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SINGLE-OBJECT AUCTION THEORY

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Part I

Introduction

Since I started college I have heard about the dissertation and four years have passed and I have this project right in front of me.

The truth is that it has been difficult to choose a topic interesting as well as related with economic theory because of our inexperience in this field. However, after looking for an appropriate subject, I found that auction was a good candidate since it mixes economics and mathematics at a quite high level.

The chosen option seemed the most appropriated for developing a project where I could apply what I have learned over the past four years. Moreover, another reason I chose this area was because of my interest in probability and applied mathematics and how to apply my knowledge to the real world.

But, how the auctions work?

Usually, objects in the markets have a defined price where buyers accept it purchasing the object. However, in some cases, objects are sold in the markets using a competitive game among the potential, few of them generally, buyers pretending that the buyer who likes it more will acquire it. Notice that, if the mechanism is appropriated, the one who likes it more is the one who is willing to pay more for it. This game is called *an auction* where the buyers are called *bidders* and what they offer for the object is a *bid*. The auction (that is, the proposed game) should satisfy certain properties like, for instance, who is willing to pay more for the object needs to be the one getting it. In other words, an auction is not a lottery since in this case the one getting the prize is not necessarily the one who has bought more tickets.

In general, in game theory, a game is defined by the number of players, their strategies and the payoff functions of the players (that are the payments of each player for any given strategy profile) (see, [9] *An Introduction to Game Theory*, Osborne, M.J.). Here, we could differentiate between complete and incomplete information games if we have all the information to know what strategy follows each player is a complete information game and otherwise an incomplete information game. Returning to the auctions, the payoff functions are unknown to the bidders except his own payoff function because if some buyer knows what would be the payoff function of another competitor he would change his bid to win. Thus, an auction is an incomplete information game.

Let us propose an easy example to differentiate between complete and incomplete information games. Suppose that a seller wants to auction a painting and there are two interested buyers Anne and Bart. Suppose that Anne's value for the painting is $v_A = 10000\$$ and Bart's value is $v_B = 8000\$$. The auction mechanism establishes that each buyer must bid in a sealed envelope then, the seller assigns the painting to the highest bid and the winner pays what he has bid. If the Anne's offer is equal to Bart's, the painting is assigned to Anne or Bart with probability $1/2$.

As we can see, the values of the players are what they think the object costs. The strategy set is any non negative offer. Suppose that the buyers are risk neutral, which means that if the bidder is between choices he could choose what he thinks even if one is riskier. To begin with, Anne's payoff function is determined by her value, her offer, b_A, and Bart's offer, b_B. To define exactly what is her payoff function, we must differentiate what offer is higher than the other. Firstly, if Anne wins the painting, her offer is higher than Bart's, then her payoff function will be $v_A - b_A$. In case she does not win, her payoff function would be 0. Finally, if the offers are equal then, we must do a raffle between Anne and Bart to know who would win the object. Thus,

her payoff function would be $\frac{1}{2}(v_A - b_A)$. Thus,

$$\Pi_A(b_A, b_B) = \begin{cases} v_A - b_A & \text{if } b_A > b_B \\ \frac{1}{2}(v_A - b_A) & \text{if } b_A = b_B \\ 0 & \text{if } b_A < b_B \end{cases}$$

Bart's payoff function will be Anne's payoff function but changing her value and offer by Bart's value and offer.

$$\Pi_B(b_A, b_B) = \begin{cases} v_B - b_B & \text{if } b_B > b_A \\ \frac{1}{2}(v_B - b_B) & \text{if } b_B = b_A \\ 0 & \text{if } b_B < b_A \end{cases}$$

As a conclusion, when the players know certainly all the game's elements, particularly, the payoff functions, we say this is a complete information game.

Secondly, the dissertation differentiates between private (Part II) and interdependent (Part III) value auctions. Private-value auctions correspond to the case where the actual value of the object for a bidder i only depends on its signal and not on the signals (valuations) of all the other bidders. A good example of a private value auction would be the acquisition of an object in www.ebay.com. On the other hand, interdependent-value auctions correspond to the case where the actual value of the object for a bidder i depends not only on his own value but on the signals of all bidders. As example of interdependent values, we could think about the radio-frequency auction. Let us suppose that a Company A had the frequency 100.00 for almost 30 years and now a Company B wants this frequency for himself. Company A's value could or could not depend on Company B's value, but not viceversa because the Company B knows the relation between Company A and his frequency. Here, we have that the Company B's value depends on information of the other bidder.

When we talk about auctions, people, usually, think about selling art or fish, but nowadays, a seller could auction almost everything. Let us focus on selling art and fish auctions. Firstly, when a painting is auctioned, the seller starts with an entrance price so that people who can not afford the painting do not enter to the auction. Then, the price of the painting starts rising until only one buyer remains. This one wins the object and pays the seller an amount equal to the price at which the second-last bidder dropped out. This kind of auction is called English Auction and is the oldest and prevalent auction form. Secondly, in the sale of fish, the product starts with a price high enough so that presumably no bidder is interested in buying the object at that price. Then, the price is gradually going under until some bidder indicates his interest. This one is called the Open Descending Price or Dutch Auction. Although, these two auctions are the most common, we could imagine other mechanisms like Third Price Auctions etc. In fact, it is well known that an English Auction is strategically equivalent to a sealed-bid Second Price Auction while under reasonable assumptions, the Open Descending Auction is equivalent to a sealed-bid First Price Auction.

In Chapter 5 we will prove that for a huge class of auctions the expected revenue of the auction (that is what the seller expects to win before the game starts) is the same. This result is known as *Expected revenue equivalence Theorem* (see [8] *Counterspeculation, Auctions, and Competitive Sealed Tenders.* Vickrey, W.).

The key assumptions for this theorem are symmetry among players (and so bids), risk neutral players, values are independently and identically distributed and the expected payment of a bidder with value zero is zero.

As de reader can imagine, this dissertation is a theoretical rather than practical approach on auction theory, it does not improve or revolutionize it. The objective is to understand and venture into the world of auction theory following Vijay Krishna's Auction Theory [6].

Chapter 1

Introduction to Auctions

Collins Definition of Auction

A public sale of goods or property, especially one in which prospective purchasers bid against each other until the highest price is reached. [1]

1.1 History of Auctions

The origin of auctions

The firsts references about auctions are dated around the 500 B.C. in Babylon, where women were auctioned off as wives. They used a descending method of sale and also they accepted returns if the winner were not satisfied with what he just won. Even thought, nowadays, this practice is considered awful, in ancient Babylon selling a daughter outside of the auction method was considered illegal.

In ancient Rome we find some auction cases too. One of the most important auction events occurred in 193 A.D., when the Praetorian Guard killed the Emperor Pertinax, in an action of revenge, and they auctioned the Empire to the highest bidder. The winner, Didius Julianus, was declared emperor, but due to the anger of the people, in just one month erupted a civil war and two months after, Didius Julianus was assassinated when Septimus Severus conquered Rome. [2]

18th century

Late 18th century in France, both before and after the French Revolution, the auctions were held daily in bars to sell art. These auctions were accompanied by printed catalogs announcing what they were going to auction. As a curiosity, the most important auction houses were founded in the 18th century, Christie's in 1766 and Sotheby's in 1744.

Recent history, 20th century

Many auction schools were opened in America and they taught general merchandise, real estate and fine stock auctioneering. Even thought, many auctioneers believed that an auctioneer could not be trained; auctioneering was a gift.

In the 50s, the sale of goods and real estate by auctions was growing exponentially. There was a need for real estate and personal property to be sold faster than the private market would allow. It was the birth of the modern auction business.

The 1990s and nowadays
The technology has an important role in modern auctions. Anybody can bid for any product only by phone call, making this process faster and bearable. But technology is not important only for the bidders, also it can be used by auctioneers to take pictures of small auction items and send them to the bidders and thereby they can take a closer look to the object which they are going to bid for.

In the last years, an important application of auction theory to a relevant economic problem has taken place when governments have auctioned to telecommunication companies the right to use the different bands in the radio-electric space. Their need to obtain a high price as possible has largely contributed to the development of the theory.

Even though we have not defined the auction types yet, it is important to know how the most important auction websites work. To begin with, the most important and famous auctioning website is www.ebay.com which consists on sell any kind of objects by a First Ascending Price but sometimes the buyer has an option to acquire de object buying it for a price given. In these auctions, the seller could have a reserve price that allows him to set a low starting price to generate interest and bidding, but protects him from having to sell his item at a price that he feels is too low. Also the seller can remove his reserve price when the first bid is received. [3]

1.2 Definitions and Basic Properties

Auctions are a particular case of games, since buyers compete, by means of their bids, to obtain the object on sale and the success of a buyer's bid depends on the bids of the other buyers.

A game is a situation where several agents must take individual decisions or strategies to obtain an outcome. What characterizes a game is the strategic interdependence, that is the outcome for each agent depends not only on his own strategy but also on the strategies chosen by the other agents.

1.2.1 Game Theory

We include a brief introduction to game theory to show where the auctions could be classified as games.

Definition 1.1. A game is a triple $\left(\mathfrak{N}, \{S_i\}_{i\in\mathfrak{N}}, \{\Pi_i\}_{i\in\mathfrak{N}}\right)$ where:

- $\mathfrak{N} = \{1, 2, \ldots, N\}$ is the players set.
- S_i for $i = 1, 2, \ldots, N$ is the strategy set of each player.
- $\Pi_i(s_1, s_2, \ldots, s_N) \in \mathbb{R}$ for $i \in 1, 2, \ldots, N$ are the payoff functions. Π_i represents the payoff that the player i obtains if the strategy combination $(s_1, s_2, \ldots, s_N) \in S_1 \times S_2 \times \cdots \times S_N$ occurs.

A strategy profile is a combination $(s_1, s_2, \ldots, s_N) \in S_1 \times S_2 \times \cdots \times S_N$ formed by one strategy for each player.

Notice that, in general, the strategy set is not easy to determine.

1.2.1.1 Complete Information Games

We have a game with complete information when every player knows exactly the payoff function of each player. Then, if a player thinks of an strategy profile he could evaluate the payoff function of each player and know what it would be the consequences of using this strategy.

Solving a game consists on finding an equilibrium, which is a strategy profile where each player observes that given the strategies of the other players, his strategy is the best he could choose. For that reason, in equilibrium, no player wants to change his strategy, because if he changes he would be worse off.

Now, we define mathematically an equilibrium.

Definition 1.2. The strategy profile $\left(s_1^*, s_2^*, \ldots, s_N^*\right) \in S_1 \times S_2 \times \cdots \times S_N$ is a Nash equilibrium of a game with complete information $(\mathfrak{N} = \{1, 2, \ldots, N\}, \{S_i\}_{i \in \mathfrak{N}}, \{\Pi_i\}_{i \in \mathfrak{N}})$,

$$\Pi_i\left(s_1^*, s_2^*, \ldots s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_N^*\right) \quad \geq \quad \Pi_i\left(s_1^*, s_2^*, \ldots s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_N^*\right), \ \forall s_i \in S_i$$

1.2.1.2 Incomplete Information Games

On the other hand, in most cases, we do not know the payoff functions, then we have a game of incomplete information.

Definition 1.3. To define what an incomplete information game is let us describe the following elements:

- The players set $\mathfrak{N}$.
- The strategy set of each player, S_i for $i \in \mathfrak{N}$. We denote a player's i action as $s_i \in S_i$.
- The possible types set T_i for each player i. We denote as $t_i \in T_i$ the type of the player i. Given a types vector, one for each player, $(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_N)$ we write t_{-i} to represent the vector of types of players different from i.
- The conjecture that thinks each player i about the other players type that maybe could be conditioned by his type, $p_i(t_{-i} \mid t_i)$.
- The payoff functions $\Pi_i(s_1, s_2, \ldots, s_N; t_1, \ldots, t_N) \in \mathbb{R}$ for $i = 1, 2, \ldots, N$.

A strategy s_i for player i is a function of his type, $s_i = s_i(t_i)$, which means that this strategy could recommend player i a different action in each of his types.

Notice that here, the strategies of the other players also depend on their types which the player i does not know with certainty. Then, what we want to do is to maximize expected payoff.

This Nash equilibrium in incomplete information games is known as Bayesian Nash equilibrium.

