm eff}[(\partial_z\Theta)^2 + \sin^2\Theta(\partial_z\Phi)^2] \tag{15}$$

being the elastic energy in which $A_{\rm eff}$ is a constant. It describes contribution of the exchange and dipolar forces to the elasticity of the vortex line, with the exchange playing a dominant role. Consequently, with good accuracy, $A_{\rm eff}$ can be identified with the exchange constant A.

The set of dynamical equations for (Θ,Φ) is

$$D_t\left[\frac{\delta\tilde{\mathcal{L}}}{\delta(D_t\Phi)}\right] + \partial_z\left[\frac{\delta\tilde{\mathcal{L}}}{\delta(\partial_z\Phi)}\right] - \frac{\delta\tilde{\mathcal{L}}}{\delta\Phi} = 0,$$
$$-2A_{\rm eff}\big(\sin 2\Theta\partial_z\Theta\partial_z\Phi + \sin^2\Theta\partial_z^2\Phi\big)$$
$$+\frac{M_s}{\gamma}\frac{d\cos\Theta}{dt} + \frac{\delta\mathcal{E}_{XY}}{\delta\Phi} = 0, \tag{16}$$

and

$$D_t\left[\frac{\delta\tilde{\mathcal{L}}}{\delta(D_t\Theta)}\right] + \partial_z\left[\frac{\delta\tilde{\mathcal{L}}}{\delta(\partial_z\Theta)}\right] - \frac{\delta\tilde{\mathcal{L}}}{\delta\Theta} = 0,$$
$$-2A_{\rm eff}\left[\partial_z^2\Theta - \frac{\sin 2\Theta}{2}(\partial_z\Phi)^2\right]$$
$$+\frac{M_s}{\gamma}\sin\Theta\frac{d\Phi}{dt} + \frac{\delta\mathcal{E}_{XY}}{\delta\Theta} = 0. \tag{17}$$

Performing a Fourier transform

$$\Phi(t,z;\vec{r}) = \frac{1}{\sqrt{2\pi}}\int dq\;\bar{\Phi}(t,q;\vec{r})e^{iqz},$$
$$\Theta(t,z;\vec{r}) = \frac{1}{\sqrt{2\pi}}\int dq\;\bar{\Theta}(t,q;\vec{r})e^{iqz}, \tag{18}$$

we obtain the following set of equations for the pair $(\bar{\Theta},\bar{\Phi})$:

$$\frac{A_{\rm eff}q^2}{\pi}(\overline{\sin 2\Theta}\star\bar{\Theta}\star\bar{\Phi} + \overline{\sin\Theta}\star\overline{\sin\Theta}\star\bar{\Phi})$$
$$+\frac{M_s}{\gamma}\frac{d\,\overline{\cos\Theta}}{dt} + \frac{\overline{\delta\mathcal{E}_{XY}}}{\delta\Phi} = 0, \tag{19}$$

$$\frac{1}{\sqrt{2\pi}}\frac{M_s}{\gamma}\overline{\sin\Theta}\star\frac{d\bar{\Phi}}{dt} + \frac{\overline{\delta\mathcal{E}_{XY}}}{\delta\Theta}$$
$$+2A_{\rm eff}q^2\left(\bar{\Theta} - \frac{\overline{\sin 2\Theta}}{4\pi}\star\bar{\Phi}\star\bar{\Phi}\right) = 0, \tag{20}$$

where $\star$ means Fourier convolution.

For a small bending of the vortex core, the boundary conditions on the angle Θ are the same as in the rigid VC case, i.e.: $\Theta \simeq 0$ or π in the limit $\tilde{r} \ll \Delta_0$ and $\Theta \simeq \pi/2$ in the limit $\tilde{r} \gg \Delta_0$, with $\Delta_0 = \sqrt{A/M_s^2}$ being the exchange length of the material and where $\tilde{r} = ||\vec{r} - \vec{X}(t,z)||_2$ is the radial distance from the VC center at any height z. Considering these two limits, Eq. (19) changes as follows. (1) Limit $\tilde{r} \ll \Delta_0$. In this case, $\sin\Theta \simeq 0$ and thus $\sin 2\Theta\partial_z\Theta\partial_z\Phi + \sin^2\Theta\partial_z^2\Phi \simeq 0$. So we have the following equation in the Fourier space:

$$\frac{M_s}{\gamma}\frac{d\,\overline{\cos\Theta}}{dt} + \frac{\overline{\delta\mathcal{E}_{XY}}}{\delta\Phi} = 0. \tag{21}$$

(2) Limit $\tilde{r} \gg \Delta_0$. In this case, $\sin\Theta \simeq 1$ and thus $\sin 2\Theta\partial_z\Theta\partial_z\Phi + \sin^2\Theta\partial_z^2\Phi \simeq \partial_z^2\Phi$. So we have the equation

$$\frac{M_s}{\gamma}\frac{d\,\overline{\cos\Theta}}{dt} + 2A_{\rm eff}q^2\bar{\Phi} + \frac{\overline{\delta\mathcal{E}_{XY}}}{\delta\Phi} = 0. \tag{22}$$

Notice that in both limits $\sin 2\Theta \simeq 0$ and so $\partial_z^2\Theta - \frac{\sin 2\Theta}{2}(\partial_z\Phi)^2 \simeq \partial_z^2\Theta$. Consequently, in the Fourier space Eq. (20) becomes

$$\frac{1}{\sqrt{2\pi}}\frac{M_s}{\gamma}\overline{\sin\Theta}\star\frac{d\overline{\Phi}}{dt} + 2A_{\rm eff}q^2\bar{\Theta} + \overline{\frac{\delta\mathcal{E}_{XY}}{\delta\Theta}} = 0. \quad (23)$$

Finally, in the limit of weak bending ($A_{\rm eff}q^2 \ll 1$), we can neglect the terms of the form $2A_{\rm eff}q^2\bar{\xi}$ in the above equations. In doing so, we recover the standard Landau-Lifshitz equations for (Θ,Φ) at any z layer, with the VC center depending on the value of z. Introducing now the perturbative series (18) into the Landau-Lifshitz equation and splitting it into $O(|\dot{\vec{X}}|^n)$ terms, we obtain the equations of motion for the $\Phi^{(n)}/\Theta^{(n)}$ terms. In the case of the zeroth and first-order terms, we recover the static solution and the first perturbative solution for the rigid vortex (see Sec. VI). For the particular case of the zeroth order, we obtain

$$\Phi_0(x,y) = n_v\tan^{-1}(y-y_v/x-x_v)+\phi_0,$$
$$\cos\Theta_0(\tilde{r}) = \begin{cases} p\left[1-C_1\left(\frac{\tilde{r}}{\Delta_0}\right)^2\right], & \tilde{r}\ll\Delta_0,\\ C_2\left(\frac{\Delta_0}{\tilde{r}}\right)^{1/2}\exp(-\tilde{r}/\Delta_0), & \tilde{r}\gg\Delta_0,\end{cases} \quad (24)$$

where $n_v = \pm 1$ is the vorticity of the magnetization of the disk, $\phi_0 = \pm\pi/2$ corresponds to counter clockwise, respectively clockwise rotation of the magnetization vector in the dot plane, and C_1, C_2 are constants that can be obtained by imposing the smoothness condition on $\cos\Theta_0$ at $\tilde{r} = \Delta_0$ up to its first derivative. The corresponding values are $C_1 = \frac{3}{7}$ and $C_2 = \frac{4}{7}pe$. From all this, we straightforwardly deduce that

$$\nabla_{\tilde{r}}\times\nabla_{\tilde{r}}\Phi_0 = 2\pi n_v\delta^{(2)}[\vec{r}-\vec{X}(t,z)]\hat{e}_z, \quad \nabla_{\tilde{r}}^2\Phi_0 = 0. \quad (25)$$

