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# On Lie groups and algebras


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*In mathematics, the world of ideas is way more important than the world of calculations.*

Dr. Àngel Jorba

# Contents

# Presentation

## Foreword

The present document is my thesis for the obtention of a Master's Degree on Advanced Mathematics. As the title says, it has the pretension to be a brief overview on elementary algebraic Lie theory. My interest on such field arised practically by chance, when I was looking for a subject in Algebraic Geometry to work with and I came across a talk with my current advisor, Dr. Ricardo García. His recommendation of a couple of books ([H] and [Bak]), which I read with unexecpted joy and interest, helped me on taking my decission to produce the work you are going to read.

Lie groups arose from the need to study certain sets of symmetries and give them structure. One can approach algebraic Lie theory by two different paths: the differential one and the matricial one. The differential one defines a Lie group as a smooth manifold endowed with a product and inversion operations which are smooth with the underlying differential structure. The matricial approach deals on matrix Lie groups, which are matrix subgroups of the General Linear group over a field $\mathbb{K}$ with the property that any converging sequence of matrices within the subgroup either has its limit in the subgroup itself or lies outisde of $\mathsf{GL}(n, \mathbb{K})$. When comparing both approaches one finds that the differential definition gives rise to more Lie groups than just the matrix ones, so other Lie groups were later algebraically built starting from matrix Lie groups which are not matrix Lie groups themselves in order to be sure that both definitions are equivalent.

Whatever the approach, Lie groups incorporate the notion of Lie alegbra; another structure that somehow comes attached to them. The link between Lie groups and Lie algebras is the exponential map, which can be regarded as the biggest possible generalization of the complex or real exponential function. When dealing with Lie groups, it is often convenient to work with their Lie algebras instead. This has a good side, namely that all Lie algebras are isomorphic to some matrix algebra no matter the nature of the underlying Lie group, but on the downside, two different Lie groups may have the same Lie algebra, leaving the problem of having a complete classification of Lie groups with no definitive answer to the present day.

With the above, the objectives of this work are fundamentally three: the first is to give the basic definitions of Lie groups, Lie algebras, the exponential map and the morphisms relating them. The second is to give an elementary introduction to Lie group representation theory as well as some criteria on how to classify certain Lie groups. The third and final goal is to study a couple of specific Lie groups, namely $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$, in order to apply all the concerning theory and have an insight into the applications of Lie groups into Physics.

## Content summary

The above objectives shall be approached along five different chapters, whose contents are to be summarized in the following paragraphs.

The goal of **chapter one** is to introduce Lie groups and Lie algebras, to show the classical Lie groups and construct their respective Lie algebras. We start by giving the definitions and the detailed description of the most classical Lie groups with brief insight on an example of a non-matrix Lie group. The corresponding Lie algebras are also described in detail once the exponential map has been introduced as the linking path between the two class of objects. The chapter ends with some considerations on the differential approach, especially that of giving Lie groups a notion of dimension.

**Chapter two** begins studying the morphisms between Lie groups and the corresponding morphisms between Lie algebras, paying attention to the adjoint map. After that, a part which studies Lie groups and Lie algebras from the scope of category theory follows. The objective is to study the Lie functor and study under which conditions becomes a fully faithful functor. Then, we present some topological considerations on Lie groups, especially those concerning Lie groups which are coverings of other Lie groups. Finally, all the theory developed up to the point is applyied on a detailed description of the groups $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$.

The main content of **chapter three** is the introduction to representation theory. From the overview of general group representation theory we move into Lie group representations, studying the effects that the definition of Lie groups and their morphisms produce on their representations as mere group morphisms. As a sidestory, we get the chance to study centers, maximal tori and ranks of Lie groups.

In **chapter four** we deal on compact Lie groups, whose rich structure allows us to have a more accurate knowledge of them. It is worth to note the introduction of the Haar integral, which both provides a way of integration on Lie groups and some tools for compact Lie group representations, namely weights and roots. The chapter ends with a second review of $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ plus an insight on $\mathsf{SU}(3)$ to complete the application of the theory given in chapters three and four.

Finally, **chapter five** deals completely with the applications of Lie groups into physics. We pay a visit to particle physics via $\mathsf{SU}(2)$ and the isospin symmetry, and to the Quantum Model thanks to the representations of $\mathsf{SU}(3)$. All of this is done with the classical physicist's notation and terminology, most notably that regarding the bra-ket symbology.

## Future work

As a final wish, this author hopes that the present work will give ground to further studies on the matter. Some subjects have been ruled out of the program due to time and priorities. Among these subjects it is woth mentioning the following: to give some insight on Semisimple Theory, a deeper study of the weights and roots attached to a Lie group representation, the developement of Dynkin Diagrams and even a quick peek on infinite-dimensional Lie groups. I hope to be able to work on all of these in the future.

Barcelona, June 30, 2014.

# Chapter 1

# Fundamentals

## 1.1 Lie groups

Recall from linear geometry the notion of *General Linear Group* over a field $\mathbb{K}$, being the set of all invertible matrices with coefficients in $\mathbb{K}$, and denoted $\mathsf{GL}(n, \mathbb{K})$. It is important to retain that $\mathsf{GL}(n, \mathbb{K})$ is a group with the operation of matrix product and that it contains a number of notorious subgroups which are usually attached to linear transformations.

Topics in linear geometry like this and many others can be reviewed in [C-Ll], while for the algebric issues [Lang] and [Qey] are our preferred suggestions.

We begin our discussions here studying some subgroups of $\mathsf{GL}(n, \mathbb{C})$ which fulfill what will be called the Lie property. The two main sources for all this section and some of the following ones are [H] and [Bak].

### 1.1.1 Matrix Lie groups: definition and classical examples

**Definition 1.1.** Let $\{A_m\}_{m\in\mathbb{N}} \subset M_n(\mathbb{C})$ be a sequence of $n \times n$ matrices with complex entries. We say that the sequence $A_m$ **converges** to a matrix $A \in M_n(\mathbb{C})$ as $m \longrightarrow \infty$ if, for each $1 \leq i, j \leq n$, $a_{ij}^m \longrightarrow a_{ij}$ as $m \longrightarrow \infty$. That is, if the sequences of entries of each $A_m$ converge to the corresponding entry of $A$ as sequences of complex numbers.

**Definition 1.2.** A **matrix Lie Group** is any subgroup $G \subset \mathsf{GL}(n, \mathbb{C})$ for which whenever a sequence $\{A_m\}_{m\in\mathbb{N}} \subset G$ converges to some matrix $A$, then either $A \in G$ or $A$ is not invertible.

This condition can be reduced to stating that $G$ is a closed subgroup of $\mathsf{GL}(n, \mathbb{C})$. We will be coming back to this later in section 1.1.4, where we will be able to explore the equivalence between these two properties.

Our first task is to introduce many subgroups of $\mathsf{GL}(n, \mathbb{C})$ which are actually matrix Lie groups. Many of these are previously known for those familiar with Linear Geometry as groups of transformations, whereas some others might be less familiar. Whichever the case, the following examples sum up to what are considered to be the *classical matrix Lie groups.*

**Proposition 1.1.1.1** (The General Linear group)**.** For each $n \in \mathbb{N}$, $\mathsf{GL}(n, \mathbb{C})$ and $\mathsf{GL}(n, \mathbb{R})$ are matrix Lie groups.

*Proof.* We shall check that this are subgroups of $\mathsf{GL}(n, \mathbb{C})$ and that they fulfill the Lie group property.

Take $\mathsf{GL}(n,\mathbb{C})$ itself, which is obviously its own subgroup. If $A_m$ is a sequence of matrices of $\mathsf{GL}(n,\mathbb{C})$ which converge to a certain matrix $A$, then either $A$ is invertible, and thus $A \in \mathsf{GL}(n,\mathbb{C})$, or $A$ is not invertible. Therefore, $\mathsf{GL}(n,\mathbb{C})$ is a matrix Lie group almost by definition.

Since $\mathbb{R}$ is a subgroup of $\mathbb{C}$, it is clear that $\mathsf{GL}(n,\mathbb{R})$ is a matrix subgroup of $\mathsf{GL}(n,\mathbb{C})$ and, by completeness, any convergent sequence of real numbers has a limit within the real numbers (in the sense that it cannot have nonzero imaginary part). Therefore, if $\{A_m\}_{m\in\mathbb{N}} \subset \mathsf{GL}(n,\mathbb{R})$ is a convergent sequence of real invertible matrices whose limit is a matrix $A$, then $A \in M_n(\mathbb{R})$ and, just like in the previous case, either $A \in \mathsf{GL}(n,\mathbb{R})$ or $A$ is not invertible. Thus, $\mathsf{GL}(n,\mathbb{R})$ is a matrix Lie group. □

Many subgroups of $\mathsf{GL}(n,\mathbb{C})$ which are matrix Lie groups follow now. It is important to be familiar with them as usually the applications of Lie groups work fine with matrix Lie groups. For illustrative purposes we are giving the proof of the Lie property for the orthogonal group, as for the rest of the groups the technique is always the same but using the relevant property each one has.

**Definition 1.3** (The Special Linear group)**.** Given a field $\mathbb{K}$, the **Special Linear group** $\mathsf{SL}(n,\mathbb{K})$ is the subgroup of $\mathsf{GL}(n,\mathbb{K})$ of invertible matrices with determinant one.

**Proposition 1.1.1.2.** The Special Linear groups $\mathsf{SL}(n,\mathbb{C})$ and $\mathsf{SL}(n,\mathbb{R})$ are matrix Lie groups.

*Proof.* Just like with the General Linear group, take a sequence $\{A_n\} \subset \mathsf{SL}(n,\mathbb{K})$ which converges to a matrix $A$. Then, either $A$ is not invertible or, by the continuity of the determinant, $\det A = 1$, thus $\mathsf{SL}(n,\mathbb{K})$ is a matrix Lie group. □

Recall now that given a real vector space $E$ together with a metric $\|\cdot\|$, two vectors $u, v \in E \setminus \{0\}$ are said to be **orthogonal** if, and only if, $\langle u, v\rangle = 0$. Then, a set of vectors $\{v_1,\dots,v_k\} \subset E$ is said to be orthogonal if, and only if, $\langle v_i, v_j\rangle = 0$ for each $1 \le i, j \le k$ such that $i \ne j$. Here, $\langle\cdot,\cdot\rangle$ denotes the scalar product induced by the metric $\|\cdot\|$. If, additionally, $\langle v_j, v_j\rangle = 1$ for each $j$, then we have an **orthonormal** set.

**Definition 1.4** (The Orthogonal group)**.** The **Orthogonal group** $\mathsf{O}(n)$ is the set of $n \times n$ matrices $A$ with real entries such that $A^T A = I$. The elements of $\mathsf{O}(n)$ are called **orthogonal matrices**.

This is equivalent to say that the columns of a matrix $A \in \mathsf{O}(n)$ for an orthonormal set. See [C-Ll] for more details on this matter.

**Proposition 1.1.1.3.** $\mathsf{O}(n)$ is a matrix Lie group.

*Proof.* The general procedure for showing that a certain group of matrices is a matrix Lie group is always the same: first we have to check that it is a subgroup of $\mathsf{GL}(n,\mathbb{C})$ and then that the Lie property of matrix sequences hold.

Let $A \in \mathsf{O}(n)$ . First of all, note that $\det A$ has to be nonzero, as $\det A \cdot \det A^T = \det I \ne 0$. This shows that $\mathsf{O}(n) \subset \mathsf{GL}(n,\mathbb{R})$.

Now, if $A, B \in \mathsf{O}(n)$ then $(AB)^T \cdot AB = B^T A^T \cdot AB = B^T B = I$, so $AB \in \mathsf{O}(n)$. Moreover, $(A^{-1})^T A^{-1} = (A^T)^T A^{-1} = AA^{-1} = I$, so $A^{-1} \in \mathsf{O}(n)$. Thus, $\mathsf{O}(n)$ is a subgroup of $\mathsf{GL}(n,\mathbb{R})$.

With the property of being a subgroup of $\mathsf{GL}(n,\mathbb{R})$, and consequently of $\mathsf{GL}(n,\mathbb{C})$, already shown, it remains to check that $\mathsf{O}(n)$ satisfies the Lie property. So, let $\{A_m\}_{m\in\mathbb{N}} \subset \mathsf{O}(n)$ be a sequence which converges to a certain matrix $A \in M_n(\mathbb{R})$. It is clear that the sequence $A_m^T$ converges to $A^T$, as the only thing we are doing is modifying the position of a finite number of real sequences, so consider the sequence $B_m := A_m^T A_m$ which converges to the the matrix $B := A^T A$ and observe that, in fact, $B_m = I$ for each $m$ so, by continuity, $B = A^T A = I$. Therefore, $A \in \mathsf{O}(n)$ and this is a matrix Lie group. □

**Definition 1.5** (Special Orthogonal group)**.** Let $\mathsf{SO}(n) \subset \mathsf{O}(n)$ be the set of orthogonal matrices with determinant one. We call $\mathsf{SO}(n)$ the **Special Orthogonal group**.

It is immediate that $\mathsf{SO}(n)$ is a subgroup of $\mathsf{O}(n)$ via the multiplicativity of the determinant, so in the end it is also a subgroup of $\mathsf{GL}(n,\mathbb{C})$. Then, checking that it is a matrix Lie group is done in the same fashion as with $\mathsf{O}(n)$

It is often said that $\mathsf{SO}(n)$ is "half" $\mathsf{O}(n)$, because orthogonal matrices have determinant $\pm 1$ and the special ones are those with determinant one. We will give a formal description of this fact later on section 1.2.2.

Orthonormality extends into complex vector spaces in the following way: if $\{v_1, \dots, v_k\} \subset \mathbb{C}^n \setminus \{0\}$ is a set of complex vectors and $\langle \cdot, \cdot \rangle$ is a hermitian product, then this vector set is called an orthonormal (or hermitic) set if, and only if,

$$\begin{aligned} \langle \bar{v}_i, v_j \rangle &= 0, \ i \neq j, \\ \langle \bar{v}_i, v_i \rangle &= 1, \ 1 \leq i \leq k. \end{aligned}$$

Here, $\bar{v}$ means the complex conjugate of the vector $v$, that is, a vector formed by the conjugates of $v$'s components. Complex orthonormality gives rise to two new matrix Lie groups.

**Definition 1.6** (The Unitary group)**.** We call **Unitary group** $\mathsf{U}(n)$ to the set of complex $n \times n$ matrices whose columns form an orthonormal set.

**Definition 1.7** (The Special Unitary group)**.** The **Special Unitary group** $\mathsf{SU}(n)$ is the set of unitary $n \times n$ matrices with determinant one.

Just like in the real case, it is easy to check that $\mathsf{U}(n)$ is a subgroup of $\mathsf{GL}(n, \mathbb{C})$ by noting that if $A \in \mathsf{U}(n)$ then its determinant has to be nonzero. Equivalently to transposition in the real case, we may say that a matrix $A$ is unitary if, and only if, $A^* A = I$; that is, if its inverse is its adjoint matrix. $\mathsf{SU}(n)$ is, on its turn, a subgroup of $\mathsf{U}(n)$ and hence a subgroup of $\mathsf{GL}(n, \mathbb{C})$. Both of them are, indeed, matrix Lie groups.

One particularly relevant property of these groups is that their elements are distance-preserving, meaning that if $A$ is an orthogonal or unitary matrix then, for each $u, v \in \mathbb{R}^n$ or $\mathbb{C}^n$, $\langle u, v \rangle = \langle Au, Av \rangle$. Thus, any linear map whose associated matrix is of one of these kinds is an isometry, because in particular $\langle v, v \rangle = \langle Av, Av \rangle$, so $\|v\|^2 = \|Av\|^2$ and then $\|v\| = \|Av\|$ (recall that $\|v\| \geq 0$).

The last of the classical matrix Lie groups are the forecoming symplectic groups. Let us set in $\mathbb{R}^{2n}$, $n \geq 2$, and let us consider a basis $\mathcal{B} := \{u_1, \dots, u_n, v_1, \dots, v_n\}$. Given any two vectors $x, y \in \mathbb{R}^{2n}$, $x = (x_1, \dots, x_{2n})$, $y = (y_1 \dots, y_{2n})$ we say that a bilinear form $B : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is **skew-symmetric** if

$$B\,[x, y] = \sum_{k=1}^{2n} \left( x_k y_{n+k} - x_{n+k} y_k \right).$$

We then say that $\mathcal{B}$ is a **symplectic basis** of $\mathbb{R}^{2n}$ if, and only if,

$$\begin{aligned} B\,[u_i, v_i] &= 1, \\ B\,[v_i, u_i] &= -1, \ 1 \leq i \leq n, \end{aligned}$$

and zero for any other pairing. Any set of nonzero vectors –not necessarily a basis– with that property is called a **symplectic set**.