Definition 1.4. In a incomplete information game

$$G = \left(N; S_1, \ldots, S_N; T_1, \ldots, T_N; p_1, \ldots, p_N; \Pi_1, \ldots, \Pi_N\right)$$

the strategies $s^* = \left(s_1^*, \ldots, s_N^*\right)$ are a Bayesian Nash equilibrium if for each $i = 1, 2, \ldots, N$ and each type, $t_i \in T_i$, the strategy $s_i^*(t_i)$ is a solution of

$$\max_{s_i \in S_i} \sum_{t_{-i} \in T_{-i}} p_i(t_{-i} \mid t_i) \Pi_i\left(s_1^*(t_1), \ldots, s_{i-1}^*(t_{i-1}), s_i, s_{i+1}^*(t_{i+1}), \ldots, s_N^*(t_N); t_1, \ldots, t_N\right)$$

Notice that an auction is a game where the buyers are the players and the bids are the actions. The payoff functions are the profits that each buyer obtains. Moreover, an auction is an incomplete information game because we do not know how the other buyers valuate the object on sale and hence we do not know his profit with certainty. Then, the players' types are the possible valuations of the objects. We assume the possible types are the same for every player. The conjectures of each player are a distribution function of each player over the set of types of the others.

1.2.2 Classification of the Auctions

In this subsection we review the most common auction forms. They can be classified either because of their format, simultaneous sealed bid auctions or open auctions, or because of the information buyers have of the valuations of their competitors.

1.2.2.1 According to the Format

Sealed-bid auctions

The players submit bids in sealed envelopes, this means that it is a simultaneous auction. We can divide the sealed-bid auctions in two parts:

1. First Price Sealed-bid auctions. The winner is the highest bid and pays what he bid.

2. Second Price Sealed-bid Auctions. The winner is the highest bid too, but pays the second highest bid.

Open format auctions

The open format auctions are the most common auction that we see in harbors or in fine art auctions. A player bids and the other know what he bid, then he can bid again if the auction goes on. This is a sequential auction that we can divide in two types:

1. Open ascending price or English auction. It starts with a low price defined by the auctioneer and raises it as long as there are at least two bidders. The auction ends when only one of theses two bidders remains in the auction. Then, he pays what the last player that abandons the game has bid.

2. Open descending price or Dutch auction. It starts with a very high price which no bidder is interested in, and, over time, the price is descending until some bidder shows some interest. Then, the object is sold to this bidder and pays what he bid.

Notice that there is an equivalence between the First Price Sealed-bid Auctions and the Dutch auctions because what the winner pays is the same in the two auctions, what he has bid. Also, we have this equivalence between the Second Price Sealed-bid Auctions and the English Auction due to the same reason: the winner pays the second highest bid.

1.2.2.2 According to the Valuations

Private values

The valuation for buyer i of the object or objects only depends on a signal that buyer i receives:

$$V_i(x_1,\ldots,x_N) = V_i(x_i)$$

where V_i denotes the value that the bidder i assigns to the object. In fact, notice that with private values we use $V_i(x_i) = x_i$ and it does not depend on the signals (or valuations) of the other buyers.

Interdependent values

The valuation of the object or objects depends on the signals received by all buyers:

$$V_i(x_1,\ldots,x_N)$$

Then, V_i depends on the N variables which could be independent of each other.

If $V_i(x_1,\ldots,x_N) = V(x_1,\ldots,x_N)$ for all i, then, it is defined like a common value auction.

Part II

Private-value Auctions

Chapter 2

Second Price Auctions

Let us suppose we have a symmetric model which means that we have:

Players. We suppose we have N players.

Valuations. The value that the player i assigns to the object is X_i. Each X_i is a random variable independently and identically distributed on some interval $[0, \omega]$, $\omega \in \mathbb{R}^+$. Also, we can assume $\omega = \infty$, abusing of notation. Let us suppose that X_i follows a distribution $F(x_i)$ and density $f(x_i)$.

Offer. It is the function which assigns to any possible valuation the offer or bid of each buyer i:

$$\begin{aligned} \beta_i : [0, \omega] &\longrightarrow \mathbb{R}^+ \\ x_i &\mapsto \beta_i(x_i) \end{aligned}$$

Notice this is a game of incomplete information where the players are the buyers, the valuations are the types of the agents and the offers their available strategies. Recall that a strategy of a player i is a simultaneous game of incomplete information consists in specifying one action for each of his types. In the auction game, the strategy of a buyer specifies a bid for each of his possible valuations.

The objective is to find the strategies that form an equilibrium in the two auction formats.

We denote by β^* the strategies in equilibrium, understanding that $* = I$ for a First Price Auction and $* = II$ for a Second Price Auction.

2.1 Equilibrium bids

We begin with the study of the Second Price Auctions due to its easiness compared to the First Price Auctions.

In a Second Price Auction the payoff function of a bidder i is:

$$\Pi_i(b_1, \ldots, b_N) = \begin{cases} x_i - \max_{j \neq i} b_j & \text{if } b_i > \max_{j \neq i} b_j \\ 0 & \text{if } bi < \max_{j \neq i} b_j \end{cases}$$

where b_i is what the bidder i bids. In case we have a tie, $b_i = \max_{j \neq i} b_j$, then the object auctioned is going to be raffled between the people who bid b_i.

Proposition 2.1. *In a Second Price Sealed-bid Auction, it is a weakly dominant strategy to bid according to $\beta^{II}(x) = x$.*

Proof. Suppose we are the player i, then our expected payoff when bidding b_i is obtained by the next formula:

$$\Pi_i = \left(x_i - \max_{j \neq i} b_j\right) \Pr\left(b_i > \max_{j \neq i} b_j\right)$$

We want to maximize our expected payoff, given the bids of the remaining bidders.

To see that $\beta^{II}(x_i) = x_i$ is an equilibrium, we want to see that if $\beta^{II}(x_i) \neq x_i$ we are going to be worse than with $\beta^{II}(x_i) = x_i$. Let's suppose that $\beta^{II}(x_i) = x_i \pm c$ with $c \in \mathbb{R}^+$ and we define b_k as $b_k := \max_{j \neq i} b_j$.

Bid	**Win**	**Lose**	**Win/Lose**
x_i	$x_i - b_k$	0	0
$x_i + c$	$x_i - b_k$	0	$x_i - b_k$

Notice that when $x_i < b_k$ we prefer not to win because we are going to pay more than our valuation.

Bid	**Win**	**Lose**	**Win/Lose**
x_i	$x_i - b_k$	0	$x_i - b_k$
$x_i - c$	$x_i - b_k$	0	0

So, each buyer i bids his valuation x_i and the buyer with the highest valuation wins. □

The following example shows that the equilibrium we have presented in the Second Price Auction is not unique.

Example 2.1. The next table illustrates an example with the valuation of the object for each player and what they really bid.

i	1	2	3
x_i	0,1	0,4	0,9
b_i	1	0	0

We have that the bidder 1 wins the auction although he is not paying anything for the object and, actually, he is the one who values less the object. Notice that this is an equilibrium since given what bidder 1 does, the others do not have incentives to deviate.

On the other hand, going back to the equilibrium, presented in Proposition 2.1 each agent bids his own valuation. Let us remark that the expected revenue for the seller increases and goes to 1 as the numbers of bidders grows larger.

2.2 Order statistics

Now we are introducing the order statistics, an important tool for this section because all calculations we do are going to be very useful for auctions.

Let $X_{1,}\ldots,X_N$ be independent and identically distributed random variables with distribution function F and density function f. We define $Y_{1:N,}\ldots,Y_{N:N}$ as a rearrangement of $X_{1,}\ldots,X_N$. For all $1 \leq k \leq N$, $Y_{k:N}$ is the random variable that follows the bigger kth realization over N realizations. This $Y_{k:N}$ are called order statistics.

Highest Order Statistic
Now, we calculate the distribution function of the highest order statistic, $Y_{1:N}$.

$$\begin{aligned} F_{1:N}(y) &= \Pr\left(Y_{1:N} \leq y\right) = \Pr\left(\max\{X_1,\ldots,X_N\} \leq y\right) = \prod_{i=1}^{N} \Pr\left(X_i \leq y\right) \\ &= \prod_{i=1}^{N} F(y) = F(y)^N \end{aligned}$$

In the second equality we use that $X_1,\ldots,X_N$ are independent and identically distributed.

Then, the density function of $Y_{1:N}$ is the derivative of its distribution function.

$$f_{1:N}(y) = \frac{\partial F_{1:N}}{\partial y}(y) = NF(y)^{N-1} f(y) \tag{2.1}$$

Second Highest Order Statistic
Now, we calculate the distribution function of the second highest order statistics, $Y_{2:N}$.

$$\begin{aligned} F_{2:N}(y) &= \Pr\left(Y_{2:N} \leq y\right) \\ &= \Pr\left(\max\{X_1,\ldots,X_N\} \leq y\right) + \sum_{i=1}^{N} \Pr\left(X_i > y\right) \prod_{\substack{j=1 \\ j\neq i}}^{N} \Pr\left(X_j \leq y\right) \\ &= F(y)^N + \sum_{i=1}^{N} \left(1 - F(y)\right) F(y)^{N-1} \\ &= (1-N)\, F(y)^N + NF(y)^{N-1} \end{aligned}$$

In the second equality we use that $\Pr\left(Y_{2:N} \leq y\right)$ is equal to the probability of all $X_1,\ldots,X_N$ are being smaller than y plus the probability of X_i being bigger than y, times the probability of the other $X_1,\ldots X_{i-1}, X_{i-1},\ldots,X_N$ being smaller than y. Also we use that $X_1,\ldots,X_N$ are independent and identically distributed.

Then, the density function of $Y_{2,N}$ is the derivative of its distribution function.

$$\begin{aligned} f_{2:N}(y) &= \frac{\partial F_{2:N}}{\partial y}(y) = N(1-N)F(y)^{N-1} f(y) + N(N-1)F(y)^{N-2} f(y) \\ &= N(N-1)F(y)^{N-2} f(y)\left(1 - F(y)\right) \end{aligned}$$

Remark. We see that

$$f_{2:N}(y) = N f_{1:N-1}(y)\left(1 - F(y)\right) \tag{2.2}$$

Joint and Conditional Distributions of Order Statistics
Even though $X_1, \ldots, X_N$ are independently drawn, the order statistics $Y_{1:N}, Y_{2:N}, \ldots, Y_{N:N}$ are not independent. Then, the joint density of $Y = (Y_{1:N}, Y_{2:N}, \ldots, Y_{N:N})$ is

$$f_{Y:N}(y_1, y_2, \ldots, y_N) = \begin{cases} N! f(y_1) f(y_2) \ldots f(y_N) & \text{if } y_1 \geq \cdots \geq y_N \\ 0 & \text{otherwise} \end{cases}$$

Then, we can know what it will be the density in general of $Y' = (Y_{1:N}, Y_{2:N}, \ldots, Y_{k:N})$

$$f_{Y':N}(y_1, y_2, \ldots, y_k) = \begin{cases} N(N-1)\ldots(N-k) f(y_1) f(y_2) \ldots f(y_k) F(y_k)^{N-k} & \text{if } y_1 \geq \cdots \geq y_k \\ 0 & \text{otherwise} \end{cases}$$

What interests us is the density of the first and second order statistics that will be

$$f_{Y_{1:N}, Y_{2:N}}(y_1, y_2) = \begin{cases} N(N-1) f(y_1) f(y_2) F(y_2)^{N-2} & \text{if } y_1 \geq y_2 \\ 0 & \text{otherwise} \end{cases}$$

Now the density of $Y_{2:N}$ conditional on $Y_{1:N} = y$ is

$$\begin{aligned} f_{2:N}(z \mid Y_{1:N} = y) &= \begin{cases} \frac{f_{Y_{1:N}, Y_{2:N}}(y,z)}{f_{1:N}(y)} & \text{if } y \geq z \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{N(N-1) f(y) f(z) F(z)^{N-2}}{N f(y) F(y)^{N-1}} & \text{if } y \geq z \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{(N-1) f(z) F(z)^{N-2}}{F(y)^{N-1}} & \text{if } y \geq z \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

On the other hand, the density of $Y_{1:N-1}$ conditional on $Y_{1:N-1} < y$ is

$$\begin{aligned} f_{1:N-1}(z \mid Y_{1:N-1} < y) &= \frac{f_{1:N-1}(z)}{F_{1:N-1}(y)} \\ &= \frac{(N-1) f(z) F^{N-2}(z)}{F^{N-1}(y)} \\ &= f_{2:N}(z \mid Y_{1:N} = y) \end{aligned} \tag{2.3}$$

Since notice that $y \geq z$.

