Unit IV: Optimization. Solutions

Exercise 1.

\[ A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & 2 \\ 2 & 9 & 0 \end{pmatrix} \]

Exercise 2.

\[ q(x, y, z) = 2x^2 + 5y^2 + 2xy + 8xz + 3yz \]

Exercise 3. Positive definite

Exercise 4. Positive semidefinite

Exercise 5.

\[
\begin{align*}
\text{Positive definite} & \quad \text{if } -2\sqrt{6} < a < 2\sqrt{6} \\
\text{Positive semidefinite} & \quad \text{if } a = \pm 2\sqrt{6} \\
\text{Indefinite} & \quad \text{if } a < -2\sqrt{6} \text{ or } a > 2\sqrt{6}
\end{align*}
\]

Exercise 6.

\[
\begin{align*}
\text{Negative definite} & \quad \text{if } -\sqrt{15} < a < \sqrt{15} \\
\text{Negative semidefinite} & \quad \text{if } a = \pm \sqrt{15} \\
\text{Indefinite} & \quad \text{if } a < -\sqrt{15} \text{ or } a > \sqrt{15}
\end{align*}
\]

Exercise 7.

a) \( B \) is an indefinite quadratic form. Then, by definition there is a production level at which the benefits of the company will be negative.

b) Setting \( y = 4x \) we have that \( B(x, y) = 19x^2 \) which is a positive definite form. Then, the benefits of the company will always be positive.

Exercise 8.

a) The Extreme Value Theorem can be applied because \( f \) is continuous in \( A \) and \( A \) is a compact set. Then, \( \exists p, q \in A \) such that \( \forall x \in A, f(p) \leq f(x) \leq f(q) \), in other words, we know that there are both a global maximum and a global minimum in \( A \).

b) The Extreme Value Theorem cannot be applied because \( f \) is not continuous in \( B \).

c) The Extreme Value Theorem cannot be applied because \( C \) is not compact (it is not bounded).
Exercise 9.

a) The Extreme Value Theorem cannot be applied because $A$ is not compact (it is not bounded).

b) The Extreme Value Theorem can be applied because $f$ is continuous in $\mathbb{R}^2$ (in particular, it is continuous in $B$) and $B$ is a compact set. Then, $\exists p, q \in B$ such that $\forall x \in B$, $f(p) \leq f(x) \leq f(q)$, in other words, we know that there are both a global maximum and a global minimum in $B$.

c) The Extreme Value Theorem cannot be applied because $C$ is not compact (it is not closed).

Exercise 10. The Extreme Value Theorem cannot be applied because the domain of $f$:

$$\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + (y - 1)^2 - 9 \geq 0\}$$

is not compact (it is not bounded).

Exercise 11. If $-6 \leq a \leq 6$ ($a \in [-6, 6]$) the Extreme Value Theorem cannot be applied because $f$ is not continuous in $A$. In any other case the Extreme Value Theorem can be applied, that is, if $a < -6$ or $a > 6$ we can ensure the existence of a global maximum and a global minimum of $f$ in $A$.

Exercise 12.

a) $(0, 0)$ and $(0, 4)$ are saddle points and $(-1, 2)$ is a local minimum.

b) $(0, 0)$ is the only stationary point of $f$ but we cannot classify it precisely. We can only say that it is either a local minimum or a saddle point.

c) There is no stationary point.

d) $(-6/5, 2/5)$ is a saddle point.

e) $(-2/5, -6/5, 0)$ is a saddle point.

Exercise 13. Studying the Hessian matrix of $f$ we find out that $f$ is concave in $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \text{ and } xy - 1 \geq 0\}$ and convex in $\{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0, \text{ and } xy - 1 \geq 0\}$. Then, we conclude that

a) $f$ is neither concave nor convex in $A$.

b) $f$ is neither concave nor convex in $B$.

c) $f$ is concave in $C$ because $C \subseteq \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \text{ and } xy - 1 \geq 0\}$.

d) $f$ is convex in $D$ because $D \subseteq \{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0, \text{ and } xy - 1 \geq 0\}$.
Exercise 14.

a) $(0, 0)$ is the global minimum of $f$ in $\mathbb{R}^2$.

b) $(1, 0)$ is the global minimum of $f$ in $\mathbb{R}^2$.

c) $(8/3, 8/3)$ is the global minimum of $f$ in $\mathbb{R}^2$.

d) $(8, 8)$ is the global minimum of $f$ in $\mathbb{R}^2$.

Exercise 15. The profit of the monopolist is maximum when $q = 60$ units are produced and sold at price $p = 590$.

Exercise 16.

a) The profit function of the company is given by $B(q_A, q_B) = -7q_A^2 - 6q_B^2 + 70q_A + 60q_B - 2$, where $q_A$ and $q_B$ are the quantities of goods A and B produced, respectively.

b) The profit of the company is maximum when $q_A = q_B = 5$ units of each commodity are produced and sold at prices $p_A = 55$ and $p_B = 42$.

Exercise 17. The global minimum of $f$ in $A$ is attained at point $(-1, 2)$. 