Set now $J$ to be the $2n \times 2n$ real matrix defined as

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

it can be proved that $B\,[x, y] = B\,[x, Jy]$ for each $x, y \in \mathbb{R}^{2n}$. This leads us to the definition of the Real Symplectic group.

**Definition 1.8** (The Real Symplectic group)**.** The **Real Symplectic group** $\mathsf{Sp}(n; \mathbb{R})$ is the group of matrices $A \in M_{2n}(\mathbb{R})$ such that, for each $x, y \in \mathbb{R}^{2n}$, $B\,[x, y] = B\,[Ax, Ay]$. Such matrices are called **skew-symmetric matrices**.

A convenient characterization of the symplectic groups is that a matrix $A$ is skew-symmetric if, and only if, $A^T JA = J$. Once this property has been established, it is easy to check that $\mathsf{Sp}(n;\mathbb{R})$ is a matrix Lie group but taking care that it is so as a subgroup of $\mathsf{GL}(2n, \mathbb{C})$.

These discussions work exactly the same if we replace $\mathbb{R}$ by $\mathbb{C}$ because a skew-symmetric complex bilinear form does not involve complex conjugates but is defined exactly as its real version. Thus, the property $A^T JA = J$ holds for complex skew-symmetric matrices and we get the matrix Lie group $\mathsf{Sp}(n;\mathbb{C})$, which we call the **Complex Symplectic group**. A complex skew-symmetric matrix $A$ is then defined by the relation $B\,[x, y] = B\,[Ax, Ay]$.

There is one last group related to symplectic basis, the Compact Symplectic group.

**Definition 1.9** (The Compact Symplectic group)**.** We define the **Compact Symplectic Group** as the group $\mathsf{USp}(n) := \mathsf{Sp}(n;\mathbb{C}) \cap \mathsf{U}(2n)$.

It is clear by the definition that $\mathsf{USp}(n) \subset \mathsf{GL}(2n, \mathbb{C})$ and that it is implied that if $A \in \mathsf{USp}(n)$ then both $A^T JA = J$ and $A^*A = I$ do hold, so if $A, B \in \mathsf{USp}(n)$ our discussions for both the unitary and symplectic groups already grant that $AB \in \mathsf{USp}(n)$ and that $A^{-1} \in \mathsf{USp}(n)$, so in the end $\mathsf{USp}(n)$ is a subgroup of $\mathsf{GL}(2n, \mathbb{C})$. In a similar fashion, the Lie property is almost immediate with the inherited properties from $\mathsf{Sp}(n;\mathbb{C})$ and $\mathsf{U}(2n)$.

### 1.1.2 More examples

**Definition 1.10** (The Heisenberg group)**.** Let $H$ be the matrix group defined as

$$H := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R}) \right\}.$$

We call $G$ the **Heisenberg group**.

By the definition itself we can note that $H \subset \mathsf{GL}(3;\mathbb{C})$ and direct computations show that it is actually a subgroup (furthermore, it is a subgroup of $\mathsf{SL}(3, \mathbb{C})$). It is also immediate that it is a matrix Lie group, as any convergent matrix sequence $\{A_m\}_{m\in\mathbb{N}} \subset H$ necessarily has an upper triangular matrix $A$ with ones on its diagonal as a limit.

**Definition 1.11** (The Euclidean group)**.** The **Euclidean group** $\mathsf{E}(n)$ is the group of one-to-one maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that for each $x, y \in \mathbb{R}^n$, $\|x - y\| = \|f(x) - f(y)\|$; that is, the group of distance-preserving, one-to-one, onto maps.

It is known that $\mathsf{E}(n)$ with the map composition operation has a group structure, but our concern here is to determine if it is actually a subgroup of $\mathsf{GL}(\mathbb{K}, \mathbb{C})$ for some $\mathbb{K}$. For that purpose, consider the set of translations of $\mathbb{R}^n$, $T_x = T_x(y) := x + y$, $x \in \mathbb{R}^n$, which is itself a subgroup of $\mathsf{E}(n)$. It can be showed that any element $T \in \mathsf{E}(n)$ can be represented as

$$T = T_x R,$$

for $x \in \mathbb{R}^n$ and $R \in \mathsf{O}(n)$, which sums up to saying that *any distance-preserving, one-to-one and onto map of* $\mathbb{R}^n$ *can be expressed as a translation followed by an orthogonal linear transformation.* However, our problem here is that translations are not linear maps, so $\mathsf{E}(n)$ is not a subgroup of $\mathsf{GL}(n, \mathbb{C})$; but let us write the elements of $\mathsf{E}(n)$ in the form $(x, R)$, so we can associate them to the matrices

$$(x, R) = \begin{pmatrix} & & x_1 \\ & R & \vdots \\ & & x_n \\ 0 & \dots & 1 \end{pmatrix}.$$

This is a one-to-one relation and it is clear that these matrices have nonzero determinant ($\det(x, R) = \det R \neq 0$), so let these matrices be *the* elements of $\mathsf{E}(n)$. Then, $\mathsf{E}(n) \subset \mathsf{GL}(n+1, \mathbb{C})$ and it is a subgroup and then the Lie property becomes clear.

We can take the orthonormal group and generalize it in the following way: take two vectors $u, v \in \mathbb{C}^{p+q}$ and define the bilinear form

$$\langle u, v\rangle = \sum_{k=1}^{p} u_k v_k - \sum_{k=p+1}^{q} u_k v_k,$$

then define an orthonormal set in the same fashion as we did in the classical case. The set of orthogonal matrices with respect to that scalar product is called the **generalized Orthogonal group** $\mathsf{O}(p; q) \subset \mathsf{GL}(p+q, \mathbb{C})$. Important examples are the **Lorentz group** $\mathsf{O}(3; 1)$ and the spanning **Poincaré group** $\mathsf{P}(n; 1)$ of matrices $(x, A)$ with $A \in \mathsf{O}(n; 1)$, all of them being matrix Lie groups.

With a greater generality, any group which is isomorphic to a matrix Lie group can be thought as such. This gives structure of Lie group to some important sets as $\mathbb{R}^*$ and $\mathbb{C}^*$ which are isomorphic, respectively, to $\mathsf{GL}(1, \mathbb{R})$ and $\mathsf{GL}(1, \mathbb{C})$; or $\mathbb{S}^1$ thought as the multplicative group of complex numbers with modulus one, which is isomorphic to $\mathsf{U}(1)$. $\mathbb{R}^n$ is another example as a group isomorphic to $\mathsf{GL}(n, \mathbb{R})^+ \subset \mathsf{GL}(n, \mathbb{R})$, the group of real matrices with positive determinant.

### 1.1.3 Not all matrix groups are matrix Lie groups

Despite all the preceding, there are easy examples of matrix groups which are not matrix Lie groups. Let's take a look at a couple of them.

**Example 1.1.3.1.** The group $G := \{A \in \mathsf{GL}(n; \mathbb{C}) | a_{ij} \in \mathbb{Q}\}$ is not a matrix Lie group because although it has a group structure, it is possible to take a sequence of converging matrices $A_m$ whose limit *is not* in $G$, say for instance

$$A_m = \begin{pmatrix} (1+\frac{1}{m})^m & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & (1+\frac{1}{m})^m \end{pmatrix},$$

which converges to a nonzero determinant matrix with irrational entries on its diagonal. In fact, *any* nonzero determinant irrational matrix can be written as a limit of rational matrices.

**Example 1.1.3.2.** Let $a \in \mathbb{R} \setminus \mathbb{Q}$ be fixed. Define the group

$$G := \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{ita} \end{pmatrix} \mid t \in \mathbb{R} \right\} \subset \mathsf{GL}(2, \mathbb{R}).$$

Observe that $-I \notin G$ because, if so, $ta$ has to be an odd integer multiple of $\pi$ in which case $a$ cannot be an odd integer multiple of $\pi$; but by properties of real numbers we can take $t$ such that $ta$ is arbitrarily close to an integer multiple of $\pi$ and hence define a converging sequence of matrices with limit $-I$. Hence, $G$ cannot be a matrix Lie group, as $\det(-I) \neq 0$.

### 1.1.4 General Lie groups

Whereas most applications of Lie groups work fine with matrix Lie groups, the concept of Lie group is actually *much* more. A different approach to Lie groups can be attained through the path of Differential Geometry, which gives rise to Lie groups that are not matrix Lie groups (in the sense that there is no isomorphism between them and a subgroup of $\mathsf{GL}(n, \mathbb{C})$). We take a peek on that matter in here for the sake of completeness.

The general theory of manifolds leads us to the concepts of smooth manifold, charts, atlas and smooth maps. We will assume them for the discussions to come.

**Definition 1.12.** A Lie group is a group $G$ which is also a smooth manifold, meaning that the operations

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (x, y) &\longmapsto x \cdot y \end{aligned}$$

and

$$\begin{aligned} \iota : G &\longrightarrow G \\ x &\longmapsto x^{-1} \end{aligned}$$

are smooth.

The matrix Lie group theory can be accessed this way by just checking that $\mathsf{GL}(n;\mathbb{C})$ is a Lie group in the sense of definition 1.12. The easiest way to do this is to note that there exists a natural embedding

$$\mathsf{GL}(n, \mathbb{C}) \hookrightarrow \mathbb{R}^{2n^2}$$

which arises identifying $\mathsf{GL}(n;\mathbb{C})$ with the open subset $\{(z_1, \ldots, z_{n^2}) \mid \det(z_1, \ldots, z_{n^2}) \neq 0\}$. Since $\mathbb{C}^{n^2}$ is a smooth manifold, any open subset is a smooth submanifold and hence $\mathsf{GL}(n, \mathbb{C})$ is a Lie group. This pretty much means that any matrix Lie group is a Lie group, a fact that can be proved, for instance, via the **Regular Value** theorem. The converse is also true, that is, if a Lie group is a matrix group, then it is a matrix Lie group in the sense of definition 1.2, meaning that both definitions are equivalent in the background of matrix Lie groups.

One particular advantage of taking this into account when dealing with matrix Lie groups are its topological aspects. For instance, when showing that a particular matrix group is a matrix Lie group it is often claimed that the determinant *is a continuous function* so it can be carried on into limits to prove that a limit matrix remains inside the group. For that purpose, one can assume the group $G \subset \mathsf{GL}(n, \mathbb{C})$ to be a smooth submanifold, which consequently possesses an underlying topology, and then show that the determinant is a smooth function $\mathsf{GL}(n, \mathbb{C}) \longrightarrow \mathbb{C}$, so in particular it is continuous with respect to the underlying topology.

Now, the following question arises: are there Lie groups which are not matrix Lie groups? The answer is yes, but they are not always easy to construct.

**Example 1.1.4.1.** Let $G := \mathbb{R} \times \mathbb{R} \times \mathbb{S}^1 \subset \mathbb{R} \times \mathbb{R} \times \mathbb{C}$ with the group operation

$$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) := (x_1 + x_2, y_1 + y_2, e^{ix_1x_2}u_1u_2),$$

which gives rise to the inversion

$$(x, y, u)^{-1} = (-x, -y, e^{iyx}u^{-1}).$$

It can be checked that this operation is associative, thus making $(G, \cdot)$ into a group.

Observe that $G$ is a manifold because it is the product of three manifolds, then considering the product and the inversion as maps $G \times G \longrightarrow G$ and $G \longrightarrow G$ respectively we see that they are smooth maps because they are so componentwise. Therefore, $G$ is a Lie group.

However, there is no continuous homomorphism which identifyes $G$ with any matrix Lie group, and thus this is an example of a Lie group which is not a matrix Lie group.

## 1.2 Topological aspects

This section explores some features a matrix Lie group might (or might not) have. The forecoming terminology might be familiar with that of general topology, and it is indeed so, meaning that the things discussed in here are actually topology matters "translated" into matrix Lie group theory, with the advantage of not having to delve deep into abstract topology in order to get by with them.

### 1.2.1 Compactness

Recall from basic geometry the notion of norm of a vector. In the same sense, we can define the **norm of a matrix** to be a map $\|\cdot\| : M_n(\mathbb{K}) \longrightarrow \mathbb{R}$ with the usual properties. Let $A \in M_n(\mathbb{K})$ be a matrix whose elements are $a_i^j$, so sub-indeces denote rows and super-indeces denote columns. The following are some matrix norms:

- $\|A\|_0 = \max\limits_{1\leq k\leq n} \sum_{j=1}^n |a_k^j|$.
- $\|A\|_1 = \max\limits_{1\leq k\leq n} \sum_{i=1}^n |a_i^k|$.
- $\|A\|_2 = \sqrt{\rho(A^T A)}$, where $\rho(A)$ denotes the spectral ratio of the matrix $A$.

We are not making any distinctions with the norms used for what is to follow because the identification of $M_n(\mathbb{K})$ with $\mathbb{K}^{n^2}$, grants it to be a finite-dimensional Banach space, so all norms on it are equivalent ([Rud]). Of course, all concepts about convergence and limits of matrix sequences used in the previous section can be achieved by means of any of these norms. Also, whichever the norm, the following properties hold for each $X, Y \in M_n(\mathbb{K})$:

(1). $\|X\| \geq 0$ and $\|X\| = 0 \Leftrightarrow X = 0$.

(2). $\|X + Y\| \leq \|X\| + \|Y\|$.

(3). $\|XY\| \leq \|X\|\|Y\|$.

With the notion of a norm, a matrix set $T \subset M_n(\mathbb{K})$ is said to be **bounded** if, for each $A \in M$, there exists some constant $k \in \mathbb{R}$ such that $\|A\| \leq k$. This is completely analogous to the boundedness notion in $\mathbb{C}^n$. We can now define what a compact matrix Lie group is.

**Definition 1.13.** A matrix Lie group $G \subset \mathsf{GL}(n, \mathbb{C})$ is said to be **compact** if it is closed and bounded.

By the defintion of matrix Lie group the closedness is automatic, so the definition can be lowered to simply ask $G$ to be bounded.

**Example 1.2.1.1.** $\mathsf{O}(n)$ is a compact matrix Lie group. Indeed, if $A \in \mathsf{O}(n)$ then its columns are unitary vectors, meaning that $|a_i^j| \leq 1$ for each $i, j$. Then, $\|A\|_1 = \max\limits_{1\leq k\leq n} \sum_{i=1}^n |a_i^k| \leq \max\limits_{1\leq k\leq n} n = n$.

**Example 1.2.1.2.** $\mathsf{USp}(n)$ is a compact matrix Lie group. If $A \in \mathsf{USp}(n)$, in particular $A \in \mathsf{U}(n)$ so just like in the previous example, $\|A\|_1 = \max\limits_{1\leq k\leq n} \sum_{i=1}^n |a_i^k| \leq \max\limits_{1\leq k\leq n} n = n$. That is the reason this group is called *compact* symplectic group.

In the other hand, $\mathsf{GL}(n, \mathbb{C})$ is not a compact group because it is not closed in $M_{n\times n}(\mathbb{C})$, meaning that a sequence of matrices with nonzero determinant might have a limit with zero determinant, which lies outside $\mathsf{GL}(n, \mathbb{C})$.

### 1.2.2 Connectedness

Another topological concept we want to use is that of a path. A **matrix path** linking the matrix $A$ with the matrix $B$ is a continuous (with respect to some topology, namely one induced by a norm) map $\gamma : [0,1] \longrightarrow M_n(\mathbb{K})$ such that $\gamma(0) = A$ and $\gamma(1) = B$. A path is called **simple** if it is injective.

**Definition 1.14.** A matrix Lie group $G$ is said to be connected if for each couple of elements $A, B \in G$ there exists a path $\gamma$ in $G$ linking $A$ and $B$.

Topologically, this is rather the definition of path-connectedness, which is not always the same as connectedness, but it is so in the background of matrix groups.

**Example 1.2.2.1.** The groups $\mathsf{GL}(n, \mathbb{C})$, $\mathsf{SL}(n, \mathbb{C})$, $\mathsf{U}(n)$ and $\mathsf{SU}(n)$ are connected, while $\mathsf{GL}(n, \mathbb{R})$ is not.

When a group is not connected then it has **connected components**, which are subsets such that they are themselves connected. An interesting result on this matter is the following.

**Theorem 1.2.2.2.** If $G$ is a matrix Lie group which is not connected, then the connected component containing the identity matrix is a subgroup of $G$.