2.3 Revenue and Expected Payment in Second Price Auctions

Let us find what is the expected payment of any player in a Second Price Auction, $m^{II}(x)$. This payment is the product of probability of winning and the amount he pays when he wins.

$$
\begin{aligned}
m^{II}(x) &= \Pr(\text{Win})E\left[\text{2nd highest bid} \mid x \text{ is the highest bid}\right] \\
&= \Pr(Y_{1:N-1} < x)E\left[Y_{1:N-1} \mid Y_{1:N-1} < x\right] \\
&= F(x)^{N-1}\frac{1}{F(x)^{N-1}}\int_0^x z f_{1:N-1}(z)\,dz \\
&= \int_0^x z f_{1:N-1}(z)\,dz
\end{aligned}
$$

In the third equality we use the definition of the conditional expectation of a random variable X given that $X < x$ which is $E[X \mid X < x] = \frac{1}{F(x)}\int_0^x z f(z) dz$.

Then, the ex ante (that is, before he knows his type) expected payment of any player is:

$$
\begin{aligned}
E\left[m^{II}(X)\right] &= \int_0^\omega m^{II}(x) f(x)\,dx \\
&= \int_0^\omega \left(\int_0^x z f_{1:N-1}(z)\,dz\right) f(x)\,dx
\end{aligned}
$$

The expected revenue is what the auctioneers expects to win for this auction and it is sum of the ex ante expected payments of each player, then:

$$
\begin{aligned}
E\left[R^{II}\right] &= \sum_{i=1}^{N} E\left[m^{II}(X)\right] \\
&= NE\left[m^{II}(X)\right]
\end{aligned}
$$

Proposition 2.2. *The expected revenue is the expectation of the second highest value, namely*

$$
E\left[R^{II}\right] = \int_0^\omega z f_{2:N}(z)\,dz = E\left[Y_{2:N}\right] \tag{2.4}
$$

Proof. We know that:

$$
\begin{aligned}
E\left[m^{II}(X)\right] &= \int_0^\omega \left(\int_0^x z f_{1:N-1}(z)\,dz\right) f(x)\,dx \\
&= \left[F(x)\int_0^x z f_{1:N-1}(z)\,dz\right]_0^\omega - \int_0^\omega x F(x) f_{1:N-1}(x)\,dx \\
&= \int_0^\omega z f_{1:N-1}(z)\,dz - \int_0^\omega x F(x) f_{1:N-1}(x)\,dx \\
&= \int_0^\omega x f_{1:N-1}(x)\,dx - \int_0^\omega x F(x) f_{1:N-1}(x)\,dx \\
&= \int_0^\omega x f_{1:N-1}(x)(1 - F(x))\,dx \\
&= \int_0^\omega x f_{2:N}(x)\frac{1}{N}\,dx \\
&= \frac{1}{N}\int_0^\omega x f_{2:N}(x)\,dx
\end{aligned}
$$

In the second equality we use the integration by parts, where $u = \int_0^x z f_{1:N-1}(z)\, dz$ and $dv = f(x)\, dx$. In the sixth, we use (2.2). Knowing that $E\left[R^{II}\right] = NE\left[m^{II}(X)\right]$, then:

$$\begin{aligned} E\left[R^{II}\right] &= NE\left[m^{II}(X)\right] \\ &= N \cdot \frac{1}{N} \int_0^{\omega} x f_{2:N}(x)\, dx \\ &= \int_0^{\omega} x f_{2:N}(x)\, dx \end{aligned}$$

□

Next example computes the ex ante expected revenue of a Second Price Auction when valuations follow a uniform distribution.

Example 2.2. Suppose that the signals of the players follow a continuous uniform distribution $U(0,1)$, which means $w = 1$ and $F(x) = x$ and $f(x) = 1$. We want to calculate the expected payment and the expected revenue of this distribution. Firstly, we begin with the expected payment:

$$\begin{aligned} m^{II}(x) &= \int_0^x z f_{1:N-1}(z)\, dz = \int_0^x z(N-1) z^{N-2}\, dz \\ &= \left[(N-1)\frac{z^N}{N}\right]_0^x = (N-1)\frac{x^N}{N} \end{aligned}$$

Let's calculate the ex ante expected payment:

$$\begin{aligned} E\left[m^{II}(X)\right] &= \int_0^1 (N-1)\frac{x^N}{N}\, dx = \left[\frac{N-1}{N}\frac{x^{N+1}}{N+1}\right]_0^1 \\ &= \frac{N-1}{N(N+1)} \end{aligned}$$

Then, the expected revenue will be:

$$E\left[R^{II}\right] = NE\left[m^{II}(X)\right] = \frac{N-1}{N+1}$$

Chapter 3

First Price Auctions

3.1 Equilibrium bids

As we did in the Second Price Auctions we start defining which is its payoff function. Here, each player bids b_i and, if he wins, he pays his bid.

$$\Pi_i\left(b_{1,}\ldots,b_N\right) \quad = \quad \begin{cases} x_i - b_i & \text{if } b_i > \max_{j\neq i} b_j \\ 0 & \text{if } b_i < \max_{j\neq i} b_j \end{cases}$$

In case we had a draw between k bidders, $b_i = \max_{j\neq i} b_j$, we must do a raffle between these k players with the same probability of winning.

Remark. If we use the strategy of the Second Price Auction, $\beta^{II}(x) = x$, if we win we are going to have 0 profit. Then, the equilibrium will be different to the Second Price Auction because we want to win money with the auction.

Lemma 3.1. *If β_i is a symmetric increasing and differentiable equilibrium strategy, then:*

1. $\beta_i(0) = 0$
2. $\beta_i(\omega) = \beta_j(\omega) = \bar{b} < \omega \ \forall i, j$

Proof. Firstly, we proof the first statement. Fixed $x \in [0, \omega]$, it is better to do $\beta_i(x) = x$ than $\beta_i(x) = y > x$ because if we do $\beta_i(x) = y > x$ then if we win the auction we lose money since $x - y < 0$. Then, $\beta_i(0) = 0$. Also, we prefer to do $\beta_i(\omega) = \omega$ than $\beta_i(\omega) > \omega$ due to the same reasoning than before.

To proof that $\beta_i(\omega) = \beta_j(\omega) = \bar{b} < \omega \ \forall i, j$, suppose that player i follows the strategy β_i and it happens $\beta_i(\omega) > \beta_j(\omega)$. This strategy is not optimal for i because he wins, but with $\beta_i(\omega) = \beta_j(\omega) + \varepsilon$, $\varepsilon > 0$ he also wins but he pays less. Then, in the limit $\beta_i(\omega) = \beta_j(\omega)$ and due to the symmetric and increasing strategies we have that $\beta_i(\omega) = \beta_j(\omega) = \bar{b} < \omega$. □

Like in the Second Price Auctions, our achievement is to find the First Price Auction symmetric and increasing equilibrium. We want to know which would be the optimal bid β. Then, if the bidder i has value x and bids b_i while the remaining bidders bid according to β, his payoff function will be

$$
\begin{aligned}
\Pi_i &= (x-b_i)\Pr\left(b_i \geq \max_{j\neq i} b_j\right) = (x-b_i)\Pr\left(b_i \geq \max_{j\neq i} \beta\left(X_j\right)\right) \\
&= (x-b_i)\Pr\left(b_i \geq \beta\left(\max_{j\neq i} X_j\right)\right) \\
&= (x-b_i)\Pr\left(b_i \geq \beta\left(Y_{1:N-1}\right)\right) \\
&= (x-b_i)\Pr\left(\beta^{-1}(b_i) \geq Y_{1:N-1}\right) \\
&= (x-b_i)F\left(\beta^{-1}(b_i)\right)^{N-1}
\end{aligned}
$$

Our purpose is to find a b_i that optimizes the payoff function. Then, assuming differentiability, we derive with respect to b_i.

$$
\begin{aligned}
0 &= \frac{\partial \Pi_i}{\partial b_i} \\
0 &= (N-1)F\left(\beta^{-1}(b_i)\right)^{N-2}\left(\beta^{-1}\right)'(b_i)f\left(\beta^{-1}(b_i)\right)(x-b_i) - F\left(\beta^{-1}(b_i)\right)^{N-1} \\
0 &= (N-1)F\left(\beta^{-1}(b_i)\right)^{N-2}\frac{1}{\beta'\left(\beta^{-1}(b_i)\right)}f\left(\beta^{-1}(b_i)\right)(x-b_i) - F\left(\beta^{-1}(b_i)\right)^{N-1}
\end{aligned}
$$

In the third equality we use the Inverse Function Theorem. In a symmetric equilibrium we have that $\beta(x) = b_i$, Then,

$$
\begin{aligned}
(N-1)F(x)^{N-2}\frac{1}{\beta'(x)}f(x)\left(x-\beta(x)\right) - F(x)^{N-1} &= 0 \\
(N-1)F(x)^{N-2}f(x)\left(x-\beta(x)\right) &= F(x)^{N-1}\beta'(x) \\
\beta'(x)F(x)^{N-1} + \beta(x)(N-1)F(x)^{N-2}f(x) &= (N-1)F(x)^{N-2}f(x)x \\
\frac{d}{dx}\left(\beta(x)F(x)^{N-1}\right) &= (N-1)F(x)^{N-2}f(x)x
\end{aligned}
$$

Now, using the Lemma 3.1, we have the initial condition $\beta(0) = 0$ to solve the differential equation.

$$
\begin{aligned}
\beta(x)F(x)^{N-1} &= \int_0^x (N-1)F(z)^{N-2}f(z)z\,dz \\
\beta(x) &= \frac{1}{F(x)^{N-1}}\int_0^x (N-1)F(z)^{N-2}f(z)z\,dz \\
\beta(x) &= E\left[Y_{1:N-1} \mid Y_{1:N-1} < x\right]
\end{aligned}
$$

Hence, we have obtained that if there is a symmetric and increasing equilibrium $\beta(x)$ it must be $E\left[Y_{1:N-1} \mid Y_{1:N-1} < x\right]$. Let us now prove that this is, indeed, an equilibrium.

Now, we want to see that $\beta(x) = E\left[Y_{1:N-1} \mid Y_{1:N-1} < x\right]$ is an equilibrium strategy in a First Price Auction.

Proposition 3.1. *$\beta^I(x) = E\left[Y_{1:N-1} \mid Y_{1:N-1} < x\right]$ is a symmetric equilibrium strategy in a First Price Auction, where $Y_{1:N-1}$ is the highest of $N-1$ independently drawn values.*

Proof. Suppose that $N-1$ bidders follow the strategy β^I. Let's see that the missing bidder i follows this strategy too. Firstly, β^I is an increasing and continuous function. Then, in equilibrium the player with the highest value bids the highest price to win the auction. In the other hand, it is not optimal for the player i bid $b > \beta^I(\omega)$. Suppose bidder i with value x bids something different from $\beta^I(x)$, assume he bids $b = \beta^I(z)$. Then, the payoff function will be

$$\begin{aligned}
\Pi(x,b) &= \Pi\left(x,\beta^I(z)\right) = \left(x-\beta^I(z)\right)\Pr(Y_{1:N-1}\leq z) \\
&= (x - E[Y_{1:N-1} \mid Y_{1:N-1} < z])\,F(z)^{N-1} \\
&= xF(z)^{N-1} - E[Y_{1:N-1} \mid Y_{1:N-1} < z]\,F(z)^{N-1} \\
&= xF(z)^{N-1} - \int_0^z s(N-1)F(s)^{N-2}f(s)\,ds \\
&= xF(z)^{N-1} - \left[sF(s)^{N-1}\right]_0^z + \int_0^z F(s)^{N-1}\,ds \\
&= (x-z)F(z)^{N-1} + \int_0^z F(s)^{N-1}\,ds \qquad (3.1)
\end{aligned}$$

In the fourth equality, we use $E[X \mid X < z] = \frac{1}{F(z)}\int_0^x sf(s)\,ds$ and in the sixth equality we use the integration by parts.

