# Disorder-Induced Critical Phenomena in the 2-D Random Field Ising Model

Author: Elisenda Ortiz Castillo.
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

**Abstract:** This paper discusses Barkhausen noise in magnetic systems in terms of avalanches near a disorder-induced critical point. We simulate the dynamics of a non-equilibrium zero-temperature Random Field Ising Model in two dimensions. Critical behaviour is analyzed from numerical simulations through scaling techniques. In addition, analytical approaches are briefly discussed.

## I. INTRODUCTION

In recent years, condensed matter research has focused in the study of complex materials. A new range of interesting phenomena have appeared as new tools for understanding non-equilibrium systems have been developed. A crutial turn-about was the addition of a new ingredient: *disorder*.

Disorder plays an important role in the dynamics of far from equilibrium systems. Usually, such systems occupy metastable[1] states that have been selected according to the history of the system. Disorder can change the free energy landscape of the system and introduce metastable states with large energy barriers. Due to this effect, when external conditions are changed non-equilibrium systems will move from one metastable state to another through a series of discrete events, which exhibit a broad range of scales. These events are collective processes that receive the name of *avalanches*. Systems exhibiting avalanches are said to produce *crackling noise*. Through the years, several systems that crackle have been studied: earthquakes [1], fluctuations in the stock market [2], fracture in disordered materials [3], the dynamics of vortex in type II superconductors [4], fluids invading porous materials [5], and many more. All this research has led to the developement of new methodologies both experimental and analytical. Particularly important was the contribution of D. Fisher [6] towards the interpretation of the charge-density wave deppining transition as a dynamic critical phenomenon. It opened the doors to the study of other non-equilibrium systems and led to the use of renormalization group methods. Despite all the effort done, we still do not have an established mathematical framework for treating complex non-equilibrium systems. Moreover, adding disorder to the picture usually makes things even more difficult. For this reason, this field in physics remains still full of open challenges and is so appealing for researchers.

In this paper we intend to study the *athermal* dynamical response of ferromagnets to an applied increasing external field $H$. For this purpose we will use the zero-temperature *Random Field Ising Model* (RFIM). In this model we can set temperature to zero since we consider that the magnetic domains have barriers to flipping that are large enough that thermal activation can be ignored. Therefore, temperature has no importance and the rellevant parameters are disorder ($R$) and the driving external field $H$.
We will see that there exist two different regimes, one for small disorders and another for large disorders. At the critical disorder ($R_c$), that separates the two phases, numerical simulations show avalanches distributed as power laws and scaling behaviour. Moreover, nearby $R_c$ the behaviour of the model is very similar to that of equilibrium systems at critical points and the critical behaviour can be studied by analytical methods such as mean-field theory and the renormalization group.

Our main objective will be to analyze this disorder-induced phase transition by using the results from the simulations obtained from the program we have developed. We compare them with preliminary studies in the literature [10,11,12,13]. At an undergraduate level, analyzing this simple model represents a great opportunity for getting familiar with new procedures and concepts that may approach us to the complex reality of disordered materials.

## II. EXPERIMENTAL WORK

In experiments, avalanches are often give rise to crackling noise as in the case of accoustic emission during Barkhausen noise experiments. In such experiment, the magnetic domains in a ferromagnetic material (ie. a slab of iron) flip over to align with an increasing external field $H(t)$. The field is produced by approaching a magnet to the ferromagnetic material. The magnetic domains will flip in avalanches making the magnetization in the material increase in sharp jumps. These pulses can then be turned into electrical signals that can be listened with loudspeakers. Appart from being listened, avalanches can also be seen in the hysteretic[2] $M(H)$ curve of the ferro-

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*Electronic address: `eortizca8@ub.edu`

[1] A metastable state is one in which the system is in a local minimum of free energy different from the absolute minimum.

[2] Hysteresis arises from the fact that the metastable state occupied by the non-equilibrium system depends on previous history.

magnet, see Fig.1.
Analogous effects are found in ferroelectric materials in response to a changing external electric field, and also in elastic transformations, for example in athermal shape-memory alloys when ramping stress or temperature [7]. Indeed, the model we discuss in this paper is presumably one of the most paradigmatic models for explaining several experimental systems.