*Proof.* Take $A, B \in G$ so they both lie in the connected component that contains $I$, which we will be calling $G_I$. Then, there exist paths $\gamma_A$ and $\gamma_B$ such that $\gamma_A(0) = \gamma_B(0) = I$ and $\gamma_A(1) = A$ and $\gamma_B(1) = B$. Therefore, the path $\gamma_{AB}(t) := \gamma_A(t)\gamma_B(t)$ satisfies $\gamma_{AB}(0) = I$ and $\gamma_{AB}(1) = AB$, so $AB \in G_I$. Now, for a given $A \in G_I$, if we take the path $\gamma_A(t)A^{-1}$ we have the following: $\gamma_A(0)A^{-1} = A^{-1}$ and $\gamma_A(1)A^{-1} = AA^{-1} = I$, so there is a path connecting $I$ with $A^{-1}$ for each $A$ connected to $I$, hence $A^{-1} \in G_I$ and $G_I$ is a subgroup. $\square$

This provides one alternative way of proving that $\mathsf{SO}(n)$ is a matrix Lie group by checking that it is a subgroup of $\mathsf{O}(n)$. First of all, note that $\mathsf{O}(n)$ is not connected, as there is no continuous path $\gamma$ linking $I$ with $-I$ because if there was one, the continuity of the determinant leads to the existence of some $t \in (0,1)$ such that $\det(\gamma(t)) = 0$, which cannot be. The same idea shows that any matrix $A$ with determinant one is linked with $I$, so using the theorem, the set of orthogonal matrices with determinant one is a subgroup of $\mathsf{O}(n)$, therefore $\mathsf{SO}(n)$ is a matrix Lie group.

**Definition 1.15.** A **loop** is a path $\gamma$ such that $\gamma(0) = \gamma(1)$.

The topological notion of **contractibility** tells us about how "minimal" can a loop be rendered by performing continous changes into it. The concept of "minimal" is of course blurry, but the idea is to determine wether if in a given topological space any loop can be brought into a point by such continuous manipulations or there exist certain ones that do not. For instance, in $\mathbb{S}^1$ such thing is impossible since any path connecting a point $x \in \mathbb{S}^1$ with itself necessarily has to go through the *whole* of $\mathbb{S}^1$ at least once in order to be complete. This idea gives rise to the concept of **fundamental group** which is a way of measuring the contractibility of a topological space. In the case of $\mathbb{S}^1$ it is well-known that its fundamental group $\pi_1(\mathbb{S}^1)$ is isomorphic to $\mathbb{Z}$, whereas those topological spaces whose fundamental group is trivial are called **contractible** (meaning that they can be "shrunk" into a single point).

Note that in order to be contractible a topological space has to be necessarily connected, but it is not a sufficient condition, as provided by the example with $\mathbb{S}^1$.

**Definition 1.16.** A matrix group $G$ is said to be **simply connected** if its fundamental group is trivial. That is, if any loop can be shrunk into a point.

**Example 1.2.2.3.** $\mathsf{SL}(n, \mathbb{C})$, $\mathsf{SU}(n)$, $\mathsf{USp}(n)$ and $\mathsf{Sp}(n, \mathbb{C})$ are simply connected matrix groups.

Of particular interest is the case $\mathsf{SU}(2)$. It can be proved that the points in $z := (z_1, z_2) \in \mathbb{C}^2$ such that $|z_1|^2 + |z_2|^2 = 1$, which can be thought as points in $\mathbb{S}^3$, can be represented as matrices in the form

$$z = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix},$$

which are precissely the elements of $\mathsf{SU}(2)$. Indeed, observe that

$$|-\bar{z_2}|^2 + |\bar{z_1}|^2 = |z_2|^2 + |z_1|^2 = 1$$

and that

$$\langle (z_1, z_2), (-\bar{z_2}, \bar{z_1}) \rangle = z_1(-\bar{z_2}) + z_2\bar{z_1} = -z_1z_2 + z_1z_2 = 0,$$

so the two columns form a hermitic basis and the identification $\mathsf{SU}(2) \cong \mathbb{S}^3$ follows. Hence, since we know from topology that $\mathbb{S}^3$ is a simply connected space, we get that $\mathsf{SU}(2)$ is simply connected.

In the other hand, a case of a non-simply connected matrix Lie group is $\mathsf{SO}(3)$, which can be identifyed with $\mathbb{S}^2/R$, where $R$ is the equivalence relation that identifies antipodal points, being that one of the classical constructions of the real projective space $\mathbb{P}^3_{\mathbb{R}}$, whose fundamental group is known to be $\mathbb{Z}/2$ ([N-P], sections 4.6 and 4.7, regarding $\pi_1$ as $H_1$, the 1st homology group).

## 1.3 Lie algebras

We end this first chapter studying the so-called Lie algebras, which are vector spaces endowed with an extra operation called Lie bracket, and its relation with Lie groups. In chapter 3 we will need them because in some sense, a Lie algebra tells us some things about its underlying Lie group that perhaps wouldn't be as easy to approach if working directly on the group itself (though for some cases it works the other way around). In the process of its definition we get to define the exponential of a matrix and pay a visit to the one-parameter subgroups.

### 1.3.1 The matrix exponential

Consider the discussions made on matrix norms in 1.2.1. Then, just like in $\mathbb{R}^n$ and $\mathbb{C}^n$, the notions of **Cauchy sequence** and **absolute convergence** hold for elements in $M_n(\mathbb{C})$. We want to define an analogous of the exponential function for matrix spaces, and thinking of $M_n(\mathbb{C})$ as a metric space there is a reasonable way to do so.

**Definition 1.17.** Let $A \in M_n(\mathbb{C})$. The **exponential matrix** of $A$, denoted $e^A$ or $\exp(A)$, is the matrix defined as

$$e^A := \sum_{k\geq 0} \frac{A^k}{k!}.$$

The exponential matrix is well-defined, as this is an absolutely convergent sequence of matrices. Indeed, use the properties seen in 1.2.1, and check the following:

$$\|e^A\| = \left\| \sum_{k\geq 0} \frac{A^k}{k!} \right\| \leq \sum_{k\geq 0} \frac{\|A^k\|}{k!} \leq \sum_{k\geq 0} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty.$$

The matrix exponential has the following properties:

(1). $e^A \in M_n(\mathbb{C})^*$.

(2). For $\alpha, \beta \in \mathbb{C}$, $e^{\alpha A}e^{\beta A} = e^{(\alpha+\beta)A}$.

(3). $e^0 = I$.

(4). $(e^A)^* = e^{A^*}$.

(5). If $A$ and $B$ commute, then $e^Ae^B = e^{A+B}$.

(6). $\|e^A\| \leq e^{\|A\|}$.

(7). If $C$ is invertible, then $Ce^AC^{-1} = e^{CAC^{-1}}$.

(8). If $J := \begin{pmatrix} J_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & J_r \end{pmatrix}$, $r \leq n$, then $e^J = \begin{pmatrix} e^{J_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{J_r} \end{pmatrix}$, where the $J_i$ are smaller matrices.

In particular, (7) and (8) tell us that one can compute the matrix exponential of $A$ by computing its Jordan canonical form $J$ together with the invertible eigenvector matrix $P$ and then use the equality

$$e^A = e^{PJP^{-1}} = Pe^JP^{-1}.$$

It is easy to check that the map $\exp : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \setminus \{0\}$ is continuous (as a consequence of property (6)) and injective.

Once the exponential matrix has been defined, it becomes natural to ask wether if we can get back the matrix $A$ from the matrix $e^A$ or not. The answer comes with the following definition.

**Definition 1.18.** For any matrix $A \in M_n(\mathbb{C})$, we define the **matrix logarithm** as the matrix

$$\log A := \sum_{k\geq 0} (-1)^{k+1} \frac{(A-I)^k}{k}.$$

Of course this is based in the complex logarithm in the same way the matrix exponential is based in the complex exponential. However, recall that in the complex case the $\log z$ series converges whenever $|z-1| < 1$, whereas here the series might be convergent even when this is not satisfied, as there are matrices such that $\|A\|^{m+1} < \|A\|^m$, e.g. nilpotent matrices. The map $\log : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ is also continuous.

It can be shown ([H], section 2.3) that $\log(I + A) = A + \mathcal{O}(\|A\|^2)$ and that $e^{\log A} = A$. This is useful for the theorems to come.

**Theorem 1.3.1.1** (Lie product formula)**.** For $X, Y \in M_n(\mathbb{C})$,

$$e^{X+Y} = \lim_{m\to\infty} \left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right)^m.$$

*Proof.* Performing the product $e^{\frac{X}{m}} e^{\frac{Y}{m}}$ yields a series in the form

$$e^{\frac{X}{m}} e^{\frac{Y}{m}} = I + \frac{X}{m} + \frac{Y}{m} + \mathcal{O}\left(\frac{1}{m^2}\right).$$

As $m$ grows, RHS gets close to $I$, so $\left\|e^{\frac{X}{m}} e^{\frac{Y}{m}} - I\right\| < 1$ for $m$ sufficiently large. Hence, we can perform the logarithm and write

$$\begin{aligned} \log\left(e^{\frac{X}{m}} e^{\frac{Y}{m}}\right) &= \log\left(I + \frac{X}{m} + \frac{Y}{m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right) \\ &= \frac{X}{m} + \frac{Y}{m} + \mathcal{O}\left(\frac{1}{m^2}\right). \end{aligned}$$

Exponentiating again, we get

$$e^{\frac{X}{m}} e^{\frac{Y}{m}} = \exp\left(\frac{X}{m} + \frac{Y}{m} + \mathcal{O}\left(\frac{1}{m^2}\right)\right),$$

and taking powers

$$\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m = \exp\left(X + Y + \mathcal{O}\left(\frac{1}{m}\right)\right).$$

As $m \longrightarrow \infty$ the $\mathcal{O}$ term fades and we get

$$e^{X+Y} = \lim_{m\to\infty}\left(e^{\frac{X}{m}}e^{\frac{Y}{m}}\right)^m.$$

□

.

**Theorem 1.3.1.2.** For each $A \in M_n(\mathbb{C})$, $\det(e^A) = e^{\mathrm{tr}(A)}$.

*Proof.* The strategy here is to prove the property for nilpotent and diagonalizable matrices because the general case follows then by noting that any matrix $X$ can be factorised into the product of a diagonalizable matrix $A$ and a nilpotent one $S$ which, additionally, commute. Therefore, if $X = SA$ and the property holds in these cases, then $e^X = e^{S+A}$, so

$$\det(e^X) = \det(e^{S+A}) = \det(e^S)\det(e^A) = e^{\mathrm{tr}(S)}e^{\mathrm{tr}(A)} = e^{\mathrm{tr}(S)+\mathrm{tr}(A)} = e^{\mathrm{tr}(X)}.$$

Now, if $A$ is diagonalizable, then there exists an invertible matrix $P$ such that $A = PDP^{-1}$, with $D$ diagonal. Then, $\det(e^A) = \det(Pe^DP^{-1}) = \det(e^D) = \prod_{i\leq n} e^{\lambda_i} = e^{\sum_{i\leq n}\lambda_i} = e^{\mathrm{tr}(D)} = e^{\mathrm{tr}(A)}$, where the $\lambda_i$ are the eigenvalues of $A$.

For a nilpotent matrix $S$, there exist an invertible matrix $C$ such that $S = CZC^{-1}$, where $Z$ is an upper-triangular matrix with zeros on the diagonal. Observe that if such is the case, then

$$e^Z = I + Z + \frac{Z^2}{2} + \cdots + \frac{Z^k}{k!} = \begin{pmatrix} 1 & \dots & t_1^n \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

Therefore, $e^S = Ce^ZC^{-1}$ and $\det(e^S) = \det(Ce^ZC^{-1}) = \det(e^Z) = 1 = e^0 = e^{\mathrm{tr}(S)}$. This completes our proof. □

### 1.3.2 One-parameter subgroups

We now study a class of real-valued matrix functions which sets us one step ahead of defining what a Lie algebra is.

**Definition 1.19.** A **one-parameter subgroup** of $\mathsf{GL}(n,\mathbb{C})$ is a function $\gamma : \mathbb{R} \longrightarrow \mathsf{GL}(n,\mathbb{C})$ satisfying the following:

(1). $\gamma$ is continuous.

(2). $\gamma(0) = I$.

(3). For each $s, t \in \mathbb{R}$, $\gamma(s+t) = \gamma(s)\gamma(t)$.

We want to prove that if $A(t)$ is a one-parameter subgroup then $A(t) = e^{tA}$ for a complex matrix $A$. For that purpose we need to set on some terminology first.

**Definition 1.20.** Just like a path (1.2.2), a **differential curve** in $M_n(\mathbb{K})$ is a function

$$\gamma : (a,b) \longrightarrow M_n(\mathbb{K})$$

for which the **derivative** exists for each $t \in (a,b)$. We define the derivative in the usual way as

$$\gamma'(t) := \lim_{s\to t}\frac{\gamma(s)-\gamma(t)}{s-t},$$

whenever this limit exists.

**Theorem 1.3.2.1.** The first order **differential equation**

$$\begin{aligned}\gamma'(t) &= A\gamma(t),\\ \gamma(0) &= C,\end{aligned}$$

$A, C \in M_n(\mathbb{K})$, has a unique solution $\gamma(t) = Ce^{tA}$. Furthermore, if $C$ is invertible, then so is $\gamma(t)$.

Details on that matter and a proof for this theorem lie in section 2.3 of [Bak]. We are interested on a direct consequence of it, as it provides a *good* definition for the Lie algebra of a Lie group.

**Proposition 1.3.2.2.** If $\gamma$ is a one-parameter subgroup, then $\gamma(t)$ is differentiable for each $t \in \mathbb{R}$ and

$$\gamma'(t) = \gamma'(0)\gamma(t) = \gamma(t)\gamma'(0).$$

*Proof.* Pick a small $\varepsilon \in \mathbb{R}$. Then, $\gamma(\varepsilon)\gamma(t) = \gamma(\varepsilon + t) = \gamma(t + \varepsilon) = \gamma(t)\gamma(\varepsilon)$. Therefore,

$$\begin{aligned}\gamma'(t) &= \lim_{\varepsilon\to 0}\frac{\gamma(t+\varepsilon)-\gamma(t)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\gamma(t)\gamma(\varepsilon)-\gamma(t)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{\gamma(\varepsilon)-I}{\varepsilon}\gamma(t)\\ &= \gamma'(0)\gamma(t).\end{aligned}$$

Analogously we can prove $\gamma'(t) = \gamma(t)\gamma'(0)$. □

**Theorem 1.3.2.3.** If $\gamma(t)$ is a one-parameter subgroup then there exists a unique matrix $A \in M_n(\mathbb{C})$ such that

$$\gamma(t) = e^{tA}.$$

*Proof.* Let $A = \gamma'(0)$. By proposition 1.3.2.2, $\gamma$ satisfies the differential equation

$$\begin{aligned}\gamma'(t) &= A\gamma(t),\\ \gamma(0) &= I.\end{aligned}$$

Then, by theorem 1.3.2.1, this has a unique solution $\gamma(t) = e^{tA}$, which completes the proof. □

Perhaps one might wonder what a one-parameter subgroup looks like. Let us give an example that, additionally, will clarify the link between them and matrix Lie groups.

**Example 1.3.2.4.** Set $t \in \mathbb{R}$ and consider a vector $v \in \mathbb{R}^2$. Recall that a rotation of angle $t$ comes defined by a matrix in $\mathsf{O}(2)$ of the form

$$R(t) = \begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix},$$

so the vector $u = R(t)v$ comes from $v$ by rotating it by an angle of $t$. Check that $R(0) = I$ and that $R(t)$ is continuous; furthermore, if $s, t \in \mathbb{R}$ are given angles, then

$$\begin{aligned}R(s+t) &= \begin{pmatrix}\cos(s+t) & -\sin(s+t)\\ \sin(s+t) & \cos(s+t)\end{pmatrix} = \begin{pmatrix}\cos s\cos t - \sin s\sin t & -\sin s\cos t - \cos s\sin t\\ \sin s\cos t + \cos s\sin t & \cos s\cos t - \sin s\sin t\end{pmatrix}\\ &= \begin{pmatrix}\cos s & -\sin s\\ \sin s & \cos s\end{pmatrix}\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix} = R(s)R(t).\end{aligned}$$

Thus, the group of rotations is a one-parameter subgroup. According to theorem 1.3.2.3 the expression of $R$ in terms of exponential matrices is $R(t) = e^{tJ}$, the exponent being the $J$ matrix we defined when talking about the symplectic groups. Sometimes, one-parameter subgroups are also called **one-parameter groups of transformations**.

### 1.3.3 Lie algebras associated to matrix Lie groups

Once we are done discussing one-parameter subgroups, let us define the Lie algebra associated to a Lie group. Lie algebras are important because we can work on them using just linear algebra, and it turns out than many features of the underlying Lie groups can be studied through its Lie algebra.

**Definition 1.21.** Set $G$ to be a matrix Lie group. The **Lie algebra** $\mathfrak{g}$ of $G$ (or $\mathrm{Lie}(G)$) is the set of all matrices $X \in M_n(\mathbb{C})$ such that $e^{tX} \in G$ for all $t \in \mathbb{R}$.