Then we obtain that the difference between the payoffs when bidding $\beta^I(x)$ and bidding $b = \beta^I(z)$ is

$$\begin{aligned}
\Pi\left(x,\beta^I(x)\right) - \Pi\left(x,\beta^I(z)\right) &= \int_0^x F(s)^{N-1}\,ds - (x-z)F(z)^{N-1} - \int_0^z F(s)^{N-1}\,ds \\
&= (z-x)F(z)^{N-1} - \int_x^0 F(s)^{N-1}\,ds - \int_0^z F(s)^{N-1}\,ds \\
&= (z-x)F(z)^{N-1} - \int_x^z F(s)^{N-1}\,ds
\end{aligned}$$

We see that $\Pi\left(x,\beta^I(x)\right) - \Pi\left(x,\beta^I(z)\right) \geq 0$ independently if $z - x \geq 0$ or otherwise, as the Figure 3.1 shows. We use that $F(x)$ is an increasing function; if $z - x \geq 0$ and $F(x)$ is as increasing function then, $\int_x^z F(s)^{N-1}\,ds \leq F(z)^{N-1}\int_x^z ds$. Then, we can conclude that due to $\Pi\left(x,\beta^I(x)\right) - \Pi\left(x,\beta^I(z)\right) \geq 0$, player i does not have incentives to bid $b = \beta^I(z)$ instead of $\beta(x)$. □

Remark. $\beta^I(x) = E[Y_{1:N-1} \mid Y_{1:N-1} < x]$ is a Nash equilibrium strategy but it could not be the only equilibrium; we do not have unicity in the equilibrium strategies.

Corollary 3.1. $\beta^I(x) = x - \int_0^x \frac{F(z)^{N-1}}{F(x)^{N-1}}\,dz$

Proof.

$$\begin{aligned}
\beta^I(x) &= \frac{1}{F(x)^{N-1}}\int_0^x (N-1)F(z)^{N-2}f(z)z\,dz \\
&= \frac{1}{F(x)^{N-1}}\left[F(z)^{N-1}z\right]_0^x - \int_0^x \frac{F(z)^{N-1}}{F(x)^{N-1}}\,dz \\
&= x - \int_0^x \frac{F(z)^{N-1}}{F(x)^{N-1}}\,dz
\end{aligned}$$

□

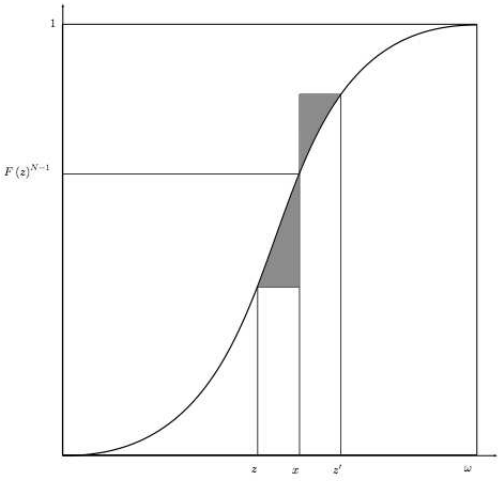

Figure 3.1: $\Pi\left(x, \beta^I(x)\right) - \Pi\left(x, \beta^I(z)\right) \geq 0$, if $z - x \geq 0$ or otherwise

The above result shows that in the symmetric and increasing equilibrium of the First Price Auction, each buyer bids below his valuation. How much below depends on the number N of bidders.

Example 3.1. Suppose, again, that the signals of the players follow a continuous uniform distribution $U(0,1)$, which means $w = 1$ and $F(x) = x$ and $f(x) = 1$. We could then calculate the equilibrium bid in a First Price Auction.

$$\beta^I(x) \;=\; x - \int_0^x \frac{z^{N-1}}{x^{N-1}}\, dz \;=\; x - \frac{x}{N} \;=\; \frac{N-1}{N} x < x \;=\; \beta^{II}(x)$$

3.2 Revenue Comparison

Now, having found the equilibrium strategy for the First Price Auctions, we can find the expected payment to the auctioneer.

$$\begin{aligned} m^I(x) &= \Pr(\text{Win}) \times \text{Amount bid} \\ &= \Pr(Y_{1:N-1} \leq x)\, E\left[Y_{1:N-1} \mid Y_{1:N-1} < x\right] \\ &= m^{II}(x) \end{aligned}$$

Thus, we have the same expected payment in the two price auctions. Then, the ex ante expected payment and the expected revenue in the two auctions are the same. Let us denote A as the number of the price auction, $A = I$ or II

$$\begin{aligned} E\left[m^A(X)\right] &= \int_0^\omega \left(\int_0^x z f_{1:N-1}(z)\, dz \right) f(x)\, dx \\ &= \int_0^\omega \left(\int_z^\omega f(x)\, dx \right) z f_{1:N-1}(z)\, dz \\ &= \int_0^\omega z\,(1 - F(z))\, f_{1:N-1}(z)\, dz \end{aligned}$$

Thus, the expected revenue will be $E\left[R^A\right] = \int_0^\omega x f_{2:N}(x)\ dx = E\left[Y_{2:N}\right]$, as we calculated in (2.4). Then, we can write a proposition to remember what we have found.

Proposition 3.2. *Let $X_1, \ldots, X_N$ be independently and identically distributed random variables that represent the private values of N bidders. Then, the ex ante expected payment and the expected revenue in a First Price Auction and in a Second Price Auction is going to be the same.*

We can say more about the distribution of prices in the two auctions. The revenues in a Second Price Auction are more variable than in its First Price Auction. In the Second Price Auction the prices can range between 0 and ω but in the First Price Auction the prices can only range between 0 and $E\left[Y_{1:N-1}\right]$.

The notion of mean-preserving spread that we introduce now shows the fact that the Second Price Auction is riskier than the First Price Auction.

Definition 3.1. Suppose X is a random variable with distribution function F. Let Z be a random variable with distribution function H such that $E\left[Z \mid X = x\right] = 0\ \forall x$. Suppose $Y = X + Z$ with distribution function G, then we say that G is a mean-preserving spread of F.

Let L^A be the distribution of the equilibrium prices in a A Price Auction, $A = I$ or II. Now, we are going to define what is a mean-preserving spread.

Now, we show that the distribution L^{II} is a mean-preserving spread of L^I, which means that the perspective of the auctioneer a Second Price Auction is riskier than the First Price Auction because the bidder only will pay the second price.

Proposition 3.3. *With independently and identically distributed private values, the distribution L^{II} of equilibrium prices in a Second Price Auction is a mean-preserving spread of the distribution L^I of equilibrium prices in a First Price Auction.*

Proof. We see that $L^{II} = Y_{2:N}$, the price paid will be the second highest order statistic, and $L^I = \beta^I\left(Y_{1:N}\right)$, the price paid will be the offer of the highest order statistic.

Then,

$$\begin{aligned}
E\left[L^{II} \mid L^I = p\right] &= E\left[Y_{2:N} \mid \beta^I\left(Y_{1:N}\right) = p\right] \\
&= E\left[Y_{2:N} \mid Y_{1:N} = \left(\beta^I\right)^{-1}(p)\right] \\
&= E\left[Y_{1:N-1} \mid Y_{1:N-1} < \left(\beta^I\right)^{-1}(p)\right] \\
&= \left(\beta^I\right)\left(\left(\beta^I\right)^{-1}(p)\right) \\
&= p
\end{aligned}$$

In the third equality we use (2.3). Then, there exists a random variable Z such that the distribution of L^{II} is the same as that of $L^I + Z$ and $E\left[Z \mid L^I = p\right] = 0$, Then, L^{II} is a mean-preserving spread of L^I. □

Chapter 4

Reserve Prices

An auction has a reserve price $r > 0$ if the auctioneer would not sell the object in case the object's price obtained in the auction is lower than the reserve price. Now, let us see what would be the impact of the reserve prices in the First and Second Price Auctions.

Reserve Prices in Second Price Auctions
Suppose that the seller sets a reserve price, $r > 0$. Then, no bidder with a value $x < r$ is going to have positive payoff, hence, is not going to enter to the auction. Let us find, now, the expected payment of a bidder with $r \leq x$.

$$\begin{aligned} m^{II}(x,r) &= \int_0^x y f_{1:N-1}(y)\ dy \\ &= \int_0^r y f_{1:N-1}(y)\ dy + \int_r^x y f_{1:N-1}(y)\ dy \\ &= r F^{N-1}(r) + \int_r^x y f_{1,N-1}(y)\ dy \end{aligned}$$

Notice that, at least, the seller wins r even if the second highest bid is lower than r.

Reserve Prices in First Price Auctions
Knowing that the equilibrium in First Price Auctions is $\beta^I(x) = E[Y_{1:N-1} \mid Y_{1:N-1} < x]$ and that we need $\beta^I(r) = r$, because β^I is a symmetric equilibrium of the First Price Auction, then the strategy for any bidder with $r \leq x$ is

$$\begin{aligned} \beta^I(x) &= E[\max\{r, Y_{1:N-1}\} \mid Y_{1:N-1} < x] \\ &= E[Y_{1:N-1} \mid Y_{1:N-1} < x] \Pr(Y_{1:N-1} \geq r) + \frac{r \Pr(Y_{1:N-1} \leq r)}{F^{N-1}(x)} \\ &= E[Y_{1:N-1} \mid r \leq Y_{1:N-1} < x] + \frac{r}{F^{N-1}(x)} F^{N-1}(r) \\ &= \frac{1}{F^{N-1}(x)} \int_r^x y f_{1:N-1}(y)\ dy + \frac{r}{F^{N-1}(x)} F^{N-1}(r) \end{aligned} \tag{4.1}$$

Hence, the expected payment of a bidder with value $x \leq r$

$$\begin{aligned} m^I(x,r) &= \Pr(\text{Win}) \times \text{Amount bid} \\ &= \Pr(Y_{1:N-1} \leq x)\, E\left[\max\{r, Y_{1:N-1}\} \mid Y_{1:N-1} < x\right] \\ &= F^{N-1}(x)\, E\left[\max\{r, Y_{1:N-1}\} \mid Y_{1:N-1} < x\right] \\ &= \int_r^x y f_{1:N-1}(y)\ dy + r F^{N-1}(r) \end{aligned}$$

Where the last equality follows from (4.1).

Notice that $m^I(x,r) = m^{II}(x,r)$ and also the ex ante expected payment and the expected revenue are the same.

Revenue Effects of Reserve Prices

As we did before, we are going to calculate de ex ante expected payment.

$$\begin{aligned} E\left[m^A(x,r)\right] &= \int_r^\omega m^A(x,r) f(x)\ dx \\ &= \int_r^\omega \left(\int_r^x y f_{1:N-1}(y)\ dy + r F^{N-1}(r)\right) f(x)\ dx \\ &= r F^{N-1}(r) \int_r^w f(x)\ dx + \int_r^\omega \left(\int_r^x y f_{1:N-1}(y)\ dy\right) f(x)\ dx \\ &= r F^{N-1}(r)(1-F(r)) + \int_r^\omega y\left(1-F(y)\right) f_{1:N-1}(y)\ dy \end{aligned}$$

In the last equality we use Fubini's Theorem.

Is there an optimal auction?

Let us suppose that the seller attaches a value $x_0 \in [0, w)$. Then, if the object is unsold the auctioneer would set a reserve price r greater than x_0, $r \geq x_0$. What would be the expected payoff?

$$\begin{aligned} \Pi_0(x_0, r) &= N \times E\left[m^A(X,r)\right] + F(r)^N x_0 \\ &= N\left(r F^{N-1}(r)(1-F(r)) + \int_r^\omega y\left(1-F(y)\right) f_{1:N-1}(y)\ dy\right) + F(r)^N x_0 \end{aligned}$$

Now, we want to find out which reserve price r would be optimal for this seller.

Differentiating the above expression with respect to r, we obtain

$$\begin{aligned} \frac{d\Pi_0}{dr}(x_0, r) &= x_0 N F(r)^{N-1} f(r) + N\left(F^{N-1}(r)(1-F(r)) - r F^{N-1}(r) f(r)\right) \\ &= N\left(1 - F(r) - (r - x_0) f(r)\right) F^{N-1}(r) \\ &= N\left(1 - (r - x_0)\frac{f(r)}{1-F(r)}\right) F^{N-1}(r)(1-F(r)) \end{aligned}$$

Then, knowing that the hazard rate is defined as $\lambda(x) = \frac{f(x)}{1-F(x)}$, we have that

$$\frac{d\Pi_0}{dr} = N(1-(r-x_0)\lambda(r)) F^{N-1}(r)(1-F(r))$$

Remark. Notice that if $x_0 = 0$ then, $\frac{d\Pi_0}{dr}(0,0) = N(1)F^{N-1}(0)(1-F(0)) = 0$, but if $r \gtrsim 0$, then $\frac{d\Pi_0}{dr}(0,r) = N(1-r\lambda(r))F^{N-1}(r)(1-F(r))$. If $\lambda(r)$ is bounded and due to r is small, then $\frac{d\Pi_0}{dr}(0,r) > 0$, then at $r = 0$ there is a minimum of the expected payment.