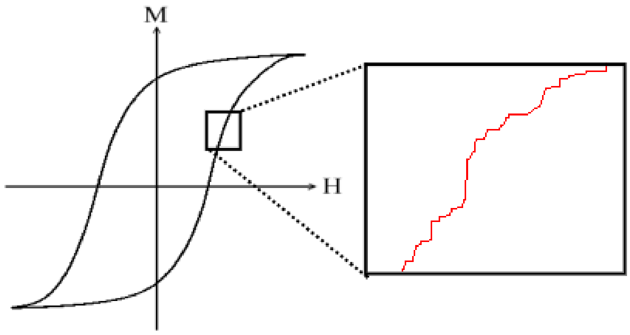


FIG. 1: **Barkhausen noise.** The magnetization curve looks smooth, but under magnification one can see it grows in a series of small jumps which indicate avalanches.

## III. MODEL

The non-equilibrium RFIM was first proposed by Dahmen and Sethna in 1993 [8]. The model is defined in a $d-dimensional$ Euclidean lattice, which for us takes $d=2$. Like in pure Ising model, at each site $i$ of the lattice there is a spin $s_i = \pm 1$ which interacts with its neighbours and with an (homogeneous) external magnetic field $H$. We consider the simple case in which the spins are coupled only to their nearest neighbours ferromagnetically. That means the coupling constant is $J > 0$. The collective behaviour is then incorporated through the nearest neighbour interactions. Now, we only have to add disorder. We model disorder by coupling each spin $s_i$ to a local random magnetic field $h_i$. The values of every $h_i$ are chosen randomly and independently from a Gaussian probability distribution

$$P(h) = \frac{1}{\sqrt{2\pi}R} e^{-(h^2/2R^2)}$$

whose standard deviation $R$ quantifies the disorder in the system. Once disorder is added we have no longer a pure Ising model but a RFIM. Consequently, the Hamiltonian of the system is

$$H = -J \sum_{<i,j>} s_i s_j - H \sum_i s_i - \sum_i h_i s_i$$

where $< i, j >$ denotes that the sum runs over nearest neighbour pairs of spins on sites $i$ and $j$.
Initially we take that the external field is $H = -\infty$ and all spins are pointing downwards ($s_i = -1$, for all $i$). Then $H$ is gradually increased until all spins are pointing up ($s_i = +1$). Our rules for evolving the spin configuration are as follows: each spin $s_i$ will flip only when doing so will decrease the energy. This occcurs at a site $i$ when the local effective field at this site, given by

$$h_i^{eff} = J \sum_j{}' s_j + H + h_i \qquad (1)$$

changes from negative to positive. Note that subindex $j$ refers only to nearest neighbours of $i$. Since $h_i$ is an initially fixed value, there are two mechanisms that can make a spin flip up: its neighbours and the external field $H$. The spin can flip because one of its neighbours flips (then it participates in a propagating avalanche). Or the spin can flip because of the slow increase of $H$ (the spin can start a new avalanche).
Therefore, when we start increasing $H$ some spin will flip first. When it does so, it makes the local field at neighbouring sites increase, and in turn, it may cause some of its neighbours to flip. The neighbours which flip can push other neighbours to flip and so on, generating an avalanche of spin flips. The avalanche finishes when there are no more spins with positive local effective fields. During an avalanche the external field $H$ is kept constant. After the avalanche has finished we will continue increasing $H$ and more avalanches of different *sizes* will take place (we call the size of an avalanche the number of spins that flip during it). Thus, the spins flip in clusters of variable sizes, which typically have fractal shapes. Eventually, all spins will have flipped and further increase of $H$ will have no effect in the system.

## IV. ALGORITHM

For simulating the system we have set up a 2-D lattice with periodic boundary conditions. Linear size of the lattice is denoted as $L$, and the total number of spins is $N = L\text{x}L$. For easiness we set $J = 1$.
We start by filling the lattice, assigning to each site $i$ a spin $s_i = -1$, a value of $h_i$ and a number $(1, 2, 3..., N)$. After that, it is time to slowly increase the external field from $H = -\infty$. It is an unnecessary waste of time to increase $H$ in small fixed increments, since it involves searching through the lattice even when there are no spins that can flip. Instead, we look at all the spins finding the next one that will flip and then increase $H$ to the necessary value so that the flip occurs. This rule is used for starting the first avalanche, and also after every other. Precisely, this means that after a finished avalanche we increase $H$ by the exact amount that will make another spin flip and then start a new one. The spins starting avalanches after an increment of $H$ are called *triggering* spins. But, how do we identify which will be the triggering spin? The answer is in Eq.(1). A spin will flip if $h_i^{eff}$ changes to positive, so the spin with larger internal field

