Large-D Gravity and Low-D Strings

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We show that in the limit of a large number of dimensions a wide class of nonextremal neutral black holes has a universal near-horizon limit. The limiting geometry is the two-dimensional black hole of string theory with a two-dimensional target space. Its conformal symmetry explains the properties of massless scalars found recently in the large-D limit. For black branes with string charges, the near-horizon geometry is that of the three-dimensional black strings of Horne and Horowitz. The analogies between the α' expansion in string theory and the large-D expansion in gravity suggest a possible effective string description of the large-D limit of black holes. We comment on applications to several subjects, in particular to the problem of critical collapse.

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The theory of general relativity is notoriously difficult to solve in the situations where it becomes most interesting, namely, the nonlinear regimes where black holes are involved. It has been proposed that the situation can be improved if one views the number of spacetime dimensions D as a parameter that can vary continuously [1,2]. Then, if D can be regarded as a large number, a perturbative expansion in $1/D$ is feasible. References [1,2] show through explicit examples that this large-D expansion can be a very efficient method for analytic calculations, achieving surprisingly accurate results even for relatively low dimensions.

This success stems from the fact that general relativity and, in particular, its black hole solutions, simplify drastically in the limit $D \to \infty$: when D is very large; the gravitational field is strongly localized within a region very close to the black hole horizon. Near a horizon of radius r_0, the potential develops a very large gradient, $\sim D/r_0$, with the result that the geometry further than a distance $\sim r_0/D$ from the horizon is essentially a flat spacetime [2].

The appearance of two separate scales $r_0/D \ll r_0$ can be used to identify two different regions in the black hole geometry: a "far region," defined by $r - r_0 \gg r_0/D$, and a "near-horizon region," where $r - r_0 \ll r_0$. The dynamics in each of them is quite different: in the far region there are waves that propagate in flat spacetime; the near region contains the dynamics intrinsic to the black hole. These two sets of degrees of freedom interact in an "overlap region," $r_0/D \ll r - r_0 \ll r_0$, common to both.

In Ref. [2] it has been argued that the existence of a far region in which the metric becomes exactly flat in the limit $D \to \infty$ is generic for very wide classes of black holes—essentially, all black holes whose horizon length scales, as well as their asymptotic gravitational field, remain finite as $D \to \infty$.

In this Letter, we investigate the properties of the near-horizon region. We find that for many neutral black holes—including all known vacuum and anti–de Sitter (AdS) nonextremal black holes whose size, again, remains finite in the limit $D \to \infty$—this region is universally described by a well-known geometry: the two-dimensional (2D) string-theory black hole of [3].

This solution played a prominent role in black hole research in string theory in the early 1990s, since its exact string-theoretical description is known. The meaning of its appearance in the context of this Letter is unclear but tantalizing. One consequence is that the 2D conformal symmetry of this geometry explains the properties of the amplitudes for massless scalar fields in these backgrounds found in [2]. Our results imply that the same symmetry is also present in the limit $D \to \infty$ near the horizon of many other neutral black holes, including rotating black holes and AdS black holes. We shall present further clues to a possible role of string theory in the large-D limit of black holes, including the fact that solutions with an electric string charge have as their near-horizon region the 3D black string of Horne and Horowitz [4].

Let us begin with the Schwarzschild-Tangherlini solution in $D = 3 + n$ dimensions

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n+1}^2, \qquad f(r) = 1 - \left(\frac{r_0}{r}\right)^n, \tag{1}$$

where $d\Omega_{n+1}^2$ is the line element of the round $(n+1)$ sphere, and introduce the coordinate $\mathsf{R} = (r/r_0)^n$ in terms of which

$$ds^2 = -\frac{\mathsf{R}-1}{\mathsf{R}}dt^2 + \frac{r_0^2}{n^2}\mathsf{R}^{2/n}\frac{d\mathsf{R}^2}{\mathsf{R}(\mathsf{R}-1)} + r_0^2\mathsf{R}^{2/n}d\Omega_{n+1}^2. \tag{2}$$

The near-horizon (nh) region at large n is defined by $\ln\mathsf{R} \ll n$, so we find

$$ds_{\rm nh}^2 = -\frac{\mathsf{R}-1}{\mathsf{R}}dt^2 + \frac{r_0^2}{n^2}\frac{d\mathsf{R}^2}{\mathsf{R}(\mathsf{R}-1)} + r_0^2 d\Omega_{n+1}^2. \tag{3}$$

