Derivation of Friedman equations

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Abstract: In this report we make a detailed derivation of Friedman Equations, which are the dynamical equations of a homogeneous and isotropic universe. First, we derive them in the framework of the General Relativity keeping the relativistic expressions as a reference for the rest of the report. Then, using the Newtonian formalism and solving some fundamental problems, we reach some dynamical equations for the universe and compare the results with the relativistic ones, focusing our attention on the meaning of the expressions and on the cosmological constant.

I. INTRODUCTION

We study the universe that follows from the *cosmological principle*. The cosmological principle states that the universe is isotropic and homogeneous at a large scale. Due to the symmetries that this principle implies, we can set a cosmological time which allows us to have a reference time to study the universe dynamics.

The Friedman equations, the dynamical equations of a homogeneous and isotopic universe, were first derived using General Relativity, so the question is, why had not Newton found the dynamical equations for the universe at his time? The problem was that classical mechanics is a global theory that involves the gravitational potential which diverges in a homogeneous and isotropic universe.

Einstein also had problems when he tried to apply his equations to the universe even though General Relativity is a local theory as it uses differential geometry, instead of differential calculus. At his time it was believed that the universe was static but he found dynamical equations that involved acceleration terms. Thus, the only solution he found to impose a static universe was to add a constant term called the *cosmological constant*. In the end, when he accepted the non-static universe because Friedman reached that conclusion, he said that the cosmological constant was the biggest mistake of his life. However, nowadays the cosmological constant has been introduced again to study the effect of exotic components, for instance the dark energy. Because of that, we are going to add the cosmological constant to the Newtonian derivation to see what it implies.

II. RELATIVISTIC FRIEDMAN EQUATIONS

Our aim in this chapter is to derive the general dynamic equations of a homogeneous and isotropic universe. Working in the framework of General Relativity it is essential to achieve that purpose, as General Relativity is the accepted gravitational theory nowadays. In fact, what we are going to do is to adapt a nature law, as Einstein equations are, to a universe that follows the cosmological principle. We are going to work in the frame where $c = 1$ and use Einstein notation.

A. Einstein equations

Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

The left-hand member of (1) is the Einstein tensor ($G_{\mu\nu}$) in which we have plugged an extra term that includes the cosmological constant. This extra term only gathers importance when we are working in the cosmology field, which is the case.

In the right-hand member G is the universal gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor, its explicit expression is

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu} \tag{2}$$

$g_{\mu\nu}$ is the metric of the manifold where the equations apply and u_α is the macroscopic speed of the medium.

B. Robertson-Walker metric

Notice that in Einstein tensor ($G_{\mu\nu}$) there is a Ricci tensor and a Ricci escalar. The metric with which we are going to calculate them is the one that we need to particularise our final expressions for the homogeneous and isotropic universe. Therefore now we have to find a metric ($g_{\mu\nu}$) that includes all the different aspects of the cosmological principle. The answer is the Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t) \left(\frac{1}{1 - \frac{r^2}{K^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right) \tag{3}$$

We see that the Robertson-Walker metric describes an isotropic universe, because it does not have crossed terms between time and space so there is not any privileged direction. And it also describes homogeneous universe because of the spherical symmetry.

The factor $a(t)$ is called the scale factor and it is the temporal dependence between the relative distance of two

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points of the universe. The scale factor is defined to be 1 in the present time. From now on the time dependence of the scale factor can be implicit, so $a(t) \equiv a$.

K^{-2} is directly related to the curvature radius of the spatial hypersurface. Notice that if $K^{-2} = 0$ and $a^2(t) = 1$ we have the usual euclidean metric for spherical symmetry. It can be seen that if $K^{-2} > 0$ we have a close universe (the volumic integral converges) and if $K^{-2} < 0$ we have an open universe (the volumic integral diverges).

C. Calculation of the Ricci tensor and the Ricci scalar

Remember that we need the Ricci tensor and the Ricci scalar to particularise Einstein equations for a homogeneous and isotropic universe.