$J\sum_j s_j + h_i$ will need a lower $H$ to achieve this change of sign. Thus, the spin that will be about to flip is the one with the largest internal field of all.
It is important to make clear that after a spin has flipped it can not flip again. That being said, the algorithm for evolving the system consists on the following steps:

*i*) Find the triggering spin $i$ for the next avalanche.
*ii*) Increment $H$ to minus the internal field of this spin.
*iii*) Put that spin as the first of a list that we call the *flipping list* (this list contains all the spins that will flip during the avalanche).
*iv*) Flip the spin, and push to the flipping list all unflipped neighbours with positive local fields.
*v*) Move to the next position in the flipping list and repeat (*iv*) until the list finishes.

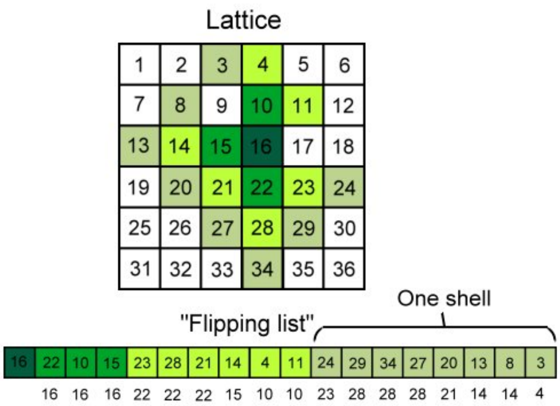


FIG. 2: **Avalanche propagation.** Spin 16 is the overall triggering of this avalanche. It will make spins 22, 10 and 15 flip so they are included next into the flipping list. These spins form the first *shell*. The second shell will consist of the spins triggered by the spins of the first shell. In the list, the number under a bin represents its triggering. In fact, there could be more than one triggering for the same spin in a bin.

The flipping list is erased after every finished avalanche so after step (*v*) follows (*i*) again. The same story is repeated until all spins in the lattice are flipped. Fig.2 aims to illustrate a little bit more how the steps from above produce the avalanche propagation.

## V. ANALYTICAL APPROACHES

Appart from simulating the model, the non-equilibrium zero-temperature RFIM can be studied analitically. The first approaches were done by K. Dahem and J.P. Sethna [9] using Mean Field Theory and Renormalization Group (RG) methods. We will not discuss in here how to perform such approaches, but present their results.
In the framework of mean field theory the interaction between any two spins, no matter how far apart, has the same strength. Every spin is then coupled to all other $N$ spins with coupling constant $J/N$. This limit of infinite-range interactions describes the model accurately for sufficiently high spatial dimensions. For dimensions under the upper critical dimension, which for the RFIM is $d_{c,up} = 6$, this theory gives inaccurate predictions of the *critical exponents* that describe the critical behaviour near the critical point. However, with a RG description of the model it is possible to improve the accuracy of the exponents predicted with mean field theory. In the RG context the exponents are calculated through power series expansions around their mean field values. These are the so-called $\epsilon-$expansions [10]. The results for the critical exponents from RG predictions compare well with large scale numerical simulations (specially when dimension tends to 6), and more importantly both methodologies agree well with the experimental measurements [11].

## VI. CRITICAL BEHAVIOUR

There is a second order phase transition in the dynamics of our model. There exist two different regimes separeted by a critical disorder at which the transition occurs. For small values of disorder one large avalanche flips most of the spins. In the opposite limit of a large disorder only small avalanches are observed. Close to the critical point there are avalanches of all sizes, that is no characteristic size exists. We define the critical disorder $R_c$ as the value of disorder for which one first finds *spanning avalanches* that, as $L \to \infty$, extend from one side of the lattice to the other. For finite size systems of lenght $L$, the transition occurs at an effective critical disorder $R_c^{eff}(L)$ larger than $R_c$. We have that for $L \to \infty$, $R_c^{eff}(L) \to R_c$. This is the reason why one should simulate systems with the largest sizes possible.

