The a-serial cost sharing rule

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Abstract

A new family of cost sharing rules for cost sharing problems is proposed. This family generalizes the family of α-serial cost sharing rules (Albizuri, 2010) which contains the serial cost sharing rule (Moulin and Shenker, 1992) among others. Every rule of the family is characterized by means of two properties.

Keywords: Cost sharing problems; Serial cost sharing rule

1 Introduction

Cost sharing problems arise in many real life situations, which include the production of private goods by a group of people using a jointly owned production facility, the allocation of overhead expenses of a company among its divisions or charging individuals for any service that may be provided for groups of people. In all such situations each of the individuals involved, $i \in \{1, \ldots, n\}$, demand a quantity, q_i, of the output and the cost of producing the jointly demanded amount, $C(\sum_{i \in N} q_i)$, has to be paid by the individuals. Cost sharing rules associate each demand profile, $(q_1, \ldots, q_n)$, with a sharing of the overall cost.

In this paper we introduce a family of rules which is related to the serial cost sharing rule defined by Moulin and Shenker (1992). Since its origin the serial cost sharing rule has caught great attention and several related rules have been defined. Inter alia we can mention the decreasing serial mechanism defined by de Frutos (1998), the mixed cost sharing introduced by Hougaard and Thorlund-Petersen (2001), and the concave serial rule and the convex serial rule introduced by Koster (2002). There are also generalizations of the serial cost sharing rule to heterogeneous cost sharing models (Koster, 2006, 2007) and when agents require bundle of goods (Koster et al., 1998).

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Being more precise, in this paper we introduce a generalization of the α-serial cost sharing rule defined by Albizuri (2010). This latter rule can be seen as a combination of the serial cost sharing rule (Moulin and Shenker, 1992) and the the dual serial cost sharing rule (Albizuri and Zarzuelo, 2007). The parameter α tells us how this combination is made.

We describe the α-serial cost sharing rule with only two agents, i and j. Assume that $q_i \leq q_j$. When the production of the output starts, each unit is equally divided between the agents until i receives q_i, at this point he has received his entire demand. Agent i leaves the picture and the cost of the produced amount (that is, of $2q_i$) is divided in the following way. Agents i and j pay a portion of $2q_i$ (precisely $\alpha 2q_i$) taking into account the cost increment associated with this portion. So both i and j pay $C(\alpha 2q_i)/2$ and the rest ($(1-\alpha)2q_i$) is paid by both agents taking into account the cost increment of this quantity to quantities to be produced (with respect to $q_i + q_j$). Therefore, both i and j pay $\frac{C(q_i+q_j)-C(q_i+q_j-(1-\alpha)2q_i)}{2}$ for the rest. Hence, in total agent i pays $\frac{C(\alpha 2q_i)+C(q_i+q_j)-C(q_i+q_j-(1-\alpha)2q_i)}{2}$. Agent j pays in addition the rest of the production, that is, j pays in addition $C(q_i + q_j - (1-\alpha)2q_i) - C(\alpha 2q_i)$.

By generalizing this procedure to n agents, the α-serial cost sharing rule is obtained (the explicit definition is presented in the preliminaries). If $\alpha \in (0,1)$, by means of this allocation rule, agents with low demands have to pay cost increments associated with low outputs and cost increments associated with high outputs. Since the quantity of low outputs is determined by parameter α (and therefore, the quantity of high outputs as well), the proportion fixed by α does not depend on the quantities demanded by the agents. In this paper we allow for the possibility of taking different proportions for different demands. For doing so, instead of taking a parameter $\alpha \in [0,1]$ we take a function $a : \mathbb{R}_+ \to \mathbb{R}_+$ that determines the proportion for each production level. The resulting cost sharing rule is called the a-serial cost sharing rule.

For example, suppose that the cost increment in high demands is very high and that there are two agents with low demands such that the agent with the lowest demand has hardly contributed to the total demand of all agents in comparison with the other agent. Then, it might be sensible to allocate a greater proportion of the cost associated with low demands to the agent with less demand. As another example, we can suppose that the cost increment in high demands is higher than in small ones and that there are two agents with low demands, but now the second one not having contributed so much to high demands. In this case, it could be sensible to allocate a smaller proportion of the cost of high demands to the second agent, since otherwise he would pay a great portion of the cost of high demands.

In this paper we characterize the a-serial cost sharing rule by means of two properties. Namely, Anonymity and a generalization of the Independence of higher demands used by Moulin and Shenker (1992) to characterize the serial cost sharing rule.