If $x_0 > 0$, then $\frac{d\Pi_0}{dr}(x_0,x_0) = N(1)F^{N-1}(x_0)(1-F(x_0)) > 0$.

Returning to our initial problem, we want to find the optimal reserve price r^*, then

$$\begin{aligned}
\frac{d\Pi_0}{dr}(x_0,r) &= 0 \\
1-\left(r^*-x_o\right)\lambda\left(r^*\right) &= 0 \\
r^*-\frac{1}{\lambda(r^*)} &= x_0
\end{aligned}$$

Notice that if $\lambda(\cdot)$ is increasing, then $x_0 < r$. Also, it is remarkable that the optimal reserve prices does not depend on the number of bidders. It seems that the reserve price comes into play only when there is a single bidder with a value that exceeds the reserve price, $x > r$.

Chapter 5

The Revenue Equivalence Principle

In the previous chapter we have seen that a risk neutral seller is indifferent between the First and Second Price Auctions, but these two auctions are not strategically equivalent since the equilibrium strategy for both of them is different

5.1 A General Auction Format

By now, the auction forms we consider all have the feature that buyers must submit bids. These amounts of money are the only thing that determines who wins the object and how much is going to pay. Then, we can define what an standard auction is.

Definition 5.1. An auction is standard if the rules of an auction determines that the person who bids the highest amount wins the object.

Notice that Definition 5.1 does not say that the person who values the object the most wins.

Also, notice that the First and Second Price Auctions are standard auctions but a lottery is a nonstandard method because a player with one ticket could win the lottery against a person who has eight tickets.

Let A be an standard auction, β^A a symmetric equilibrium of the auction and let $m^A(x)$ be the equilibrium expected payment by a bidder with value x. We must emphasize that the expected payment with value 0 is 0.

Next proposition shows that the expected payment does not depend on the standard auction form.

Proposition 5.1. *Suppose that values are independently and identically distributed and all bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue to the seller.*

Proof. Let A be an standard auction, with a symmetric equilibrium β^A and expected payment $m^A(x)$. Suppose that $m^A(0) = 0$. We must remember that

$$m^A(x) = \Pr(\text{Win}) \times \text{Amount bid}$$

Suppose that all players except one follow $\beta^A(x)$, we could assume that the one that is not following $\beta^A(x)$ bids $\beta^A(z)$. What would be this buyer's expected payoff?

$$\begin{aligned}
\Pi(z,x) &= (x-p) \times \Pr(\text{Win}) \\
&= x \times \Pr(\text{Win}) - p \times \Pr(\text{Win}) \\
&= x F_{1:N-1}(z) - m^A(z)
\end{aligned}$$

The maximization condition is:

$$\begin{aligned}
\frac{\partial \Pi(z,x)}{\partial z} &= 0 \\
x f_{1:N-1}(z) - \frac{d\, m^A(z)}{d z} &= 0
\end{aligned}$$

and since $\beta(x)$is an equilibrium strategy, the maximum is attained at $z = x$.

$$\begin{aligned}
\frac{d\, m^A(x)}{d x} &= x f_{1:N-1}(x) \\
m^A(x) &= m^A(0) + \int_0^x y f_{1:N-1}(y)\ dy \\
&= \int_0^x y f_{1:N-1}(y)\ dy \qquad (5.1)
\end{aligned}$$

Notice that $m^A(x)$ does not depend on A and hence the expected payment of a bidder does not depend on the auction format, and the same happens with the expected revenue of the seller. □

5.2 Some Applications

All-Pay Auctions

Each bidder submits a bid and the highest bidder wins the object auctioned, but everybody pays his bid. Suppose that we have a symmetric, increasing equilibrium of the All-Pay Auction such that the expected payment of a bidder with value 0 is 0. Due to (5.1) we know that the expected payment is $m^{AP}(x) = m^A(x) = \int_0^x y f_{1:N-1}(y)\ dy$. On the other hand, we know that what the bidders pay is exactly what they offer, then

$$\begin{aligned}
\beta^{AP}(x) &= m^{AP}(x) \\
&= \int_0^x y f_{1:N-1}(y)\ dy
\end{aligned}$$

But, is $\beta^{AP}(x)$ an equilibrium? To see it we proceed as usual in these cases, we suppose that one bidder does not follow $\beta^{AP}(x)$ but follows $\beta^{AP}(z)$, then the expected payoff of a bidder with value x is

$$
\begin{aligned}
\Pi_i(x,z) &= (x-p)\Pr(\text{Win}) - p\Pr(\text{Lose}) \\
&= x\Pr(\text{Win}) - p(\Pr(\text{Win}) + \Pr(\text{Lose})) \\
&= x\Pr(\text{Win}) - p \\
&= xF_{1:N-1}(z) - \beta^{AP}(z) \\
&= xF_{1:N-1}(z) - \int_0^z y f_{1:N-1}(y)\ dy \\
&= (x-z)F_{1:N-1}(z) - \int_0^z F_{1:N-1}(y)\ dy
\end{aligned}
$$

In the last equality we integrate the second term by parts. We have the same result as we have in (3.1), then if we do the same procedure we reach that we maximize $\Pi_i(x,z)$ when $z = x$. Thus, β^{AP} is a symmetric and increasing equilibrium.

Equilibrium of Third Price Auctions

Suppose that we have at least three bidders. Consider a sealed-bid auction where the highest bidder wins the object but, as the name suggests, pays the third highest bid.

As we know, the expected payment is $m^{III}(x) = m^A(x) = \int_0^x y f_{1:N-1}(y)\ dy$, but we must answer to the question: what is the strategy to follow?

To begin with, if the bidder 1 wins in equilibrium with value x means that the signal of the second best player is lower than his signal, x, then $Y_{1:N-1} < x$. The price bidder 1 pays is the second highest bid of the other $N-1$ players, which means $\beta^{III}(Y_{2:N-1})$. Then, we must calculate $f_{2:N-1}(y \mid Y_1 < x)$ before we continue our procedure.

First of all, we recall that $f_{2:N-1}(y) = (N-1)(1-F(y))f_{1:N-2}(y)$ due to (2.2). Then,

$$
f_{2:N-1}(y \mid Y_1 < x) = \frac{1}{F_{1:N-1}(x)}(N-1)(F(x)-F(y))f_{1:N-2}(y)
$$

We have the factor $\frac{1}{F_{1:N-1}(x)}$ because it is the distribution function of $Y_{1:N-1}$ and $(N-1)(F(x)-F(y))$ is de probability that $Y_{1:N-1}$ exceeds $Y_{2:N-1}$ but is less than x.

On the other hand, the expected payment would be

$$
\begin{aligned}
m^{III}(x) &= \Pr(\text{Win}) \times \text{Amount bid} \\
&= F_{1:N-1}(x)E\left[\beta^{III}(Y_{2:N-1}) \mid Y_{1:N-1} < x\right] \\
&= F_{1:N-1}(x)\int_0^x \beta^{III}(y) f_{2:N-1}(y \mid Y_1 < x)\ dy \\
&= \int_0^x \beta^{III}(y)(N-1)(F(x)-F(y))f_{1:N-2}(y)\ dy
\end{aligned}
$$

Then, if we equate the two expected payments in a Third Price Auction then, we have

$$
\int_0^x \beta^{III}(y)(N-1)(F(x)-F(y))f_{1:N-2}(y)\ dy = \int_0^x y f_{1:N-1}(y)\ dy
$$

Differentiating with respect to x then,

$$(N-1)f(x)\int_0^x \beta^{III}(y)\, f_{1:N-2}(y)\ dy \quad = \quad x f_{1:N-1}(x)$$

We have used the Leibniz Formula which says: Let $F(x)=\int_{g(x)}^{h(x)} f(y,x)\ dy$ then differentiating with respect to x we have $F'(x)=h'(x)f(h(x),x)-g'(x)f(g(x),x)+\int_{g(x)}^{h(x)} \frac{\partial f(y,x)}{\partial x}\ dy$. Rearranging the equality we arrive to

$$\int_0^x \beta^{III}(y)\, f_{1:N-2}(y)\ dy \quad = \quad x(F(x))^{N-2}$$

Notice that we only use that $f_{1:N-1}=(N-1)f(x)(F(x))^{N-2}$.
If we differentiate, again, with respect to x

$$\beta^{III}(x)\, f_{1:N-2}(x) \quad = \quad (F(x))^{N-2}+x(N-2)(F(x))^{N-3}f(x)$$

and rearranging once again

$$\begin{aligned}
\beta^{III}(x) &= \frac{(F(x))^{N-2}+x(N-2)(F(x))^{N-3}f(x)}{f_{1:N-2}(x)} \\
&= \frac{(F(x))^{N-2}+x(N-2)(F(x))^{N-3}f(x)}{(N-2)(F(x))^{N-3}f(x)} \\
&= \frac{F(x)}{(N-2)f(x)}+x
\end{aligned}$$

This derivation is valid only if $\beta^{III}(x)$ is increasing thus, we need that $\frac{F}{f}$ is increasing. This condition is the same as F is log-concave.

Remark. Notice that $\beta^{III}(x) > x$ which means that the player bids more than he values the object.

Chapter 6

Risk-Averse Bidders

In this chapter, we analyze what happens if bidders are risk-averse. We maintain all assumptions we have been working with, like independence of values, symmetry among bidders and absence of reserve prices.

The expected payoff of a risk-neutral bidder is just the difference between his expected gain and his expected payment, i.e. $\Pi(x) = (x-p)\Pr(\text{Win}) = x\Pr(\text{Win}) - p\Pr(\text{Win})$, but an averse or risk-loving bidder does not have this expected payoff because depends on his utility function.

Suppose that the bidder is risk-averse, then he must have an utility function strictly concave

$$\begin{aligned} u : \mathbb{R}^+ &\longrightarrow \mathbb{R} \\ x &\longmapsto u(x) \end{aligned}$$

i.e. $u' > 0$, $u'' < 0$ and satisfies $u(0) = 0$. Now, the expected payment will be $\Pi(x) = u(x-p)\Pr(\text{Win})$.[4]

Proposition 6.1. *Suppose that bidders are risk-averse with the same utility function. If we suppose symmetric, independent private values then the expected revenue in a First Price Auction is greater than that in a Second Price Auction.*

Proof. Firstly, notice that in a Second Price Auction with a risk-averse bidder it is a dominant strategy to bid his value, due to the same argument we used in the Proposition 2.1. Thus, the expected payment would be the same as for a risk-neutral bidder.