Lie algebras are often called **the set of infinitesimal group objects**, especially in physics, where they play an important role. In physicists' literature the distinction between Lie group and Lie algebra is not always well regarded.

Now that the definition is set, we want to know which are the Lie algebras associated to those matrix Lie groups introduced in section 1.1.1.

**Example 1.3.3.1** (General Linear group)**.** If $A \in \mathsf{GL}(n,\mathbb{R})$ all we ask is that $\det A \neq 0$, so if $e^{tX} = A$ for all $t$ then $\det(e^{tX}) = e^{\mathrm{tr}(tX)} \neq 0$, but this is true for all $t \in \mathbb{R}$ and all $X \in M_n(\mathbb{R})$, thus $\mathfrak{gl}(n,\mathbb{R}) = M_n(\mathbb{R})$. Analogously, we find that $\mathfrak{gl}(n,\mathbb{C}) = M_n(\mathbb{C})$. In particular, $\mathrm{Lie}(\mathbb{R}^*) = \mathbb{R}$ and $\mathrm{Lie}(\mathbb{C}^*) = \mathbb{C}$.

**Example 1.3.3.2** (Special Linear group)**.** We are looking now for matrices $X \in M_n(\mathbb{C})$ such that $\det(e^{tX}) = 1$ for each $t \in \mathbb{R}$, that is, $e^{\mathrm{tr}(tX)} = 1$. Then, $\mathrm{tr}(tX) = t \cdot \mathrm{tr}(X) = 2k\pi i$, $k \in \mathbb{Z}$. However, this must hold for all $t \in \mathbb{R}$, so it is actually only possible if $k = 0$. Hence, $\mathfrak{sl}(n,\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}$. Similarly, $\mathfrak{sl}(n,\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{tr}(X) = 0\}$.

**Example 1.3.3.3** (Orthogonal groups)**.** We saw in section 1.1.1 that $A \in \mathsf{O}(n)$ if, and only if, $A^T A = I$ which basically stands for $A^T = A^{-1}$. Hence, its Lie algebra will be formed by the real matrices $X$ such that $(e^{tX})^T = (e^{tX})^{-1}$, that is, $e^{tX^T} = e^{-tX}$. Of course, this will be fulfilled if $X^T = -X$, that is, if $X$ is **anti-symmetric**. Additionally, since this has to hold for each $t$, we just have to differentiate (using theorems 1.3.2.2 and 1.3.2.3) and evaluate at $t = 0$ to get that the condition is also necessary:

$$\left.\frac{d}{dt}\right|_{t=0} e^{tX^T} = \left.\frac{d}{dt}\right|_{t=0} e^{-tX} \Rightarrow \left. X^T e^{tX^T}\right|_{t=0} = \left. -Xe^{-tX}\right|_{t=0} \Rightarrow X^T = -X.$$

Therefore, $\mathfrak{o}(n)$, as well as $\mathfrak{so}(n)$, is the set of anti-symmetric matrices, $\mathsf{so}(n)$. Note that if $X \in \mathfrak{o}(n)$ then its trace has to be zero, so $\mathfrak{o}(n) \subset \mathfrak{sl}(n,\mathbb{R})$.

**Example 1.3.3.4** (Unitary groups)**.** Much like in the orthogonal case, for $\mathsf{U}(n)$ we end up getting that elements on its Lie algebra must satisfy $X^* = -X$, being the set of **anti-hermitic** matrices $\mathsf{su}(n)$. For the Special Unitary group, the condition $\mathrm{tr}(X) = 0$ adds to the previous one, so $\mathfrak{su}(n) \subset \mathfrak{u}(n)$.

**Example 1.3.3.5** (Symplectic groups)**.** As seen in section 1.1.1, we are looking for matrices $X$ such that $e^{tX}$ is skew-symmetric, that is, such that $(e^{tX})^T J e^{tX} = e^{tX^T} J e^{tX} = J$. This can be also written as $J^{-1} e^{tX^T} J = (e^{tX})^{-1}$ which finally yields $e^{tJ^{-1}X^T J} = e^{-tX}$. Analogous arguments as in the previous examples yield that this is only possible when $J^{-1} X^T J = -X$, so

$$\mathfrak{sp}(n;\mathbb{C}) = \left\{ X \in M_n(\mathbb{C}) \mid J^{-1} X^T J = -X \right\},$$

the same happening with $\mathbb{R}$ instead of $\mathbb{C}$. For $\mathsf{USp}(n)$, we add the condition $X^* = -X$, giving the set $\mathfrak{usp}(n)$ of matrices in the form

$$X = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix},$$

where $A$ is arbitrary and $B$ and $C$ are symmetric matrices so the new condition holds. Naturally, $\mathfrak{usp}(n) = \mathfrak{sp}(n;\mathbb{C}) \bigcap \mathfrak{u}(n)$.

**Example 1.3.3.6** (The Heisenberg group)**.** This is straightforward with what we saw in the proof of theorem 1.3.1.2, namely

$$\mathfrak{h} = \left\{ X \in M_3(\mathbb{R}) \mid X = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \right\} \cong \mathbb{R}^3,$$

as the exponential of an upper-triagular nilpotent matrix is an upper-triangular matrix with ones on its main diagonal (recall that this is seen by writing the first $k$ terms of the Taylor expansion).

### 1.3.4 Properties

Lie algebras possess a considerable amount of properties. We list and prove some of them in this section with the objective of proving that a Lie algebra is a vector space which is closed under a certain operation. This property will give Lie algebras its own entity, meaning that they need not to be necessarily attached to a Lie group.

**Proposition 1.3.4.1.** Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $A \in G$ and $X \in \mathfrak{g}$. Then

(1). $e^X$ lies in the component of $G$ containing $I$.

(2). $AXA^{-1} \in \mathfrak{g}$.

*Proof.* Property (1) is trivial if $G$ is connected, so let's assume it is not. Now, since $e^{tX}$ is a one-parameter subgroup and hence continuous, there is a continuous path connecting $e^X = e^{1X}$ with $I = e^{0X}$. For (2), let $M \in G$ such that $M = e^X$, then $G \ni AMA^{-1} = Ae^X A^{-1} = e^{AXA^{-1}}$, thus $AXA^{-1} \in \mathfrak{g}$. $\square$

**Example 1.3.4.2.** A known example illustrating property (1) comes from the field of real numbers thought as the Lie algebra of $\mathbb{R}^*$; indeed, for each $x \in \mathbb{R}$, $e^x \in \mathbb{R}_+$, which is the connected component of $\mathbb{R}^*$ containing the element 1. The same goes with the complex case, though unlike its real counterpart $\mathbb{C}^*$ is a connected Lie group.

**Corollary 1.3.4.3.** There is no element $X \in M_n(\mathbb{R})$ such that $\det(e^X) < 0$.

The following theorem gives Lie algebras their own algebraic structure aside from having its elements attached to elements of a certain Lie group.

**Theorem 1.3.4.4.** Let $G$ be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$. Then

(1). $sX \in \mathfrak{g}$ for each $s \in \mathbb{R}$.

(2). $X + Y \in \mathfrak{g}$.

(3). $XY - YX \in \mathfrak{g}$.

*Proof.* Property (1) is straightforward taking advantage of the exponential being a one-parameter subgroup: let $M(t) \subset G$ such that $M(t) = e^{tX}$ for each $t \in \mathbb{R}$, then $e^{t(sX)} = e^{(ts)X} = M(ts) \in G$ since $ts \in \mathbb{R}$. Property (2) is easy if $X$ and $Y$ commute; if not, the problem can be solved using the Lie product formula:

$$e^{t(X+Y)} = \lim_{m\to\infty} \left( e^{\frac{tX}{m}} e^{\frac{tY}{m}} \right)^m . \tag{1.1}$$

It is clear that $t/m \in \mathbb{R}$, so $e^{\frac{tX}{m}} e^{\frac{tY}{m}} \in G$ because $G$ is a group. Furthermore, $G$ is a *matrix Lie* group, so the limit 1.1 is in $G$ for each $t$, and it is still in when taking the $m$-th power. Thus, $e^{t(X+Y)} \in G$ for each $t$ and $X + Y \in \mathfrak{g}$. For property (3) we need to derive at zero and use the product rule so

$$\left.\frac{d}{dt}\right|_{t=0} e^{tX} Y = XY,$$

then

$$\left.\frac{d}{dt}\right|_{t=0} (e^{tX}Ye^{-tX}) = (XY)e^0 + e^0(Y)(-X)$$
$$= XY - YX.$$

Therefore, since part (2) of 1.3.4.1 grants that $e^{tX}Ye^{-tX} \in \mathfrak{g}$ and noting that (1) and (2) imply that $\mathfrak{g}$ is a real subspace of $M_n(\mathbb{C})$ and hence topologically closed, we get that

$$XY - YX = \lim_{h\to 0} \frac{e^{tX}Ye^{-tX} - Y}{h}$$

lies in $\mathfrak{g}$. □

**Definition 1.22.** The **commutator**, or Lie bracket, of a matrix Lie algebra $\mathfrak{g}$ is the map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ defined as

$$[A, B] = AB - BA.$$

According to theorem 1.3.4.4, a Lie algebra is closed under commutators. The commutator terminolgy comes from the fact that it somehow measures how much do two given matrices in $\mathfrak{g}$ commute, namely $[A, B] = 0$ if $A$ and $B$ commute. It is clear then that $[A, A] = 0$ for all $A \in \mathfrak{g}$.

**Proposition 1.3.4.5.** The commutator fulfills the Jacobi Identity ([C-Ll], proposition 8.4).

*Proof.* Take $X, Y, Z \in \mathfrak{g}$. Then

$$\begin{aligned}
[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] &= X\,[Y,Z] - [Y,Z]\,X + Y\,[Z,X] - [Z,X]\,Y + Z\,[X,Y] - [X,Y]\,Z \\
&= XYZ - XZY - YZX + ZYX \\
&+ YZX - YXZ - ZXY + XZY \\
&+ ZXY - ZYX - XYZ + YXZ = 0.
\end{aligned}$$

□

### 1.3.5 General Lie algebras

Using a generalization of the commutator, so we regardi it as a class of operation satisfying a series of properties, we may give rise to the following notion of Lie algebra.

**Definition 1.23.** A **Lie algebra** $\mathfrak{g}$ over a field $\mathbb{K}$ is a vector space endowed with an operation $[\cdot,\cdot]$ called **Lie bracket**, such that $[X, Y] \in \mathfrak{g}$ for each $X, Y \in \mathfrak{g}$. The Lie bracket must fulfill the following:

(1). $[\cdot,\cdot]$ is bilinear.

(2). $[X, Y] = -[Y, X]$ (skew-symmetry).

(3). $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, for each $X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

We shall note that the Jacobi Identity property for the Lie bracket makes Lie algebras such a general notion, as the commutator version (that is, $XY - YX$) is not the only possible way to define a Lie bracket satisfying it. In fact, this commutator needs not to make sense, as a Lie algebra might be a vector space with no product operation whatsoever. One such example is $\mathbb{R}^3$ with the usual vector product, which is known to satisfy the Jacobi identity.

Even if the elements of a Lie group $G$ have complex entries, note that elements of its Lie algebra need not to. For instance, $X \in \mathfrak{su}(n)$ must fulfill $X^* = -X$ so, in particular, $X \neq 0$. Now, is it true that if $X \in \mathfrak{su}(n)$ then $iX \in \mathfrak{su}(n)$? Observe that $iX^* = -iX$ which is true if and only if $X = 0$, so $iX$ is not in $\mathfrak{su}(n)$.

**Definition 1.24.** A Lie group $G$ is called **complex** if its Lie algebra $\mathfrak{g}$ is a complex vector subspace of $M_n(\mathbb{C})$.

Complex Lie groups and complex Lie algebras play an important role in the representation theory that we will be regarding starting at chapter 3.

From the scope of differential geometry, given a (general) Lie group $G$, its Lie algebra $\mathfrak{g}$ is the tangent space at the the identity element, $\mathfrak{g} := T_{id}G$, which we know to be a vector space, endowed with a Lie bracket. This leads us to the definition of dimension of a Lie group.

**Definition 1.25.** The **dimension** of a Lie group $G$ is the dimension of its Lie algebra as a real vector space.

We shall remark that the dimension is over $\mathbb{R}$, meaning that if the underlying manifold is complex, then the dimension should be regarded as two times the complex dimension. The notion of dimension, however, is the same regardless of wether if we treat Lie groups as differential manifolds or as matrix Lie groups.

**Example 1.3.5.1.** For the classical Lie groups we have the following:

- $\dim \mathsf{GL}(n,\mathbb{R}) = n^2$, $\dim \mathsf{GL}(n,\mathbb{C}) = 2n^2$.
- $\dim \mathsf{SL}(n,\mathbb{C}) = \dim_{\mathbb{R}} \{X \in M_n(\mathbb{C}) \mid \mathrm{tr}(X) = 0\} = 2(n^2-1)$.
- $\dim \mathsf{O}(n) = \dim_{\mathbb{R}} \left\{X \in M_n(\mathbb{R}) \mid X^T = -X\right\} = \binom{n}{2} = \frac{n^2-n}{2}$.
- $\dim \mathsf{U}(n) = \dim_{\mathbb{R}} \{X \in M_n(\mathbb{C}) \mid X^* = -X\} = 2\binom{n}{2} = n^2-1$.
- $\dim \mathsf{Sp}(n;\mathbb{C}) = \dim_{\mathbb{R}} \left\{X \in M_n(\mathbb{C}) \mid J^{-1}X^TJ = -X\right\} = 4\binom{n}{2} = 2n^2-2$.
- $\dim \mathsf{USp}(n) = \dim_{\mathbb{R}} \left\{X \in M_n(\mathbb{C}) \mid J^{-1}X^TJ = -X,\ X^* = -X\right\} = \dim \mathsf{Sp}(n;\mathbb{C}) - n = 2n^2-n-2$.

Regardless of being a definition inherited from Differental Geometry, we can set the dimension of a matrix Lie group to be the dimension of its Lie algebra as a real vector space without paying any attention to its geometric properties. Nonetheless, the notion of dimension from Differential Geometry is acquired by showing that the tangent space to a manifold in a point is isomorphic to the space of derivations of the algebra of smooth functions over the underlying manifold.

Moving in the background of general Lie groups and algebras a natural question may arise: is there a notion of exponential of the elements in a Lie algebra $\mathfrak{g}$ whatever their kind? Does it lie within the underlying Lie group, should it exist? We want to give a first hint on that matter here, though we will be talking about it widely in the next chapter.

**Definition 1.26.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The **exponential map** $\exp : \mathfrak{g} \longrightarrow G$ is the map such that, for a given $X \in \mathfrak{g}$, $\exp(X) = \gamma(1)$; where $\gamma : \mathbb{R} \longrightarrow G$ is a one-parameter subgroup whose tangent vector at the identity of $G$ is precisely $X$.

Actually, one can prove that $\gamma$ is unique, so we might as well say that the exponential map is defined by *the* one-parameter subgroup with such property.

The properties of the exponential map should match those of the exponential matrix and the analytic exponential function, namely $\exp(\mathfrak{g}) \subset G_I$, the connected component of $G$ containing the identity element;

$\exp(X+Y) = \exp(X)\exp(Y)$ (the aforementioned Trotter formula) and $\exp(0) = Id$; rendering the exponential map as a generalization of the exponential function. In fact, one can define the exponential map only in terms of manifolds and tangent spaces allowing it to go further than the scope of Lie groups and algebras.

Of course, this definition coincides with the one we gave for matrix Lie groups, and the link is clear thanks to the use we made of one-parameter subgroups.

The underlying theory of Differential Geometry discussed here can be conveniently explored in [Cur] and [DC].

# Chapter 2

# Morphisms of Lie groups and algebras

## 2.1 Morphisms and continuity

We begin our path towards representation theory by studying maps between Lie groups and its algebras. As one might expect, a Lie group homomorphism will be a group homomorphism which additionally preserves some kind of Lie-ness. Whereas the same goes for Lie algebra homomorphisms, our main goal is to regard the link between the two of them. Note that, unless the opposite is stated, we will be doing little difference between matrix Lie groups and general Lie groups.

The following discussions take [Qey] and [Lang] for the group matters and [McL] for the category issues as their main references.

### 2.1.1 Lie group and Lie algebra homomorphisms

**Definition 2.1.** Let $G$ and $H$ be Lie groups. A **Lie group homomorphism** is a map $\phi : G \longrightarrow H$ such that $\phi$ is a group homomorphism and, additionally, it is continuous with respect to the underlying topologies of $G$ and $H$.