We would like to point out that in this paper we introduce a family of cost sharing rules. Different functions $a : \mathbb{R}_+ \to \mathbb{R}_+$ will give rise to different a-

serial cost sharing rules. At the end of the paper we present precisely different examples to better understand our new rules.

The rest of the paper is organized as follows. After some preliminaries in Section 2, we define the a-serial cost sharing rule in Section 3 and in Section 4 we characterize the a-serial cost sharing rules. Finally, we illustrate the use of the rules by several examples.

2 Preliminaries

A *cost sharing problem* is a triple (N, q, C) where $N = \{1, \ldots, n\}$ is the set of agents, $q \in \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0\}$ is the vector of demands or demand profile and $C : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function such that $C(0) = 0$, where $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Notice that there may be several homogeneous goods or a single good with demands of a number of agents. We denote by $\mathcal{C}^N$ the set of all cost sharing problems with player set N and by $\mathcal{C}$ the set of all such problems with an arbitrary set of players. We also write $Q = \sum_{i \in N} q_i$ to ease notation.

A *cost sharing rule* is a mapping f associating to each cost sharing problem, a vector of cost shares, i.e., $f : \mathcal{C}^N \to \mathbb{R}^n_+$ such that

$$\sum_{i \in N} f_i(N, q, C) = C(Q). \qquad \text{(Efficiency)}$$

In Albizuri (2010), the α-serial cost sharing rule, φ^α, is proposed and characterized. The α-serial cost sharing rule generalizes several cost sharing rules in the literature. For instance, if $\alpha = 1$ it coincides with the serial cost sharing rule of Moulin and Shenker (1992), if $\alpha = 0$ then φ^α is the dual serial cost sharing rule (Albizuri and Zarzuelo, 2007), and when $\alpha = 1/2$ it turns out to be the self dual serial cost sharing rule proposed by Albizuri (2009).

Formally, let $(N, q, C) \in \mathcal{C}$ such that $q_1 \leq q_2 \leq \cdots \leq q_n$ and $\alpha \in [0, 1]$. Define,

$$\begin{array}{llll} q^0 = 0, & q^1 = nq_1, & q^2 = q_1 + (n-1)q_2, & \ldots \\ q^i = q_1 + \cdots + q_{i-1} + (n+1-i)q_i, & & \ldots, & q^n = Q. \end{array} \qquad (1)$$

The *α-serial cost sharing rule*, φ^α, is given for each $i \in N$ by

$$\varphi^\alpha_i(N, q, C) = \sum_{j=1}^{i} \frac{C(\alpha q^j) - C(\alpha q^{j-1})}{n - j + 1} + \sum_{j=1}^{i} \frac{C(Q - (1-\alpha)q^{j-1}) - C(Q - (1-\alpha)q^j)}{n - j + 1}.$$

The cost sharing rule consists of two terms, the first one is based on the cost increments of the quantities produced (where α determines the portion taken into account) and the second one on the cost increments of the quantities left to produce.

The serial cost sharing rule of Moulin and Shenker (1992) is characterized by means of two properties. The first one, Anonymity, states that the labeling of the agents should not affect the cost sharing. The second one, the so called Independence of higher demands requires that the payoff of an agent does not depend on demands which are higher than his own. Formally, let $q, p \in \mathbb{R}^n_+$ such that $p_1 \leq p_2 \leq \cdots \leq p_n$, $p \leq q$, and $i \in N$ such that $p_j = q_j$ for every $j \leq i$, then a cost sharing rule f satisfies *Independence of higher demands* if,

$$f_i(N, q, C) = f_i(N, p, C).$$

Given a demands profile q, if the agents with greater demand than agent i's decrease their demands, then the demands profile becomes p. This property requires that the saved cost, that is, $C(Q) - C(P)$, should only affect those agents who have decreased their demand. Therefore, the production costs on the interval $[P, Q]$ do not affect agent i's payoff.

The α-serial cost sharing rule of Albizuri (2010) is characterized by means of three properties. The Anonymity property described above. Scale invariance, which states that the scale by which the good is measured should not affect agents payoffs. And a modification of Independence of higher demands introduced by Albizuri (2010).

