Unit III: Real Functions of Several Variables
(Extra Material)

Exercise 1. Compute the domain of f and discuss if it is an open set for

$$f(x,y) = \ln\left(\frac{xy}{x^2+y^2-4}\right).$$

Exercise 2. Draw and describe the topological properties of the domains of the following functions

a) $f(x,y) = \sqrt{\dfrac{2x-y}{x+y}}$

b) $f(x,y) = \sqrt{x^2+y^2-4}$

c) $f(x,y) = \dfrac{x+5y}{\sqrt{y-x^2}}$

d) $f(x,y) = \sqrt{y-x^2}$

Exercise 3. Describe the domain of the real function $f(x,y) = \dfrac{\ln(y)}{\sqrt{x}}$ in terms of its topological properties.

Exercise 4. Study the topological properties of the domain of the real function of two variables $f(x,y) = \ln(1-y)\cdot\sqrt{1-x^2-y^2}$

Exercise 5. Given $f(x,y) = x(y-2)$, decide whether the following claims are true or false:

a) The 0-level curve, $C_0(f)$, consists of the single point $(0,2)$.

b) All level curves are lines.

c) The point $(10,12)$ lies on the 100-level curve, $C_{100}(f)$.

d) The level curves of f are circles centered at $(0,2)$.

Exercise 6. Let $f(x,y) = (x-2)^2+(y-1)^2$, decide whether the following claims are true or false:

a) The 0-level curve, $C_0(f)$, consists of the single point $(2,1)$.

b) The 1-level curve passes through the points $(2,0)$, $(1,1)$, $(3,1)$, and $(2,2)$.

c) The point $(22,-3)$ lies on the 416-level curve, $C_{416}(f)$.

d) The level curves of f are circles centered at $(-2,-1)$.

The first five Exercises are about real functions of one variable. This topic is a previous requirement that all students are supposed to know.

Exercise 7. Draw the level curves of the following real functions:

a) $f(x,y) = x + 2y$

b) $g(x,y) = x^2 - 4x + 4 + y^2$

c) $h(x,y) = xy$

Exercise 8. Let $f(x,y,z) = 4y^2 + \cos(z-x)$. Compute the norm of the gradient vector of the function f at point $(0, \pi, \pi/2)$.

Exercise 9. Find the gradient vector of the function $f(x,y) = x^2 + \dfrac{y^2}{2}$ at point $(2,1)$.

Exercise 10. Let $f(x,y) = \sqrt{y - x^2 + 1}$.

a) Draw the domain and level curves of the function f.

b) Compute the gradient of the function f at point $(0,0)$.

c) Compute the directional derivative of f along the vector $(0,-1)$ at point $(0,0)$.

Exercise 11. Let $f(x,y) = x \ln\left(\dfrac{y}{ax}\right)$, with $a > 0$. Study when (for what value of a) does the function f satisfy

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f$$

Exercise 12. Let $f(x,y) = \ln\left(\dfrac{x}{y}\right)$.

a) Draw the domain and level curves of the function f.

b) Compute the gradient of the function f at point $\left(\frac{1}{2}, 1\right)$.

c) Compute the directional derivative of f along the vector $(1,1)$ at point $\left(\frac{1}{2}, 1\right)$.

d) Compute the directional derivative of f along the direction of the vector $(4,3)$ at point $\left(\frac{1}{2}, 1\right)$.

Exercise 13. Let $f(x,y) = \sqrt{\dfrac{x}{y}}$.

a) Draw the domain and level curves of the function f.

b) Compute the gradient of the function f at point $(1,1)$.

c) Compute the directional derivative of f along the vector $(-3,4)$ at point $(1,1)$.

d) Compute the directional derivative of f along the direction of the vector $(4,3)$ at point $(1,1)$.

Exercise 14. Given the function $f(x,y) = 2x^2y + 3y$ compute the directional derivative of the function along the vector $(4,5)$ at point $(0,3)$.

Exercise 15. Given $f(x,y,z) = (x-y)(y-z)(z-x)$, compute $\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} + \dfrac{\partial f}{\partial z}$.

Exercise 16. Given $f(x,y) = e^x \cos y$, compute $\nabla f(x,y)$ and $\nabla f(0,\pi)$.

Exercise 17. Given $f(x,y) = x^2y - 5xy$, compute the Hessian matrix, $Hf(x,y)$, and evaluate it at point $(1,1)$.

Exercise 18. Given $f(x,y) = e^{x^3y}$, compute the Hessian matrix, $Hf(x,y)$, and evaluate it at point $(1,0)$.

Exercise 19. Given $f(x,y,z) = x^2y - y^2z^2 + xyz$, compute the Hessian matrix, $Hf(x,y,z)$, and evaluate it at point $(-2,1,0)$.

Exercise 20. Given $f(x,y) = xy$, find the equation of the tangent plane of f at $p = (a,b)$

Exercise 21. Given $f(x,y) = xy^3 + x^2y^2$ and $p = (1,1)$. Find the equation of the tangent plane of f at p. Use the equation to estimate $f(1, 1'05)$.