We see that the size along the radial direction is very small when n is large. As observed in [2], this region would be traversed very quickly, on a time $\sim r_0/n$, by freely falling observers. It is then convenient to rescale the time to $\hat{t} = nt/(2r_0)$. Furthermore, if we change $\mathsf{R} = \cosh^2\rho$ the near-horizon metric becomes

$$ds_{\rm nh}^2 = \frac{4r_0^2}{n^2}(-\tanh^2\rho d\hat{t}^2 + d\rho^2) + r_0^2 d\Omega_{n+1}^2. \tag{4}$$

The part of the metric in parentheses is the 2D string black hole of [3], which is the coset manifold $SL(2,\mathbb{R})/U(1)$.

The appearance of this geometry in this context is actually expected from an observation made in [5]. It is well known that the dimensional reduction of the Einstein-Hilbert action on a sphere,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + r_0^2 e^{-4\Phi/(n+1)}d\Omega_{n+1}^2, \tag{5}$$

where $g_{\mu\nu}(x^\lambda)$ is a 2D metric and $\Phi(x^\lambda)$ a scalar field, yields a specific 2D dilaton gravity action,

$$I = \frac{\Omega_{n+1}r_0^{n+1}}{16\pi G}\int d^2x\sqrt{-g}e^{-2\Phi} \times\left(R + \frac{4n}{n+1}(\nabla\Phi)^2 + \frac{n(n+1)}{r_0^2}e^{4\Phi/(n+1)}\right). \tag{6}$$

Maybe less well known [5] is that in the limit $n \to \infty$ the action (6) is equivalent to the 2D string action

$$I = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}e^{-2\Phi}[R + 4(\nabla\Phi)^2 + 4\lambda^2], \tag{7}$$

with $G_2 = \lim_{n\to\infty} G/(\Omega_{n+1}r_0^{n+1})$ after we identify the cosmological constant parameter $\lambda = n/(2r_0)$. The dilaton field for the 2D string black hole solution, $\Phi = -\ln\cosh\rho$, is correctly obtained by comparing Eqs. (2) and (5) and taking $n \to \infty$. Keeping r_0/n finite amounts to keeping finite the Hawking–temperature $T_H = \lambda/2\pi$ of both the large-D Schwarzschild-Tangherlini and the 2D string black hole.

The presence of the 2D string-theory black hole geometry in (4) implies that the amplitudes for the propagation of waves in this background will realize the conformal symmetry $SL(2,\mathbb{R})$, thus providing a rationale for the results of [2]. The Minkowski vacuum in the limit of large n corresponds, after rescaling t by a factor of n, to the linear dilaton vacuum of the 2D theory.

Other features of fields in the background, Eq. (1), also have a nice interpretation upon dimensional reduction on the sphere. The action for a minimal scalar $\Psi = \psi(t,r)Y_{n+1}^{(l)}(\Omega)$, where $Y_{n+1}^{(l)}(\Omega)$ are spherical harmonics on S^{n+1}, in the geometry, Eq. (5), is

$$\begin{aligned} I[\Psi] &= \frac{1}{2}\int d^Dx\sqrt{-g(D)}(\nabla_{(D)}\Psi)^2 \\ &= \frac{\kappa}{2}\int d^2x\sqrt{-\hat{g}}e^{-2\Phi} \\ &\quad\times\left[(\hat{\nabla}\psi)^2 + 4e^{4\Phi/(n+1)}\frac{l}{n}\left(\frac{l}{n}+1\right)\psi^2\right] \\ &\to \frac{\kappa}{2}\int d^2x\sqrt{-\hat{g}}e^{-2\Phi}\left[(\hat{\nabla}\psi)^2 + 4\frac{l}{n}\left(\frac{l}{n}+1\right)\psi^2\right], \end{aligned} \tag{8}$$

with $\kappa = \Omega_{n+1}r_0^{n+1}$, where we have rescaled the 2D metric $g_{\mu\nu} = \lambda^{-2}\hat{g}_{\mu\nu}$. First consider the fields with $l \sim \mathcal{O}(n^0)$. These propagate in the 2D black hole geometry as massless, nonminimally coupled scalars. If their frequency in the far region time t is $\omega r_0 \sim \mathcal{O}(n^0)$, then in the near-horizon time $\hat{t}$ it is $\hat{\omega} = 2\omega r_0/n \sim \mathcal{O}(n^{-1})$. These excitations encounter a dilaton barrier much higher than their energy and have a vanishingly small amplitude for tunneling between the near and far regions, so that they can be said to decouple. Instead, waves of frequency $\hat{\omega} \sim \mathcal{O}(n^0)$ have a nonzero probability to penetrate or to pass above the barrier [2]. Thus, the large-D limit is not a decoupling limit. In this sense, these near-horizon geometries are similar to the near-horizon region of a near-extremal Neveu-Schwarz five-brane [6].