When we ask for Independence of higher demands, if the demand profile changes from p to q, the agents with greater demand than agent i's have to pay the cost increment associated with $[P, Q]$. But this might not always be desirable. On the one hand, this cost increment could be the highest one among all the cost increments associated with $Q - P$ outputs and therefore it could be sensible to allocate this increment also to the agents with lower demands. Consider for instance, a two agents cost sharing problem where the first few units are free and the next units have a given unitary cost. As an example, let $N = \{1, 2\}$, $p = (3, 3)$, $q = (3, 5)$, and $C(x) = \max\{x - 6, 0\}$. If Independence of higher demands holds then agent 2 pays the last two units and agent 1 pays nothing. This could be not sensible since the stand-alone cost of both agents is 0 and the non zero cost arises when both agents are present. On the other hand, consider another situation in which the cost increment associated with $[P, Q]$ is very low and therefore it could be reasonable to allocate it to all agents, since all of them have contributed to this low cost increment. As an example, take $N = \{1, 2\}$, $p = (3, 3)$, $q = (3, 5)$, and $C(x) = \min\{x, 6\}$.

The modification of Independence of higher demands by Albizuri (2010) reflects these alternative situations, and agent i can also pay for the increment $C(Q) - C(P)$. According to the modified property, agents who change their demand do not have to pay $C(Q) - C(P)$ less for asking for $Q - P$ less but a cost increment associated with an intermediate interval in $[0, Q]$. This intermediate interval is denoted by $[a(P), a(P) + Q - P]$, where a is a mapping from $\mathbb{R}_+$ to $\mathbb{R}_+$ which satisfies $a(P) \leq P$. Therefore, in the modified property the production costs on the interval $[a(P), a(P) + Q - P]$ do not affect agent i's payoff.

To formalize this property we consider the following cost function obtained from the original one when an interval of length K is deleted at point λ preserving the continuity.

$$C^{\lambda,K}(t) = \begin{cases} C(t) & \text{if} \quad t \leq \lambda \\ C(t+K) - C(\lambda + K) + C(\lambda) & \text{if} \quad t \geq \lambda. \end{cases}$$

Let f be a cost sharing rule. We say that f satisfies *a-Independence of higher demands* if there exists a mapping $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(P) \in [0, P]$ for all $P \in \mathbb{R}_+$ such that for every $(N, q, C), (N, p, C) \in \mathcal{C}$ with $p_1 \leq p_2 \leq \cdots \leq p_n$, for which there is an $i \in N$ such that $p_j = q_j$ for every $j \leq i$ and $p \leq q$, it holds

$$f_i(N, q, C) = f_i(N, p, C^{a(P), Q-P}).$$

Observe that if a cost sharing rule f satisfies *a-Independence of higher demands* then this property determines one function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(P) \in [0, P]$ for which the required equalities are satisfied.

Notice also that when $a(P) = P$, then a-Independence of higher demands coincides with Independence of higher demands.

When the three properties, Anonymity, Scale invariance, and a-Independence of higher demands are considered together, then $a(P) = \alpha P$ for some $\alpha \in [0, 1]$, that is, the function $a : \mathbb{R}_+ \to \mathbb{R}_+$ is linear.

3 The a-serial cost sharing rule

We seek for a generalization of the α-serial cost sharing rule of Albizuri (2010), by considering a more general function $a : \mathbb{R}_+ \to \mathbb{R}_+$ than the linear $a(x) = \alpha x$ with $\alpha \in [0, 1]$. Following the idea behind the α-serial cost sharing rule, suppose x is to be produced. Then, $a(x)$ is paid taking into account the cost increment associated with this quantity and the remaining $x - a(x)$ is paid taking into account the quantities left to produce.

First of all, the procedure that gives rise to the a-serial cost sharing rule is described. Afterwards, the conditions that the mapping a has to satisfy in order to obtain a cost sharing rule are discussed. Let $(N, q, C) \in \mathcal{C}$ such that $q_1 \leq q_2 \leq \cdots \leq q_n$. The *a-serial cost sharing rule* is defined as follows.

When the production starts each unit is divided equally among the agents, who equally share the cost incurred. When $q^1 = nq_1$ is produced the first agent has fulfilled his demand and stops receiving more. Then, $a(q^1)$ is paid by all the agents by means of cost increments associated with this quantity, and $q^1 - a(q^1)$ is paid by all the agents, but now taking into account cost increments of quantities left to produce. Hence, each agent pays

$$\frac{C(a(q^1)) - C(0)}{n} + \frac{C(Q) - C(Q - (q^1 - a(q^1)))}{n}.$$

At this point the first agents drops out.

Now, the rest of the good is shared among the remaining $n-1$ agents until the second agent's demand has been met. This happens when $q^2 = (n-1)q_2 + q_1$ has been produced. As before, quantities produced and quantities left to produce are considered. So, the remaining agents share the cost increment associated with quantities $a(q^1)$ and $a(q^2)$ and the cost increment associated with quantities $Q-(q^1-a(q^1))$ and $Q-(q^2-a(q^2))$. Thus, the remaining $n-1$ players also pay

$$\frac{C(a(q^2))-C(a(q^1))}{n-1}+\frac{C(Q-(q^1-a(q^1)))-C(Q-(q^2-a(q^2)))}{n-1}.$$

Next, the second player drops out and the process continues until all agents obtain the demanded amount.

