Measurement of low impedances with a lock-in amplifier

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Abstract: This document details an experiment for measuring the frequency response of the impedance of cylindrical conductors, which is using a lock-in amplifier and the four-terminal sensing technique. Resistivity has been determined with high accuracy. Experimental results reveal the dependency of impedance on frequency, and the experimental data fit predictions derived from Maxwell-Faraday equations for a cylindrical conductor in a wide range of frequencies.

I. INTRODUCTION

This document details an experiment for measuring the frequency response of the impedance of cylindrical conductors, which is using a lock-in amplifier due to its capability of suppressing electromagnetic noise and providing high precision results.

A theoretical background is explained in the following sections as well as specific description of the experiment and the conclusion that can be derived from it.

The impedance of any conductor is dependent on the frequency of the applied voltage, and this fact has enormous importance whenever a signal is to be transmitted. In practice, the fact that impedance increases with frequency makes the transmission medium behave as a low pass filter.

This attenuation of the signal has a strong impact in technology application wherein the frequency is relatively high, which is the case of high frequency signal such as coaxial antenna cables or even audio cables.

II. SKIN EFFECT

A. Fundamentals of the skin effect

The current distribution in any conductor with constant conductivity can only be considered uniform at very low frequencies. An alternating current distribution produces an alternating magnetic field. This alternating magnetic field also produces another current distribution which opposes the original current distribution and is concentrated in the center of the conductor. This is produced by the superposition of magnetic field lines, which has the biggest magnitude in the center of the conductor. Therefore the current distribution is pushed outwards, resulting in a distribution whit a maximum value in the crust of the conductor and nearly zero in the center.

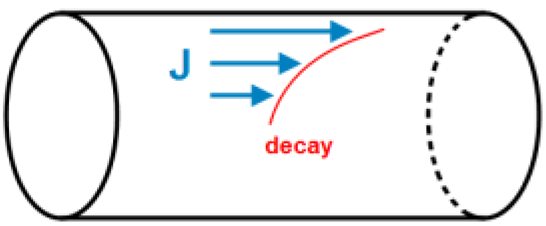

FIG. 1: Representation of the current distribution of a cylindrical conductor [1].

The exact shape of the curve is determined by its parameters that are dependent on frequency and change with the material.

$$\delta = \sqrt{\frac{2\rho}{\omega\mu}} \tag{1}$$

δ is the so called skin depth, ρ is the resistivity, ω is the angular frequency and μ is the magnetic permeability. In the case of a cylindrical conductor, the skin depth describes the thickness of an equivalent ring where the current density is constant. Since the effective area of conduction decreases with frequency, the skin effect shows up as an increase of the resistance with frequency. When the skin depth is greater than the radius of a cylindrical conductor ($\delta \gg r_0$), the current distribution can be considered uniform in all the cross section of the wire. Therefore, the resistance is approximately constant over the frequency. No changes in the resistance will be noticeable until the skin depth becomes smaller than the radius of the wire ($\delta \gg r_0$). For frequencies above this point, changes in resistance will start being more significant. This phenomenon is taken into account for manufacturing cables that must carry high frequency signals. It is the case of the Litz wire, which consists of multiple thin strands insulated electrically from each other. It is designed in such a way that the skin effect is reduced, so that it can be used in electronics to carry alternating currents with frequencies above 1MHz [2].

B. Mathematical model of the skin effect

According to [3], equation (3) describes the frequency dependence of the impedance of an arbitrary cylindrical conductor of radius r_0.

$$Z_i = R + j\omega L_i \tag{2}$$

$$Z_i = \frac{j\rho}{\delta\pi r_0\sqrt{2}}\left[\frac{\mathrm{Ber}(q) + j\mathrm{Bei}(q)}{\mathrm{Ber}'(q) + j\mathrm{Bei}'(q)}\right] \tag{3}$$

where Z_i is the internal impedance per unit length, $\mathrm{Ber}(q)$ and $\mathrm{Bei}(q)$ are the real and imaginary parts, respectively, of the Bessel function of the first kind of a complex argument and

$$q = \frac{\sqrt{2}r_0}{\delta} \tag{4}$$

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To obtain the per-unit-length resistance and internal reactance of the cylindrical conductor one must take the real and imaginary part, respectively, of equation (3). These equations are obtained by solving the Maxwell-Faraday equations in cylindrical coordinates.