Waves with large angular momentum $\hat{l} = 2l/n = \mathcal{O}(n^0)$ have an effective mass $\sim\hat{l}/r_0$. They must have frequency $\hat{\omega} > \hat{l} + 1$ in order to escape to the asymptotic region [2]. These excitations probe scales much smaller than the radius of the $(n+1)$ sphere and effectively see the geometry:

$$ds_{\rm nh}^2 = \frac{4r_0^2}{n^2}(-\tanh^2\rho d\hat{t}^2 + d\rho^2 + d\mathbf{x}_{n+1}^2). \tag{9}$$

We can verify the presence of the same geometry in other neutral black holes. Consider the Myers-Perry solutions with (for simplicity) a single rotation [7]. In terms of the coordinate R introduced above, they take the form

$$\begin{aligned} ds^2 &= -dt^2 + \frac{1}{\sigma\mathsf{R}}(dt + \alpha r_0\sin^2\theta d\phi)^2 \\ &\quad + \mathsf{R}^{2/n}r_0^2\left(\frac{\sigma}{n^2\delta}\frac{d\mathsf{R}^2}{\mathsf{R}^2} + \sigma d\theta^2 \right.\\ &\quad \left. + (1+\alpha^2\mathsf{R}^{-2/n})\sin^2\theta d\phi^2 + \cos^2\theta d\Omega_{n-1}^2\right), \end{aligned} \tag{10}$$

where $\sigma = 1 + \alpha^2\cos^2\theta/\mathsf{R}^{2/n}$ and $\delta = 1 + \alpha^2/\mathsf{R}^{2/n} - 1/\mathsf{R}$. The usual rotation parameter is $a = \alpha r_0$. We see again that the metric in radial directions is small. So we consider waves of frequency $\omega = \mathcal{O}(n)$. The metric that a partial wave sees will be different depending on the value of the component l_ϕ of angular momentum along the rotation direction ϕ. When the wave has small impact parameter l_ϕ/ω in the plane of rotation, as is the case when $l_\phi = \mathcal{O}(n^0)$, it probes the region near the pole at $\theta = 0$. At larger impact parameters, with $l_\phi = \mathcal{O}(n)$, it probes the region around a finite angle θ_0. Denoting $v = \alpha \sin\theta_0/\sqrt{1+\alpha^2}$, the appropriate rescalings are

$$\hat{t} = \frac{n}{2r_0}\frac{t}{\sqrt{1-v^2}}, \qquad \hat{y} = \frac{n(1+\alpha^2)}{2\alpha}\frac{v}{\sqrt{1-v^2}}\phi, \tag{11}$$

$$\hat{\theta} = \frac{n\sqrt{1+\alpha^2}}{2}(\theta_0 - \theta). \tag{12}$$

If we set $\mathsf{R} = (1+\alpha^2)^{-1}\cosh^2$ we find

$$ds^2 = \frac{4r_0^2}{n^2}(1-v^2)\bigg(-d\hat{t}^2 + d\hat{y}^2 + \frac{(d\hat{t} + v d\hat{y})^2}{(1-v^2)\cosh^2\rho} + d\rho^2 + d\hat{\theta}^2 + \frac{n^2}{4(1-v^2)}\cos^2\theta d\Omega_{n-1}^2\bigg). \tag{13}$$

The $(\hat{t}, \hat{y}, \rho)$ part of the metric is the result of adding a line $\hat{y}$ to the 2D string black hole and performing a boost of velocity v along $\hat{y}$. So locally this geometry is equivalent to the 2D string black hole.

These conclusions generalize to the case in which the black hole has several nonzero spins: for $\mathcal{O}(n)$ frequencies and angular momenta along a direction in which the black hole rotates, we find the 2D string black hole with a boost along the rotation direction. For small, $\mathcal{O}(n^{-1})$, impact parameters we recover the static 2D black hole.