As expected for a continuous phase transition, nearby $R_c$ the correlation lenght $\xi$ is found to diverge. Besides, power law distributions emerge for avalanche sizes, durations, energies, etc. This kind of behaviour clearly indicates the presence of a critical point. Consequently, one can make an analogy with critical phenomena in equilibrium and find scaling laws for this far from equilibrium system. By doing so, one finds the *critical exponents* that define the non-equilibrium *universality class* to which our model belongs . The finite size scaling analysis involves making an *ansätze* for the form of $\xi$. Here we use the traditional scaling form

$$\xi \sim r^{-\nu}$$

where $r \equiv (R - R_c)/R_c$ is the reduced disorder. Other scaling forms were initially tested in other studies [12]. The form of the correlation length determines the scaling laws obeyed by all other physical magnitudes of the system. For example, for the magnetization and the distribution of avalanche sizes we have

$$M \sim |r|^{\beta} \mathcal{M}_{\pm}(h/|r|^{\beta\delta}) \tag{2}$$

$$D(S,r,h) \sim S^{-\tau} \mathcal{D}_{\pm}(S/|r|^{-1/\sigma}, h/|r|^{\beta\delta}) \tag{3}$$

respectively, where $h \equiv (H - H_c)/H_c$ is the reduced external field, $\sigma$ is the cutoff exponent (describing the scaling of the largest avalanche size with reduced disorder: $S_{max} \sim |r|^{-1/\sigma}$), and $\delta$ is the reduced magnetisation exponent ($(M - M_c)/M_c \sim h^{\delta}$). Notice that the field and the magnetization at the critical point, $H_c$ and $M_c$, are both different from zero. Actually, in a non-equilibrium model these magnitudes change with dimension $d$.

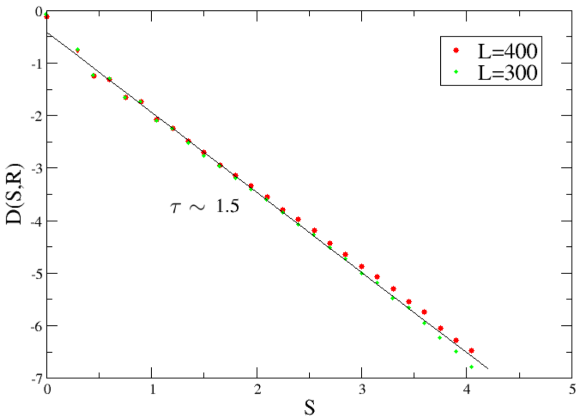


FIG. 3: **Avalanche size distribution curves** in log-log scale for system sizes $L = 300 - 400$ and $R_c = 0.54$ (mean field prediction). The black straight line is our best fit for the slope of the power law with $\tau$ exponent. In the graphic we do not include data for system sizes $L = 100 - 200$, because we finally have not used it for calculating the exponent.

In addition, we can also define the distribution of durations $T$. The duration of an avalanche is the number of steps that it takes to finish it. Considering that an avalanche propagates in shells of spins (see Fig.2), each shell constitutes one step, and the total number of shells of an avalanche is its duration. The durations distribution takes the form

$$D(T,r,h) \sim T^{-\alpha} \mathcal{D}_{\pm}(T/|r|^{-1/\sigma}, h/|r|^{\beta\delta}) \tag{4}$$

which is analogous to that of the sizes distribution.

The power law behaviour of the avalanche distributions of sizes and durations can be seen from the plots of our simulations in Fig.3 and Fig.4 respectively. Straight lines should appear when plotting data in a log-log scale whenever a good power law behaviour is found. Hence, one finds critical exponents from the slope of the best

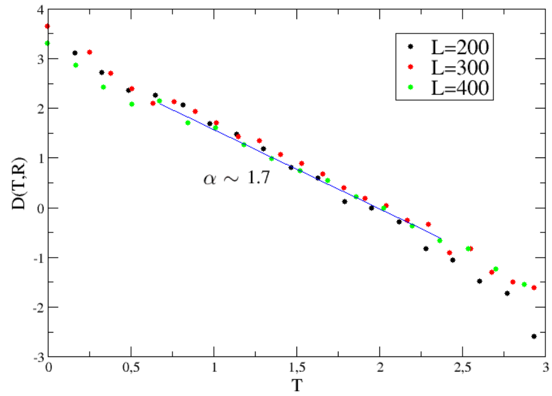


FIG. 4: **Avalanche duration distribution curves** in log-log scale for system sizes $L = 200 - 400$ and $R_c = 0.54$. The straight line corresponds to the linear regression we adjusted for extracting $\alpha$ exponent.