Formally, let $a:\mathbb{R}_+\to\mathbb{R}_+$ and $(N,q,C)\in\mathcal{C}$ such that $q_1\le q_2\le\cdots\le q_n$. Let $q^1,q^2,\ldots,q^n$ be as defined in (1). Then, the *a-serial cost sharing rule*, φ^a, is given for every $i\in N$ by,

$$\begin{aligned}\varphi_i^a(N,q,C)=&\sum_{j=1}^{i}\frac{C(a(q^j))-C(a(q^{j-1}))}{n-j+1}\\&+\sum_{j=1}^{i}\frac{C(Q-(q^{j-1}-a(q^{j-1})))-C(Q-(q^j-a(q^j)))}{n-j+1}.\end{aligned}\tag{2}$$

We consider the following conditions that the mapping a should satisfy in order to obtain a cost sharing rule following the procedure described above.

$$a(x)\le x \qquad \text{for every } x\in\mathbb{R}_+. \tag{3}$$
$$a(x)\le a(y) \qquad \text{for every } 0\le x\le y. \tag{4}$$
$$a(y)-a(x)\le y-x \qquad \text{for every } 0\le x\le y. \tag{5}$$

Note that condition (3) requires the image of any quantity under a to be a portion of that quantity. The second one, (4), requires a to be nondecreasing. Finally, condition (5) states that the increment rate of a with respect to any point is never bigger than one. Note that if a is derivable and satisfies (4), condition (5) is equivalent to the slope of a being smaller than one. It can be easily checked that any map $a:\mathbb{R}_+\to\mathbb{R}_+$ that satisfies (4) and (5) is continuous.

Proposition 3.1. *The right hand side of (2) defines a cost sharing rule if and only if the mapping $a:\mathbb{R}_+\to\mathbb{R}_+$ satisfies conditions (3), (4), and (5).*

Proof. It is straightforward to prove the sufficiency of the three conditions. So, let us prove the necessity. Let $a:\mathbb{R}_+\to\mathbb{R}_+$ be a mapping and let φ^a be as defined in (2).

(i) If a does not satisfy condition (3), then there exists $x\in\mathbb{R}_+$ such that $a(x)>x$. Let $q_1\in\mathbb{R}_+$ be such that $nq_1=x$ and $q_n\in\mathbb{R}_+$ be such that $x+q_n-q_1>a(x)$. Consider the demand profile $q=(q_1,\ldots,q_1,q_n)\in\mathbb{R}_+^n$ and the cost function $C(t)=\begin{cases}0 & t\le Q\\ 1 & t>Q\end{cases}$. Then, from (2) we have

$$\varphi_1^a(N,q,C)=\frac{C(a(x))}{n}+\frac{C(Q)-C(Q-(x-a(x)))}{n}=\frac{-1}{n},$$

since $a(x) < Q = x + q_n - q_1 < Q - (x - a(x))$. Hence, φ^a is not a cost sharing rule.

(ii) If a does not satisfy condition (4), then there exist $x, y \in \mathbb{R}_+$ such that $x < y$ and $a(x) > a(y)$. Let $q_1, q_2, q_n \in \mathbb{R}_+$ such that $nq_1 = x$, $q_1 + (n-1)q_2 = y$, and $q_n > q_2$. Consider the demand profile $q = (q_1, q_2, \ldots, q_2, q_n) \in \mathbb{R}^n_+$ and the cost function $C(t) = \begin{cases} 0 & t \leq a(y) \\ 1 & t > a(y) \end{cases}$. Notice that $q^1 = x$, $q^2 = y$, and $q^1 < q^2$. Then, from (2) we have

$$\begin{aligned}\varphi^a_2(N, q, C) = \frac{C(a(x))}{n} + \frac{C(Q) - C(Q - (x - a(x)))}{n} + \frac{C(a(y)) - C(a(x))}{n-1} \\ + \frac{C(Q - (x - a(x))) - C(Q - (y - a(y)))}{n-1} = \frac{1}{n} - \frac{1}{n-1},\end{aligned}$$

since $a(y) < a(x) < x < y$, $Q - (x - a(x)) > a(x) > a(y)$, and $Q - (y - a(y)) > a(y)$. Hence, φ^a is not a cost sharing rule.