First we need to calculate the Christoffel symbols of Robertson-Walker metric (3).

$$\Gamma^l_{ji} = \frac{1}{2} g^{lm} (\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij}) \tag{4}$$

Fortunately, Robertson-Walker metric is diagonal and has a symmetric connection, so the majority of the Christoffel symbols will be symmetric or null. The ones which are different from zero are

- $\Gamma^t_{rr} = \frac{a\dot{a}}{1-\frac{r^2}{K^2}}$
- $\Gamma^t_{\theta\theta} = r^2 a\dot{a}$
- $\Gamma^t_{\varphi\varphi} = r^2 a\dot{a} \sin^2\theta$
- $\Gamma^r_{tr} = \Gamma^r_{rt} = \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \Gamma^\varphi_{t\varphi} = \Gamma^\varphi_{\varphi t} = \frac{\dot{a}}{a}$
- $\Gamma^r_{rr} = \frac{r}{K^2(1-\frac{r^2}{K^2})}$
- $\Gamma^r_{\theta\theta} = -r(1-\frac{r^2}{K^2})$
- $\Gamma^r_{\varphi\varphi} = -r(1-\frac{r^2}{K^2})\sin^2\theta$
- $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}$
- $\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$
- $\Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \frac{1}{\tan\theta}$

Once the Christoffel symbols have been calculated, we can calculate the Riemann tensor

$$R^l_{kji} = \partial_i \Gamma^l_{kj} - \partial_j \Gamma^l_{ki} + \Gamma^m_{kj}\Gamma^l_{mi} - \Gamma^m_{ki}\Gamma^l_{mj} \tag{5}$$

In fact, we are only interested in the Riemann tensor components that have the same top index as the middle bottom one. These components are enough to calculate the Ricci tensor (R^m_{imj}). The only components of the Ricci tensor that are different from 0 are

- $R_{tt} = R^m_{tmt} = R^r_{trt} + R^\theta_{t\theta t} + R^\varphi_{t\varphi t} = -3\frac{\ddot{a}}{a}$
- $R_{rr} = R^m_{rmr} = \frac{a\ddot{a}}{1-\frac{r^2}{K^2}} + \frac{2\dot{a}^2}{1-\frac{r^2}{K^2}} + \frac{2}{K^2(1-\frac{r^2}{K^2})}$
- $R_{\theta\theta} = R^m_{\theta m\theta} = r^2 a\ddot{a} + 2r^2\dot{a}^2 + 2\frac{r^2}{K^2}$
- $R_{\varphi\varphi} = R^m_{\varphi m\varphi} = r^2 a\ddot{a}\sin^2\theta + 2r^2\dot{a}^2\sin^2\theta + 2\frac{r^2}{K^2}\sin^2\theta$

We can see that the Ricci tensor is diagonal, to summarize the result we can state

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{6}$$

$$R_{ii} = \frac{-g_{ii}}{a^2}(a\ddot{a} + 2\dot{a}^2 + 2K^{-2}) \tag{7}$$

Finally we can get the Ricci scalar:

$$R = g^{ik}R_{ik} = -6\frac{\ddot{a}}{a} - 6\left(\frac{\dot{a}}{a}\right)^2 - 6\frac{1}{K^2a^2} \tag{8}$$

D. Energy-momentum tensor for a perfect fluid

By definition a perfect fluid is the one that is isotropic, which means that it has to look equal to us in every direction we can move. Then, the macroscopic speed of the fluid cannot have a privileged direction, so it has only temporal component: $u^\alpha = (1, 0, 0, 0)$.

Notice that $u^t = 1$ because of the restricted relativity.

$$(u^\alpha)^2 = g_{\alpha\beta}u^\alpha u^\beta = c^2 = 1 \Rightarrow g_{tt}(u^t)^2 = c^2 \Leftrightarrow u^t = 1$$

Now, taking into account the expression (2), we can find the energy-momentum tensor for a perfect fluid. We see that it is diagonal and its components are

$$T_{tt} = \rho g_{tt} \tag{9}$$

$$T_{ii} = -p g_{ii} \tag{10}$$

We see that for our derivation we can think about the universe being filled by a perfect fluid as this kind of fluid follows the cosmological principle.

E. Friedman Equations

In the previous sections we have calculated and derived all the elements that we need to reach our goal. Now we only have to plug all the elements to Einstein equations (1).