Now, let us see what happens with a First Price Auction. Suppose that exists an equilibrium strategy given by an increasing and differentiable function

$$\begin{aligned} \gamma^I : [0,\omega] &\longrightarrow \mathbb{R}^+ \\ x &\longmapsto \gamma^I(x) \end{aligned}$$

satisfying $\gamma^I(0) = 0$. If all bidders except one follow this strategy, the one that is not following this strategy, with a signal x would bid $\gamma^I(z)$ where $z \in [0,\omega]$ and $\gamma^I(z) < \gamma^I(\omega)$. Now, we want to maximize his expected payoff

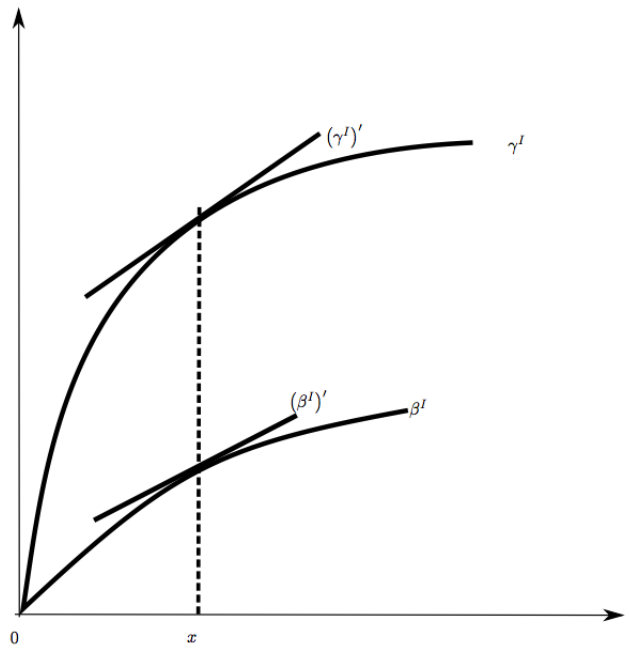

Figure 6.1: Plot of β^I and γ^I

$$\begin{aligned}\Pi(x,z) &= u\left(x-\gamma^I(z)\right)\Pr(\text{Win}) \\ &= u\left(x-\gamma^I(z)\right)F_{1:N-1}(z)\end{aligned}$$

The first order condition is

$$\begin{aligned}\frac{\partial\Pi(x,z)}{\partial z} &= 0 \\ -\left(\gamma^I\right)'(z)\,u'\left(x-\gamma^I(z)\right)F_{1:N-1}(z)+u\left(x-\gamma^I(z)\right)f_{1:N-1}(z) &= 0\end{aligned}$$

In a symmetric equilibrium, it must be optimal to do $z = x$. Rearranging the equation we have

$$\left(\gamma^I\right)'(x) = \frac{u\left(x-\gamma^I(x)\right)}{u'\left(x-\gamma^I(x)\right)}\frac{f_{1:N-1}(x)}{F_{1:N-1}(x)}$$

If the bidder were risk-neutral, $u(x)=x$, then

$$\left(\beta^I\right)'(x) = \left(x-\gamma^I(x)\right)\frac{f_{1:N-1}(x)}{F_{1:N-1}(x)}$$

Now, due to the fact that u is a strictly concave function and $u(0)=0$, $\frac{u(x)}{u'(x)} > x\ \forall x>0$ we could say

$$\left(\gamma^I\right)'(x) = \frac{u\left(x-\gamma^I(x)\right)}{u'\left(x-\gamma^I(x)\right)}\frac{f_{1:N-1}(x)}{F_{1:N-1}(x)} > \left(x-\gamma^I(x)\right)\frac{f_{1:N-1}(x)}{F_{1:N-1}(x)}$$

Now if $\beta^I(x) > \gamma^I(x)$ then

$$\begin{aligned}\left(\gamma^I\right)'(x) &> \left(x-\gamma^I(x)\right)\frac{f_{1:N-1}(x)}{F_{1:N-1}(x)} \\ &> \left(x-\beta^I(x)\right)\frac{f_{1:N-1}(x)}{F_{1:N-1}(x)} \\ &= \left(\beta^I\right)'(x)\end{aligned}$$

Also we have that $\beta^I(0) = \gamma^I(0) = 0$.

To summarize, we have that $\beta^I(x) > \gamma^I(x) \Rightarrow \left(\gamma^I\right)'(x) > \left(\beta^I\right)'(x)$ and $\beta^I(0) = \gamma^I(0) = 0$. But is it really happening $\beta^I(x) > \gamma^I(x)$?

If we suppose that, as the Figure 6.1 shows us, we have that $\beta^I(x) < \gamma^I(x)$. Thus, in a First Price Auction with risk aversion, the expected price in equilibrium bids increases. Due to the unaffection by risk aversion in a Second Price Auction the expected revenue in a First Price Auction is higher than that in a Second Price Auction. □

Part III

Interdependent-value Auctions

Chapter 7

Introduction and preliminaries

In this latest part we are going to analyze how the First and Second Price Auctions react when the valuation of the bidders does not depend only on their own value (that was the case in Part II) but on the others too.

Interdependent values means that a bidder i could only have partial information and the information of other bidders affect the value he assigns to the object. Then, bidder i's value of the object, V_i, can be expressed as

$$V_i \quad = \quad v_i(X_1, \ldots, X_N)$$

where $X_j \in \left[0.\omega_j\right]$ is the random variable called j's signal that is bidder's j private information; v_i is bidder i's valuation which is a nondecreasing function in all its variables and we will assume for technical reasons that it is twice continuously differentiable. Also, it is assumed that v_i is strictly increasing in X_i. Thus, at this stage, if bidder i would know the signal of all buyers, including its own value, then he would deduce the actual value of the object. In this new scenario is not always the natural one; for instance, for the art auction one could think that each bidder has its own opinion no matter the value of the others. However, in other situations like the companies interested in a radioelectrical license where one frequency is more important than the others for one specific radio station and everybody knows it.

However, a general formulation could start supposing that $V_1, \ldots, V_N$ are the unknown N values of the bidders; $X_1, \ldots, X_N$ are, as before, the N bidders signals. In that case, we could think that the function that allows us to calculate the value of the object depending on the signals of all bidders it could be defined as the expected value to i conditional on all the information available to bidders

$$v_i(x_1, \ldots, x_N) \quad \equiv \quad E\left[V_i \mid X_1 = x_1, \ldots, X_N = x_N\right]$$

Then, due to the knowing of $x_1, \ldots, x_N$ the bidder i could find an estimator of V_i but not the exact value. For example, if $v_i(x_1, x_2) = \frac{1}{2}(x_1 + x_2)$ this would be an estimator of V_i.

Remark. We suppose that

- $v_i(0, \ldots, 0) = 0$
- $E\left[V_i\right] < +\infty$

- Bidders are risk neutral which means that each bidder maximizes the expectation of $V_i - p_i$ where p_i is the price the bidder i pays.

Also, notice that in private values

$$\begin{aligned} v_i(x_1,\ldots,x_N) &\equiv E[V_i \mid X_1 = x_1,\ldots,X_N = x_N] \\ &= E[V_i \mid X_i = x_i] \\ &= x_i \end{aligned}$$

for all bidders which means that the value of the object for a bidder i only depends on him.

In the case we had common values, the value of the object V is obtained evaluating the same function v; $v_i(x_1,\ldots,x_N) = v_j(x_1,\ldots,x_N)$ for all $i,j = 1,\ldots,N$ then, we could write $V = v(X_1,\ldots,X_N)$.

7.1 Affiliation

In some cases, we consider that the bidders' signal is correlated, for example, if a bidder i receives a higher signal it is because the other bidders also have a higher signal with high probability. Thus, there is no reason to the joint density of the bidders' signals be equal to the product of individual signals' densities, $f_{X_1,\ldots,X_N}(x_1,\ldots,x_N) \neq \prod_{j=0}^{N} f_{X_j}\left(x_j\right)$.

Definition 7.1. Let $X_1,\ldots,X_N$ be the random variables distributed on subspace of $\mathbb{R}^N$,$\mathfrak{X} \subset \mathbb{R}^N$, according to the joint density of $X_1,\ldots,X_N$, denoted by $f_{X_1,\ldots,X_N}$. We say that $X_1,\ldots,X_N$ are affiliated if for all $x', x'' \in \mathfrak{X}$

$$f_{X_1,\ldots,X_N}\left(x' \vee x''\right) f_{X_1,\ldots,X_N}\left(x' \wedge x''\right) \quad \geq \quad f_{X_1,\ldots,X_N}\left(x'\right) f_{X_1,\ldots,X_N}\left(x''\right)$$

where

$$x' \vee x'' \quad = \quad \left(\max\left\{x_1', x_1''\right\},\ldots,\max\left\{x_N', x_N''\right\}\right)$$

and

$$x' \wedge x'' \quad = \quad \left(\min\left\{x_1', x_1''\right\},\ldots,\min\left\{x_N', x_N''\right\}\right)$$

Let $X_1,\ldots,X_N$ be the bidders' signals; $Y_{1:N},\ldots,Y_{N-1:N}$, the highest, the second highest,... smallest random variable from among $X_2, X_3 \ldots, X_N$. If $X_1,\ldots,X_N$ are affiliated, then $X_1, Y_{1:N},\ldots,Y_{N-1:N}$ are also affiliated.

Let $G(\cdot \mid x)$ be the distribution of $Y_{1:N}$ conditional on $X_1 = x$. Then, if $x' > x$ in terms of the reverse hazard rate, for all y,

$$\frac{g\left(y \mid x'\right)}{G\left(y \mid x'\right)} \quad \geq \quad \frac{g\left(y \mid x\right)}{G\left(y \mid x\right)}$$

Furthermore, if γ is an increasing function then, if $x' > x$ then

$$E\left[\gamma(Y_{1:N}) \mid X_1 = x'\right] \quad \geq \quad E\left[\gamma(Y_{1:N}) \mid X_1 = x\right] \tag{7.1}$$

Chapter 8

Different Auctions Formats

As we have considered in the case of private value auctions, with interdependent values also we split in different scenarios depending on the precise mechanism we use.

8.1 Second Price Auctions

We want to give a symmetric equilibrium in a Second Price Sealed-bid Auction with interdependent values.

Firstly, with interdependent values and affiliated signals there are two aspects to symmetry:

- The bidders' signals, X_i for $i = 1, \ldots, N$, live in the same interval $[0, \omega]$.
- $v_i(X_1, \ldots, X_N) = u(X_i, X_{-i})$ where the function u is the same for all bidders and is symmetric in the last $N-1$ components. As an example, suppose $N = 3$ then $u(x, y, z) = u(x, z, y)$ where x, y, z are the signals of the the first, second and third bidder, respectively.

Also, we assumed that the joint density of the signals, f, is defined on $[0, \omega]^N$, is a symmetric function of its arguments.

Definition 8.1. We define the function below as the expectation of the value to bidder 1 conditioned by the signal received by him, x, and also the highest signal among the other bidders is y

$$v(x, y) = E\left[V_1 \mid X_1 = x, Y_{1:N} = y\right]$$

Notice that:

- $v(x, y)$ is the same function for all bidders because of its symmetry.
- Due to (7.1) $v(x, y)$ is nondecreasing function of x and y.

Proposition 8.1. *Symmetric equilibrium strategies in a Second Price Auction are given by:*

$$\beta^{II}(x) = v(x, x)$$

Proof. Suppose for all bidders except one, this one we call it i, follow the strategy β^{II}. As we have proceed before, the bidder i with x signal bids b. Then, his expected payoff would be

$$\begin{aligned}\Pi(b,x) &= \int_0^{\left(\beta^{II}\right)^{-1}(b)} \left(v\left(x,y\right)-\beta^{II}\left(x\right)\right) g\left(y\,|\,x\right)\ dy \\ &= \int_0^{\left(\beta^{II}\right)^{-1}(b)} \left(v\left(x,y\right)-v\left(y,y\right)\right) g\left(y\,|\,x\right)\ dy\end{aligned}$$

where $v\left(x,y\right)$ is the bidder i's expected valuation, $v\left(y,y\right)$ is the price that the bidder i is going to pay if the others follow β^{II} and $g\left(y\,|\,x\right)$ is the probability of success.

We know that $v\left(x,y\right)$ is increasing in the first argument, then for all $y > x$, $v\left(x,y\right)-v\left(y,y\right) > 0$ and for all $y > x$, $v\left(x,y\right)-v\left(y,y\right) < 0$. Thus, we choose b such tat $\left(\beta^{II}\right)^{-1}(b) = x \Rightarrow b = \beta^{II}(x)$. □

What does really mean $\beta^{II}(x) = v(x,x)$?

The bidder i makes an offer such that if the second highest bidder offers the same as him, then his profit would be 0. If the second highest bid were higher than the i's offer he could lose.

Proposition 8.2. *The equilibrium strategy $\beta^{II}(x) = v(x,x)$ is unique in the class of symmetric equilibrium.*

Proof.

$$\Pi(b,x) = \int_0^{\left(\beta^{II}\right)^{-1}(b)} \left(v\left(x,y\right)-\beta^{II}\left(y\right)\right) g\left(y\,|\,x\right)\ dy$$

Then, to maximize the expected payoff we must differentiate with respect to b:

$$\begin{aligned}\frac{\partial \Pi}{\partial b}(b,x) &= 0 \\ \left(v\left(x,\left(\beta^{II}\right)^{-1}(b)\right)-\beta^{II}\left(\left(\beta^{II}\right)^{-1}(b)\right)\right) g\left(\left(\beta^{II}\right)^{-1}(b)\,|\,x\right) \quad \left(\left(\beta^{II}\right)^{-1}\right)'(b) &= 0 \\ \left(v\left(x,\left(\beta^{II}\right)^{-1}(b)\right)-b\right)\frac{g\left(\left(\beta^{II}\right)^{-1}(b)\,|\,x\right)}{\left(\left(\beta^{II}\right)'\right)\left(\left(\beta^{II}\right)^{-1}(b)\right)} &= 0 \\ v\left(x,\left(\beta^{II}\right)^{-1}(b)\right) &= b\end{aligned}$$

□

Example 8.1. Suppose that there are three bidders with a common value V that is uniformly distributed on $[0,1]$. Given $V = v$ bidders' signal X_i are uniformly and independently distributed on $[0,2v]$. Find the equilibrium strategy for a Second Price Auction.