When the number of nonzero spins grows like $n/2$, the situation requires slight modifications. As an illustrative case we take a black hole in odd dimensions with all possible rotation parameters turned on and equal to each other, $a_i = \alpha r_0$. The radial coordinate of the 2D black hole, in terms of the Boyer-Lindquist radius r [7], is $\cosh^2\rho = (r/r_H)^{n(1-\alpha^2)}$. The horizon radius r_H is

$$r_H = r_0\sqrt{1-\alpha^2}\bigg(1 + \frac{1}{n}\frac{\ln(1-\alpha^2)}{1-\alpha^2} + \mathcal{O}(n^{-2})\bigg). \tag{14}$$

We assume that $\alpha < 1$ remains fixed as $n \to \infty$. Then, rescaling t and the angles appropriately as before, we recover the boosted 2D string black hole. In the cases where all the spins of the black hole are turned on, the solution admits an extremal limit. This would correspond in this example to $\alpha \to 1$, for which the above expansion breaks down. Extremal rotating black holes have zero temperature and we expect their near-horizon geometry to be locally inequivalent to the 2D string black hole. We shall not discuss in this Letter the detailed analysis required to study their near-horizon geometry.

Let us now consider nonrotating AdS black holes. Their metric is of the same form as Eq. (1) but with $f(r) = 1 - (r_0/r)^n + r^2/L^2$. Thus, when we take the large n limit keeping the coordinate $\mathsf{R} = (r/r_0)^n$ finite we find

$$ds^2 = -\frac{\mathsf{R}/\mathsf{R}_0 - 1}{\mathsf{R}}dt^2 + \frac{r_0^2}{n^2}\frac{d\mathsf{R}^2}{\mathsf{R}(\mathsf{R}/\mathsf{R}_0 - 1)} + r_0^2 d\Omega_{n+1}^2, \tag{15}$$

where $\mathsf{R}_0 = L^2/(r_0^2 + L^2)$. The only change relative to Eq. (4) is a shift in the location of the horizon to $\mathsf{R} = \mathsf{R}_0 < 1$. It can be absorbed in a rescaling of the time coordinate and angles as we have done in the rotating case. It amounts to a change in the mass of the 2D string black hole. The result also extends to black branes in AdS. When rotation is added, the results parallel the ones we have obtained for vacuum black holes. Our result suggests that the flat space limit, $L \to \infty$, of AdS holography could be particularly simple in the large-D limit.

We conjecture that the appearance of the 2D string black hole is a universal feature of the large-D limit of nonextremal neutral black holes, at least when the limit is taken in such a manner that the length scales of the horizon remain finite. Waves of frequencies and angular momenta $\omega, l \sim \mathcal{O}(D)$ perceive the geometry, Eq. (9).

The geometry near the horizon of other black holes and black branes also simplifies considerably in the large-D limit. In general, however, it is not obvious if or how they bear any relationship to low-dimensional string theory. Such a relationship does occur for black p-branes that source a three-form field strength (and no dilaton) (see Appendix A.2 of [8] for details of the solutions). Their near-horizon geometry is

$$n^2 ds^2 = -\bigg(1 - \frac{1}{\mathsf{R}}\bigg)d\hat{t}^2 + \bigg(1 - \frac{u^2}{\mathsf{R}}\bigg)d\hat{y}^2 + \frac{d\mathsf{R}^2}{(\mathsf{R}-u^2)(\mathsf{R}-1)} + n^2 d\Omega_{n+1}^2 + n^2 d\mathbf{z}_{p-1}^2. \tag{16}$$

The sector $(\hat{t}, \hat{y}, \mathsf{R})$ is the geometry of the 3D black strings of [4], which are a solution of 3D string theory with a known exact conformal field theory. This is less of a surprise considering that it can be obtained by adding a line y to the 2D string black hole followed by a boost and T duality along y [7]. One can easily find that the large-D limit of the extremal Reissner-Nordström black holes results in the replacement of the nonextremal horizon of the 2D string black hole with an extremal throat characteristic of the solution at finite D. We expect this type of replacement to be a general feature of large-D extremal charged solutions.