Exercise 22. Given the differentiable function $f(x,y) = x^2y^2 - 3xy$ compute the equation of the tangent plane of f at the point $(1,2)$. Use the equation to approximate the value of $f(1'1, 2'01)$.

Exercise 23. Compute the elasticity of the real function of one variable, f, at point x_0 in the following cases:

a) $f(x) = (10-3x)^2$ and $x_0 = 5$.

b) $f(x) = (x^2-1)^3$ and $x_0 = 0$.

c) $f(x) = \dfrac{3}{x}$ and $x_0 \in \{0,1,2\}$.

d) $f(x) = x^4 + 3$ and $x_0 = 3$.

Exercise 24. Find the partial elasticity of f with respect to x in the following cases:

a) $f(x) = xy$

b) $f(x) = x^2y^5$

c) $f(x) = x^n e^x y^n e^y$, with $n \in \mathbb{N}$

d) $f(x) = x + y$

Exercise 25. Given the real function

$$f(x,y) = 2x^2y - xy^2$$

a) Compute the image of $(10,5)$ according to f.

b) Find the equation of the tangent plane of f at $(10,5)$. Use the equation to estimate $f(10'1, 5)$.

c) Assume that the second variable is fixed $y = 5$. Study the relative change of f at $(10, 5)$ using the partial elasticity of f.

d) Compare the results obtained in b) and c) with the output of f at $(10'1, 5)$.

Exercise 26. Let $f(x, y, z) = 6x^2y + e^{y-z}$. Compute the following partial elasticities:

a) $E_x f(1, 1, 1)$. *b*) $E_y f(1, 1, 1)$. *c*) $E_z f(1, 1, 1)$.

Exercise 27. A given company produces certain good. The daily production of the good is given by the following function $Q(K, L) = 60K^{\frac{1}{2}}L^{\frac{1}{3}}$, where K is the invested capital (in units of 1.000 €) and L is the working time (in hours a day). Suppose that the current investment is 900.000 € and that 1.000 hours of working time are used daily. Carry out a marginal analysis to estimate the effect of investing 1.000 additional euros on the production function while maintaining the working time fixed.

Exercise 28. A company imports a given product and sells it back using its own staff. Each one of the sellers sells approximately $\left(\frac{d^2}{2000p} + \frac{v^2}{100} - v\right)$ units a month, where v represents the amount of sellers, p stands for the price of each item, and d is the money spent monthly in advertising. Currently, the company has 10 sellers, spends 6.000 € in advertising, and charges 800 € for each of the items sold. Moreover, the company pays 80 € for each of the imported items and the sellers earn 600 € a month. Study how does the total benefit of the company change if one more seller is hired using a marginal analysis.

Exercise 29. Let $z = x^2e^y$, where $x = u + v$ and $y = uv$. Compute $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ in terms of u, v, and z.

Exercise 30. Let $z = f(x, y)$, where $x = uw$ and $y = vw$. Compute

$$\frac{\partial z}{\partial u}(1, 1, -1) + \frac{\partial z}{\partial v}(1, 1, -1) + \frac{\partial z}{\partial w}(1, 1, -1)$$

Exercise 31. The relation $y^2 - x^2e^{y^2} = 1$ implicitly defines y as a function of x in a neighborhood of the point $(0, 1)$. Find the value of $\frac{dy}{dx}$ at point $(0, 1)$.

Exercise 32. The equation $[\ln(xy)]^2 - \ln(xy)^2 = 0$ implicitly defines y as a function of x in a neighborhood of the point $(1, e^2)$. Find the value of $\frac{dy}{dx}$ at point $(1, e^2)$.

Exercise 33. The equation $\ln(x + y) - x - y + 1'5 = 0$ implicitly defines y as a function of x in a neighborhood of the point $(0, 0'3017)$. Compute the real function $\frac{dy}{dx}$.

Exercise 34. Let $z = z(x, y)$ be implicitly defined as a function of x and y by the equation $y^2 + xz + z^2 - e^z - c = 0$. Find the value of c for which $z(0, e) = 2$. Compute the values of the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at point $(0, e)$.

Exercise 35. Compute, whenever possible, the degree of homogeneity of the following real functions:

a) $f(x, y) = \dfrac{x^4 \cdot \sin\left(\frac{x}{y}\right)}{2y} \cdot e^{\frac{x}{y}}$

b) $f(x, y) = \dfrac{\sqrt{x} \cdot \cos\left(\frac{x}{y}\right)}{2y}$

Exercise 36. Compute the degree of homogeneity of the following real functions and check that Euler's Theorem is satisfied:

a) $f(x_1, \ldots, x_n) = Ax_1^{m_1} x_2^{m_2} \cdot \cdots \cdot x_n^{m_n}$ where $n, m_1, \ldots, m_n \in \mathbb{N}$ (Cobb-Douglas function).

b) $f(K, L) = K^\alpha L^\beta$ where $\alpha + \beta = 1$.

c) $f(x, y, z) = \dfrac{yz^2}{x^2}$.

d) $f(x, y, z) = \dfrac{yz^2}{x^3}$.