III. ELASTIC THIELE EQUATION

We now proceed to the computation of the gyrovector and the mass density tensor. Using (25), we obtain that the first term of Eq. (11) equals $(\pi n_v pM_s/\gamma)\hat{e}_z$, where $p = \cos\Theta(\vec{0}) = \pm 1$ defines the direction of the polarization of the vortex core $[\Theta(\vec{0}) = 0$ or $\pi]$. The second term is evaluated at the zeroth order of the perturbative expansion of the angular coordinates in the low-dynamics regime, Eq. (24), taking into account that in the weak-bending regime, the deformation of the vortex core is small and one can consider $\tilde{r}\simeq r$ because $||\vec{X}(t,z)||_2 \ll 1$. In doing so, we obtain $(\pi n_v pM_s/\gamma)\hat{e}_z$ again.[32] Thus the gyrovector becomes $\vec{\rho}_G = \rho_G p n_v\hat{e}_z$ with

$$\rho_G = 2\pi M_s/\gamma. \quad (26)$$

Notice that $\vec{\rho}_G$ is the gyrovector linear density as compared to the gyrovector in the Thiele equation for a rigid vortex.[33]

A computation of the mass density tensor will be performed in Sec. VI. For circular polarized disks, we show that this tensor reduces to a scalar, $M_{ij} = \rho_M\delta_{ij}$, with the vortex core mass density given by $\rho_M = \frac{1}{4\gamma^2}\ln(R/\Delta_0)$, where R is the radius of the disk. Only $\omega(\vec{X},\partial_z\vec{X})$ contributes to the partial derivative $\delta\tilde{\mathcal{L}}/\delta\vec{X}$ in the slow dynamics regime, because $\vec{\Pi}_t = \rho_M\dot{\vec{X}}$ and so the term $\dot{\vec{X}}\cdot\vec{\Pi}_t$ equals $\rho_M\dot{\vec{X}}^2$. Consequently, the generalized Thiele equation becomes

$$\rho_M\ddot{\vec{X}}(t,z) + \dot{\vec{X}}(t,z)\times\vec{\rho}_G + \partial_z\vec{\Pi}_z + \nabla_{\vec{X}}\omega = 0. \quad (27)$$

The linear energy density $\omega(\vec{X},\partial_z\vec{X})$ is the sum of the magnetostatic and exchange contributions in the z cross section, $\omega_{XY}(\vec{X})$, and an elastic contribution due to the deformation of the vortex core line, $\omega_{\rm el}(\partial_z\vec{X})$. Zeeman contribution will be considered later. The dependence on the vortex core coordinates on the $\omega_{XY}(\vec{X})$ term for small displacements is[12,14]

$$\omega_{XY}(\vec{X}) = \tfrac{1}{2}\rho_M\omega_M^2\epsilon_0\vec{X}^2, \quad (28)$$

where $\omega_M = \rho_G/\rho_M$ is the characteristic frequency of the system and $\epsilon_0 = \omega_G/\omega_M$ is a dimensionless parameter. Recall that the conventional gyrofrequency ω_G is defined as[14,15]

$$\omega_G = \frac{\omega''_{XY}(\vec{X}=\vec{0})}{\rho_G} \simeq \frac{20}{9}\gamma M_s\beta, \quad (29)$$

where $\beta = L/R$ is the ratio of the thickness and the radius of the disk. This last expression is valid in the limit $\beta \ll 1$.

From the continuous spin-field model, we know that

$$\omega_{\rm el}(\partial_z\vec{X}) = A_{\rm eff}\int d^2\vec{r}[(\partial_z\Theta)^2 + \sin^2\Theta(\partial_z\Phi)^2]. \quad (30)$$

Noticing that $\partial_z\Theta = -\nabla_{\tilde{r}}\Theta\cdot\partial_z\vec{X}$ and $\partial_z\Phi = -\nabla_{\tilde{r}}\Phi\cdot\partial_z\vec{X}$ and taking into account the vector identity, $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$, with either $\vec{A}=\vec{D}=\nabla_{\tilde{r}}\Theta$, $\vec{B}=\vec{C}=-\partial_z\vec{X}$, or $\vec{A}=\vec{D}=\nabla_{\tilde{r}}\Phi$, $\vec{B}=\vec{C}=-\partial_z\vec{X}$, we obtain the following relations:

$$(\nabla_{\tilde{r}}\Theta\cdot\partial_z\vec{X})^2 = (\nabla_{\tilde{r}}\Theta)^2(\partial_z\vec{X})^2 - (\nabla_{\tilde{r}}\Theta\times\partial_z\vec{X})^2, \quad (31)$$

$$(\nabla_{\tilde{r}}\Phi\cdot\partial_z\vec{X})^2 = (\nabla_{\tilde{r}}\Phi)^2(\partial_z\vec{X})^2 - (\nabla_{\tilde{r}}\Phi\times\partial_z\vec{X})^2. \quad (32)$$

The main contribution to the integral comes from the zero-th order in the perturbative expansion (18). Notice that $\Theta_0(\vec{r}) = \Theta_0(\tilde{r})$ and that $\nabla_{\tilde{r}}\Phi_0 = \frac{n_v}{\tilde{r}}\hat{e}_\phi$. As discussed before, in the weak-bending regime, we can use the approximation $\tilde{r}\simeq r$, so that $\nabla_{\tilde{r}}\Phi_0\times\partial_z\vec{X} = -\frac{n_v}{r}\partial_z X_r\hat{e}_z$ and $\nabla_{\tilde{r}}\Theta_0\times\partial_z\vec{X} = \frac{d\Theta_0}{dr}\partial_z X_\phi\hat{e}_z$, where $X_r = \hat{e}_r\cdot\vec{X} = x\cos\theta + y\sin\theta$ and $X_\phi = \hat{e}_\phi\cdot\vec{X} = -x\sin\theta + y\cos\theta$. The elastic energy density finally becomes

$$\begin{aligned}\omega_{\rm el}(\partial_z\vec{X}) &= A_{\rm eff}\int d^2\vec{r}\left[(\nabla_{\tilde{r}}\Theta_0)^2 + \frac{\sin^2\Theta_0}{r^2}\right]\left(\frac{\partial\vec{X}}{\partial z}\right)^2\\ &\quad - A_{\rm eff}\int d^2\vec{r}\left(\frac{d\Theta_0}{dr}\right)^2\left(\frac{\partial X_\phi}{\partial z}\right)^2\\ &\quad - A_{\rm eff}\int d^2\vec{r}\,\frac{\sin^2\Theta_0}{r^2}\left(\frac{\partial X_r}{\partial z}\right)^2\\ &= \pi A_{\rm eff}\int r\,dr\left[\left(\frac{d\Theta_0}{dr}\right)^2 + \frac{\sin^2\Theta_0}{r^2}\right]\left(\frac{\partial\vec{X}}{\partial z}\right)^2,\end{aligned} \quad (33)$$

where the angular dependence of $(\partial_z X_r)^2$ and $(\partial_z X_\phi)^2$ has been integrated on θ. We can recast this energy density as $\omega_{\rm el}(\partial_z\vec{X}) = \frac{1}{2}\lambda(\frac{\partial\vec{X}}{\partial z})^2$, where λ is the elastic constant given by

$$\lambda = 2\pi A_{\rm eff}\int r\,dr\left[\left(\frac{d\Theta_0}{dr}\right)^2 + \frac{\sin^2\Theta_0}{r^2}\right]. \quad (34)$$