III. Lock in

A lock-in amplifier is a very effective device used to measure weak signals having pink or white noise. It is commonly used in laboratories. The lock-in, besides the signal input, needs a reference input at a known frequency, phase and amplitude, typically 1V, to measure the weak signal. Any signal which frequency differs from the reference, like a noise signal, is filtered and does not affect the measurement.

The lock-in amplifier first amplifies the weak signal (5), that can be as low as an order of magnitude of 10nV. It analyzes the reference signal, which can be a square wave, to generate a new reference sine-wave signal (6) with a phase that the user can manipulate.

$$V_{signal} = V_0 \cos(\omega t + \varphi) \tag{5}$$

$$V_{ref} = V_{0,ref} \cos(\omega_{ref} t + \varphi_{ref}) \tag{6}$$

This new reference signal and the amplified signal are multiplied and then passed through a low pass filter, so that only the DC component of the signal product is left. If the reference signal has the same frequency as the noisy signal the DC component is proportional to the clean signal. One last low-filter is applied and we obtain a precise measure of the signal without its noise (7).

$$< V_{signal}(t) V_{ref}(t) >_t = \frac{1}{2} V_0 V_{0,ref} \tag{7}$$

IV. Four-terminal sensing

This technique suppresses the influence of the wiring resistances as well as the Galvani potential in the connections between metals.

The four-terminal sensing technique uses two linked circuits (Fig.2). A current *I* is passed through the external one. Since all common voltmeters in laboratories have high impedance, above 10MΩ, almost no current is circulating for the tension measurement circuit.

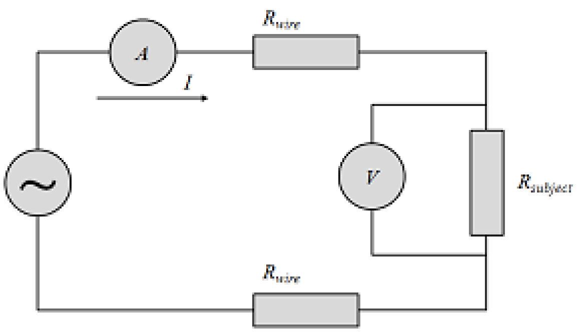

FIG. 2: Circuit diagram of the four terminal sensing measurement method.

In the case of an AC source, it is advisable to use a device capable of filtering the DC component of the signal. The lock-in amplifier is an example of an instrument with that possibility. The measured AC voltage at the resistor is

$$V_{AC} = I_{AC} R \tag{8}$$

Since the contact voltages are DC, they are added to the DC component of the measurement and they are thus filtered by the measurement instrument.

V. Experimental methodology

The impedance of a cable has been measured using the four-terminal sensing with a lock-in amplifier as described above.

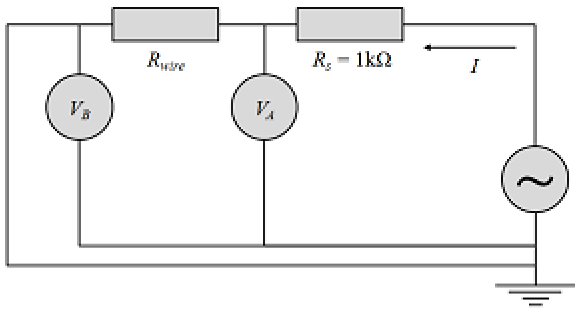

FIG. 3: Circuit diagram of the experiment.