The only equations that will be different from the null one are those which have the same indexes, since our metric is diagonal.

Therefore we start with the temporal part.

$$R_{tt} - \frac{1}{2}Rg_{tt} - \Lambda g_{tt} = 8\pi G\rho u_t u_t$$

$$-3\frac{\ddot{a}}{a}+3\frac{\ddot{a}}{a}+3\left(\frac{\dot{a}}{a}\right)^2+3\frac{1}{K^2a^2}-\Lambda=8\pi G\rho(t)$$

We arrive at

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2=\frac{8\pi G}{3}\rho(t)+\frac{\Lambda}{3}-\frac{1}{K^2a^2(t)} \tag{11}$$

Now we can study the spacial part. For each spacial component we reach the same equation

$$\frac{-g_{ii}}{a^2(t)}\left(a\ddot{a}+2\dot{a}^2+\frac{2}{K^2}\right)-\frac{1}{2}Rg_{ii}-\Lambda g_{ii}=8\pi G(-p)g_{ii}$$

Removing the metric from both sides we obtain

$$-\frac{\ddot{a}}{a}-2\left(\frac{\dot{a}}{a}\right)^2-\frac{2}{K^2a^2}+3\frac{\ddot{a}}{a}+3\left(\frac{\dot{a}}{a}\right)^2+\frac{3}{K^2a^2}-\Lambda=-8\pi Gp$$

$$\frac{\ddot{a}(t)}{a(t)}+\frac{1}{2}\left(\frac{\dot{a}(t)}{a(t)}\right)^2=-4\pi Gp+\frac{\Lambda}{2}-\frac{1}{2}\frac{1}{K^2a^2(t)} \tag{12}$$

If we pay attention, we realize that making a linear combination between (11) and (12) we can get an equation without the $(\frac{\dot{a}}{a})^2$ term, whose meaning will be easier to interpret. If we do $2\cdot(12)-(11)$ we reach

$$\frac{\ddot{a}(t)}{a(t)}=-\frac{4\pi G}{3}(\rho(t)+3p)+\frac{\Lambda}{3} \tag{13}$$

Notice that there are only two independent Friedman equations. The ones we are going to take as the reference are (11) and (13).

In conclusion we can state that Friedman equations say that in general conditions the universe is not static. To know the physical meaning of these equations first we have to make the Newtonian derivation comparing both results.

III. NEWTONIAN FRIEDMAN EQUATIONS

In this chapter we want to derive again Friedman Equations but now following Newton mechanics. We want to adapt a global theory, in this case classical mechanics, to the cosmological principle. To reach our goal we are going to solve different fundamental problems that we have not had with General Relativity, which is a local theory.

A. Divergence of the Newton potential

From classical mechanics we know that the Newton potential using spherical symmetry for a differential piece of mass (m) in a homogeneous space has the following expression:

$$V=-G\int_M\frac{1}{r}dm=-G\int_V\frac{\rho}{r}dv \tag{14}$$

We see that we have to make the integral over all the homogeneous matter distribution, which brings us to find a singularity. The total mass and volume of a Newtonian homogeneous universe has to be infinite so that the universe does not collapse, therefore necessarily any imaginary matter sphere of the universe has to have more matter outside it, to offset the internal attractive force.

To solve that problem we have to take into account that all the points of an homogeneous universe are equivalent, so with that we can study the relative movement between two random points of the universe.

We consider one point as the origin and we describe an sphere passing through the other point. It is clear that the external shell of the sphere only feels the internal force because the external one is compensated by the spherical symmetry, as the Gauss theorem states.

Now we can integrate the potential.

$$V=-4\pi G\int_0^{R(t)}\frac{\rho(t)r^2}{r}dr=-4\pi G\rho(t)\frac{R(t)^2}{2} \tag{15}$$

Where R(t) is the relative distance between the two points of the universe.

B. Necessity of a reference frame

In the previous section we have talked about distances so, like in any system in classical mechanics, we need a reference point and a reference frame to define the distances.