(iii) If condition (5) does not hold, then there exist $x, y \in \mathbb{R}_+$ such that $x < y$ and $a(y) - a(x) > y - x$. As before, let $q_1, q_2 \in \mathbb{R}_+$ such that $nq_1 = x$ and $q_1 + (n-1)q_2 = y$. This time, let $q_n \in \mathbb{R}_+$ such that $a(y) - a(x) \leq (n-2)(q_2 - q_1) + q_n - q_1$. Consider the demand profile $q = (q_1, q_2, \ldots, q_2, q_n) \in \mathbb{R}^n_+$ and the cost function $C(t) = \begin{cases} 0 & t \leq Q - x + a(x) \\ 1 & t > Q - x + a(x) \end{cases}$. Then, from (2) we have

$$\begin{aligned}\varphi^a_2(N, q, C) = \frac{C(a(x))}{n} + \frac{C(Q) - C(Q - (x - a(x)))}{n} + \frac{C(a(y)) - C(a(x))}{n-1} + \\ \frac{C(Q - (x - a(x))) - C(Q - (y - a(y)))}{n-1} = \frac{1}{n} - \frac{1}{n-1},\end{aligned}$$

since $a(x) < a(y) \leq Q - x + a(x) < Q - y + a(y)$ implies that all the costs above are zero with the exception of $C(Q) = C(Q - y + a(y)) = 1$. Hence, φ^a is not a cost sharing rule. □

It can be easily checked that $\varphi^a(N, q, C) = \varphi^{a'}(N, q, C)$ for all $(N, q, C) \in \mathcal{C}$ if and only if $a = a'$.

Notice that in this paper we define a family of cost sharing rules. One rule for each $a : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfies (3), (4), and (5).

4 Characterization of the a-serial cost sharing rule

In this section, we characterize each a-serial cost sharing rule. The a-serial cost sharing rule is characterized by means of Anonymity and a-Independence of higher demands. As a result of that, eventually the a-serial cost sharing rule, defined on page 6, is only considered for mapppings $a : \mathbb{R}_+ \to \mathbb{R}_+$ that are determined by the a-Independence of higher demands property.

Next, the standard anonymity property is formally introduced. Let f be a cost sharing rule and $\Pi(N)$ the set of permutations over N.

Anonymity. For every $\pi \in \Pi(N)$ and $i \in N$,

$$f_i(N, q, C) = f_{\pi(i)}(N, \pi q, C),$$

where $\pi q \in \mathbb{R}^n_+$ is given by $(\pi q)_i = q_{\pi^{-1}(i)}$ for each $i \in N$.

Anonymity requires a rule to be independent of the order in which the players are arranged. The second property considered is the a-Independence of higher demands introduced in Section 2.

First let us see that these two properties imply the mapping a associated with a-Independence of higher demands to satisfy conditions (4) and (5). Recall that condition (3) is required in the definition of a-Independence of higher demands.

Lemma 4.1. *Let f be a cost sharing rule that satisfies Anonymity and a-Independence of higher demands. Then, the mapping $a : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies conditions (4) and (5).*

Proof. (i) If a does not satisfy condition (4), then there exist $x, y \in \mathbb{R}_+$ such that $x < y$ and $a(x) > a(y)$. Let $q_1, q_2 \in \mathbb{R}_+$ such that $nq_1 = x$ and $nq_1 < y < (n-1)q_1 + q_2$. Let also

$$C(t) = \begin{cases} 0 & \text{if } t < a(x) \\ t - a(x) & \text{if } a(x) \leq t \leq a(x) + q_2 - q_1 \\ q_2 - q_1 & \text{else} \end{cases}.$$

Let $N = \{1, \ldots, n\}$ and $q = (q_1, \ldots, q_1, q_2) \in \mathbb{R}^n_+$. We are assuming that f is a cost sharing rule satisfying Anonymity and a-Independence of higher demands, then

$$f_1(N, q, C) = f_1(N, (q_1, ..., q_1), C^{a(x), q_2 - q_1}) = \frac{C^{a(x), q_2 - q_1}(x)}{n}.$$

Since a satisfies condition (3),

$$C^{a(x), q_2 - q_1}(x) = C((n-1)q_1 + q_2) - C(a(x) + q_2 - q_1) + C(a(x)) = 0.$$