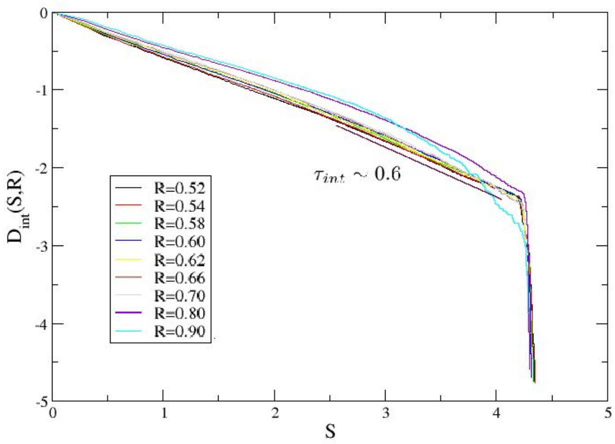


FIG. 5: **Integrated avalanche size distribution curves** for disorders $R = 0.52 - 0.90$ and system size $L = 300$ in log-log scale. For intermediate avalanche sizes, the graphic shows power law behaviour for curves with various disorders. There is a final cutoff due to finite size effectes.

linear regression adjustment. With this method we have obtained the following values for the critical exponents:

$$\tau = 1.5 \pm 0.05 \qquad \alpha = 1.7 \pm 0.1 \tag{5}$$

In Fig.5 we represent the integrated probability distribution of the avalanche sizes. This means we represent

$$D_{int}(S,r,h) = \int_0^S D(S',r,h)dS' \qquad D_{int} \sim S^{-\tau_{int}} \tag{6}$$

where $\tau_{int} = \tau - 1$. Fig.5 confirms the fact that even thought each specific $L$ has its own $R_c^{eff}(L)$, the range of

$R$'s for which we observe critical features can be relatively large.

Our program provides also data for $\xi$ and our first intention was to calculate the exponent $\nu$ as well. However, the time for running the simulations exeeded our predictions and we have not been able to compile enough data. We chose to focus on the exponents $\nu$, $\tau$ and $\alpha$, but others may also be of interest. In fact, between critical exponents there exist relationships that allow to calculate one from another [13]. For example, for our particularly case in $d = 2$ there exists the useful connectivity hyperscaling relation:

$$\frac{1}{\sigma\nu} = d - \frac{\beta}{\nu} \tag{7}$$

It seems that Eq.(7) might not hold in higher dimensions. The 2-dimensional RFIM has been extensively simulated over the years. Recently extremely large systems sizes ($L$ up to 130000) have been used [16]. In Table I we provide the values for the critical exponents obtained from large system simulations. One may want to use this information to compare it with our results (5).

| $\tau$ | $\alpha$ | $\nu$ | $\beta$ | $\sigma$ |
|---|---|---|---|---|
| $1.54 \pm 0.05$ | $1.87 \pm 0.06$ | $5.15 \pm 0.20$ | $0.15 \pm 0.04$ | $0.10 \pm 0.01$ |

TABLE I: Critical exponents for 2D non-equilibrium zero-temperature RFIM. This results were tanken from [16].

## VII. CONCLUSIONS

We have used the non-equilibrium zero-temperature 2-dimensional RFIM with Gaussian distribution of random fields to study the disorder-induced phase transition that undergo ferromagnets when driven by an increasing external field $H$. We have reproduced the dynamics by performing numerical simulations. From finite size scaling analysis we have obtained the values for two critical exponents ($\tau$, $\alpha$). Furthermore, we have found evidence that the critical behaviour can be observed even far from the critical point ($R_c, H_c$). We have also discussed that this model is solvable analytically in the frameworks of mean field theory and the renormalization group. Our results could have improved spending more time simulating larger system sizes. Moreover, other critical exponents, such as $\nu$ could have been measured.
In conclusion, we have achieved a global idea of the methods that are currently being used for the study of disordered systems and materials.


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