Unit IV: Optimization. Solutions

Exercise 1.

$$A = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 5 & \frac{9}{2} \\ 2 & \frac{9}{2} & 0 \end{pmatrix}$$

Exercise 2.

$$q(x, y, z) = 2x^2 + 5y^2 + 2xy + 8xz + 3yz$$

Exercise 3. Positive definite

Exercise 4. Positive semidefinite

Exercise 5.

$$\begin{cases} \text{Positive definite} & \text{if } -2\sqrt{6} < a < 2\sqrt{6} \\ \text{Positive semidefinite} & \text{if } a = \pm 2\sqrt{6} \\ \text{Indefinite} & \text{if } a < -2\sqrt{6} \text{ or } a > 2\sqrt{6} \end{cases}$$

Exercise 6.

$$\begin{cases} \text{Negative definite} & \text{if } -\sqrt{15} < a < \sqrt{15} \\ \text{Negative semidefinite} & \text{if } a = \pm\sqrt{15} \\ \text{Indefinite} & \text{if } a < -\sqrt{15} \text{ or } a > \sqrt{15} \end{cases}$$

Exercise 7.

a) B is an indefinite quadratic form. Then, by definition there is a production level at which the benefits of the company will be negative.

b) Setting $y = 4x$ we have that $B(x, y) = 19x^2$ which is a positive definite form. Then, the benefits of the company will always be positive.

Exercise 8.

a) The Extreme Value Theorem can be applied because f is continuous in A and A is a compact set. Then, $\exists p, q \in A$ such that $\forall x \in A$, $f(p) \leq f(x) \leq f(q)$, in other words, we know that there are both a global maximum and a global minimum in A.

b) The Extreme Value Theorem cannot be applied because f is not continuous in B.

c) The Extreme Value Theorem cannot be applied because C is not compact (it is not bounded).

Exercise 9.

a) The Extreme Value Theorem cannot be applied because A is not compact (it is not bounded).

b) The Extreme Value Theorem can be applied because f is continuous in $\mathbb{R}^2$ (in particular, it is continuous in B) and B is a compact set. Then, $\exists p, q \in B$ such that $\forall x \in B$, $f(p) \leq f(x) \leq f(q)$, in other words, we know that there are both a global maximum and a global minimum in B.

c) The Extreme Value Theorem cannot be applied because C is not compact (it is not closed).

Exercise 10. The Extreme Value Theorem cannot be applied because the domain of f:

$$Dom(f) = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 - 9 \geq 0\}$$

is not compact (it is not bounded).

Exercise 11. If $-6 \leq a \leq 6$ ($a \in [-6, 6]$) the Extreme Value Theorem cannot be applied because f is not continuous in A. In any other case the Extreme Value Theorem can be applied, that is, if $a < -6$ or $a > 6$ we can ensure the existence of a global maximum and a global minimum of f in A.

Exercise 12.

a) $(0, 0)$ and $(0, 4)$ are saddle points and $(-1, 2)$ is a local minimum.

b) $(0, 0)$ is the only stationary point of f but we cannot classify it precisely. We can only say that it is either a local minimum or a saddle point.

c) There is no stationary point.

d) $(-6/5, 2/5)$ is a saddle point.

e) $(-2/5, -6/5, 0)$ is a saddle point.

Exercise 13. Studying the Hessian matrix of f we find out that f is concave in $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0$, and $xy - 1 \geq 0\}$ and convex in $\{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0$, and $xy - 1 \geq 0\}$. Then, we conclude that

a) f is neither concave nor convex in A.

b) f is neither concave nor convex in B.

c) f is concave in C because $C \subseteq \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0$, and $xy - 1 \geq 0\}$.

d) f is convex in D because $D \subseteq \{(x, y) \in \mathbb{R}^2 : x \leq 0, y \leq 0$, and $xy - 1 \geq 0\}$.

Exercise 14.

a) $(0,0)$ is the global minimum of f in $\mathbb{R}^2$.

b) $(1,0)$ is the global minimum of f in $\mathbb{R}^2$.

c) $(8/3,8/3)$ is the global minimum of f in $\mathbb{R}^2$.

d) $(8,8)$ is the global minimum of f in $\mathbb{R}^2$.

Exercise 15. The profit of the monopolist is maximum when $q = 60$ units are produced and sold at price $p = 590$.

Exercise 16.

a) The profit function of the company is given by $B(q_A, q_B) = -7q_A^2 - 6q_B^2 + 70q_A + 60q_B - 2$, where q_A and q_B are the quantities of goods A and B produced, respectively.

b) The profit of the company is maximum when $q_A = q_B = 5$ units of each commodity are produced and sold at prices $p_A = 55$ and $p_B = 42$.

Exercise 17. The global minimum of f in A is attained at point $(-1,2)$.