Making use of the variable $m_0(r) = \cos\Theta_0(r)$, we can rewrite the above equation as

$$\lambda = 2\pi A_{\mathrm{eff}} \int r\,dr \left[\frac{1}{1-m_0^2}\left(\frac{dm_0}{dr}\right)^2 + \frac{1-m_0^2}{r^2}\right], \tag{35}$$

and using the spatial dependence (24), we get

$$\frac{1}{1-m_0^2}\left(\frac{dm_0}{dr}\right)^2 + \frac{1-m_0^2}{r^2} = \begin{cases} \frac{\Delta_0^2}{2C_1r^2}\left(\frac{2pC_1}{\Delta_0^2}r\right)^2 + \frac{2C_1r^2}{\Delta_0^2}\frac{1}{r^2} = \frac{4C_1}{\Delta_0^2}, & r \ll \Delta_0, \\ \frac{1}{\Delta_0^2}m_0^2 + \frac{1}{r^2} = \frac{C_2^2}{\Delta_0 r}\exp(-2r/\Delta_0) + \frac{1}{r^2}, & r \gg \Delta_0. \end{cases} \tag{36}$$

Recalling that $C_1 = \frac{3}{7}$ and $C_2 = \frac{4}{7}pe$ and computing the integral (34) by splitting into two regions, $[0,\Delta_0]$ and $[\Delta_0,R]$, we obtain

$$\lambda = 2\pi A_{\mathrm{eff}}\left[\tfrac{50}{49} + \ln(R/\Delta_0)\right]. \tag{37}$$

In the limit $R \gg \Delta_0$, the logarithmic term dominates and λ becomes

$$\lambda = 2\pi A_{\mathrm{eff}} \ln(R/\Delta_0). \tag{38}$$

Finally, for the total energy density, we obtain

$$\omega(\vec{X},\partial_z\vec{X}) = \frac{1}{2}\rho_M\omega_M^2\epsilon_0\vec{X}^2 + \frac{1}{2}\lambda\left(\frac{\partial\vec{X}}{\partial z}\right)^2, \tag{39}$$

and thus the generalized Thiele equation for an elastic vortex core line becomes

$$\rho_M\ddot{\vec{X}}(t,z) - \lambda\partial_z^2\vec{X}(t,z) + \dot{\vec{X}}(t,z)\times\vec{\rho}_G + \rho_M\omega_M^2\epsilon_0\vec{X}(t,z) = 0. \tag{40}$$

IV. SPIN WAVES IN THE VORTEX CORE

Introducing the complex variable $\chi = x - iy$ we can recast Eq. (40) as the following complex partial differential equation:

$$\rho_M\ddot{\chi} - \lambda\partial_z^2\chi + i\rho_G\dot{\chi} + \rho_M\omega_M^2\epsilon_0\chi = 0. \tag{41}$$

Let $\chi_0(z)$ be the equilibrium complex center of the straight vortex core line. In the presence of the wave, it gets perturbed and becomes $\chi(t,z) = \chi_0(z) + \chi_w(t,z)$, with $||\chi_w||_z \ll ||\chi_0||_z$. Switching to the Fourier transform,

$$\chi_w(t,z) = \frac{1}{2\pi}\int d\omega\,dq\,\chi_w(\omega,q)e^{i(\omega t - qz)}, \tag{42}$$

we obtain the following equation for $\chi_w(\omega,q)$:

$$\left(-\rho_M\omega^2 + \lambda q^2 - \rho_G\omega + \rho_M\omega_M^2\epsilon_0\right)\chi_w(\omega,q) = 0. \tag{43}$$

For nonzero amplitude of the wave, the expression in the square parenthesis must vanish. This determines the spectrum of the waves:

$$\rho_M\omega^2 - \lambda q^2 + \rho_G\omega - \rho_M\omega_M^2\epsilon_0 = 0. \tag{44}$$

At $\rho_M \neq 0$, one can normalize Eq. (44) to get

$$\omega^2 + \omega_M\omega - \omega_M^2\left(\epsilon_0 + \frac{\lambda}{\rho_M\omega_M^2}q^2\right) = 0. \tag{45}$$

Solving this equation, we obtain the spectrum of vortex core excitations:

$$\omega_\pm(q) = \frac{\omega_M}{2}\left[\sqrt{(1+4\epsilon_0) + \frac{4\lambda q^2}{\rho_M\omega_M^2}} \pm 1\right]. \tag{46}$$

In the weak-bending regime, we have $4\lambda q^2/\rho_M\omega_M^2 \ll 1$ and so we can expand the square root and obtain the following expression for the frequencies:

$$\omega_\pm(q) = \frac{\omega_M}{2}(\sqrt{1+4\epsilon_0} \pm 1) + \frac{1}{\sqrt{1+4\epsilon_0}}\frac{\lambda}{\rho_G}q^2, \tag{47}$$

where we have used the relation $\rho_M\omega_M = \rho_G$.

As will be shown in Sec. IV, the parameter $\epsilon_0 = \omega_G/\omega_M$ is normally small due to the smallness of $\beta = L/R$. Consequently,

$$\omega_-(q) \approx \omega_G + \frac{\lambda}{\rho_G}q^2, \tag{48}$$

$$\omega_+(q) \approx \omega_M + \frac{\lambda}{\rho_G}q^2. \tag{49}$$

With the help of Eqs. (26) and (38) with $A_{\mathrm{eff}} \approx A = M_s^2\Delta_0^2$, the above equations can be written in a transparent form:

$$\omega_-(q) = \omega_G + \gamma M_s(q\Delta_0)^2\ln(R/\Delta_0), \tag{50}$$

$$\omega_+(q) = \omega_M + \gamma M_s(q\Delta_0)^2\ln(R/\Delta_0). \tag{51}$$

Note that the weak-bending regime corresponds to $q\Delta_0 \ll 1$.

V. QUANTUM MECHANICS OF THE EXCITATIONS IN THE VORTEX CORE

In this section, we will show that excitations of the vortex core can be also obtained in a rather nontrivial way from the quantum theory as well. This problem is interesting on its own as it turns out to be equivalent to the problem of quantum excitations of a charged string confined in a parabolic potential and subjected to the magnetic field.