Asking $\phi$ to be continuous is almost pure formality, as it is not easy to give an example of a group homomorphism which is not continuous (unless we are talking about finite groups), but regardless of that, we might grow interested about how does this property translates into our Lie group language. For instance, in the case of matrix Lie groups, we need morphisms to be continuous because it means that, given a convergent sequence $\{A_n\}_n \subset G$ with limit $A$, the sequence $\{\phi(A_n)\}_n \subset H$ has a limit $B$ which we can identify with $\phi(A)$, thus the Lie property is carried through the morphism.

In the more general background of Lie groups as manifolds, the continuity serves the purpose of conserving the differentiability of the product and inversion operations. This might sound surprisng as being continuous is weaker than being differentiable, but it can be proved that a continuous morphism between Lie groups is always differentiable, rendering this a notorious property of Lie groups ([H], proposition 1.20).

**Definition 2.2.** A Lie group homomorphism $\phi$ is called a **Lie group isomorphism** if $\phi$ is one-to-one and onto, and the map $\phi^{-1}$ is continuous.

Since $\phi^{-1}$ is a continuous group homomorphism, it is a Lie group homomorphism as well. Whenever an isomorphism between two (matrix) Lie groups $G$ and $H$ exists, we shall say that $G$ and $H$ are **isomorphic**. If that is the case, we will write $G \cong H$ as usual and regard both groups as essentially the same.

Note also that if $\phi_1 : G \longrightarrow H$ and $\phi_2 : H \longrightarrow K$ are Lie group homomorphisms, then, as continuity holds through composition, the map $\phi_1 \circ \phi_2 : G \longrightarrow K$ as a Lie group homomorphism. In particular, if $G = H = K$, the set of **Lie group automorphisms**, $\mathrm{Aut}(G)$, is a group itself with the composition operation.

Recall from general algebraic theory the notions of kernel and image of a map $\phi : G \longrightarrow H$, as well as the Isomorphism Theorem, stating that $\mathrm{Im}\phi \cong G/\ker\phi$.

**Proposition 2.1.1.1.** If $G$ and $H$ are matrix Lie groups and $\phi : G \longrightarrow H$ is a Lie group homomorphism, then $\ker\phi$ is a matrix Lie group.

*Proof.* Set $\{A_n\}_n \subset \ker\phi \subset G$ to be a converging matrix sequence. Since a Lie group homomorphism is a continuous map, we have

$$\phi\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \phi(A_n) = 0,$$

so $\lim\limits_{n\to\infty} A_n \in \ker\phi$. Therefore, $\ker\phi$ is a closed subgroup of $G$, which on its turn is a closed subgroup of $\mathsf{GL}(k,\mathbb{C})$, hence $\ker\phi$ is a matrix Lie group. □

**Definition 2.3.** A **Lie algebra homomorphism** is a linear map $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that for each pair $X, Y \in \mathfrak{g}$, $\psi[X,Y] = [\psi(X), \psi(Y)]$.

The following property is crucial for the forecoming theorem.

**Proposition 2.1.1.2.** Any Lie algebra homomorphism commutes with the derivative.

*Proof.* Set $\gamma(t) : [0,1] \longrightarrow \mathfrak{g}$ to be a differential curve and consider a Lie algebra homomorphism $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$. Since $\psi$ is continuous we can put it into limits and write

$$\psi(\gamma'(t)) = \psi\left(\lim_{s\to t}\frac{\gamma(s)-\gamma(t)}{s-t}\right) = \lim_{s\to t}\left(\psi\left(\frac{\gamma(s)-\gamma(t)}{s-t}\right)\right),$$

and, by linearity,

$$\begin{aligned} &= \lim_{s\to t}\left(\frac{1}{s-t}\psi\left(\gamma(s)-\gamma(t)\right)\right) \\ &= \lim_{s\to t}\frac{\psi(\gamma(s))-\psi(\gamma(t))}{s-t} = \left(\psi(\gamma(t))\right)'. \end{aligned}$$

□

Note that we are not using the Lie algebra homomorphism property. Indeed, the property holds for all continuous linear maps, but we are interested in its consequences for Lie algebra homomorphisms.

We are in the conditions to prove an important theorem:

**Theorem 2.1.1.3.** Set $\phi : G \longrightarrow H$ to be a Lie group homomorphism. Then $\phi$ induces a unique Lie algebra homomorphism $\psi : \mathfrak{g} \longrightarrow \mathfrak{h}$.

*Proof.* We have to show that $\psi$ preserves Lie brackets in accordance with definition 2.3. Consider a one-parameter subgroup $\gamma(t)$ of $G$. Then, for some $X \in \mathfrak{g}$, $\gamma(t) = e^{tX} \in G$ for each $t \in \mathbb{R}$, and $\phi(e^{tX}) \in H$. Since $\phi$ is continuous, $\phi(\gamma(t))$ is continuous; and since $\phi$ is a group homomorphism,

$$\phi(\gamma(t+s)) = \phi(\gamma(t)\gamma(s)) = \phi(\gamma(t))\phi(\gamma(s)),$$

and $\phi(\gamma(0)) = \phi(\mathrm{Id}) = \mathrm{Id}$; thus $\gamma \circ \phi$ is a one-parameter subgroup of $H$. If that is the case, then theorem 1.3.2.3 tells that there is a unique matrix $Z \in \mathfrak{h}$ such that $\phi(e^{tX}) = e^{tZ}$. Set then $Z = \psi(X)$, so $\phi(e^X) = e^{\psi(X)}$. This implies that, if $A \in G$ and $X \in \mathfrak{g}$, then we have the following:

$$\begin{aligned} e^{t\psi(Ae^X A^{-1})} &= \phi(Ae^{tX}A^{-1}) = \phi(A)\phi(e^{tX})\phi(A^{-1}) \\ &= \phi(A)e^{t\psi(X)}\phi(A)^{-1}. \end{aligned}$$

Differentiating at $t = 0$, this yields $\psi(Ae^X A^{-1}) = \phi(A)\psi(X)\phi(A)^{-1}$. Now, in order to check that $\psi$ preserves Lie brackets we must follow the strategy used in theorem 1.3.4.4, namely writing

$$[X, Y] = \frac{d}{dt} e^{tX} Y e^{-tX}\bigg|_{t=0}.$$

Since $\psi$ is a linear map, it commutes with the derivative, so

$$\begin{aligned} \psi\,[X, Y] &= \psi\left(\frac{d}{dt} e^{tX} Y e^{-tX}\bigg|_{t=0}\right) = \frac{d}{dt}\psi\left(e^{tX} Y e^{-tX}\right)\big|_{t=0} \\ &= \frac{d}{dt}\,\phi(e^{tX})\psi(Y)\phi(e^{-tX})\big|_{t=0} = \frac{d}{dt}\,e^{t\psi(X)}\psi(Y)e^{-t\psi(X)}\bigg|_{t=0} \\ &= [\psi(X), \psi(Y)]\,. \end{aligned}$$

For the uniqueness, suppose that there was another map $\psi'$ induced by $\phi$, then $\phi(e^{tX}) = e^{t\psi(X)} = e^{t\psi'(X)}$, and differentiating at $t = 0$ we get $\psi(X) = \psi'(X)$, so the induced homomorphism is unique. □

From the scope of Differential Geometry, $\psi$ is nothing else than the differential map taken at the identity element of $G$,

$$d_{id}\phi : \mathfrak{g} = T_{id}G \longrightarrow T_{id}H = \mathfrak{h}.$$

For that reason, we will be writting $d\phi$ whenever we refer to the Lie algebra homomorphism induced by the Lie group homomorphism $\phi$. Note that viewing the induced Lie algebra homomorphism as a differential map the above description of the Lie bracket as a derivative becomes much more natural.

Theorem 2.1.1.3 has been developed with the implicit languaje of matrix Lie groups, although the statement talks about plain Lie groups. The fact is that the proof goes the same way for general Lie groups replacing the matrix exponential by the exponential mapping defined in section 1.3.5. We are to talk about this map in the next section.

### 2.1.2 The adjoint map and further properties

We have already defined the exponential map in the end of section 1.3.5, now it's the time to study it in a more detailed way and to take a look at its properties, especially those concerning Lie group and algebra homomorphism. Prior to that, though, we shall define another map.

**Definition 2.4.** Let $G$ be a matrix Lie group with Lie algebra $\mathfrak{g}$. For each $A \in G$ we define the map

$$\mathrm{ad}_A : \mathfrak{g} \longrightarrow \mathfrak{g}$$

through the formula $\mathrm{ad}_A(X) = AXA^{-1}$. This is called the **adjoint map** of $A$.

The properties shown in proposition 1.3.4.1 grant that $\mathrm{ad}_A(X) \in \mathfrak{g}$, so this map is well-defined. The adjoint map has some notorious properties, namely stability through Lie brackets. Take $A \in G$, then

$$\mathrm{ad}_A\,[X, Y] = \mathrm{ad}_A\left(\frac{d}{dt} e^{tX} Y e^{-tX}\bigg|_{t=0}\right).$$

Observe that $\mathrm{ad}_A$ is a linear transformation, so we can put it into the derivative

$$\begin{aligned}\mathrm{ad}_A[X,Y] &= \frac{d}{dt}\left(\mathrm{ad}_A\left(e^{tX}Ye^{-tX}\right)\right)\Big|_{t=0} = \frac{d}{dt}\left(Ae^{tX}Ye^{-tX}A^{-1}\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(Ae^{tX}A^{-1}AYA^{-1}Ae^{-tX}A^{-1}\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(e^{tAXA^{-1}}AYA^{-1}e^{-tAXA^{-1}}\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(e^{t\mathrm{ad}_A(X)}\mathrm{ad}_A(Y)e^{-t\mathrm{ad}_A(X)}\right)\Big|_{t=0} = [\mathrm{ad}_A(X), \mathrm{ad}_A(Y)]\,.\end{aligned}$$

We have proved the following:

**Proposition 2.1.2.1.** For each $A \in G$, $\mathrm{ad}_A$ is a Lie algebra homomorphism.

Now, $\mathfrak{g}$ is a real vector space, so the set of invertible transformations of $\mathfrak{g}$ is a group and can be regarded as some $\mathsf{GL}(k,\mathbb{R})$, so this group, denoted as $\mathsf{GL}(\mathfrak{g})$, is a Lie group itself. We can consider then the map

$$\begin{aligned}\mathrm{ad}: G &\longrightarrow \mathsf{GL}(\mathfrak{g})\\ A &\longmapsto \mathrm{ad}_A\end{aligned}$$

which is continuous and satisfies, for a given $X \in \mathfrak{g}$,

$$\begin{aligned}\mathrm{ad}(AB) = \mathrm{ad}_{AB}(X) &= ABX(AB)^{-1} = ABXB^{-1}A^{-1}\\ &= A\mathrm{ad}_B(X)A^{-1} = \mathrm{ad}_A(\mathrm{ad}_B(X)) = \mathrm{ad}(A)\mathrm{ad}(B),\end{aligned}$$

where in the RHS we understand the "product" operation as the map composition. Therefore, ad is a Lie group homomorphism, then theorem 2.1.1.3 grants that there is an associated Lie algebra homomorphism

$$\begin{aligned}\mathfrak{ad}: \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g})\\ X &\longmapsto \mathfrak{ad}_X\end{aligned}$$

satisfying that $e^{\mathfrak{ad}_X} = \mathfrak{ad}(e^X)$. Check that $\mathfrak{gl}(\mathfrak{g}) \cong \mathrm{End}(\mathfrak{g})$, the group of endomorphisms of the vector space $\mathfrak{g}$.

**Proposition 2.1.2.2.** In the conditions and notations of the above we have that

$$\mathfrak{ad}_X(Y) = [X,Y]\,.$$

*Proof.* It will be easier if we take hand of the differential geometry language here. Since $\mathfrak{ad}$ is the differential map of ad, we can write

$$\mathfrak{ad}_X = \frac{d}{dt}\mathrm{ad}(e^{tX})\Big|_{t=0}.$$

Therefore,

$$\begin{aligned}\mathfrak{ad}_X(Y) &= \frac{d}{dt}\mathrm{ad}(e^{tX})(Y)\Big|_{t=0}\\ &= \frac{d}{dt}e^{tX}Ye^{-tX}\Big|_{t=0} = [X,Y]\,.\end{aligned}$$

□

**Corollary 2.1.2.3.** For any $X \in \mathbb{C}$, let $\mathfrak{ad}_X : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ be given by $\mathfrak{ad}_X(Y) = [X,Y]$, $Y \in M_n(\mathbb{C})$. Then, $e^{\mathfrak{ad}_X}(Y) = \mathrm{ad}_{(e^X)}(Y) = e^X Y e^{-X}$.

We come back now to the exponential map to study some of its aspects.

**Definition 2.5.** Given a Lie group $G$ with Lie algebra $\mathfrak{g}$ the **exponential mapping** is the map

$$\exp : \mathfrak{g} \longrightarrow G.$$

Much can be said about the exponential mapping that is out of our interests here. For instance, it can be proved ([H], section 2.7) that exp is a local homeomorphism between a Lie group and its Lie algebra, with a local inverse called, of course, the **logarithm mapping**.

It can also be proved that if $G$ is a connected matrix Lie group, then for each $A \in G$, there exists a finite sequence $X_1, \ldots, X_m \in \mathfrak{g}$ such that

$$A = e^{X_1} \cdots e^{X_m}.$$

These tools lead to the following theorem which is important in the differential geometry approach of Lie groups.

**Theorem 2.1.2.4.** Every matrix Lie group is a smooth embedded submanifold of $\mathsf{GL}(n,\mathbb{C})$. That is, every matrix Lie group is a Lie group.

Despite our disregard for the proof of this theorem, it spans a corollary for which we care much about, because it tells us about the functoriality (or lack of) between Lie group homomorphisms and Lie algebra homomorphisms that we will be studying in the next section of this chapter.

**Corollary 2.1.2.5.** If $G$ is a connected matrix Lie group and $H$ is a matrix Lie group, and we have two Lie group homomorphisms $\phi_1$ and $\phi_2$ that map $G$ onto $H$, let $d\phi_1$ and $d\phi_2$ be the (unique) associated Lie algebra homomorphisms. Then, if $d\phi_1 = d\phi_2$, we have that $\phi_1 = \phi_2$.

*Proof.* Since $G$ is connected, then for $A \in G$ we may write $A = e^{X_1} \cdots e^{X_m}$, so

$$\begin{aligned}
\phi_1(A) &= \phi_1(e^{X_1} \cdots e^{X_m}) = \phi_1(e^{X_1}) \cdots \phi_1(e^{X_m}) \\
&= e^{d\phi_1(X_1)} \cdots e^{d\phi_1(X_m)} = e^{d\phi_2(X_1)} \cdots e^{d\phi_2(X_m)} \\
&= \phi_2(e^{X_1}) \cdots \phi_2(e^{X_m}) = \phi_2(e^{X_1} \cdots e^{X_m}) \\
&= \phi_2(A).
\end{aligned}$$

□

The fact that exp is just a local homeomorphism hints that the relation between Lie groups and Lie algebras is not one-to-one, but corollary 2.1.2.5 gives the idea under the condition of connectedness it may be. As stated above, we will regard this fact in the next section of this chapter.

### 2.1.3 Structure of finite-dimensional Lie algebras

We end this section learning that, unlike Lie groups, any Lie algebra is a matrix algebra (Ado's theorem).

**Proposition 2.1.3.1.** The spaces $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ are, respectively, real and complex Lie algebras with respect to the commutator version of the Lie bracket operation. If $V$ is a finite dimensional real or complex vector space, then $\mathfrak{gl}(V)$ is a real or complex Lie algebra with respect to the same Lie bracket operation.

*Proof.* The first part of the statement has been already proved in proposition 1.3.4.5, so let $V$ be a real (resp. complex) vector space. Then, $\mathfrak{gl}(V)$ is the vector space of endomorphisms of $V$, which can be thought as a matrix space. Since both $M_n(\mathbb{R})$ and $M_n(\mathbb{C})$ are closed under Lie brackets, the only thing that remains to be proved is that the Lie bracket taken over $\mathfrak{gl}(V)$ remains within $\mathfrak{gl}(V)$, but this is true, as if $A, B \in \mathfrak{gl}(V)$ are linear maps over $V$, then $AB - BA = [A, B] \in \mathfrak{gl}(V)$. □

**Proposition 2.1.3.2.** The Lie algebra $\mathfrak{g}$ of a matrix Lie group $G$ is a real Lie algebra.

*Proof.* Here comes again a time when using the language of differential geometry comes in handy. Indeed, a Lie group $G$ can always be thought as a real differential manifold, whose tangent space at the identity is its Lie algebra, which in such case is always a real vector space. Thus, $\mathfrak{g}$ is a real Lie algebra. □

We are goign to state Ado's theorem, which is the structure theorem for finite-dimensional Lie algebras. Let us first define the concept of Lie subalgebra.