We want to find $v\left(x,y\right) = E\left[V\,|\,X_i = x, Y_{1:2} = y\right]$, then for that, we need to find the expectation of the value to a bidder i and conditional densities.

Let be $X = (X_1, X_2, X_3)$ and $Z = \max\{X_1, X_2, X_3\}$ a random variable.

To begin with, the density of X_i conditional on $V = v$ is $f(X_i \mid V = v) = \frac{1}{2v}$, $v \in [0, 2v]$. The joint density of (V, X) is $f_{(V,X)}(x, v) = 1 \times \left(\frac{1}{2v}\right)^3$ on the set $\{(V, X) \mid \forall i, X_i \leq 2V\}$. But, notice that what we know about V is only that $V \geq \frac{1}{2}Z$. Then, the joint density of X is:

$$\begin{aligned} f(x_1, x_2, x_3) &= \int_{z/2}^{1} f_{(V,X)}(x, v)\, dv = \int_{z/2}^{1} \left(\frac{1}{2v}\right)^3 dv \\ &= \frac{4 - z^2}{16z^2} \end{aligned}$$

where $z = \max\{x_1, x_2, x_3\}$. Thus, on the interval $v \in \left[\frac{z}{2}, 1\right]$

$$\begin{aligned} f(v \mid X = x) &= f(v \mid Z = z) = \frac{f_{(V,X)}(x, v)}{f(z)} \\ &= \frac{\left(\frac{1}{2v}\right)^3}{\frac{4-z^2}{16z^2}} = \frac{1}{8v^3} \times \frac{16z^2}{4 - z^2} \end{aligned}$$

Thus,

$$\begin{aligned} E[V \mid X = x] &= E[V \mid Z = z] = \int_{z/2}^{1} v f(v \mid Z = z)\, dv \\ &= \int_{z/2}^{1} \frac{1}{8v^2} \times \frac{16z^2}{4 - z^2}\, dv = \frac{16z^2}{4 - z^2} \int_{z/2}^{1} \frac{1}{8v^2}\, dv \\ &= \frac{16z^2}{4 - z^2} = \frac{2 - z}{z} \frac{2z}{2 + z} \end{aligned}$$

Due to $z = \max\{x_1, x_2, x_3\}$ and $Y_{1:2} = \max\{X_2, X_3\}$ and $Z = \max\{X_1, X_2, X_3\}$, $Z = \max\{X_1, Y_{1:2}\}$ then

$$\begin{aligned} v(x, y) &= E\left[V \mid X_1 = x, Y_{1:2} = y\right] \\ &= E\left[V \mid Z = \max\{x, y\}\right] \\ &= \frac{2\max\{x, y\}}{2 + \max\{x, y\}} \end{aligned}$$

Now, from Proposition 8.2 we have

$$\beta^{II}(x) = v(x, x) = \frac{2x}{2 + x}$$

8.2 English Auctions

In an Open Ascending Auction, or English Auction, the bidders are known and also what they are bidding in each moment.

Due to the variety of open ascending price formats, we accept the one that the auctioneer sets the price at zero and the bidders signal to the auctioneer of accepting this price is by raising a hand, pushing a button, holding a sign, etc. Thus, each

bidder knows what the others are doing in every moment. A bidder may drop out whenever he want but in this case he could not reenter the auction. The auction ends when only remains one bidder and he pays the last price the auctioneer said.

Notice that in an English Auction there is not a unique equilibrium because the auction is in a constant change. Then, a symmetric equilibrium strategy is a collection of $N-1$ functions

$$\boldsymbol{\beta} = \left(\beta^N, \beta^{N-1}, \ldots, \beta^2\right)$$

where

$$\begin{aligned} \beta^k : [0,1] \times \mathbb{R}_+^{N-k} &\longrightarrow \mathbb{R}_+ \\ (x; p_{k+1}, \ldots, p_N) &\longmapsto \beta^k (x; p_{k+1}, \ldots, p_N) \end{aligned}$$

for $1 < k \leq N$. $\beta^k (x; p_{k+1}, \ldots, p_N)$ is the price at which the player 1 would drop out if there are k bidders who are still active and $p_{k+1} \geq \cdots \geq p_N$ are the dropped out prices of the other $N-k$ players.

Now, we give the equilibrium strategies for the bidders. Suppose that when all bidders are in the auction

$$\beta^N (x, \ldots, x) = u(x, x, \ldots, x)$$

where $\beta^N(\cdot)$ is a continuous and increasing function.

Now, suppose that bidder N is the first to abandon the auction at a price p_N and let x_N be the unique signal such that $\beta^N(x_N, \ldots, x_N) = p_N$. Then, when some bidder drops out at this price, p_N, now the other $N-1$ bidders follow the strategy

$$\beta^{N-1}(x, p_N) = u(x, \ldots, x, x_N)$$

where $\beta^N(x_N, \ldots, x_N) = p_N$. Also, $\beta^{N-1}(\cdot, p_N)$ is a continuos and increasing function.

Now, proceeding recursively,

$$p_{k+1} := \beta^{k+1}(x_{k+1}, p_{k+2}, \ldots, p_N) \tag{8.1}$$

where p_{k+1} is the price when the bidder $k+1$ drops out of the auction.

Now let the remaining k bidders in the auction follow the strategy

$$\beta^k (x, p_{k+1}, \ldots, p_N) = u(x, \ldots, x, x_{k+1}, \ldots, x_N)$$

This function gives us the abandon price when there are only k bidders in the auction and the other $N-k$ dropped out with prices $p_N, \ldots, p_{k+1}$.

But, why is these strategies a symmetric equilibrium?

To begin with, let us suppose the bidders $k+1, k+2 \ldots, N$ have dropped out, then, there are only k bidders remaining in the auction. Knowing that the $k+1, k+2 \ldots, N$ abandoned the auction we know in which price they did it, then, their signals $x_{k+1}, x_{k+2} \ldots, x_N$ are known by the other bidders.

Suppose that bidder 1 has a signal x and the other $k-1$ bidders are following β^k. The bidder 1 thinks about dropping out at the current price p or continue. But, what happens if he does not drop out and pays p? If this is to occur, the other $k-1$ players drop out at p and the only way that this could happen

is if the $k-1$ bidders their signal is y, the same for each bidder, such that $\beta^k(y, p_{k+1}, \ldots, p_N) = p = u(y, \ldots, y, p_{k+1}, \ldots, p_N)$, then the value of the object auctioned is $u(y, \ldots, y, p_{k+1}, \ldots, p_N)$. Thus, bidder 1 would continue in the auction if and only if $u(y, \ldots, y, p_{k+1}, \ldots, p_N) > p$, which means only if $x = y$.

Proposition 8.3. *In an English Auction the symmetric equilibrium strategies are given by* $\beta^N(x, \ldots, x) = u(x, x, \ldots, x)$ *and* $\beta^k(x, p_{k+1}, \ldots, p_N) = u(x, \ldots, x, x_{k+1}, \ldots, x_N)$.

Proof. Suppose that bidder 1 has a signal $X_1 = x$, the other $N-1$ bidders follow the strategy $\boldsymbol{\beta}$.

Suppose the realizations of $Y_{1:N-1}, \ldots, Y_{N-1:N-1}$ denoted by $y_1, \ldots, y_{N-1}$, are such that the bidder 1 wins the auction following $\boldsymbol{\beta}$. Then, $x > y_1$. The price that he pays is the price that the second highest signal, y_1, drops out and that would be $u(y_1, y_1, y_2, \ldots, y_N)$. But he would win $u(x, y_1, y_2, \ldots, y_N)$, thus he would want to follow $\boldsymbol{\beta}$.

If the bidder 1 does not win following $\boldsymbol{\beta}$ means that $x < y_1$ and this argument makes that

$$u(x, y_1, y_2, \ldots, y_N) - u(y, y_1, y_2, \ldots, y_N) < 0$$

□

Remark. Notice that the equilibrium strategy does not depend on the distribution of signals f, but on the distribution of values u.

8.3 First Price Auctions

Let us suppose that $j \neq 1$ bidders follow an increasing and differentiable strategy β. Everybody, including the bidder 1, makes an offer in the interval $[\beta(0) = 0, \beta(w)]$.

Let $G(\cdot \mid x)$ be the distribution function of $Y_{1:N-1} = \max_{i \neq 1} X_i$ conditional on $X_1 = x$. And, also, let $g(\cdot \mid x)$ be its density function.

Now, the bidder 1 with signal x but he bids $\beta(z)$, then, his expected payoff would be:

$$\begin{aligned}
\Pi(x, z) &= \int_0^z \left(v(x, y) - \beta(z)\right) g(y \mid x) \, dy \\
&= \int_0^z v(x, y) \, g(y \mid x) \, dy - \beta(z) G(z \mid x)
\end{aligned}$$

The first order condition is

$$v(x, z) g(z \mid x) - \beta'(z) G(z \mid x) - \beta(z) g(z \mid x) = 0$$

At a symmetric equilibrium, the optimal $z = x$, then, rearranging and setting $z = x$ the equation we have

$$\beta'(x) = \left(v(x, x) - \beta(x)\right) \frac{g(x \mid x)}{G(x \mid x)} \tag{8.2}$$

Now, we have that linear differential equation that we must that $v(x, x) - \beta(x) \geq 0$ because otherwise β would not be an increasing function. Also, due to $v(0, 0) = 0$ we have that $\beta(0) = 0$. Thus, the Cauchy problem that we have is:

$$\begin{cases} \beta'(x) &= \left(v(x, x) - \beta(x)\right) \frac{g(x \mid x)}{G(x \mid x)} \\ \beta(0) &= 0 \end{cases}$$

$$
\begin{aligned}
\beta(x) &= e^{-\int_0^x \frac{g(y|y)}{G(y|y)}\,dy} \int_0^x v(y,y)\,\frac{g(y\mid y)}{G(y\mid y)} e^{\int_0^y \frac{g(s|s)}{G(s|s)}\,ds}\,dy \\
&= \int_0^x v(y,y)\,\frac{g(y\mid y)}{G(y\mid y)} e^{\int_0^y \frac{g(s|s)}{G(s|s)}\,ds} e^{-\int_0^x \frac{g(s|s)}{G(s|s)}\,ds}\,dy \\
& \int_0^x v(y,y)\,\frac{g(y\mid y)}{G(y\mid y)} e^{-\int_t^x \frac{g(s|s)}{G(s|s)}\,ds}\,dy
\end{aligned}
$$

Now, we define

$$
L(y\mid x) := e^{-\int_y^x \frac{g(s|s)}{G(s|s)}\,ds}
$$

Then,

$$
\frac{d}{dy} L(y\mid x) = \frac{g(y\mid y)}{G(y\mid y)} e^{-\int_y^x \frac{g(s|s)}{G(s|s)}\,ds}
$$

Thus,

$$
\beta(x) = \int_0^x v(y,y)\,dL(y\mid x)
$$

But, by now we have not said that this is a symmetric equilibrium strategy.

Proposition 8.4. *In a Sealed-bid First Price Auction the symmetric equilibrium strategy is given by*

$$
\beta^I(x) = \int_0^x v(y,y)\,dL(y\mid x)
$$

where

$$
L(y\mid x) = e^{-\int_y^x \frac{g(s|s)}{G(s|s)}\,ds}
$$

Proof. Firstly, notice that you could think $L(y\mid x)$ as a distribution function with support $[0,x]$.

$$
\begin{aligned}
\frac{g(y\mid y)}{G(y\mid y)} &\geq \frac{g(y\mid 0)}{G(y\mid 0)} \qquad (8.3) \\
-\int_0^x \frac{g(y\mid y)}{G(y\mid y)}\,dy &\leq -\int_0^x \frac{g(y\mid 0)}{G(y\mid 0)}\,dy \\
&= -\int_0^x \frac{d}{dy}\left(\ln G(y\mid 0)\right)\,dy \\
&= -\int_0^x \frac{d}{dy}\left(\ln G(y\mid 0)\right)\,dy \\
&= \ln G(0\mid 0) - \ln G(x\mid 0) \\
&= \ln 0 - \ln G(x\mid 0) \\
&= -\infty
\end{aligned}
$$

In then, in (8.3) we use affiliation. Then,

$L(0\,|\,x) = e^{-\int_0^x \frac{g(s|s)}{G(s|s)}\,ds} = e^{-\infty} = 0$
$L(x\,|\,x) = e^{-\int_x^x \frac{g(s|s)}{G(s|s)}\,ds} = e^{0} = 1$
$L(\cdot\,|\,x)$ is a non-decreasing function then, $L(\cdot\,|\,x)$ is a distribution function.