It is straightforward to prove that the generalized Thiele equation (40) is the Euler-Lagrange equation associated with the following effective Lagrangian density that can be derived from Eq. (4):

$$\tilde{\mathcal{L}}(t,z;\vec{X},\dot{\vec{X}},\partial_z\vec{X}) = \tfrac{1}{2}\rho_M\dot{\vec{X}}^2 + \dot{\vec{X}}\cdot\vec{A}_{\rho_G} - \omega(\vec{X},\partial_z\vec{X}), \tag{52}$$

where $\vec{A}_{\rho_G}$ is the gyrovector potential satisfying $\nabla_{\vec{X}}\times\vec{A}_{\rho_G} = -\vec{\rho}_G$. Thus the total Lagrangian becomes

$$\mathcal{L} = \int dz\,\tilde{\mathcal{L}} = \int dz\left[\frac{1}{2}\rho_M\dot{\vec{X}}^2 + \dot{\vec{X}}\cdot\vec{A}_{\rho_G} - \omega(\vec{X},\partial_z\vec{X})\right]. \tag{53}$$

Noticing that $\{\varphi_n(z)\}_{n\in\mathbf{N}} = \{\sqrt{\frac{2}{L}}\sin(q_nz)\}_{n\in\mathbf{N}}$, with $q_n = \frac{2\pi}{L}n$, is a Hilbert basis of the function subspace $\mathcal{W} = \{\varphi \in \mathcal{L}^2(0,L),\ \varphi(0) = \varphi(L) = 0\}$, we can expand $\vec{X}$ as

$$\vec{X}(t,z) = \vec{X}_0(t) + \sum_n \vec{X}_n(t)\varphi_n(z), \tag{54}$$

where $\vec{X}_0(t)$ is the center of the undisturbed vortex and $\vec{X}_n(t) = <\vec{X}(t,z),\varphi_n(z)>_{\mathcal{L}^2(0,L)}$. Introducing this expansion in Eq. (53) and taking into account the orthonormality of the Hilbert basis (and its spatial derivatives), we obtain the following identity:

$$\begin{aligned}\mathcal{L}(t,\{\vec{X}_n\}_{n\in\mathbf{Z}^+},\{\dot{\vec{X}}_n\}_{n\in\mathbf{Z}^+}) &= \left[\frac{1}{2}M\dot{\vec{X}}_0^2+\dot{\vec{X}}_0\cdot\vec{A}_0-\frac{1}{2}M\omega_M^2\epsilon_0\vec{X}_0^2\right]\\ &\quad+\sum_{n>0}\left[\frac{1}{2}\rho_M\dot{\vec{X}}_n^2+\dot{\vec{X}}_n\cdot\vec{A}_n-\frac{1}{2}\rho_M\omega_M^2\epsilon_0\vec{X}_n^2-\frac{1}{2}\lambda q_n^2\vec{X}_n^2\right],\end{aligned} \tag{55}$$

where $\mathbf{Z}^+=\{0\}\cup\mathbf{N}$, $M=\rho_M L$ is the total mass of the rigid vortex line and $\vec{A}_n$ is the gyrovector potential associated to the nth coordinate $\vec{X}_n$, which satisfies $\nabla_{\vec{X}_0}\times\vec{A}_0=-\vec{G}$ and $\nabla_{\vec{X}_n}\times\vec{A}_n=-\vec{\rho}_G$, $n>0$, with $\vec{G}=\vec{\rho}_G L$ being the gyrovector of the rigid vortex.

Applying the Laguerre transformation to the above Lagrangian, we obtain the following expression for the Hamiltonian:

$$\begin{aligned}\mathcal{H}(t,\{\vec{X}_n\}_{n\in\mathbf{Z}^+},\{\vec{\Pi}_n\}_{n\in\mathbf{Z}^+}) &= \left[\frac{1}{2M}(\vec{\Pi}_0-\vec{A}_0)^2+\frac{1}{2}M\omega_M^2\epsilon_0\vec{X}_0^2\right]\\ &\quad+\sum_{n>0}\left[\frac{1}{2\rho_M}(\vec{\Pi}_n-\vec{A}_n)^2\right.\\ &\quad\left.+\frac{1}{2}\rho_M\omega_M^2\left(\epsilon_0+\frac{\lambda}{\rho_M\omega_M^2}q_n^2\right)\vec{X}_n^2\right],\end{aligned} \tag{56}$$

where $\vec{\Pi}_0=M\dot{\vec{X}}_0+\vec{A}_0$ and $\vec{\Pi}_n=\rho_M\dot{\vec{X}}_n+\vec{A}_n$, $n>0$ are the corresponding canonical momenta. Notice that Eq. (56) shows that $\mathcal{H}$ splits into the direct sum $\oplus_{m\in\mathbf{Z}^+}\mathcal{H}_m$, with $\mathcal{H}_m$ being the Hamiltonian defined over the phase space $(\vec{X}_m,\vec{\Pi}_m)$. It has a structure of the form

$$\mathcal{H}'=\frac{1}{2\eta}(\vec{\Pi}-\vec{A})^2+\frac{1}{2}\eta\omega_M^2\xi\vec{X}^2, \tag{57}$$

where $(\vec{X},\vec{\Pi})$ are the canonically conjugate variables, η and ξ are constants, and $\vec{A}$ is the gyrovector satisfying $\nabla_{\vec{X}}\times\vec{A}=-\chi\hat{z}$, with χ being a constant. It is important to point out that $\chi/\eta=\omega_M$ in all cases.

From now on, we consider the case of the vortex core of a nonzero mass, $(\eta\neq 0)$. It is convenient to choose a "symmetric gauge" given by

$$\vec{A}=\frac{1}{2}(-\chi\hat{z})\times\vec{X}=\frac{\chi y}{2}\hat{x}-\frac{\chi x}{2}\hat{y}. \tag{58}$$

Firstly, we define the kinetic momentum operators as $\vec{p}=\eta\dot{\vec{X}}$, so that $\vec{\Pi}=\vec{p}+\vec{A}$. Notice the following nonvanishing commutators:

$$[p_j,p_k]=-i\hbar\chi\epsilon_{jk}\quad j,k\in\{x,y\}, \tag{59}$$

where ϵ_{jk} is the antisymmetric tensor $\epsilon_{xy}=-\epsilon_{yx}=1$. Secondly, we introduce the operators

$$a=\sqrt{\frac{1}{2\hbar\chi}}(p_y+ip_x),\quad a^\dagger=\sqrt{\frac{1}{2\hbar\chi}}(p_y-ip_x), \tag{60}$$

which satisfy standard commutation relations for Bose operators, $[a,a^\dagger]=1$. The number operator $N_a=a^\dagger a$ satisfies commutation relations $[N_a,a]=-a$, $[N_a,a^\dagger]=a^\dagger$, and we have the identity

$$\frac{1}{2\eta}(\vec{\Pi}-\vec{A})^2=\hbar\omega_M\left(N_a+\frac{1}{2}\right). \tag{61}$$

In analogy with the case of a charged particle in the electromagnetic field,[34] we obtain that the gyrotropic translational group is generated by $\vec{T}=\vec{\Pi}+\vec{A}$,

$$T_x=p_x+\chi y,\quad T_y=p_y-\chi x, \tag{62}$$

which satisfies the following commutation relations:

$$[T_j,p_k]=0,\quad [T_j,T_k]=i\hbar\chi\epsilon_{jk},\quad j,k\in\{x,y\}. \tag{63}$$

Now we introduce another set of Bose operators,

$$b=\sqrt{\frac{1}{2\hbar\chi}}(T_y-iT_x),\quad b^\dagger=\sqrt{\frac{1}{2\hbar\chi}}(T_y+iT_x), \tag{64}$$

which satisfy commutation relations

$$[b,b^\dagger]=1,\quad [M_b,b]=-b,\quad [M_b,b^\dagger]=b^\dagger,$$

where $M_b=b^\dagger b$ is the corresponding number operator. Notice that the commutation relations $[a,b]=[a,b^\dagger]=0$ also hold.