Thus,

$$f_1(N, q, C) = 0. \tag{6}$$

Next, let q'_2 be such that $(n-1)q_1 + q'_2 = y$ and $D = C^{a(y), q_2 - q'_2}$. Observe that $q_1 < q'_2 < q_2$, then

$$\begin{aligned} D^{a(x), q'_2 - q_1}(x) &= D(y) - D(a(x) + q'_2 - q_1) + D(a(x)) \\ &= C(y + q_2 - q'_2) - C(a(y) + q_2 - q'_2) + C(a(y)) \\ &\quad - C(a(x) + q_2 - q_1) + C(a(y) + q_2 - q'_2) - C(a(y)) \\ &\quad + C(a(x) + q_2 - q'_2) - C(a(y) + q_2 - q'_2) + C(a(y)), \end{aligned} \tag{7}$$

where the first equality follows from $a(x) \leq x$ and the second equality holds because $a(y) < y$, $a(y) < a(x) + q'_2 - q_1$, and $a(y) < a(x)$. Next, simplifying eq.

(7)

$$
\begin{aligned}
D^{a(x),q_2'-q_1}(x) &= C(y+q_2-q_2') - C(a(x)+q_2-q_1) \\
&\quad + C(a(x)+q_2-q_2') - C(a(y)+q_2-q_2') + C(a(y)) \\
&= (q_2-q_1) - (q_2-q_1) + (q_2-q_2') - C(a(y)+q_2-q_2') + 0 \\
&= q_2 - q_2' - C(a(y)+q_2-q_2'),
\end{aligned}
$$

where the second equality holds since $y+q_2-q_2' = x+q_2-q_1 \geq a(x)+q_2-q_1$, $a(x)+q_2-q_2' < a(x)+q_2-q_1$, and $a(y) < a(x)$. Hence, taking into account that $a(y)+q_2-q_2' < a(x)+q_2-q_1$ two cases may arise,

$$
D^{a(x),q_2'-q_1}(x) = \begin{cases} a(x)-a(y) & \text{if } a(y)+q_2-q_2' \geq a(x) \\ q_2-q_2' & \text{otherwise} \end{cases}
$$

In any case $D^{a(x),q_2'-q_1}(x) > 0$. On the other hand, applying a-Independence of higher demands twice,

$$
f_1(N,q,C) = f_1(N,(q_1,\ldots,q_1,q_2'),D) = f_1(N,(q_1,\ldots,q_1),D^{a(x),q_2'-q_1})
$$

Again, by Anonymity and Efficiency,

$$
f_1(N,(q_1,\ldots,q_1),D^{a(x),q_2'-q_1}) = \frac{D^{a(x),q_2'-q_1}(x)}{n} > 0.
$$

The last equation contradicts eq. (6). Hence, a must be nondecreasing.

(ii) Suppose that there exist $x,y \in \mathbb{R}_+$ such that $x < y$ and $a(y)-a(x) > y-x$. As before, let $0 \leq q_1 < q_2' < q_2$ such that $nq_1 = x$ and $(n-1)\,q_1 + q_2' = y < (n-1)\,q_1 + q_2$, and let C be the cost function defined above but with respect to these new q_1 and q_2. We argue as before and reach eq. (6) only by assuming that f is a cost sharing rule satisfying Anonymity and a-Independence of higher demands, and that a satisfies condition (3). Finally, let D be defined as before. Then, by the hypothesis and the fact that a satisfies (4),

$$
\begin{aligned}
&D^{a(x),q_2'-q_1}(x) = D(y) - D(a(x)+q_2'-q_1) + D(a(x)) \\
&= C((n-1)q_1+q_2) - C(a(y)+q_2-q_2') + C(a(y)) - C(a(x)+q_2'-q_1) + C(a(x)) \\
&\qquad = q_2-q_1-q_2+q_1+C(a(y)) - q_2'+q_1 = C(a(y)) - q_2' + q_1 > 0.
\end{aligned}
$$

As before, by Anonymity and Efficiency we have

$$
f_1(N,(q_1,\ldots,q_1),D^{a(x),q_2'-q_1}) > 0,
$$

which contradicts eq. (6). □

Lemma 4.2. *The a-serial cost sharing rule satisfies Anonymity and a-Independence of higher demands.*

Proof. It is straightforward to check that the a-serial cost sharing rule satisfies Anonymity. To prove that it also satisfies a-Independence of higher demands, let (N,p,C), $(N,q,C) \in \mathcal{C}$ and $i \in N$ such that $p_1 \leq p_2 \leq \cdots \leq p_n$, $p_j = q_j$ for every $j \leq i$ and $p \leq q$. Let $j \leq i$, then since a satisfies condition (4),

$$C^{a(P),Q-P}(a(p^j)) = C(a(q^j)),$$

which implies that

$$C^{a(P),Q-P}(a(p^j)) - C^{a(P),Q-P}(a(p^{j-1})) = C(a(q^j)) - C(a(q^{j-1})). \tag{8}$$