**Definition 2.6.** A **Lie subalgebra** $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ is a vector subspace closed under the Lie bracket inherited from $\mathfrak{g}$.

**Theorem 2.1.3.3** (Ado)**.** Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, and every finite-dimensional complex Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

The proof is beyond the scope of this master thesis, but check that it tells us that every Lie algebra can be realised as a matrix Lie algebra, even when it is a Lie algebra of a Lie group which is not.

We can improve that theorem stating that any Lie algebra is isomorphic to some complex Lie algebra via a process known as the complexification of a real Lie algebra.

**Definition 2.7.** Given a real vector space $V$, its **complexification** is the space $V_{\mathbb{C}}$ of formal linear combinations $v_1 + iv_2$ of elements $v_1, v_2 \in V$ with the structure of complex vector space given by the relation

$$i(v_1 + iv_2) = -v_2 + iv_1.$$

**Theorem 2.1.3.4.** If $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ is its complexification as a vector space, then the Lie bracket operation in $\mathfrak{g}$ admits a unique extension into a Lie bracket in $\mathfrak{g}_{\mathbb{C}}$.

The proof for this is straightforward and can be found in [H] (proposition 2.44). One only has to check bilinearity, skew-symmetry and, of course, that the Jacobi identity holds. Then, the uniqueness follows from the bilinearity because the Lie bracket has to satisfy

$$[X_1 + iX_2, Y_1 + iY_2] = [X_1, Y_1] - [X_2, Y_2] + i([X_1, Y_1] - [X_2, Y_2]).$$

We call $\mathfrak{g}_{\mathbb{C}}$ the **complexification** of the real Lie algebra $\mathfrak{g}$.

**Corollary 2.1.3.5.** Every finite-dimensional Lie algebra can be realised as a complex matrix subalgebra of $\mathfrak{gl}(n, \mathbb{C})$.

This is a nice result, for it tells that studying Lie algebras is not just studying vector spaces with an additional operation, but it is actually studying complex matrix vector spaces.

## 2.2 Functoriality

This section uses the language of categories in order to find out how much information can be passed from a Lie group to its Lie alegbra. More precissely, we are interested in checking if there is a faithful functor between the category of Lie groups and the category of Lie algebras and, should it not be, what additional conditions are needed in order to obtain a faithful one. The basic and not-so-basic categorical results can be found in [McL] in a rather deep and nice fashion.

### 2.2.1 The LieGr and LieAl categories

Recall that a category $\mathcal{A}$ consists of a class $\mathrm{Ob}(\mathcal{A})$ formed by the objects of $\mathcal{A}$, a class $\mathrm{Hom}(A,B)$, $\forall A,B \in \mathrm{Ob}(\mathcal{A})$, containing the morphisms or "arrows" between the objects of $\mathcal{A}$ together with a composition map for each ordered triple $A,B,C \in \mathrm{Ob}(\mathcal{A})$:

$$\begin{aligned}\mathrm{Hom}(A,B)\times\mathrm{Hom}(B,C) &\longrightarrow \mathrm{Hom}(A,C)\\ (f,g) &\longmapsto gf.\end{aligned}$$

The following must be satisfied:

(1). If $D \in \mathrm{Ob}(\mathcal{A})$ and $h \in \mathrm{Hom}(C,D)$, then $h(gf) = (hg)f$.

(2). For each $A \in \mathrm{Ob}(\mathcal{A})$, there is a unique element $1_A \in \mathrm{Hom}(A,A)$ such that $f1_A = f \in \mathrm{Hom}(A,B)$ and $1_A g = g \in \mathrm{Hom}(B,B)$. This is called the **identity element**.

**Definition 2.8. Lie group**s form a **category** LieGr whose objects are all Lie groups and whose arrows are Lie group morphisms.

This definition is correct in the sense that it fulfills all the above properties, namely there is a composition law which takes pairs of Lie group homomorphisms and produces a new Lie group homomorphism. This law is associative and has a unique identity map.

**Definition 2.9.** A subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is formed by a subcollection $\mathrm{Ob}(\mathcal{B})$ of objects of $\mathcal{A}$ and a subcollection $\mathrm{Hom}(\mathcal{B})$ of morphisms of $\mathcal{B}$ such that

(i). For each $X \in \mathrm{Ob}(\mathcal{B})$, $id_X \in \mathrm{Hom}(\mathcal{B})$.

(ii). For each morphism $f : X \longrightarrow Y \in \mathrm{Hom}(\mathcal{B})$, both $X$ and $Y$ lie within $\mathrm{Ob}(\mathcal{B})$.

(iii). For each pair $f,g \in \mathrm{Hom}(\mathcal{B})$, the composition $f \circ g$ lies within $\mathrm{Hom}(\mathcal{B})$.

**Proposition 2.2.1.1.** Compact Lie groups are a subcategory of LieGr. We denote this subcategory as CpLieGr.

*Proof.* We need to check that the composition of Lie group morphisms between compact Lie groups is again a Lie group morphism between compact Lie groups. Recall that a Lie group is compact if it is closed and bounded under certain norm $\|\cdot\|$, so consider morphisms $f,g \in \mathrm{Hom}(\mathrm{CpLieGr})$ and take $A,B,C \in \mathrm{Ob}(\mathrm{CpLieGr})$. Then, for each $X \in A$ such that $\|X\| < \varepsilon$, $f(X) \in B$ satisifies $\|f(X)\| < \delta_\varepsilon$, as $f$ is continuous. In the same fashion, the continuity of $g$ grants that $\|g(f(X))\| < \phi_\varepsilon$, $g(f(X)) \in C$. Therefore, $gf$ is continous and hence $gf \in \mathrm{Hom}(\mathrm{CpLieGr})$.

Now, the associatity is inherited from the associativity in Hom(LieGr) and the existance of the identity element is also inmediate because the map $1_A : A \longrightarrow A$ is obviously continuous. □

**Definition 2.10.** The **Lie $\mathbb{K}$-algebra category** ($\mathbb{K}$ a field) LieAl has as objects Ob(LieAl) all Lie algebras over $\mathbb{C}$ and as morphisms all linear maps which are Lie bracket-preserving.

We know from linear algebra that linear maps are closed under compositions, associative and possess an identity element, so the only thing we have to check is that a composition of Lie bracket-preserving linear maps is also Lie bracket-preserving: pick a pair $\phi$ and $\psi$ of Lie bracket-preserving linear maps, then

$$\begin{aligned}\psi\phi\,[X,Y] &= \phi\,[\psi(X),\psi(Y)] = [\phi(\psi(X)),\phi(\psi(Y))]\\ &= [\psi\phi(X),\psi\phi(Y)]\,.\end{aligned}$$

Thus, $\psi\phi \in \mathrm{Hom}(\mathrm{LieAl})$. We have proved the following:

**Corollary 2.2.1.2.** LieAl is a subcategory of Vsp, the category of vector spaces over a field.

### 2.2.2 The Lie functor

In a rough sense, a functor is a "category morphism" (stated in [McL], section 1.3), being that a name we can use in order to understand the concept but not as a technical name. The reason is that the whole point of defining categories is to be able to handle objects which in some sense generalize the concept of set by adding the arrows, which actually translates into the notion of morphism itself and hence it is unclear what a category morphism is. At least, unless one considers a "category of categories", which is sometimes called a (or *the*) **metacategory** ([McL] again), where functors can be viewed as the arrows in the metacategory.

**Definition 2.11.** A **functor** $T$ between two categories $\mathcal{A}$ and $\mathcal{B}$ consists of two related maps: for each object $c \in \mathrm{Ob}(\mathcal{A})$ there is an object $Tc \in \mathrm{Ob}(\mathcal{B})$; and for each arrow $f \in \mathrm{Hom}(\mathcal{A})$ there is an arrow $Tf \in \mathrm{Hom}(\mathcal{B})$ such that

(i). $T(1_A) = 1_{\mathcal{B}}$.

(ii). For each $f, g \in \mathrm{Hom}(\mathcal{A})$, $T(f \circ g) = Tf \circ Tg \in \mathrm{Hom}(\mathcal{B})$.

We will denote a functor with the classical map notation as $T : \mathcal{A} \longrightarrow \mathcal{B}$, with the meaning given in the above definition, that is, understanding that it does not only refers to the objects of the categories but also to their morphisms.

Much like classical morphisms, functors can be **composed** to trigger new functors; namely, if $T : \mathcal{A} \longrightarrow \mathcal{B}$ and $T : \mathcal{B} \longrightarrow \mathcal{C}$ are two functors, the composite functor $T \circ S : \mathcal{A} \longrightarrow \mathcal{C}$ sends objects in $\mathcal{A}$ to objects in $\mathcal{C}$, and arrows in $\mathcal{A}$ to arrows in $\mathcal{C}$. There is, of course, an identity functor $1_{\{\cdot\}}$ which acts as the identity both on objects and arrows.

With that composition one can define a **category isomorphism**, being a functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ such that there exists another functor $S : \mathcal{B} \longrightarrow \mathcal{A}$ with the property that $T \circ S = 1_{\mathcal{A}}$ and $S \circ T = 1_{\mathcal{B}}$. This is not an interesting property as it only provides tautologies. That's because, in a formal level, stating that the category $\mathcal{A}$ is isomorphic to the category $\mathcal{B}$ pretty much means that they are the same thing, as we are saying that objects in $\mathcal{A}$ are isomorphic to objects in $\mathcal{B}$, which translates into saying that an object $Tc \in \mathrm{Ob}(\mathcal{B})$ coming from $c \in \mathrm{Ob}(\mathcal{A})$ via the isomorphism $T$ actually lies within the objects of $\mathcal{A}$ and viceversa using the functor $S$, because, in the end, if two objects are isomorphic, then they belong to the same class of objects and thus to the same category.

**Definition 2.12.** The **Lie functor** is that one functor $\mathrm{Lie} : \mathrm{LieGr} \longrightarrow \mathrm{LieAl}$ such that if $G \in \mathrm{Ob}(\mathrm{LieGr})$, then $\mathfrak{g} := \mathrm{Lie}(G) := T_{id}G \in \mathrm{Ob}(\mathrm{LieAl})$. Additionally, if $\phi \in \mathrm{Hom}(\mathrm{LieGr})$, then $\mathrm{Lie}(\phi) := d\phi \in \mathrm{Hom}(\mathrm{LieAl})$.

Of course this is not an isomorphism, as a natural candidate for an inverse functor $\mathrm{Lie}^{-1}$ arises from the exponential map through its already known property $\exp(\mathfrak{g}) = G_I \subsetneq G$. Then, it is clear that $\mathrm{Lie} \circ \mathrm{Lie}^{-1}$ is not the identity on LieGr since it fails to be so on nonconnected Lie groups. Nonetheless, there exist weaker conditions than that of a category isomorphism that a functor can fulfill, and we are interested in which one of these are satisfied by the Lie functor.

**Definition 2.13.** A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be **full** if for every pair of objects $a, b$ of $\mathcal{A}$ and for every arrow $g : Ta \longrightarrow Tb$ of $\mathcal{B}$, there is an arrow $f : a \longrightarrow b$ of $\mathcal{A}$ such that $g = Tf$.

Matters on the fullness of the Lie functor need a theory which lies beyond what we have done to this point.

**Definition 2.14.** A functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is said to be **faithful** if for every pair of objects $a, b$ of $\mathcal{A}$ and for every pair of arrows $f, g : a \longrightarrow b$ of $\mathcal{A}$, the equality $Tf = Tg$ implies $f = g$.

A functor that is both full and faithful, sometimes called a **fully faithful** functor, need not be an isomorphism, as there may be objects in $\mathcal{A}$ which are not within the image of $T$.

With the things we know, the Lie functor **is not** a faithful functor, and this comes also from the connectedness issue. Consider two nonconnected Lie groups $G$ and $H$ and their respective identity connected components $G_I$ and $H_I$, which are Lie groups themselves. Then, consider two Lie group homomorphisms $\phi, \psi : G \longrightarrow H$ such that $\phi \neq \psi$ but $\phi_{|_{G_I}} \equiv \psi_{|_{G_I}}$. It is then clear that

$$\mathrm{Lie}(\phi) = d_{id}\phi = \mathrm{Lie}\left(\phi_{|_{G_I}}\right) = \mathrm{Lie}\left(\psi_{|_{G_I}}\right) = d_{id}\psi = \mathrm{Lie}(\psi),$$

so this provides a counterexample of the faithfulness of the Lie functor.

### 2.2.3 Restricting the Lie functor

**Definition 2.15.** SCLie $\subset$ Lie is the full subcategory of simply connected Lie groups.

**Theorem 2.2.3.1.** The **restricted** Lie **functor**, Lie : SCLie $\longrightarrow$ LieAl, is (fully) faithful.

This is a deep theorem that can be translated into saying that if $G$ and $H$ are two simply connected Lie groups and $f : \mathfrak{g} \longrightarrow \mathfrak{h}$ is a Lie algebra homomorphism then there exists a **unique** Lie group homomorphism $\phi : G \longrightarrow H$ such that $f(\exp X) = \exp(f(X))$ for each $X \in \mathfrak{g}$. This is theorem 3.7 in [Bak].

**Corollary 2.2.3.2.** A Lie algebra isomorphism induces an isomorphism between Lie groups if these Lie groups are simply connected.

All this boils down to saying that, in order to study simply connected Lie groups, we just have to study their associated Lie algebras or, putting it simple, we can deal with simply connected Lie groups by doing linear algebra. Of course, we have in particular that classifying simply connected Lie group homomorphisms is essentially classifying endomorphisms (via the restricted Lie functor).

In the next section, however, we will learn that simply connected groups are not the only classifiable Lie groups but a mean to achieve a wider classification, though a general classification cannot be performed.

## 2.3 Coverings

Once we know that the restricted Lie functor is fully faithful we begin our path towards Lie group representation theory. Since a Lie algebra homomorphism induces a Lie group homomorphism if those groups are simply connected it is reasonable to associate to a Lie group which is not simply connected another group which is. That group is the covering space of the original one, and hence we are going to study coverings in this section.

### 2.3.1 Topological background

The following topics, definitions and properties are taken from [Ms]. The omitted proofs can be found in that reference.

**Definition 2.16.** Let $X$ be a locally path-connected topological space. A **covering** of the space $X$ is a pair $(\tilde{X}, p)$ where $\tilde{X}$ is another topological space and $p$ is a continuous map with the property that for each $x \in X$ there is a path-connected neigbourhood $U$ of $x$ such that the map $p : p^{-1}(U) \longrightarrow U$ is a homeomorphism.

Such neighbourhoods $U$ are called **natural neighbourhoods** whereas the map $p$ is often called a **projection**.

**Example 2.3.1.1.** Consider $X = \mathbb{S}^1$ and $p : \mathbb{R} \longrightarrow \mathbb{S}^1$ given by $p(t) = (\sin t, \cos t)$. Then $(\mathbb{R}, p)$ is a covering of $\mathbb{S}^1$ since for each point $y := (y_1, y_2) \in \mathbb{S}^1$ there is an arc $K \subset \mathbb{S}^1$ containing $y$ such that $p^{-1}(K)$ is a countable collection of open intervals of $\mathbb{R}$, namely $p^{-1}(K) = (\arcsin y_1, k\pi - \arccos y_2)$, $k \in \mathbb{Z}$, which are path-connected subsets of $\mathbb{R}$ that are (locally) homeomorphic to $K$ (see figure 2.1).

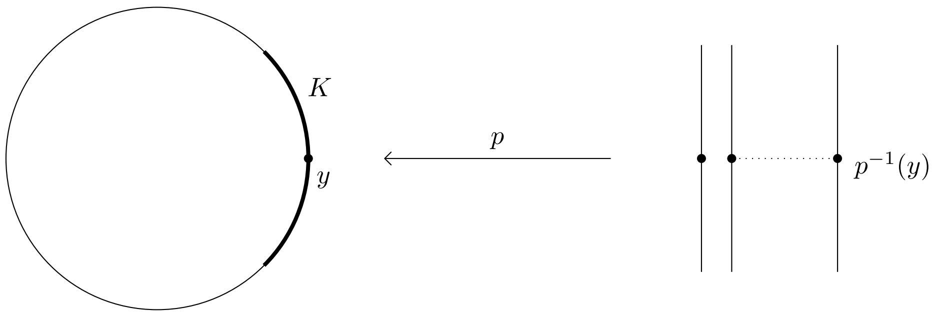



Figure 2.1: Covering of $\mathbb{S}^1$. The preimage of the ark $K$ is a countable family of copies of $\mathbb{R}$.

The following properties of covering spaces hold. We are omitting the proofs here, but they can be found in the given reference.