Coordinates x and y can be expressed in terms of the above Bose operators:

$$x=\frac{1}{\chi}(p_y-T_y),\quad y=-\frac{1}{\chi}(p_x-T_x), \tag{65}$$

so that

$$\tfrac{1}{2}\eta\omega_M^2(x^2+y^2)=\hbar\omega_M(N_a+M_b-ab-a^\dagger b^\dagger+1). \tag{66}$$

Consequently, the Hamiltonian (57) becomes

$$\mathcal{H}'=\hbar\omega_M[(1+\xi)N_a+\xi M_b-\xi(ab+a^\dagger b^\dagger)+\xi+\tfrac{1}{2}]. \tag{67}$$

It can be diagonalized with the help of Bogoliubov transformations

$$\bar{\alpha}=ua-vb^\dagger,\quad \bar{\beta}=ub-va^\dagger \tag{68}$$

with u,v being real numbers. These new operators satisfy Bose commutation relations if $u^2-v^2=1$. Substituting the above equations into Eq. (67), we obtain

$$\begin{aligned}\mathcal{H}'=\hbar\omega_M\{&\bar{\alpha}^\dagger\bar{\alpha}(u^2(1+\xi)+\xi v^2-2\xi uv)\\ &+\bar{\beta}^\dagger\bar{\beta}(v^2(1+\xi)+\xi u^2-2\xi uv)\\ &+(\bar{\alpha}^\dagger\bar{\beta}^\dagger+\bar{\alpha}\bar{\beta})[uv(1+2\xi)-\xi(u^2+v^2)]\\ &+[v^2(1+2\xi)-2\xi uv+(\xi+1/2)]\}.\end{aligned} \tag{69}$$

To get a Hamiltonian in the oscillator form, the coefficient related to $(\bar{\alpha}^\dagger\bar{\beta}^\dagger+\bar{\alpha}\bar{\beta})$ should be zero, which requires

$$uv(1+2\xi)-\xi(u^2+v^2)=0. \tag{70}$$

The solution is $u=\cosh(\theta)$, $v=\sinh(\theta)$,

$$\tanh(2\theta)=\frac{2\xi}{1+2\xi}. \tag{71}$$

Finally, the coefficients of the terms $\bar{\alpha}^\dagger\bar{\alpha}$ and $\bar{\beta}^\dagger\bar{\beta}$ become

$$\begin{aligned} &u^2(1+\xi)+\xi v^2-2\xi uv \\ &\quad=\frac{1}{2}\left[\frac{1+2\xi}{\cosh(2\theta)}+1\right]=\frac{1}{2}(\sqrt{1+4\xi}+1), \\ &v^2(1+\xi)+\xi u^2-2\xi uv \\ &\quad=\frac{1}{2}\left[\frac{1+2\xi}{\cosh(2\theta)}-1\right]=\frac{1}{2}(\sqrt{1+4\xi}-1) \end{aligned} \tag{72}$$

and, consequently, the Hamiltonian in the second quantization formalism becomes

$$\mathcal{H}'=\hbar\omega_+\big(\bar{\alpha}^\dagger\bar{\alpha}+\tfrac{1}{2}\big)+\hbar\omega_-\big(\bar{\beta}^\dagger\bar{\beta}+\tfrac{1}{2}\big), \tag{73}$$

where

$$\omega_\pm=\frac{1}{2}(\sqrt{1+4\xi}\pm 1)\omega_M. \tag{74}$$

Noticing that for any $n\in\mathbf{Z}^+$ we have $\xi=\epsilon_0+\frac{\lambda}{\rho_M\omega_M^2}q_n^2$, the second quantization procedure yields the following form of Hamiltonian (56):

$$\mathcal{H}=\sum_{n\geqslant 0}\hbar\omega_n^+\left(\bar{\alpha}_n^\dagger\bar{\alpha}_n+\frac{1}{2}\right)+\sum_{n\geqslant 0}\hbar\omega_n^-\left(\bar{\beta}_n^\dagger\bar{\beta}_n+\frac{1}{2}\right), \tag{75}$$

where $\omega_n^\pm$ are the eigenfrequencies of the vortex state given by

$$\omega_n^\pm=\frac{1}{2}\left[\sqrt{(1+4\epsilon_0)+\frac{4\lambda}{\rho_M\omega_M^2}q_n^2}\pm 1\right]\omega_M, \tag{76}$$

which coincides with Eq. (46).

VI. COMPUTATION OF THE VORTEX MASS

As it has been discussed in Sec. II, to calculate the vortex mass density tensor [see Eq. (10)], we need to find a solution $(\Phi(\vec{r},t),\Theta(\vec{r},t))$ of the Landau-Lifshitz equation in the slow dynamics regime, i.e., in the first order on $|\dot{\vec{X}}|$. A more convenient set of variables for this problem is the pair (Φ,m), where $m\equiv m_z=\frac{M_z}{M_s}=\cos\Theta$ is the projection of the magnetic moment onto the z axis. Notice that Landau-Lifshitz equation can be recast as the set of equations

$$\frac{d\Phi}{dt}=\frac{\gamma}{M_s}\frac{\delta\mathcal{E}}{\delta m},\quad \frac{dm}{dt}=-\frac{\gamma}{M_s}\frac{\delta\mathcal{E}}{\delta\Phi}. \tag{77}$$

The total energy $\mathcal{E}(\Phi,m)$ splits into the sum

$$\begin{aligned}\mathcal{E}(\Phi,m)&=\mathcal{E}_{\text{ex}}(\Phi,m)+\mathcal{E}_{\text{an}}(\Phi,m)+\mathcal{E}_{\text{demag}}(\Phi,m)\\ &=A[(\nabla\Theta)^2+\sin^2\Theta(\nabla\Phi)^2]\\ &\quad-K_\parallel\frac{M_x^2}{M_s^2}+K_\perp\frac{M_z^2}{M_s^2}-\frac{1}{2}\vec{M}\cdot\vec{H}_d\\ &=A\left[\frac{1}{1-m^2}(\nabla m)^2+(1-m^2)(\nabla\Phi)^2\right]\\ &\quad-K_\parallel\cos^2\Phi(1-m^2)+K_\perp m^2-\frac{1}{2}\vec{M}\cdot\vec{H}_d\end{aligned} \tag{78}$$

with A being the exchange constant, $K_\parallel$, $K_\perp$ being the anisotropy constants, and $\vec{H}_d$ being the demagnetizing field. Recall that $\vec{H}_d(\vec{r})=-\nabla\Phi_d(\vec{r})$, with $\nabla^2\Phi_d(\vec{r})=-4\pi\rho_d$ and $\rho_d=-\nabla\cdot\vec{M}$. Equivalently,[31]

$$\begin{aligned}\Phi_d(\vec{r})&=\int_V d^3\vec{r}'\vec{M}(\vec{r}')\cdot\nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\\ &=-\int_V d^3\vec{r}'\frac{\nabla'\cdot\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}+\int_{\partial V}d\vec{S}'\cdot\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\\ &=\int_V d^3\vec{r}'\frac{\rho_d(\vec{r}')}{|\vec{r}-\vec{r}'|}+\int_{\partial V}d^2\vec{r}'\frac{\sigma_d(\vec{r}')}{|\vec{r}-\vec{r}'|}\end{aligned} \tag{79}$$

with $\sigma_d=\vec{M}\cdot\vec{n}$ being the effective surface "charge" density and V being the volume of the system. Consequently, the demagnetizing energy can be written as

$$\mathcal{E}_{\text{demag}}=\frac{1}{2}\int_V d^3\vec{r}\rho_d(\vec{r})\Phi_d(\vec{r})+\frac{1}{2}\int_{\partial V}d^2\vec{r}\;\sigma_d(\vec{r})\Phi_d(\vec{r}). \tag{80}$$