The emergence of a 2D conformal symmetry offers, at the very least, the prospect of a significant degree of control over the classical theory of very wide classes of neutral, nonextremal large-D black holes. Even more tantalizing is the appearance of low-dimensional string geometries near the horizon of large-D black holes.

Regarding D as a parameter that can be made large is essential for the stringy interpretation. The connection to the 2D string black hole instructs us to identify $\sqrt{\alpha'} \sim r_0/D$, so the large-$D$ expansion corresponds to the α' expansion in string theory. The near-horizon geometries are "stringy geometries," of size $\sim r_0/D$. The Bekenstein-Hawking (BH) entropy $S_{\rm BH}$ of Schwarzschild-Tangherlini black holes, as a function of the mass M, is $S_{\rm BH} \propto M^{1+(1/D-3)}$, and therefore in the large D limit asymptotes to the Hagedorn behavior expected from string theory, $S_{\rm BH} \propto M$ [2]. Moreover, the black hole temperature corresponds to the string scale $T_H \sim 1/\sqrt{\alpha'}$. This may be an indication of a stringlike nature of the excitations near the horizon.

In view of this, the two-region picture of large-D black hole spacetimes has the following interpretation: in the far region we keep r_0 fixed, so when $D \to \infty$ we have $\alpha' \to 0$ and the excitations that remain in this region correspond to massless gravitons propagating in flat spacetime. In the near region we keep r_0/D finite, and we obtain a string-scale geometry with string excitations. However, α' corrections to the 2D black hole are known [10] and they do not coincide with the $1/D$ corrections to the near-horizon geometry. This is not surprising: since full quantum string theory cannot be formulated consistently in these large-D spacetimes, presumably any strings in the near-horizon region should be effective strings and not fundamental ones.

Nevertheless, it would be remarkable to find, similarly to what happens in the large N limit of Yang-Mills theories, that an effective string theory emerges in the large-D limit of gravity—in this case, one has to look for the strings near a black hole horizon. If indeed this occurs, the symmetry SO$(D-2)$ of the angular sphere will likely play a role. More work is needed to put these ideas on a firmer ground and, if they are correct, identify the kind of effective string theories that can arise in this context.

The quantum theory may also be constrained by the near-horizon conformal symmetry. The strength of quantum gravitational effects can be controlled by suitably choosing how the Planck length scales with D as the number of dimensions increases. Notice that the "effective string length" r_0/D is a purely classical length scale (as it is in string theory) which can be arbitrarily separated from the quantum Planck scale; e.g., it is possible to have large α' effects but small quantum gravity effects.

It should be interesting to study the evaporation of these black holes through quantum Hawking radiation. If the Planck length is chosen to not scale with D, then there is (at least) one significant difference with the situation for a near-extremal Neveu-Schwarz five-brane, which decays by the slow leakage of the radiation of energy $\sim T_H$ that reaches the asymptotic region of the 2D black hole. For large-D black holes (and with a D-independent Planck scale) the typical energy of Hawking quanta is much larger, $\omega \sim DT_H \sim D^2/r_0$ [2,11]. The wavelength of these quanta is so small that they do not distinguish between the near and far regions, and they accelerate enormously the decay rate of the black holes. However, if the Planck length $(G\hbar)^{1/(D-2)}$ is made to shrink like $1/D$ then typical Hawking quanta will have stringlike energies $\sim D/r_0$. This may give a better chance of controlling the evaporation process by using the near-horizon 2D conformal symmetry and, possibly, an effective string description.

Finally, from a purely pragmatic viewpoint, the effective 2D formulation can be a useful tool. As an example, consider a critical collapse [12]. So far, analytic derivations of the critical exponent γ are generally not possible. One relevant exception is the derivation in [13] of the value $\gamma = 1/2$ in the Russo-Susskind-Thorlacius model. This theory differs from the 2D model, Eqs. (7) and (8), in three aspects: it takes into account semiclassical corrections, it has a large number of scalar fields instead of just one, and the scalar fields do not couple to the dilaton field in 2D. Nevertheless, it seems plausible that their result for the critical exponent is the correct one in the large-D limit, since the geometric part of the Russo-Susskind-Thorlacius model coincides classically with the action, Eq. (7). There is also numerical evidence at $D \sim \mathcal{O}(10)$ for our conjecture [14,15]. It would be gratifying to derive our conjecture that $\gamma = 1/2$ analytically and to probe numerically the large-D regime. We believe that both avenues are accessible.

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