**Proposition 2.3.1.2.** Let $(\tilde{X}, p)$ be a covering space of $X$, $A \subset X$ a topological subspace and $\tilde{A}$ a path-connected component of $p^{-1}(A)$. Then $(\tilde{A}, p|_A)$ is a covering space of $A$.

Recall that we are interested in the connectedness of the covering space regarding the connectedness of the underlying space. This property is studied through a process known as **lifting paths**.

**Proposition 2.3.1.3.** Let $(\tilde{X}, p)$ be a covering space of $X$ and $\tilde{x_0} \in \tilde{X}$ a point. Consider $x_0 := p(\tilde{x}_0)$ and a path $f : [0,1] \longrightarrow X$ with $f(0) = x_0$. Then, there is a unique path $g : [0,1] \longrightarrow \tilde{X}$ with $g(0) = \tilde{x}_0$ such that $f = p \circ g$.

**Proposition 2.3.1.4.** If $f$ and $g$ are paths in $\tilde{X}$ such that $p \circ f \sim p \circ g$ in $X$, then $f \sim g$. In particular, $f(1) = g(1)$.

**Proposition 2.3.1.5.** If $(\tilde{X}, p)$ is a covering space of $X$ then the cardinality of $p^{-1}(x)$ is independent of $x \in X$. This cardinality $n = 1, 2, \dots, \infty$ is called the **number of leaves** of the covering.

As a consequence of this last proposition we have the following important theorem:

**Theorem 2.3.1.6.** Let $(\tilde{X}, p)$ be the covering space of $X$, let $\tilde{x}_0 \in \tilde{X}$ and $x_0 := p(\tilde{x}_0) \in X$, and consider the respective fundamental groups $\pi_1(\tilde{X}, \tilde{x}_0)$ and $\pi_1(X, x_0)$. Then, the induced group homomorphism

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$$

is a monomorphism.

Since there might be different points $\tilde{x}_0$ and $\tilde{x}_1$ in $\tilde{X}$ such that $p(\tilde{x}_0) = p(\tilde{x}_1)$, it is natural to ask if we can compare the respective induced group homomorphisms

$$p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$$
$$p_* : \pi_1(\tilde{X}, \tilde{x}_1) \longrightarrow \pi_1(X, x_0)$$

We can choose a class of paths $\gamma$ between $\tilde{x}_0$ and $\tilde{x}_1$, which induces an isomorphism

$$u : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(\tilde{X}, \tilde{x}_1)$$

through $u(\alpha) = \gamma^{-1}\alpha\gamma$. Therefore we have the following commutative diagram

$$\begin{tikzcd}
\pi_1(\tilde{X}, \tilde{x}_0) \arrow[r, "p_*"] \arrow[d, "u"'] & \pi_1(X, x_0) \arrow[d, "v"] \\
\pi_1(\tilde{X}, \tilde{x}_1) \arrow[r, "p_*"] & \pi_1(X, x_0)
\end{tikzcd}$$

where $v(\beta) := (p_* \circ \gamma)^{-1}\beta(p_* \circ \gamma)$ is the class of a closed path. Therefore, we get that the image sets of $\pi_1(\tilde{X}, \tilde{x}_0)$ and $\pi_1(\tilde{X}, \tilde{x}_1)$ by $p_*$ are conjugated subgroups (see [Lang] for details) of $\pi_1(X, x_0)$. Moreover, observe that each subgroup of this class is in the form $\alpha^{-1}\left(p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)\right)\alpha$ for some element $\alpha \in \pi_1(X, x_0)$. We may choose a closed path $f : [0, 1] \longrightarrow X$ representing $\alpha$. We obtain then a lifted path $g : [0, 1] \longrightarrow \tilde{X}$ with origin in $\tilde{x}_0$. Let $\tilde{x}_1$ be the end of this path, then we have that

$$p_*\left(\pi_1(\tilde{X}, \tilde{x}_1)\right) = \alpha^{-1}\left(p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)\right)\alpha.$$

We have proved the following:

**Theorem 2.3.1.7.** Let $(\tilde{X}, p)$ a covering space of $X$, and $x_0 \in X$. Then, the subgroups $p_*\left(\pi_1(\tilde{X}, \tilde{x})\right)$, for $\tilde{x} \in p^{-1}(x_0)$, form a conjugation class of subgroups of $\pi_1(X, x_0)$.

In the same fashion as in path lifting we can wonder wether a continuous map between topological spaces can be lifted up to covering spaces. Since we want to do that starting from paths we will denote such a map by $f : (X, x_0) \longrightarrow (Y, y_0)$, meaning that $f(x_0) = y_0$. We want a condition so that there exists $\tilde{f}$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[dd, "p"] \\
(Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[dr, "f"'] & \\
 & (X, x_0)
\end{tikzcd}$$

**Theorem 2.3.1.8.** Let $(\tilde{X}, p)$ be a covering space of $X$, $Y$ a locally path-connected and connected space. Given a continuous map $f : (Y, y_0) \longrightarrow (X, x_0)$, there exists a lifted map $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ if, and only if, $f_*\left(\pi_1(Y, y_0)\right) \subset p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$.

$$\begin{tikzcd}
 & \pi_1(\tilde{X}, \tilde{x}_0) \arrow[dd, "p_*"] \\
\pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[dr, "f_*"'] & \\
 & \pi_1(X, x_0)
\end{tikzcd}$$

For our Lie group matters we are interested in the existence of a certain canonical class of covering spaces, namely universal coverings. In order to define them we must talk about homomorphisms of covering spaces.

**Definition 2.17.** If $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are coverings of the space $X$ and $\varphi$ maps the first covering into the second, we say that $\varphi$ is a **covering space homomorphism** if the following diagram commutes

$$\begin{tikzcd}
(\tilde{X}_1, p_1) \arrow[rr, "\varphi"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, p_2) \arrow[dl, "p_2"] \\
& X &
\end{tikzcd}$$

The composition of covering space homomorphisms is again a covering space homomorphism, and the identity map is a covering space homomorphism. A covering space homomorphism of $(\tilde{X}, p)$ onto istelf is called a **covering space automorphism**, or **covering transformation**. Covering space automorphisms form a group, denoted $A(\tilde{X}, p)$ (here $X$ is the underlying space).

**Definition 2.18.** A covering space homomorphism $\varphi$ is an isomorphism if there exists a covering space homomorphism $\psi : (\tilde{X}_2, p_2) \longrightarrow (\tilde{X}_1, p_1)$ such that $\varphi \circ \psi = id_{\tilde{X}_1}$ and $\psi \circ \varphi = id_{\tilde{X}_2}$. In that case, we say that $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are **isomorphic**.

Covering space isomorphisms are easily characterized by the following theorems, which can be found in [Ms], section 5.6.

**Theorem 2.3.1.9.** A covering space homomorphism $\varphi$ is an isomorphism if, and only if, it is a homeomorphism in the usual sense.

**Theorem 2.3.1.10.** The covering spaces $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are isomorphic if, and only if, for each pair $\tilde{x}_1 \in \tilde{X}_1$ and $\tilde{x}_2 \in \tilde{X}_2$ such that $p_1(\tilde{x}_1) = p_2(\tilde{x}_2) = x_0$, the subgroups $p_*\Big(\pi_1(\tilde{X}_1, \tilde{x}_1)\Big)$ and $p_*\Big(\pi_1(\tilde{X}_2, \tilde{x}_2)\Big)$ lie in the same conjugation class of $\pi_1(X, x_0)$.

An important property of covering space homomorphisms will trigger our main tool for later Lie group representations:

**Proposition 2.3.1.11.** If $(\tilde{X}_1, p_1)$ and $(\tilde{X}_2, p_2)$ are covering spaces of the topological space $X$, and $\varphi : (\tilde{X}_1, p_1) \longrightarrow (\tilde{X}_2, p_2)$ is a covering space homomorphism, then $(\tilde{X}_1, \varphi)$ is a covering space of $\tilde{X}_2$.

As a consequence, let $(\tilde{X}, p)$ be a covering space of $X$ such that $\tilde{X}$ is simply connected. Then, by theorem 2.3.1.8, if $(\tilde{X}', p')$ is any covering space, a covering space homomorphism $\varphi : (\tilde{X}, p) \longrightarrow (\tilde{X}', p')$ always exists because $\pi_1(\tilde{X}, \tilde{x})$ is the trivial group. Therefore, $(\tilde{X}, p)$ is a covering space for all covering spaces of $X$.

**Definition 2.19.** A simply connected covering space $(\tilde{X}, p)$ of the topological space $X$ is called a **universal covering**. Two universal coverings are always isomorphic.

The last theorem of this section gives a condition under which a topological space admits a universal covering.

**Definition 2.20.** A topological space $X$ is **semi-locally simply connected** if for each point $x \in X$, there exists a neighbourhood $U \ni x$ such that $\pi_1(U, x) \cong \{0\}$.

**Theorem 2.3.1.12.** If $X$ is a connected, locally path-connected and semi-locally simply connected topological space then it admits a universal covering $\tilde{X}$.

### 2.3.2 Coverings and Lie groups

Recall that in section 2.2.3 we argued that the Lie functor is faithful when restricted to simply connected Lie groups, and that perhaps finding a way of attaching a simply connected Lie group to a wider class of Lie groups would provide a reasonable Lie group classification. This rises one questions now: is the covering space of a Lie group a Lie group too?

**Theorem 2.3.2.1.** If $G$ is a connected Lie group, then it admits a universal covering $(\tilde{G}, \varphi)$.

*Proof.* Following theorem 2.3.1.12, we have to check that $G$ is connected, locally path-connected and semi-locally simply connected, but all of this follows inmediately from the fact that a connected Lie group is a connected smooth manifold. □

**Corollary 2.3.2.2.** All Lie groups admit a universal covering.

*Proof.* It is enough to consider the universal covering for each connected component. □

It still remains to answer the question about the covering space of a Lie group being a Lie group itself.

**Theorem 2.3.2.3.** If $G$ is a connected Lie group, then its universal covering $(\tilde{G}, \varphi)$ is a Lie group. In particular, it is a simply connected Lie group.

*Proof.* This follows from a general result of topology: if $M$ and $N$ are connected manifolds, then any continuous map $f : M \longrightarrow N$ can be lifted to a map $\tilde{f} : \tilde{M} \longrightarrow \tilde{N}$ (see [Ms]). Moreover, if we choose elements $m \in M$ and $n \in N$ such that $f(m) = n$ and we choose liftings $\tilde{m}$ and $\tilde{n}$ such that $m = p(\tilde{m})$ and $n = p(\tilde{n})$ then there is a unique lifting $\tilde{f}$ of $f$ such that $\tilde{f}(\tilde{m}) = \tilde{n}$. Now, for our purposes here, note that if $G$ is a Lie group then the product operation is continuous, so we just have to apply the above to that product regarded as a continuous map between the connected manifolds $G$ and $G \times G$. Likewise, choosing some element $\tilde{1} \in \tilde{G}$ such that $p(\tilde{1}) = 1 \in G$ then there is a unique map $\tilde{\iota} : \tilde{G} \longrightarrow \tilde{G}$ which is a lifting of the inversion map in $G$ and satisfies $\tilde{\iota}(\tilde{1}) = \tilde{1}$. □

We may ask now what is the link between the Lie algebra of a connected Lie group and the Lie algebra of its universal covering. This will give rise to an important property for representation theory purposes.

**Corollary 2.3.2.4.** The Lie algebra $\mathfrak{g}$ of a connected Lie group $G$ is isomorphic to the Lie algebra $\tilde{\mathfrak{g}}$ of its universal covering group $\tilde{G}$.

*Proof.* Since $G$ and $\tilde{G}$ are Lie groups (in particular, smooth manifolds), the covering map $\varphi$ is now a local diffeomorphism, so the induced differential map $d\varphi$ between the respective Lie algebras (viewed as tangent spaces) is an isomorphism. □

This allows us to state that that universal cover of a connected Lie group $G$ is the unique simply connected Lie group $\tilde{G}$ such that $\mathrm{Lie}(\tilde{G}) \cong \mathrm{Lie}(G)$.

## 2.4 Review on $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$

It is the time to pay a visit to two important matrix Lie groups in order to apply all we have learned up to this point. Our purpose in this section is to illustrate all the properties we have seen for the groups $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$.

### 2.4.1 $\mathsf{SU}(2)$

Recall from section 1.1.1 that $\mathsf{SU}(2)$ is the group of $2 \times 2$ complex orthonormal matrices with determinant one. It can be characterized by the properties $A^*A = I$ and $\det A = 1$.

The following are a series of properties satisfyed by $\mathsf{SU}(2)$.

**Proposition 2.4.1.1.** $\mathsf{SU}(2)$ is a matrix Lie group.

*Proof.* It is immediate that $\mathsf{SU}(2) \subset \mathsf{GL}(2, \mathbb{C})$ by the determinant property. Now, consider a sequence $\{A_n\}_n$ converging to a matrix $A$. Put

$$A_n = \begin{pmatrix} z_n^1 & z_n^2 \\ z_n^3 & z_n^4 \end{pmatrix},$$

where the $z_n^i \longrightarrow z^i$ are converging sequences of complex numbers. Then,

$$A_n^* = \begin{pmatrix} z_n^4 & -z_n^3 \\ -z_n^2 & z_n^1 \end{pmatrix} \longrightarrow \begin{pmatrix} z^4 & -z^3 \\ -z^2 & z^1 \end{pmatrix} = A^*.$$

Therefore $I = {A_n}^* A_n \longrightarrow A^* A$, thus $A^* A = I$. Now, the sequence of determinants of the matrices $A_n$ is the constant sequence $a_n = 1$, so by the continuity of the determinant, $\det A = 1$ and hence $A \in \mathsf{SU}(2)$. Then, $\mathsf{SU}(2)$ is a matrix Lie group. □

It is useful to note that $\mathsf{SU}(2)$ can be written as follows:

$$\mathsf{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}; \alpha, \beta \in \mathbb{C}; |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

**Proposition 2.4.1.2.** $\mathsf{SU}(2)$ is a compact Lie group.

*Proof.* We shall check that it is closed an bounded. Take a matrix $A \in \mathsf{SU}(2)$ and consider the euclidean norm in $M_2(\mathbb{C}) \cong \mathbb{C}^4$, then

$$\|A\| = \sqrt{|\alpha|^2 + |\overline{\beta}|^2 + |\beta|^2 + |\overline{\alpha}|^2} = \sqrt{2|\alpha|^2 + 2|\beta|^2} = \sqrt{2},$$

thus $\mathsf{SU}(2)$ is bounded. The closedness comes by observing that det is a continuous function and that $D = \{(\alpha, \beta) \in \mathbb{C}^2 \mid |\alpha|^2 + |\beta|^2 = 1\} = \mathbb{D}^2$ is a closed set in $\mathbb{C}^2$. Then, $\mathsf{SU}(2) = \det^{-1}(\mathbb{D}^2)$. That is, $\mathsf{SU}(2)$ is the pre-image of a closed set by a continuous map, hence it is a closed set. □

**Proposition 2.4.1.3.** $\mathsf{SU}(2)$ is a connected Lie group.

*Proof.* Assume it was nonconnected. Then, $\mathsf{SU}(2) = G_I \cup G$, where $G$ may consist on several connected components. Let $A \in G$ and consider the path $\gamma : [0,1] \longrightarrow M_2(\mathbb{C})$ joining $A$ with $I$. Since $\gamma$ is continuous, there exists a sequence $\{t_n\}_n \subset [0,1]$ with $t_n \geq t_{n-1}$ and $\gamma(t_0) = A$ such that $\gamma(t_n)$ converges to $I$. Let

$$\gamma(t_n) := \begin{pmatrix} \gamma_1(t_n) & -\overline{\gamma_2(t_n)} \\ \gamma_2(t_n) & \overline{\gamma_1(t_n)} \end{pmatrix},$$

where $\gamma_1$ and $\gamma_2$ are continuous paths in $\mathbb{C}$, which do exist because the complex plane is connected. Now, the sequence $a_n := |\gamma_1(t_n)| + |\gamma_2(t_n)|$ converges to 1, has $a_0 = 1$ and $a_n \geq a_{n-1}$ because this is nothing but performing the determinant, which is a continuous map. Hence $\{a_n\}_n$ is the constant sequence $a_n = 1$. Then, $\gamma(t_n) \in \mathsf{SU}(2)$ for each $n \geq 0$. Now, this is does not prove that $\mathsf{SU}(2)$ is connected, but if it was not, then $\gamma$ can be taken so there exists $t \in (0,1) \setminus \{t_n\}_n$ such that $|\gamma_1(t)| + |\gamma_2(t)| \neq 1$. Then, by continuity, there exists $\varepsilon > 0$ such that in $(t - \varepsilon, t + \varepsilon) = U$ the same property holds. In particular, $\varepsilon$ can be taken so $t_n \in U$ for some $n > 0$, which triggers a contradiction. Therefore, $\mathsf{SU}(2)$ is connected. □

**Corollary 2.4.1.4.** As a direct application of theorem 2.3.2.1, we see that $\mathsf{SU}(2)$ admits a universal covering.