We are dealing with a two-dimensional micrometric object, so the surface-energy term dominates over the volume-energy term. We can approximate this surface term by an effective easy plane anisotropy contribution given by

$$\mathcal{E}_{\text{demag},S}=\int_V d^3\vec{r}\;2\pi M_z^2(\vec{r}). \tag{81}$$

This gives for the total energy the following:

$$\begin{aligned}\mathcal{E}(\Phi,m)&=A\left[\frac{1}{1-m^2}(\nabla m)^2+(1-m^2)(\nabla\Phi)^2\right]\\ &\quad-K_\parallel\cos^2\Phi(1-m^2)+\big(K_\perp+2\pi M_s^2\big)m^2,\end{aligned} \tag{82}$$

and the equations of motion (77) become

$$\begin{aligned}\frac{M_s}{\gamma}\frac{d\Phi}{dt}&=-\frac{2Am}{(1-m^2)^2}(\nabla m)^2-\frac{2A}{1-m^2}\triangle m-2Am(\nabla\Phi)^2\\ &\quad+2K_\parallel\cos^2\Phi\,m+2\big(K_\perp+2\pi M_s^2\big)m,\\ \frac{M_s}{\gamma}\frac{dm}{dt}&=-K_\parallel\sin(2\Phi)(1-m^2)-4Am\nabla m\cdot\nabla\Phi\\ &\quad+2A(1-m^2)\triangle\Phi.\end{aligned} \tag{83}$$

In the slow-dynamics regime ($|\dot{\vec{X}}|\ll 1$), solutions (Φ,m) can be split into $\Phi=\Phi_0+\Phi_1$ and $m=m_0+m_1$, where Φ_0 and m_0 are the static solutions of the Landau-Lifshitz equation (we consider the anisotropy interaction to be weak enough so that the static solutions of the Hamiltonian $\mathcal{E}_{\text{ex}}+\mathcal{E}_{\text{demag}}$ are valid for our problem) and where Φ_1 and m_1 are linear on $|\dot{\vec{X}}|$. Static solutions are given by Eq. (24). As discussed in Sec. III, in the weak-bending regime, we can approximate $\tilde{r}\simeq r$ so that $\nabla\Phi_0=n_v\frac{\hat{e}_\phi}{r}$ and $\nabla m_0=\frac{dm_0}{dr}\hat{e}_r$.

Linearizing Eq. (83) and taking into account that $\frac{d\Phi}{dt}=-\dot{\vec{X}}\cdot\nabla\Phi$ and $\frac{dm}{dt}=-\dot{\vec{X}}\cdot\nabla m$, we obtain the equations of motion:

$$\begin{aligned}\frac{M_s}{\gamma}n_v\dot{\vec{X}}\cdot\frac{\hat{e}_\phi}{r} &= \frac{2A}{1-m_0^2}\triangle m_1 + m_1\left[\frac{2A\left(1+3m_0^2\right)}{\left(1-m_0^2\right)^3}\left(\frac{dm_0}{dr}\right)^2 + \frac{4Am_0\triangle m_0}{\left(1-m_0^2\right)^2} + \frac{2A}{r^2} - 2K_\parallel\cos^2\Phi_0 - 2\left(K_\perp + 2\pi M_s^2\right)\right]\\ &\quad + \frac{4A\,m_0}{\left(1-m_0^2\right)^2}\frac{dm_0}{dr}\hat{e}_r\cdot\nabla m_1 + 4An_v\,m_0\frac{\hat{e}_\phi}{r}\cdot\nabla\Phi_1 + 2K_\parallel m_0\sin(2\Phi_0)\Phi_1,\\ \frac{M_s}{\gamma}\dot{\vec{X}}\cdot\hat{e}_r\frac{dm_0}{dr} &= -2A\left(1-m_0^2\right)\triangle\Phi_1 + 4An_v\,m_0\nabla m_1\cdot\frac{\hat{e}_\phi}{r} + 4A\,m_0\frac{dm_0}{dr}\hat{e}_r\cdot\nabla\Phi_1\\ &\quad - 2K_\parallel\sin(2\Phi_0)m_0m_1 + 2K_\parallel\cos(2\Phi_0)\left(1-m_0^2\right)\Phi_1.\end{aligned} \tag{84}$$

Asymptotic expressions for the $O(|\dot{\vec{X}}|)$ corrections to the out-of-plane vortex shape can be determined by substituting Eq. (24) into Eq. (84). In doing so, we obtain

$$m_1 = -\frac{M_s}{2\gamma}n_v\frac{\dot{\vec{X}}\cdot\hat{e}_\phi}{\left(K_\perp+2\pi M_s^2\right)+K_\parallel\cos^2\Phi_0}\frac{1}{r},\quad \Phi_1 = \frac{C_2M_s}{2\gamma A}\Delta_0^{3/2}(\dot{\vec{X}}\cdot\hat{e}_r)\frac{\exp(-r/\Delta_0)}{r^{1/2}} \tag{85}$$

for $r \gg \Delta_0$, and

$$m_1 = \frac{M_sC_1n_v}{3\gamma A\Delta_0^2}(\dot{\vec{X}}\cdot\hat{e}_\phi)r^3,\quad \Phi_1 = \frac{M_s}{\gamma}\frac{p}{12A}\dot{\vec{X}}\cdot\vec{r} \tag{86}$$

for $r \ll \Delta_0$. Computation of the mass of the vortex core can be made via $\vec{\Pi}_t$, which should be proportional to $\dot{\vec{X}}$ in this limit:

$$\begin{aligned}\vec{\Pi}_t &= -\frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi)m\\ &= -\frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_0)m_0 - \frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_0)m_1\\ &\quad - \frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_1)m_0 - \frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_1)m_1.\end{aligned} \tag{87}$$

Notice that $-\frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_0)m_0 = \vec{0}$ because it corresponds to the momentum of the static solution. The last term of Eq. (87) can be neglected because it is quadratic in $|\dot{\vec{X}}|$. Therefore it remains to calculate the second and third terms, which are given by

$$\begin{aligned}&-\frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_0)m_1\\ &= -\frac{M_s}{\gamma}\int_{r\leqslant\Delta_0} d^2\vec{r}(\nabla\Phi_0)m_1 - \frac{M_s}{\gamma}\int_{r\geqslant\Delta_0} d^2\vec{r}(\nabla\Phi_0)m_1\\ &= \frac{\pi}{\gamma^2}\left[\frac{M_s^2}{K_\perp+2\pi M_s^2}\frac{1/2}{\sqrt{1+\frac{K_\parallel}{K_\perp+2\pi M_s^2}}}\ln(R/\Delta_0) - \frac{1}{28}\right]\dot{\vec{X}}\end{aligned} \tag{88}$$

and

$$\begin{aligned}&-\frac{M_s}{\gamma}\int d^2\vec{r}(\nabla\Phi_1)m_0\\ &= -\frac{M_s}{\gamma}\int_{r\leqslant\Delta_0} d^2\vec{r}(\nabla\Phi_1)m_0 - \frac{M_s}{\gamma}\int_{r\geqslant\Delta_0} d^2\vec{r}(\nabla\Phi_1)m_0\\ &= \frac{\pi}{\gamma^2}\left[-\frac{11}{168} + \frac{4}{49}(1-\Xi\cdot e^2)\right]\dot{\vec{X}},\end{aligned} \tag{89}$$

respectively. Notice that $\Xi = \int_1^{R/\Delta_0} dx\frac{\exp(-2x)}{x} \simeq \int_1^\infty dx\frac{\exp(-2x)}{x} = 0.049$ because we are interested in the limit $R \gg \Delta_0$.