Also, if $x < x'$ then $L(\cdot\,|\,x) \geq L(\cdot\,|\,x')$ which means that $L(\cdot\,|\,x')$ stochastically dominates the distribution $L(\cdot\,|\,x)$.

Since $v\left(y,y\right) > 0$ then β^I is an increasing function.

Now, what it lasts is see that β^I is a maximum.

We proceed as usual, a bidder with signal x bids $\beta^I(z)$. Then, the expected profit would be

$$\Pi(x,z) \quad = \quad \int_0^z \left(v\left(x,y\right) - \beta^I(z)\right) f\left(y\,|\,x\right)\,dy$$

Then, the first order condition is

$$\begin{aligned}
\frac{\partial}{\partial z}\Pi(x,z) &= v(x,z)\,g(z\,|\,x) - \left(\beta^I\right)'(z)G(z\,|\,x) - \beta^I(z)\,g(z\,|\,x) \\
&= \left(v(x,z) - \beta^I(z)\right) g(z\,|\,x) - \left(\beta^I\right)'(z)G(z\,|\,x) \\
&= G(z\,|\,x)\left[\left(v(x,z) - \beta^I(z)\right)\frac{g(z\,|\,x)}{G(z\,|\,x)} - \left(\beta^I\right)'(z)\right]
\end{aligned}$$

We want to see that $\frac{\partial}{\partial z}\Pi(x,z) > 0$. If $z < x$ then, $v(x,z) > v(x,x)$ due to v is an increasing function, and because of affiliation $\frac{g(z|x)}{G(z|x)} > \frac{g(z|z)}{G(z|z)}$.

$$\frac{\partial}{\partial z}\Pi(x,z) \quad > \quad G(z\,|\,x)\left[\left(v(z,z) - \beta^I(z)\right)\frac{g(z\,|\,z)}{G(z\,|\,z)} - \left(\beta^I\right)'(z)\right] = 0$$

Using (8.2) . Now, let us see the case $x < z$ then, $v(z,z) > v(x,z)$ due to v is an increasing function, and because of affiliation $\frac{g(z|x)}{G(z|x)} > \frac{g(z|z)}{G(z|z)}$.

$$\frac{\partial}{\partial z}\Pi(x,z) \quad < \quad G(z\,|\,x)\left[\left(v(z,z) - \beta^I(z)\right)\frac{g(z\,|\,z)}{G(z\,|\,z)} - \left(\beta^I\right)'(z)\right] = 0$$

Thus, $\frac{\partial}{\partial z}\Pi(x,z) = 0$ when $z = x$. □

Remark. The Proposition 8.2 is a generalization of Proposition 3.1, where $v\left(y,y\right) = y$ because of private values and also the signals are independent, then $G(\cdot\,|\,x) \equiv G(x)$. Thus,

$$L\left(y\,|\,x\right) \quad = \quad e^{-\int_y^x \frac{g(t)}{G(t)}\,dt} \;=\; e^{\ln\left(\frac{G(x)}{G(y)}\right)^{-1}} \;=\; \frac{G\left(y\right)}{G(x)}$$

Then,

$$\begin{aligned}
\beta^I(x) &= \int_0^x y\,\frac{g(y)}{G(x)}\,dy \\
&= E\left[Y_{1:N-1}\,|\,Y_{1:\text{N-1}} < x\right]
\end{aligned}$$

Chapter 9

Further Considerations

9.1 Revenue Comparisons

Now, what we compare is the revenue according to the format, First Price, Second Price and English Auction.

Proposition 9.1. $E\left[R^{Eng}\right] \geq E\left[R^{II}\right]$.

Proof. Remembering that $\beta^{II}(x) = v(x,x)$ is the symmetric equilibrium strategy in a Second Price Auction, if $x > y$,

$$\begin{aligned} v(y,y) &= E\left[u(X_1, Y_{1:N-1}, Y_{2:N-1}, \ldots, Y_{N-1:N-1}) \mid X_1 = y, Y_{1:N-1} = y\right] \\ &= E\left[u(Y_{1:N-1}, Y_{1:N-1}, Y_{2:N-1}, \ldots, Y_{N-1:N-1}) \mid X_1 = y, Y_{1:N-1} = y\right] \\ &\leq E\left[u(Y_{1:N-1}, Y_{1:N-1}, \ldots, Y_{N-1:N-1}) \mid X_1 = x, Y_{1:N-1} = y\right] \end{aligned} \tag{9.1}$$

In (9.1) we use that u is an increasing in all its arguments and all signals are affiliated. Then, the expected revenue in a Second Price Auction would be

$$\begin{aligned} E\left[R^{II}\right] &= E\left[\beta^{II}(Y_{1:N-1}) \mid X_1 > Y_{1:N-1}\right] \\ &= E\left[v(Y_{1:N-1}, Y_{1:N-1}) \mid X_1 > Y_{1:N-1}\right] \\ &\leq E\left[E\left[u(Y_{1:N-1}, Y_{1:N-1}, \ldots, Y_{N-1:N-1}) \mid X_1 = x, Y_{1:N-1} = y\right] \mid X_1 > Y_{1:N-1}\right] \\ &= E\left[u(Y_{1:N-1}, Y_{1:N-1}, \ldots, Y_{N-1:N-1}) \mid X_1 > Y_{1:N-1}\right] \\ &= E\left[\beta^2(Y_{1:N-1}, Y_{2:N-1}, \ldots, Y_{N-1:N-1})\right] \\ &= R\left[R^{Eng}\right] \end{aligned}$$

where in the equality we have used (9.1) and then in the next equality we used the Proposition 8.3, β^2 is the strategy used in an English Auction when only two players remain, which is the price paid by the winning bidder because is the price when the other player decides to drop out due to (8.1). □

Proposition 9.2. $E\left[R^{II}\right] \geq E\left[R^{I}\right]$.

Proof. To begin with, in a First Price Auction the payment of a bidder upon winning the object is just $\beta^I(x)$ as we defined in Proposition 8.4. On the other hand, the expected payment in a Second Price Auction is $E\left[\beta^{II}(Y_{1:N-1}) \mid X_1 = x, Y_{1:N-1}\right]$ where β^{II} is defined in Proposition 8.2.

Now, if we develop the expected payment in a Second Price Auction

$$\begin{aligned} E\left[\beta^{II}(Y_{1:N-1}) \mid X_1 = x, Y_{1:N-1}\right] &= E\left[v(Y_{1:N-1}, Y_{1:N-1}) \mid X_1 = x, Y_{1:N-1} < x\right] \\ &= \int_0^x v(y,y) \frac{g(y \mid x)}{G(x \mid x)} dy \\ &= \int_0^x v(y,y)\, dK \end{aligned}$$

where for all $y < x$,

$$K(y \mid x) = \frac{G(y \mid x)}{G(x \mid x)}$$

Notice that $K(\cdot \mid x)$ is a distribution function with support $[0, x]$ because

- $K(\cdot \mid x)$ is an increasing function due to $G(\cdot \mid x)$ is a distribution function.
- $K(0 \mid x) = \frac{G(0|x)}{G(x|x)} = 0$
- $K(x \mid x) = \frac{G(x|x)}{G(x|x)} = 1$

On the other hand,

$$\beta^I(x) = \int_0^x v(y,y)\, dL(y \mid x)$$

where

$$L(y \mid x) = e^{-\int_y^x \frac{g(s|s)}{G(s|s)}\, ds}$$

Then, what we want to see if

$$\int_0^x v(y,y)\, dL(y \mid x) \leq \int_0^x v(y,y)\, dK(y \mid x)$$

occurs. This would happen if and only if

$$\int_0^x v(y,y)\left(dK(y \mid x) - dL(y \mid x)\right) \geq 0$$

and integrating by parts, we obtain

$$\begin{aligned} -\int_0^x v'(y,y)\left(K(y \mid x) - L(y \mid x)\right) dy &\geq 0 \\ \int_0^x v'(y,y)\left(L(y \mid x) - K(y \mid x)\right) dy &\geq 0 \end{aligned}$$

since $v'(y,y) > 0$ because is an increasing function we must have that $L(y \mid x) - K(y \mid x) \geq 0 \Leftrightarrow L(y \mid x) \geq K(y \mid x)$, which means that $K(\cdot \mid x)$ stochastically dominates $L(\cdot \mid x)$.

To proof the stochastic dominance we use that because of affiliation, for all $t < x$, $G(\cdot \mid x)$ dominates $G(\cdot \mid t)$ in terms of the reverse hazard rate, then

$$\frac{g(t \mid t)}{G(t \mid t)} \leq \frac{g(t \mid x)}{G(t \mid x)}$$

then, for all $y < x$,

$$\begin{aligned}
\frac{g(t \mid t)}{G(t \mid t)} &\leq \frac{g(t \mid x)}{G(t \mid x)} \\
-\frac{g(t \mid t)}{G(t \mid t)} &\geq -\frac{g(t \mid x)}{G(t \mid x)} \\
-\int_y^x \frac{g(t \mid t)}{G(t \mid t)}\, dt &\geq -\int_y^x \frac{g(t \mid x)}{G(t \mid x)}\, dt \\
&= -\int_y^x \frac{d}{dt} \ln G(t \mid x)\, dt \\
&= -\left(\ln G(x \mid x) - \ln G\left(y \mid x\right)\right) \\
&= \ln \frac{G\left(y \mid x\right)}{G(x \mid x)}
\end{aligned}$$

Now, applying the exponential function in the inequation, we obtain that for all $y < x$,

$$\begin{aligned}
\exp\left(-\int_y^x \frac{g(t \mid t)}{G(t \mid t)}\, dt\right) &\geq \exp\left(\ln \frac{G\left(y \mid x\right)}{G(x \mid x)}\right) \\
&= \frac{G\left(y \mid x\right)}{G(x \mid x)} \\
L\left(y \mid x\right) &\geq K\left(y \mid x\right)
\end{aligned}$$

□

Summarizing Proposition 9.1 and 9.2 we can conclude that

$$E\left[R^{Eng}\right] \geq E\left[R^{II}\right] \geq E\left[R^{I}\right]$$

9.2 The Winner's Curse

Let us suppose that it is being auctioned an object using a Sealed-bid First Price Auction and, also, the only information available to a bidder 1 is his own signal $X_1 = x$; and by now his estimate of the value is $E[V \mid X_1 = x]$. Now, consider that the bidder 1 is announced as the winner of the auction. If all bidders are symmetric following the same strategy β, then $Y_{1:N-1} < x$, then his estimate of the value would be $E[V \mid X_1 = x, Y_{1:N-1} < x] < E[V \mid X_1 = x]$. The announcement of winning the auction means that the estimated value is going to decrease. Then, the possibility that the winner pays more than the value is called the winner's curse.

Suppose that there are N bidders where $X_i = V + \varepsilon_i$ is each bidder's signal with ε_i are independently and identically distributed satisfying $E[\varepsilon_i] = 0$. Then for all i, $E[X_i \mid V = v] = E[V + \varepsilon_i \mid V = v] = v$, which means X_i is an unbiased estimator but the largest of such signals is not, $E[\max X_i \mid V = v] > \max E[X_i \mid V = v] = v$, then the expectation of the highest signal is greater than the value.

Thus, if a bidder does not take the winner's curse into account would pay more than the estimated worth of the object. What the bidder must do to avoid the winner's curse is shade their bids below their initial estimates.

Now, let us verify that the equilibrium strategies in a First Price Auction could have the winner's curse.

$$\begin{aligned}
\beta^I(x) &= \int_0^x v(y,y)\, dL(y \mid x) \\
&\leq \int_0^x v(y,y)\, dK(y \mid x) \\
&< \int_0^x v(x,y)\, dK(y \mid x) \\
&= E[V_1 \mid X_1 = x, Y_{1:N-1} < x]
\end{aligned}$$

In the second inequality we use that $v(\cdot, y)$ is an increasing function. Thus, we have seen that the equilibrium strategy in a First Price Auction is less than the expected value conditional on winning.

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