The AC source provides a 1V sinusoidal signal as the reference signal of the lock-in amplifier being the measured signal the voltage at the Z_{wire} resistor. V_A and V_B are two inputs for the lock-in amplifier, which also subtracts them in order obtain the value of the voltage drop in the wire .The value of Z_{wire} is obtained by simple circuit analysis. The current intensity is

$$I = \frac{V_{source}}{R_S + R_{wire}} = \frac{V_A - V_B}{R_{wire}} = \frac{V_{AB}}{R_{wire}} \tag{8}$$

The wire resistance can then be computed as

$$R_{wire} = R_S \frac{V_{AB}}{V_{source} - V_{Ab}} \approx R_S \frac{V_{AB}}{V_{source}} \tag{9}$$

Since the voltage drop V_{AB} is very low compared to the source voltage it is possible to approximate the wire resistance as

$$R_{wire} \approx R_S \frac{V_{AB}}{V_{source}} \tag{10}$$

The lock-in amplifier amplifies, multiplies and filters the voltage drop V_{AB} and its phase. Since the RS resistance is 1kΩ and the source signal is 1V the wire resistance can be obtained simply as (only valid in SI units)

$$|Z_{wire}| \approx 1000 V_{AB} \tag{11}$$

Measurements are done with a first amplification of 50dB and a low-pass filter with a time constant of 1s, with frequencies between 10Hz and 10000Hz. These measures are performed for different lengths of the wire. Figure (4) shows the experimental results with a dotted line and the theoretical response (equation 3) of the absolute value of the internal impedance of the cylindrical wire as a solid line. It is verified that the response is flat for frequencies below 1 kHz. Above that frequency, it can be seen in all lengths that the impedance starts increasing at the same rate, due to the skin effect. The increase rate is the same because the four wires has de same diameter. A high discrepancy can be found at frequencies approximately above 20 kHz.

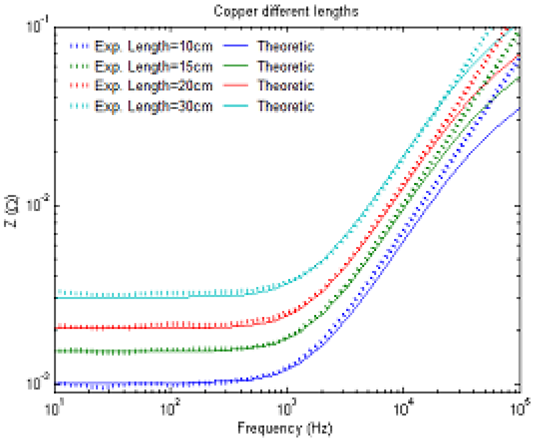

FIG. 4: Frequency response of the impedance in absolute value.

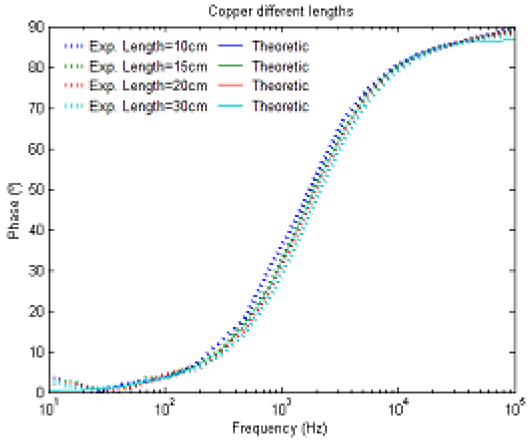

FIG. 5: Frequency dependence of the phase of the impedance.

Figure 5 shows the phase dependency on frequency and one can see that is nondependent on length as expected.