Collecting all terms for the momentum, we get for the total mass density

$$\rho_M = \frac{\pi}{\gamma^2}\left[\frac{M_s^2}{K_\perp+2\pi M_s^2}\frac{\ln(R/\Delta_0)}{2\sqrt{1+\frac{K_\parallel}{K_\perp+2\pi M_s^2}}} - 0.049\right]. \tag{90}$$

Notice that we are interested in the limit $R \gg \Delta_0$, so that the term involving $\ln(R/\Delta_0)$ is the dominant one. Furthermore, redefining the exchange length by a factor close to unity we can always absorb the small numerical constant in Eq. (90) into the logarithmic term. Magnetocrystalline anisotropies, if they are sufficiently large, destroy the circularly polarized state. Consequently, materials like permalloy, used in the studies of the vortex state, have negligible magnetocrystalline anisotropy energy as compared to the demagnetizing energy. This means that the above expression for the vortex mass density can be reduced to

$$\rho_M \simeq \frac{1}{4\gamma^2}\ln(R/\Delta_0). \tag{91}$$

With account of this formula, one obtains the following expressions for the parameters ω_M and ϵ_0 that determine eigenfrequencies in Eq. (76):

$$\omega_M = \frac{8\pi\gamma M_s}{\ln(R/\Delta_0)},\quad \epsilon_0 = \frac{5L}{18\pi R}\ln(R/\Delta_0). \tag{92}$$

VII. EFFECTS OF THE MAGNETIC FIELD AND DISSIPATION

In this section, we study the effects of a magnetic field on the excitation modes of the vortex state. Arbitrary directed magnetic field can be split into two components, one being in the plane of the disk and the other one being perpendicular to it. The effects of these two components can be investigated separately.

Consider first the case of a spatially uniform in-plane magnetic field, $\vec{H}_{\text{in}} = h_x\hat{e}_x + h_y\hat{e}_y$. For small displacements along the disk, the magnetic vortex develops an in-plane magnetization density given by[14]

$$\vec{M}(\vec{X}) = -\mu[\hat{z}\times\vec{X}], \quad \mu = (2\pi/3)M_s n_v R. \tag{93}$$

The Zeeman energy density term is

$$\begin{aligned}\omega_Z(\vec{X}) &= -M(\vec{X})\cdot\vec{H}_{\text{in}} = -\mu[\hat{z}\times\vec{H}_{\text{in}}]\cdot\vec{X}\\ &= \mu h_y x - \mu h_x y,\end{aligned} \tag{94}$$

and thus the total in-plane potential energy becomes

$$\begin{aligned}\omega_{XY}(\vec{X}) &= \frac{1}{2}\rho_M\omega_M^2\epsilon_0(x^2+y^2) + \mu h_y x - \mu h_x y\\ &= \frac{1}{2}\rho_M\omega_M^2\epsilon_0\left[\left(x+\frac{\mu h_y}{\rho_M\omega_M^2\epsilon_0}\right)^2\right.\\ &\quad\left.+\left(y-\frac{\mu h_x}{\rho_M\omega_M^2\epsilon_0}\right)^2\right] - \frac{1}{2}\frac{\mu^2}{\rho_M\omega_M^2\epsilon_0}\vec{H}_{\text{in}}^2.\end{aligned} \tag{95}$$

Notice that by shifting the origin of the coordinate system we retrieve the original in-plane term of the total energy density (39) except for the constant term $-\frac{1}{2}\frac{\mu^2}{\rho_M\omega_M^2\epsilon_0}\vec{H}_{\text{in}}^2$, which is field dependent. Consequently, the application of an in-plane magnetic field does not modify the excitation modes given by Eq. (76).

Consider now the effect of the magnetic field perpendicular to the plane of the disk. $\vec{H}_\perp = H\hat{z}$. Application of such a field results in the precession of the magnetic moment of the vortex about the direction of the field, described by the Landau-Lifshitz equation,[31]

$$\frac{\partial\vec{M}(t,\vec{X})}{\partial t} = -\gamma[\vec{M}(t,\vec{X})\times\vec{H}_\perp], \tag{96}$$

where γ is the electron gyromagnetic ratio. Formally, this effect can be accounted for by adding an extra term to the gyrovector. Indeed, integration of Eq. (27) (with no potential energy) on time gives $\dot{\vec{X}} = \alpha(\vec{X}\times\vec{\rho}_G)$, where $\alpha = -1/\rho_M$. With account of Eq. (93), we have

$$(\hat{z}\times\dot{\vec{X}}) = -\gamma(\hat{z}\times\vec{X})\times\vec{H}_\perp, \tag{97}$$

$$\alpha[\hat{z}\times(\vec{X}\times\vec{\rho}_G)] = -\gamma(\hat{z}\times\vec{X})\times\vec{H}_\perp. \tag{98}$$

The vector identity $\vec{a}\times\vec{b}\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$ leads to $\alpha\rho_G = -\gamma H$. Consequently, the precessional effect of the perpendicular field can be absorbed into the gyrovector density if one adds to it the term $\vec{\rho}_{G,\vec{H}_\perp} = -\frac{\gamma}{\alpha}\vec{H}_\perp = \rho_M\gamma\vec{H}_\perp$. This adds the Larmor frequency to ω_M:

$$\omega_M(H) = \frac{\rho_{G,\text{tot}}}{\rho_M} = \omega_M + \frac{\rho_{G,\vec{H}_\perp}}{\rho_M} = \omega_M + \gamma H, \tag{99}$$

so that the eigenfrequencies (76) become

$$\omega_n^\pm(H) = \frac{1}{2}\left\{\sqrt{[1+4\epsilon(H)] + \frac{4\lambda}{\rho_M\omega_M^2(H)}q_n^2} \pm 1\right\}\omega_M(H) \tag{100}$$

with $\epsilon(H)$ given by

$$\epsilon(H) = \frac{\omega_G(H)}{\omega_M(H)} = \frac{\omega_{XY}''(\vec{X}=\vec{0})}{\rho_M\omega_M^2(H)} = \frac{\epsilon_0}{(1+\gamma H/\omega_M)^2}. \tag{101}$$

Introducing the dimensionless variables $h = \gamma H/\omega_M$ and $\bar{\omega}_n^\pm(h) = \omega_n^\pm(H)/\omega_M$, we can rewrite Eq. (100) as

$$\begin{aligned}\bar{\omega}_n^\pm(h) &= \frac{1}{2}\left[\sqrt{1+\frac{4\epsilon_0}{(1+h)^2} + \frac{4\lambda}{\rho_M\omega_M^2(1+h)^2}q_n^2} \pm 1\right](1+h)\\ &\simeq \frac{1}{2}\left[\sqrt{1+\frac{4\epsilon_0}{(1+h)^2}} \pm 1\right](1+h)\\ &\quad + \frac{\text{sgn}(1+h)}{\sqrt{(1+h)^2+4\epsilon_0}}\frac{\lambda q_n^2}{\rho_M\omega_M^2}.\end{aligned} \tag{102}$$

The distance between ω_n^+ and ω_n^- equals $\Delta\omega = \omega_M + \gamma H$.