Next, by condition (5) and the fact that a satisfies condition (3),

$$C^{a(P),Q-P}(P-(p^j-a(p^j))) = C(Q-(q^j-a(q^j)))-C(Q+a(P)-P)+C(a(P)),$$

from which we have

$$\begin{aligned} &C(Q-(q^{j-1}-a(q^{j-1}))) - C(Q-(q^j-a(q^j))) \\ &\quad = C^{a(P),Q-P}(P-(p^{j-1}-a(p^{j-1}))) - C^{a(P),Q-P}(P-(p^j-a(p^j))) \end{aligned} \tag{9}$$

By definition of φ^a and eq.s (8) and (9), $\varphi^a_i(N,q,C) = \varphi^a_i(N,p,C^{a(P),Q-P})$ which concludes the proof. □

Theorem 4.3. *A cost sharing rule f satisfies Anonymity and a-Independence of higher demands if and only if $f = \varphi^a$.*

Proof. From Lemma 4.2 we know that φ^a satisfies the properties. For the converse, let f be a cost sharing rule that satisfies Anonymity and a-Independence of higher demands. Then, the mapping a satisfies condition (3) by a-Independence of higher demands. Moreover, from Lemma 4.1 we know that a also satisfies conditions (4) and (5). Hence, φ^a is properly defined. For any $(N,q,C) \in \mathcal{C}$ with $q_1 \leq q_2 \leq \cdots \leq q_n$ we will see that $f_i(N,q,C) = \varphi^a_i(N,q,C)$ by induction on i.

If $i = 1$, let $p = (q_1,\ldots,q_1)$. Then, by a-Independence of higher demands,

$$\begin{aligned} f_i(N,q,C) = f_1(N,p,C^{a(P),Q-P}) &= \frac{1}{n} C^{a(q^1),Q-q^1}(q^1) \\ &= \frac{C(Q) - C(Q+a(q^1)-q^1) + C(a(q^1))}{n} = \varphi^a_i(N,q,C), \end{aligned}$$

where the second equality follows from the Anonymity and Efficiency of f and using the definitions and condition (3) in the remaining equalities.

Next, by the induction hypothesis assume that $f_j(N,q,C) = \varphi^a_j(N,q,C)$ for every $j < i$ and we will prove that $f_i(N,q,C) = \varphi^a_i(N,q,C)$.

Let $p = (q_1,q_2,\ldots,q_i,\ldots,q_i) \in \mathbb{R}^n_+$. By a-Independence of higher demands,

$$f_i(N,q,C) = f_i(N,p,C^{a(q^i),Q-q^i}).$$

Next, since f satisfies Anonymity we have that for every $j \geq i$,

$$f_j(N, p, C^{a(q^i),Q-q^i}) = f_i(N, p, C^{a(q^i),Q-q^i}).$$

Applying Efficiency and the induction hypothesis we can write

$$\begin{aligned} f_i(N, p, C^{a(q^i),Q-q^i}) &= \frac{C^{a(q^i),Q-q^i}(q^i) - \sum_{j<i} \varphi_j^a(N, q, C)}{n - i + 1} \\ &= \frac{C(Q) - C(Q + a(q^i) - q^i) + C(a(q^i)) - \sum_{j<i} \varphi_j^a(N, q, C)}{n - i + 1}, \end{aligned} \tag{10}$$

where the second equality follows from $a(q^i) \leq q^i$.

In what follows, we show the last expression of eq. (10) equals $\varphi_i^a(N, q, C)$. Indeed, by definition of φ^a

$$\begin{aligned} &\sum_{j<i} \varphi_j^a(N, q, C) = \frac{i-1}{n} C(Q) \\ &+ \left(\frac{i-1}{n} - \frac{i-2}{n-1}\right) \left[C(a(q^1)) - C(Q - (q^1 - a(q^1)))\right] \\ &+ \cdots + \left(\frac{2}{n-i+3} - \frac{1}{n-i+2}\right) \left[C(a(q^{i-2})) - C(Q - (q^{i-2} - a(q^{i-2})))\right] \\ &+ \frac{1}{n+2-i} \left[C(a(q^{i-1})) - C(Q - (q^{i-1} - a(q^{i-1})))\right] \\ &= \frac{i-1}{n} C(Q) + (n - i + 1) \left[\frac{C(a(q^1)) - C(Q - (q^1 - a(q^1)))}{n(n-1)}\right. \\ &+ \cdots + \left.\frac{C(a(q^{i-1})) - C(Q - (q^{i-1} - a(q^{i-1})))}{(n-i+2)(n-i+1)}\right]. \end{aligned}$$