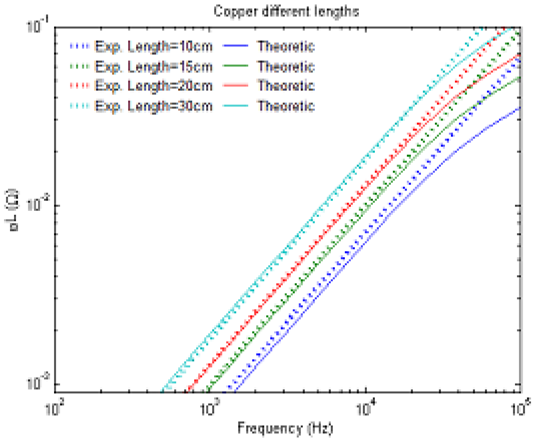

FIG. 6: Frequency dependence of the reactance.

With the data of the absolute value and its phase, the imaginary part of the impedance can be calculated. As seen in Figure (6) the imaginary part of the experimental data is linear with frequency at higher values as opposed in the theoretical response of a cylindrical conductor.

It can be seen that the experimental data is well adjusted to the theoretical model up to approximately 10 kHz. Above this value, discrepancies start to appear. Data suggest that the reactance is linear with frequency, at least in the measurement range. It would have been convenient to have more measurements, above 100kHz, in order to observe the evolution of the discrepancies and therefore give a reason to them. Albeit, the reason of this discrepancy is unclear.

The wire used was made of copper and had a diameter of 1.45mm. In order to determine the resistivity of copper, linear regressions have been carried out, using the obtained values of resistance, the diameter and the length of the wire. The obtained value is $(1.620 \pm 0.002)\times10^{-8}$ Ωm.

The dependence of the increasing rate (i.e. the slope of the straight line range in Figure 4) on the diameter of the wire has been also verified. For that purpose, the same measurements were made to different tungsten wires of 5.55cm length and different diameters. These measurements can be seen in figure (7). The experiment was slightly modified in order to reduce external influences. Experimental data visibly fits theoretical functions. Unlike the case of copper, the tungsten wires were not thick enough for experiencing the nonlinearity seen for copper at high frequencies. Instead, the measurements at the highest frequencies still fit the theory.

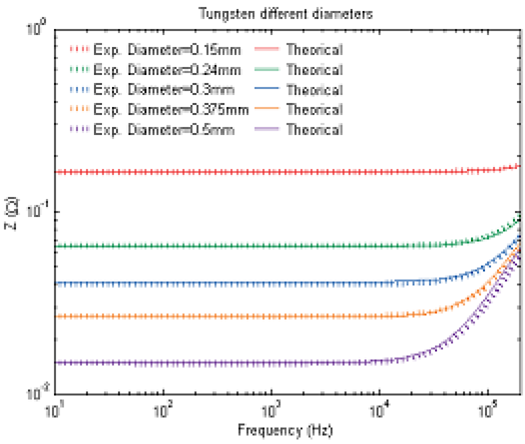

FIG. 7: Frequency response of the impedance in absolute value.

As expected, reducing the wire diameter increases the overall resistance, and makes the skin effect start happening at higher frequencies. On the contrary, increasing the diameter lowers the overall resistance but makes the skin effect happen at earlier frequencies.

VI. Conclusions

The objective was to measure low impedances using a lock-in amplifier. For that purpose, four-terminal sensing method was used and succeeded. With the obtained data, the resistivity of the copper wire used was calculated. It differs from the tabulated value for copper of high purity at 293K, which is 1.678×10^{-8} Ωm [4]. The magnitude of the discrepancy can not be evaluated since the temperature of the laboratory was not measured. However the obtained value seems to be lower than the tabulated one, and this error can be due to several reasons, like errors in the measurements of the wire length which were performed manually with low precision instruments. In worst case, assuming a temperature of 25ºC in the laboratory, the error is 3%. Nonetheless, the temperature was probably lower.

During the first test measures, a resistance increase due to the frequency was found and studied. The good adjustment between the experimental data and the theoretical functions satisfactorily prove that the reason for this increase was the skin effect. At high frequencies the experimental data does not behave as expected and would be interesting do more measurements at higher frequency.

The technique used here has an important academic value for undergraduate students in disciplines like instrumentation, electromagnetism and electronics.

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