To conclude this section, we investigate the effects of the dissipation on the excitation modes of magnetic vortices. We consider only the zero field case. Derivation of the corresponding expressions when a magnetic field is applied is straightforward. The way to introduce dissipation into our equations is by adding a damping term of the form $-D\dot{\vec{X}}$ (D being the damping constant) to Eq. (27).[14,33] Therefore the elastic Thiele's equation becomes

$$\rho_M\ddot{\vec{X}} - \lambda\partial_z^2\vec{X} + \dot{\vec{X}}\times\vec{\rho}_G - D\dot{\vec{X}} + \rho_M\omega_M^2\epsilon_0\vec{X} = 0. \tag{103}$$

Repeating the procedure of Sec. IV with the above equation in the massive vortex case ($\rho_M \neq 0$), we obtain the following equation for the frequency modes:

$$\omega^2 + (\omega_M + id)\omega - \omega_M^2\epsilon(q) = 0 \tag{104}$$

with $d = D/\rho_M$ and $\epsilon(q) = \epsilon_0 + \frac{\lambda}{\rho_M\omega_M^2}q^2$. The (complex) roots of this equation, $\omega_\pm = \text{Re}(\omega_\pm) + i\,\text{Im}(\omega_\pm)$, are given by

$$\begin{aligned}\text{Re}(\omega_\pm) &= \mp\frac{r^{1/2}}{2}\cos(\theta/2) - \frac{\omega_M}{2},\\ \text{Im}(\omega_\pm) &= \mp\frac{r^{1/2}}{2}\sin(\theta/2) - \frac{d}{2}\end{aligned} \tag{105}$$

with

$$\begin{aligned}r &= \sqrt{\{[1+4\epsilon(q)]\omega_M^2 - d^2\}^2 + 4d^2\omega_M^2},\\ \theta &= \arg\left(\{[1+4\epsilon(q)]\omega_M^2 - d^2\} + i(2d\omega_M)\right)\\ &= \arctan\left\{\frac{2d\omega_M}{[1+4\epsilon(q)]\omega_M^2 - d^2}\right\}.\end{aligned} \tag{106}$$

In the regime of weak dissipation, $d \ll \omega_M$, we have $\theta \simeq \arctan\{\frac{2d}{[1+4\epsilon(q)]\omega_M}\}$ and $r \simeq [1+4\epsilon(q)]\omega_M^2$. As $\cos[\arctan(x)/2] \simeq 1 - \frac{x^2}{8} + o(x^4)$ and $\sin[\arctan(x)/2] \simeq \frac{x}{2} + o(x^3)$ if $|x| \ll 1$, we finally obtain

$$\begin{aligned}\text{Re}(\omega_\pm) &= \mp\left\{\frac{1}{2}[\sqrt{1+4\epsilon(q)} \pm 1] - \frac{1}{4}\frac{(d/\omega_M)^2}{(1+4\epsilon(q))^{3/2}}\right\}\omega_M\\ &\simeq \mp\left\{\frac{\omega_M}{2}[\sqrt{1+4\epsilon_0} \pm 1] - \frac{\omega_M}{4}\frac{(d/\omega_M)^2}{(1+4\epsilon_0)^{3/2}}\right.\\ &\quad\left.+\frac{\lambda}{\sqrt{1+4\epsilon_0}}\left[1+\frac{3}{2}\frac{(d/\omega_M)^2}{(1+4\epsilon_0)^2}\right]\frac{q^2}{\rho_M\omega_M}\right\}\end{aligned} \tag{107}$$

and

$$\mathrm{Im}(\omega_\pm) = \left[\mp\frac{1}{\sqrt{1+4\epsilon(q)}} - 1\right]\frac{d}{2}, \tag{108}$$

$$\frac{\mathrm{Im}(\omega_+)}{\mathrm{Im}(\omega_-)} = -\frac{1+\sqrt{1+4\epsilon(q)}}{1-\sqrt{1+4\epsilon(q)}} = \frac{[1+\sqrt{1+4\epsilon(q)}]^2}{4\epsilon(q)}. \tag{109}$$

VIII. CONCLUSIONS

We have studied excitation modes of vortices in circularly polarized mesoscopic magnetic disks that correspond to the stringlike gyrotropic waves in the vortex core. This problem was studied by classical treatment based upon Landau-Lifshitz equation and by quantum treatment based upon Hamiltonian approach. The quantum problem is interesting on its own as it is equivalent to the problem of quantum oscillations of a charged string confined in a parabolic potential and subjected to the magnetic field, which in its turn, is a generalization of the problem of the field-induced orbital motion of the electron in a potential well. Both treatments rendered identical results. Our solution generalizes the expression for the frequency of the gyrotropic motion of the vortex for the case of the finite wave number q, as $\omega_-(q) = \omega_G + \gamma M_s (q\Delta_0)^2 \ln(R/\Delta_0)$, where ω_G is the conventional gyrofrequency, γ is the gyromagnetic ratio, M_s is the saturation magnetization, Δ_0 is the exchange length, and R is the radius of the disk. This expression is valid in the long-wave limit $q\Delta_0 \ll 1$. The wave number is quantized, $q_n = 2\pi n/L$, where L is the thickness of the disk and n is an integer. For, e.g., a permalloy disk ($M_s \approx 8 \times 10^5$ A/m^2, $A \approx 1.3 \times 10^{-11}$ J/m) of thickness $L = 100$ nm and diameter $2R = 1.5$ μm, the $n = 1$ mode is separated from the uniform gyrotropic mode by $f = \omega/(2\pi) \approx 7$ GHz, while for $2R = 1$ μm and $L = 50$ nm the separation is about 25 GHz. This mode could be excited by, e.g., a tip of a force microscope or a micro-SQUID placed at the center of the disk. Such measurement, while challenging, is definitely within experimental reach.

Throughout this paper, we considered of a nonzero mass of the vortex. The finite value of the mass splits the gyrotropic into two modes, one with the gap ω_G and the other with the gap $\omega_M \gg \omega_G$. The latter depends explicitly on the vortex mass. The vortex mass density has been computed by us as a coefficient of proportionality, ρ_M, in the kinetic energy of the moving vortex $\rho_M v^2/2$. It is given by $\rho_M \simeq 1/(4\gamma^2)\ln(R/\Delta_0)$, where R is the radius of the disk, Δ_0 is the exchange length, and γ is the gyromagnetic ratio. For a 25-nm thick micron-size permalloy disk, this gives the vortex mass in the ball park of 10^{-23} kg, which is close to the experimental value estimated for a comparable size permalloy ring.[25] Our result for the mass gives $\omega_M = 8\pi\gamma M_s/\ln(R/\Delta_0)$. It is higher than frequencies of the modes studied in Refs. 21 and 22 and is typically in the same frequency range with the uniform ferromagnetic resonance of the magnetic material. It would be interesting to investigate this frequency range experimentally alongside with the low-frequency gyrotropic mode. One can also test in experiment the explicit field dependence of the vortex modes, computed in this paper. So far, we have studied the low field that only slightly disturbs the vortex state formed in a zero field. However, the statement concerning the existence of the additional mode due to the finite vortex mass should apply to higher fields as well. This case, however, defies analytical study and must employ full-scale numerical micromagnetic calculations. When the field is sufficient to fully polarize the disk in the perpendicular direction, we expect the high-frequency mode to evolve into the uniform ferromagnetic resonance.

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