Pluging the expression above in eq. (10) we obtain

$$\begin{aligned} &f_i(N, p, C^{a(q^i),Q-q^i}) = \frac{C(Q)}{n} + \frac{C(a(q^i)) - C(Q + a(q^i)) - q^i)}{n - i + 1} \\ &+ \frac{C(Q + a(q^1) - q^1) - C(a(q^1))}{n(n-1)} + \cdots + \frac{C(Q + a(q^{i-1}) - q^{i-1}) - C(a(q^{i-1}))}{(n-i+2)(n-i+1)} \\ &= \varphi_i^a(N, q, C), \end{aligned}$$

where the last equality follows by definition of φ^a and the proof concludes. $\square$

Remark 4.4. We could have considered a slightly weaker version of a-Independence of higher demands to characterize the a-serial cost sharing rule since condition (3) could be omitted. It can be proved that if a cost sharing rule f satisfies Anonymity and a-Independence of higher demands, being $a : \mathbb{R}_+ \to \mathbb{R}_+$ an arbitrary mapping, then f also satisfies a^*-Independence of higher demands where $a^* : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$a^*(x) = \begin{cases} a(x) & \text{if } a(x) \leq x \\ x & \text{otherwise.} \end{cases}$$

Therefore, condition (3) would be a consequence of the properties.

Remark 4.5. Let (N, q, C) and $a^1, a^2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that $a^1(x) \leq a^2(x)$ if $x \leq Q$. Since $a^1(x)$ and $a^2(x)$ determine the portion paid by the agents with respect to the quantities produced, if the cost function C is convex on $[0, Q]$, then agents with lower demands will prefer the a^2-serial cost sharing rule to the a^1-serial cost sharing rule, and if the cost function C is concave, then agents with higher demands will prefer the a^1-serial cost sharing rule to the a^2-serial cost sharing rule.

Example 4.6. Consider the following cost sharing problems studied by Moulin and Shenker (1994): $(N, q, C_1), (N, q, C_2) \in \mathcal{C}$, where $N = \{1, 2, 3\}$, $q = (3, 5, 7)$, $C_1(p) = \max\{p - 10, 0\}$, and $C_2(p) = \min\{p, 9 + 0.1p\}$. With C_1, the first 10 units are free and the next ones cost 1 each. While with C_2, the first 10 units cost 1 each and the next ones cost 0.1 each. Table 1 depicts the sharing proposed by the a-serial cost sharing rule for the following examples of a functions: $a^1(x) = x$ (the serial cost sharing rule), $a^2(x) = 0.75x$ (the 0.75-serial cost sharing rule)

$$a^3(x) = \begin{cases} 0.75x & \text{if } x \leq 9, \\ 0.5x + 2.25 & \text{if } x \geq 9, \end{cases}$$

$a^4(x) = 0.5x$ (the self dual serial cost sharing rule), and $a^5(x) = 0$ (the dual serial cost sharing rule).

	C_1	C_2
a^1	$(0, 1.5, 3.5)$	$(3, 3.65, 3.85)$
a^2	$(0.75, 1.25, 3)$	$(2.325, 3.875, 4.2)$
a^3	$(0.75, 1.75, 2.5)$	$(2.325, 3.425, 4.75)$
a^4	$(1.5, 1.75, 1.75)$	$(1.65, 3.425, 5.425)$
a^5	$(1.67, 1.67, 1.67)$	$(1.5, 3.5, 5.5)$

Table 1: Comparison of cost sharing rules of the a-serial family

In the first problem, (N, q, C_1), it is reasonable to think that player 1 should pay more than his share according to a^1, which is nothing. But also that he should pay less than what a^4 and a^5 prescribe. The sharing according to a^3 can be seen as a compromise in this sense. Notice that player 1 takes less free units with a^3 than with a^1, and more free units than with a^4 and a^5. On the other hand, the non linearity of a^3 makes player 2 contribute a bit more than what he would have to pay with a^2.

The contrary happens in problem (N, q, C_2). In this situation it is reasonable to think that player 1 pays too much with a^1 and too few both with a^4 and with a^5. Again, a^3 can be seen as a compromise. The reason behind the assertion above is that player 1 takes less units of cost 1 each with a^3 than with a^1, and more units of cost 1 than with a^4 and a^5. Finally, the non linearity of a^3 makes player 2 pay not so much as with a^2.

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