If a continuous system is given, the essential reference point to describe the distances within the system itself is the center of mass

$$\vec{r}_{CM}=\frac{\int_M\vec{r}dm}{\int_M dm} \tag{16}$$

Notice that, if we try to calculate the center of mass, we are going to have a problem with a divergence again because we are working with an infinite system. Therefore it implies that every point of the universe is the center of mass itself.

Classical mechanics use the far stars as an inertial reference system, we are going to generalize this concept as the *cosmic fluid*, which contains the far galaxies. The observations state that locally the universe moves in conjunction with the cosmic fluid, so there is no any peculiar speed between them.

Finding the relativistic Friedman equations we have found that the universe is not static, therefore the cosmic fluid could be in expansion. This means that we have to reject the idea of finding an inertial frame for our problem, as defining something external to the universe to refer the universe itself does not make any sense.

We know that we can only use Newtonian cosmology if we do not go far in space and time because then the relativistic effects show up. Thus, to refer our system distances we can use the the cosmic fluid but bearing in mind that it is not an inertial frame. Therefore to apply the Newton law in its common form ($\vec{F} = m\vec{a}$) we have to take into account that our reference point could be changing. Because of this, we have introduce the known *scale factor* ($a(t)$) which will absorb all temporal changes that affect our reference frame.

In conclusion, from now we are going to denote the relative distances as $R(t) = a(t)R$.

C. Conservation of mass

In the relativistic derivations we have seen that the fundamental physics principle, the energy conservation, must be fulfilled like in all the physical models. Now we are working within the classical mechanics which means that the mass and the energy are independent concepts. So, apart from the energy conservation principle, another principle have to be fulfilled: the mass conservation.

We know that the mass (M) that is inside a shell of radius $R(t)$ is

$$M = \int_M dm = \frac{4\pi}{3}\rho(t)a^3(t)R^3 \tag{17}$$

Now we impose that the mass inside the shell cannot vary with the time.

$$\frac{dM}{dt} = 0 = \frac{4\pi}{3}(\rho(t)3a^2(t)\dot{a}(t)R^3 + a^3(t)R^3\frac{d\rho(t)}{dt})$$

Simplifying all the terms that we can, we find the next relation:

$$\dot{\rho}(t) = -3\rho(t)\frac{\dot{a}(t)}{a(t)} \tag{18}$$

We have found the Newtonian Friedman equation of the conservation of mass. Notice that if we do $(\dot{11}) - (13)$ for the relativistic part we reach to

$$\dot{\rho}(t) = -3(\rho(t) + p)\frac{\dot{a}(t)}{a(t)} \tag{19}$$

For the Newtonian part we have found the same expression but for a pressure-less material called the dust gas. This result makes sense because in classical mechanics the field does not carry energy, so the gravitational field cannot make pressure.

D. Acceleration equation

Having fixed the fundamentals problems we can proceed to apply the Newton equations. Thanks to the spherical symmetry we will be able to work with the modulus of the different vectorial magnitudes.

The gravitational force that a piece of mass m is going to suffer is:

$$\vec{F} = -m\vec{\nabla}V = -G\frac{M_{int}}{R^2(t)}m\hat{r} \tag{20}$$

So plugging the force to the second Newton's law:

$$m\frac{d^2a(t)R}{dt^2} = -Gm\frac{M_{int}}{a^2(t)R^2} = -Gm\frac{4\pi}{3}\frac{a^3(t)R^3}{a^2(t)R^2}\rho(t)$$

Notice that m and R can be removed. So at the and we get:

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}\rho(t) \tag{21}$$

We have reached to the acceleration Newtonian Friedman equation which implies a non-static universe again. If we compare this equation with (13) we see that it is very similar to the relativistic one but, again for a dust gas ($p = 0$). Furthermore, we see that there is not the term of the cosmological constant, which means that the cosmological constant does not come from the gravitational force.

The acceleration equation and the conservation of mass equation are independent so, again we see that we have only two linear independent Friedman equations.

