Unit IV: Optimization

Exercise 1. Find the symmetric matrix associated with the quadratic form

$$q(x, y, z) = 2x^2 + 5y^2 + 4xz + 9yz.$$

Exercise 2. Find the expression of the quadratic form associated with the matrix

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 1 & 5 & 3/2 \\ 4 & 3/2 & 0 \end{pmatrix}$$

Exercise 3. Classify the quadratic form associated with the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Exercise 4. Classify the quadratic form associated with the matrix

$$A = \begin{pmatrix} 4 & 1 & 4 \\ 1 & 4 & 1 \\ 4 & 1 & 4 \end{pmatrix}$$

Exercise 5. Classify, depending on the value of the parameter $a \in \mathbb{R}$, the quadratic form

$$q(x, y) = 3x^2 + 2y^2 + axy.$$

Exercise 6. Classify, depending on the value of the parameter $a \in \mathbb{R}$, the quadratic form

$$q(x, y, z) = -x^2 - 4y^2 - z^2 + xy + ayz.$$

Exercise 7. A company produces two goods, A and B. When x units of A and y units of B are produced, the benefits of the company (in €) are described by the following function

$$B(x, y) = 3x^2 + 5y^2 - 16xy$$

a) Check that the company can incur in losses?

b) If the production of the second good is four times the production of the first good, will the company incur in losses?

Exercise 8. Given the real function of two variables $f(x,y) = \dfrac{x^2+9}{4-y}$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2 \text{ and } -2 \leq y \leq 2\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2 \text{ and } -6 \leq y \leq 6\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : y \leq 2 \text{ or } y \geq 8\}$

Exercise 9. Given the real function of two variables $f(x,y) = e^{x^3-y^2}$, study the applicability of the Extreme Value Theorem in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : x^2 - 9 \leq y\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 1, \text{ and } x + y \leq 6\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } -2 \leq y \leq 2\}$

Exercise 10. Reason out if the real function $f(x,y) = \sqrt{(x-1)^2 + (y-1)^2 - 9}$ has global extreme points in its domain using the Extreme Value Theorem.

Exercise 11. Given the real function $f(x,y) = \dfrac{\sin(x+y)}{x-a}$ where $a \in \mathbb{R}$. Study the applicability of the Extreme Value Theorem in the set $A = \{(x,y) \in \mathbb{R}^2 : -6 \leq x \leq 6 \text{ and } -6 \leq y \leq 6\}$ with respect to the values of the parameter a.

Exercise 12. For each of the real functions below, find the stationary points and classify[1] them whenever possible.

a) $f(x,y) = 6x^2 - 3xy^2 + 12xy - 18$

b) $f(x,y) = 2x^4 + y^2$

c) $f(x,y,z) = x^2 - 8x + 6z + y^2 - 4y + 20$

d) $f(x,y) = x^2 - y^2 + xy + 2x + 2y - 2$

e) $f(x,y,z) = x^2 - y^2 + xy + 2x - 2y - z^2 - 2$

Exercise 13. Study the convexity/concavity of $f(x,y) = \dfrac{-x^3}{6} - \dfrac{y^3}{6} + xy + 9x + 12y - 2$ in the following sets:

a) $A = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$

b) $B = \{(x,y) \in \mathbb{R}^2 : x < 0 \text{ and } y < 0\}$

c) $C = \{(x,y) \in \mathbb{R}^2 : x > 1 \text{ and } y > 1\}$

d) $D = \{(x,y) \in \mathbb{R}^2 : x < -1 \text{ and } y < -1\}$

[1] Local maximum, local minimum, or saddle point

Exercise 14. Study the global extreme points of the real function f in $\mathbb{R}^2$:

a) $f(x,y) = 9x^2 + \sqrt{2}y^2$

b) $f(x,y) = x^4 + 6y^2 - 4x$

c) $f(x,y) = x^2 + y^2 + xy - 8x - 8y$

d) $f(x,y) = x^2 + y^2 - xy - 8x - 8y$

Exercise 15. The inverse demand function of a commodity is described by $p = 650 - q$, where p stands for the price and q for the quantity of product demanded. Suppose that there is a single seller in the market and that the cost function of this monopolist is $C(q) = 4q^2 + 50q + 200$. Find the production level and the retail price that maximizes the profit of the monopolist.

Exercise 16. A given company produces two different commodities. The first one, A, is sold in the domestic market and the second one, B, is sold in the foreign market. The demand function in the domestic market is described by $p_A = 60 - q_A$, where p_A stands for the price and q_A for the quantity of product A demanded. The demand function in the foreign market is described by $p_B = 52 - 2q_B$, where p_B stands for the price and q_B for the quantity of product B demanded. If the cost function of the company is $C(q_A, q_B) = 6q_A^2 + 4q_B^2 - 10q_A - 8q_B + 2$

a) Find the profit function of the company.

b) Compute the production level (of both commodities) that maximizes the profits of the company and the retail prices.

Exercise 17. Study the global extreme points of the real function

$$f(x,y) = 6x^2 - 3xy^2 + 12xy - 18$$

in the set $A = \{(x,y) \in \mathbb{R}^2 : 2x + (y-2)^2 < 0\}$.