E. Energy conservation

With the equations we have already found we have the necessity to see that in this equations there is hidden the energy conservation principle. We take the acceleration equation (21) and we integrate it to obtain expressions with only the first temporal derivatives of the scale factor, to see if we find the kinetic energy.

$$\ddot{R}(t) = -G\frac{4\pi}{3}\rho(t)R(t) = -G\frac{M_{int}}{R^2(t)}$$

Now we multiply by $\dot{R}(t)$ and we integrate the equation.

$$\dot{R}(t)\ddot{R}(t) = G\frac{M_{int}}{R^2(t)}\dot{R}(t)$$

$$\frac{1}{2}\dot{R}^2(t) = G\frac{M_{int}}{R(t)} + U \tag{22}$$

We see that we have reached an expression that have clearly the form of the energy conservation equation. It is possible to distinguish the terms of the kinetic energy and the potential energy, but we see that a wild integration constant have appeared so we have to interpret its meaning. If we manipulate the equation, we can compare the found expression with the relativistic one (11).

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{2U}{R^2a^2(t)} \tag{23}$$

Like in the other equations, we see that we have the same relativistic expression but for the dust gas and without the constant cosmological constant term. But the interesting term to analyse is the last one, the one that contains $a^{-2}(t)$.

Comparing both equations, (11) and (23), we see that the last term is related to the curvature term in the relativistic case, but in the Newtonian one we cannot talk about curvature because we are working with the euclidean metric. We see that we can associate U to some kind of mechanical energy, because it is the sum of the kinetic energy and the potential one, and also it is a dynamic constant.

So by analogy with the curvature term, we can distinguish different types of universe in function of the U sign (notice that the last term of (11) and (24) have different sign). If $U > 0$ the equations are going to a universe that re-collapses. If $U < 0$ the universe is going to expand forever. For $U = 0$ we have the intermediate case, the universe is going to expand to infinity but reaching a null speed.

F. Newtonian Friedman equations adding a cosmological constant

In the previous chapter we have seen that the cosmological constant does not appear in a natural way in the Newtonian Friedman equations, this implies that we have to see how we have to change the potential so as the cosmological constant appears. Nowadays, thanks to the astronomical observations, we know that the cosmological constant term acts as a repulsive force proportional to the radial distance (in classical terms) that suffers a mass m . So we add that new force to the second Newton law.

$$m\frac{d^2a(t)R}{dt^2} = -Gm\frac{4\pi}{3}\frac{a^3(t)R^3}{a^2(t)R^2}\rho(t) + \frac{\Lambda}{3}a(t)Rm$$

We are able to put a factor $\frac{1}{3}$ in the cosmological constant term as it is an arbitrary constant. With this we are going to compare easily the obtained result with the relativistic expressions.

After some algebra, the previous expression we reach the acceleration equation.

$$\frac{\ddot{a}(t)}{a(t)} = -G\frac{4\pi}{3}\rho(t) + \frac{\Lambda}{3} \tag{24}$$

Now, following the same steps that in the previous chapter, we can derive the energy conservation expression. The meaning of the integration constant (U) is going to be still the same, the mechanical energy.

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{\Lambda}{3} + \frac{2U}{R^2a^2(t)} \tag{25}$$

Furthermore, we can find the mass conservation equation in the same way that in the previous chapter, so now we are going to find the equation (18) again.

We see that we have obtained the same equations that in the last chapter, indeed the relativistic Friedman equations for a pressure-less fluid, but this time with the cosmological constant term as expected.

IV. CONCLUSIONS

- In both, relativistic and Newtonian Friedman equations derivation, we have arrived to dynamical expressions that describe a non-static isotropic and homogeneous universe in general conditions.
- For a homogeneous and isotropic universe there are only two independent Friedman equations.
- The Newtonian Friedman equations are the same as the relativistic ones but for a pressure-less fluid called the dust gas. In the energy conservation expression we have found a sort of curvature term which we have seen that it contains implicitly the mechanical energy of the Newtonian universe.
- Comparing the relativistic expressions with the Newtonian ones, we have seen that the relativistic expressions fulfill the energy conservation and the mass conservation principles.
- The cosmological constant does not appear in a natural way in the Newtonian derivation of the Friedman equations derived only from the gravitational potential. For the cosmological constant term to appear, we have to add an extra term to the Newtonian potential acting as a repulsive force proportional to the radial distance.

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