**Proposition 2.4.1.5.** $\mathsf{SU}(2) \cong \mathbb{S}^3$.

*Proof.* We have already seen this in subsection 1.2.2. Recall that $\mathbb{S}^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\right\}$. We can identitfy such points with matrices in the form

$$z = \begin{pmatrix} z_1 & -\bar{z_2} \\ z_2 & \bar{z_1} \end{pmatrix},$$

which are precissely the elements of $\mathsf{SU}(2)$. Indeed, observe that

$$|-\bar{z_2}|^2 + |\bar{z_1}|^2 = |z_2|^2 + |z_1|^2 = 1$$

and that

$$\langle (z_1, z_2), (-\bar{z_2}, \bar{z_1})\rangle = z_1(-\bar{z_2}) + z_2\bar{z_1} = -z_1z_2 + z_1z_2 = 0,$$

so the two columns form a hermitic basis and the identification $\mathsf{SU}(2) \cong \mathbb{S}^3$ follows. □

**Corollary 2.4.1.6.** $\mathsf{SU}(2)$ is simply connected.

**Corollary 2.4.1.7.** $\mathsf{SU}(2)$ is its own covering space, and the universal covering is given by the trivial homeomorphism.

**Proposition 2.4.1.8.** The Lie algebra of $\mathsf{SU}(2)$ is the set

$$\mathfrak{su}(2) := \{X \in M_2(\mathbb{C}) \mid X^* = -X, \operatorname{tr}(X) = 0\}.$$

*Proof.* Let $A \in \mathsf{SU}(2)$. We want to look for matrices $X \in M_2(\mathbb{C})$ such that $A = e^X$. Then, from what we know we may write

$$I = A^*A = \left(e^{tX}\right)^* e^{tX} = e^{tX^*}e^{tX}, \text{ for each } t.$$

Thus, we conclude that $e^{tX^*} = e^{-tX}$. Differentiating at $t = 0$ we obtain

$$X^*e^{tX^*}\Big|_{t=0} = -Xe^{-tX}\big|_{t=0},$$

hence $X^* = -X$. Finally, $1 = \det e^X = e^{\operatorname{tr}(X)}$, so $\operatorname{tr}(X) = 0$. □

**Corollary 2.4.1.9.** $\mathsf{SU}(2)$ has real dimension 3.

*Proof.* The dimension of a Lie group can be thought as its dimension as a smooth manifold, which is by definition the dimension of its tangent space at any element. Hence $\dim_{\mathbb{R}}(\mathsf{SU}(2)) = \dim_{\mathbb{R}}(T_I(\mathsf{SU}(2))) = \dim_{\mathbb{R}}(\mathfrak{su}(2))$. Now, if $X \in \mathfrak{su}(2)$, then

$$X = \begin{pmatrix} a & c+bi \\ c-bi & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + bi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; a, b, c \in \mathbb{R},$$

so $\dim_{\mathbb{R}}(\mathfrak{su}(2)) = \dim_{\mathbb{R}}(\mathsf{SU}(2)) = 3$. □

### 2.4.2 SO(3)

As we saw in section 1.1.1, $\mathsf{SO}(3)$ is the set of $3 \times 3$ real orthogonal matrices with determinant one, that is

$$\mathsf{SO}(3) = \left\{A \in M_3(\mathbb{R}) \mid A^TA = I, \det A = 1\right\}.$$

Just like we did with $\mathsf{SU}(2)$ in the preovious section we want to discuss some properties fulfilled by this Lie group.

**Proposition 2.4.2.1.** $\mathsf{SO}(3)$ is a matrix Lie group.

*Proof.* Take any sequence $\{A_n\}_n$ of matrices in $\mathsf{SO}(3)$ converging to a matrix $A$. Consider the sequence $\{A_n^T\}_n$ of transposed matrices. Since transposition is a polynomial map and thus continuous, this sequence converges to the matrix $A^T$. Then, by the continuity of the product operation, which is again a polynomial map, $A_n^T A_n$ converges to $A^T A$, but $A_n \in \mathsf{SO}(3)$, so this is the constant sequence $A_n^T A_n = I$, hence $A^T A = I$. Now, the real number sequence $\{a_n\}_n = \det A_n$ is the constant sequence $a_n = 1$, hence it converges to one. Since the determinant is a continuous map we get that $\det A = 1$ and thus $A \in \mathsf{SO}(3)$, so this is a matrix Lie group. $\square$

A suitable characterization of $\mathsf{SO}(3)$ is to give its elements as matrices whose entries are expressed with trigonometric functions. The following lemma tells us how:

**Lemma 2.4.2.2.** If $A \in \mathsf{SO}(3)$ then there exists an orthogonal matrix $B$ such that $A = BR(\theta)$, where $R(\theta) \in \mathsf{SO}(3)$ is given by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Check that $\det R(\theta) = 1$ for each $\theta \in [0, 2\pi)$ and that $R(\theta)^T R(\theta) = I$ by direct computation. What we want to show now is that $B^T B = I$: if $A = BR(\theta)$ then $I = A^T A = (BR(\theta))^T BR(\theta)$. Then, $I = R(\theta)^T B^T BR(\theta)$, so taking inverses we may write $B^T B = R(\theta)R(\theta)^T = I$, thus $B$ is orthogonal. With that said, observe that $B = AR(\theta)^T$, thus $\det B = 1$, hence $B \in \mathsf{SO}(3)$, which is natural since $\mathsf{SO}(3)$ is a group. $\square$

**Proposition 2.4.2.3.** $\mathsf{SO}(3)$ is a compact Lie group.

*Proof.* For the boundedness consider in $M_3(\mathbb{R})$ the norm $\|\cdot\| := \|\cdot\|_1$ introduced in 1.1.4. Take $A \in \mathsf{SO}(3)$, then $\|A\| = \|BR(\theta)\| \leq \|B\| \|R(\theta)\| = \|B\|$, for some $B \in \mathsf{SO}(3)$. In the other hand, $\|B\| = \left\|AR(\theta)^T\right\| \leq \|A\|$, which yields $\|A\| = \|B\|$. Thus, we have shown that any two matrices in $\mathsf{SO}(3)$ share the same norm. But $\|R(\theta)\| = 1$, then $\|A\| = 1$ and $\mathsf{SO}(3)$ is bounded. The closedness comes directly from the proof that $\mathsf{SO}(3)$ is a matrix Lie group: any convergent sequence of elements in $\mathsf{SO}(3)$ has a limit lying within $\mathsf{SO}(3)$, thus $\mathsf{SO}(3)$ is closed. Therefore, this is a compact Lie group. $\square$

**Proposition 2.4.2.4.** $\mathsf{SO}(3)$ is a connected matrix Lie group.

*Proof.* Consider a path $\gamma : [0,1] \longrightarrow M_3(\mathbb{R})$ joining any matrix $A \in \mathsf{SO}(3)$ with the identity element $I$. Proving that this path is completely contained in $\mathsf{SO}(3)$ will suffice, as it will provide that $\mathsf{SO}(3) = G_I$, its connected component containing $I$. Assume that there is some $t \in (0,1)$ such that $\gamma(t) \notin \mathsf{SO}(3)$. Then, by continuity, there exist some neighbourhood $U = (t-\varepsilon, t+\varepsilon)$, $\varepsilon > 0$, such that the same property holds. Therefore, for $t \in U$, $\gamma(t)^T \notin \mathsf{SO}(3)$, so $\gamma(t)^T\gamma(t) \neq I$. Now, for each $t$ outside $U$ we know that $\gamma(t)^T\gamma(t) = I$ so, again the continuity provides that $\gamma(t)^T\gamma(t)$ is the trivial path, thus $\gamma(t)^T\gamma(t) = I$ for all $t \in [0,1]$. By the continuity of the determinant we may check that $\det\gamma(t) = 1$ for each $t \in [0,1]$. Therefore, $\gamma(t) \in \mathsf{SO}(3)$ for all $t$ and hence $\mathsf{SO}(3)$ is connected. $\square$

**Corollary 2.4.2.5.** $\mathsf{SO}(3)$ admits a universal covering.

**Corollary 2.4.2.6.** $\mathsf{SO}(3)$ is the connected component of $\mathsf{O}(3)$ containing its identity element.

**Proposition 2.4.2.7.** The Lie algebra of $\mathsf{SO}(3)$ is the set of matrices

$$\mathfrak{so}(3) = \left\{X \in M_3(\mathbb{R}) \mid X^T = -X\right\}.$$

*Proof.* Let $X \in M_3(\mathbb{R})$. We want that $e^X \in \mathsf{SO}(3)$, so $(e^X)^T = (e^X)^{-1}$. Then, for each $t \in \mathbb{R}$, we have $(e^{tX})^T = (e^{tX})^{-1}$ and hence $e^{tX^T} = e^{-tX}$, so differentiating at $t = 0$ we obtain $X^T = -X$. Note that in order to be in $\mathsf{SO}(3)$ we also need that $\det(e^X) = e^{\operatorname{tr}(X)} = 1$, thus $\operatorname{tr}(X) = 0$, but this is already granted by the condition $X^T = -X$. $\square$

**Corollary 2.4.2.8.** $\mathsf{SO}(3)$ has real dimension 3.

*Proof.* Matrices in $\mathfrak{so}(3)$ are in the form

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

which are generated by three different real numbers $a, b, c$. Hence, $\dim_{\mathbb{R}}(\mathsf{SO}(3)) = \dim_{\mathbb{R}}(\mathfrak{so}(3)) = 3$. $\square$

**Proposition 2.4.2.9.** There exists a Lie group morphism $\mathsf{SU}(2) \longrightarrow \mathsf{SO}(3)$ such that it is exhaustive and has kernel $\{I, -I\}$.

*Proof.* Let us consider the adjoint representation of $\mathsf{SU}(2)$

$$\begin{array}{rccccc} \mathrm{ad}: & \mathsf{SU}(2) & \longrightarrow & \mathsf{GL}(\mathfrak{su}(2)) & & \\ & A & \longmapsto & \mathrm{ad}(A): & \mathfrak{su}(2) \longrightarrow & \mathfrak{su}(2) \\ & & & & X \longrightarrow & AXA^{-1} \end{array}$$

Consider also the following metric on $\mathfrak{su}(2)$: if

$$X = \begin{pmatrix} i\alpha & b \\ -\overline{b} & -i\alpha \end{pmatrix}, \quad Y = \begin{pmatrix} i\alpha' & b' \\ -\overline{b'} & -i\alpha' \end{pmatrix},$$

then we shall define $\langle X, Y \rangle := \alpha\alpha' + \mathrm{Re}(b\overline{b'})$. We have that

$$\|X\| = \langle X, X \rangle = \alpha^2 + \|b\|^2 = \det X.$$

Now, $\langle , \rangle$ is positive defined and it is preserved by $\mathrm{ad}(A)$, that is, it is orthogonal, since

$$\|\mathrm{ad}(A)(X)\|^2 = \det(AXA^{-1}) = \det X = \|X\|^2.$$

Then we have a Lie group morphism

$$\mathrm{ad} : \mathsf{SU}(2) \longrightarrow \mathsf{SO}(3)$$

with $\ker(\mathrm{ad}) = \{I, -I\}$. Indeed, if $\mathrm{ad}(A) = I$ then, for each $X \in \mathfrak{su}(2)$, $\mathrm{ad}(A)(X) = AXA^{-1} = X$ if, and only if, $A = \pm I$. Then it is also exhaustive as $\dim \mathsf{SU}(2) = \dim \mathsf{SO}(3)$ and its kernel is a discrete set. $\square$

**Corollary 2.4.2.10.** $\mathsf{SO}(3) \cong \mathsf{SU}(2)/\{I, -I\}$.

*Proof.* This result follows by direct application of the Isomorphism theorem for Lie groups and the proposition above. $\square$

**Corollary 2.4.2.11.** $\mathsf{SO}(3) \cong \mathbb{P}^3_{\mathbb{R}}$, the 3-dimensional real projective space.

*Proof.* Just observe that $\mathsf{SO}(3) \cong \mathsf{SU}(2)/\{I, -I\} \cong \mathbb{S}^3/\{p, \overline{p}\} \cong \mathbb{P}^3_{\mathbb{R}}$, where $\overline{p}$ means the antipodal point to the point $p \in \mathbb{S}^3$. $\square$

**Corollary 2.4.2.12.** $\pi_1(\mathsf{SO}(3)) \cong \mathbb{Z}/2$.

**Corollary 2.4.2.13.** $\mathsf{SO}(3)$ is not simply connected.

**Corollary 2.4.2.14.** $\mathsf{SU}(2)$ is the universal cover of $\mathsf{SO}(3)$.

*Proof.* Since $\mathsf{SO}(3)$ is connected, we know that it admits a covering space which is also a Lie group and such that it is simply connected, properties that $\mathsf{SU}(2)$ does fulfill. We have to show, then, that there exists a local homeomorphism $\pi$ that spans the covering. For that purpose, however, it is enough to apply the corollary 2.4.2.11, as $\mathbb{S}^3$ is the universal cover of $\mathbb{P}^3_{\mathbb{R}}$, with the local homeomorphism being the natural projection map $\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^3/\{p, \overline{p}\}$. $\square$

**Corollary 2.4.2.15.** The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic. In particular, the map $d\pi$ is Lie bracket-preserving.

*Proof.* This is a direct application of corollary 2.3.2.4. $\square$

# Chapter 3

# Representations of Lie groups

## 3.1 Group representations

We get into the matter of Lie group representations, which is one of the main purposes of this master thesis. We start by giving the basic definitions and stating Schur's lemma, which provides a simple structure for the elements of a given Lie group under the corresponding representation. In the second part we will describe the centers of some Lie groups and the section ends with the study of the adjoint representation.

### 3.1.1 Schur's lemma

The fundamental language of group representations leads immeadiately to the statement of Schur's lemma, which will be our first important result in this chapter. The following discussions assume a background on general group theory as provided in [Lang] and [Qey].

**Definition 3.1.** Let $\mathbb{K}$ be a field, $V$ a $\mathbb{K}$-vector space and $G$ a group (not necessarily a Lie group). A $\mathbb{K}$-linear **representation of the group** $G$ is a group morphism

$$\rho : G \longrightarrow \mathrm{Aut}_k(V).$$

**Example 3.1.1.1.** Let $G = (\mathbb{Z}/2, +)$ and $V = \mathbb{R}^2$. Then $\mathrm{Aut}_k(V) = \mathrm{End}(\mathbb{R}^2) \cong M_2(\mathbb{R})$. A representation $\rho$ of $G$ is given by $\rho(0) = I_2$ and $\rho(1) = -I_2$. Indeed: $\rho(0+0) = \rho(0) = I_2 = I_2 I_2 = \rho(0)\rho(0)$, $\rho(0+1) = \rho(1) = -I_2 = I_2(-I_2) = \rho(0)\rho(1)$ and $\rho(1+1) = \rho(0) = I_2 = (-I_2)(-I_2) = \rho(1)\rho(1)$. Moreover, $\rho(1^{-1}) = \rho(1) = -I_2 = (-I_2)^{-1} = (\rho(1))^{-1}$ and $\rho(0^{-1}) = \rho(0) = I_2 = (I_2)^{-1} = (\rho(0))^{-1}$. Check that here the inverse of an element is the opposite element, thus the notation $0^{-1}$ makes sense. The symbols 0 and 1 mean the representative classes of $\mathbb{Z}/2$.

**Definition 3.2.** If $G$ is a Lie group, a representation $\rho$ of $G$ is a group representation such that $\rho$ is differentiable as a map. That is, it is asked that $\rho$ is a Lie group morphism.

**Definition 3.3.** Let $G$ be a group and $\rho : G \longrightarrow \mathrm{Aut}_k(V)$ a representation of $G$. We call $\rho$ a **simple** representation (or irreducible representation) if there is no proper subspace $W \subset V$ such that $W$ is $G$-invariant (that is, for each $g \in G$, $\rho(g)(W) \subset W$).

**Example 3.1.1.2.** The representation $\rho$ of example 3.1.1.1 is not simple, as taking $W = \langle(1,0)\rangle$ both $\rho(0)(x,0) \in W$ and $\rho(1)(x,0) \in W$ for each $x \in \mathbb{R}$. An example of simple representation of $G = (\mathbb{Z}/2, +)$ is the trivial representation $\rho : G \longrightarrow \mathbb{R}$ given by $\rho(0) = 0$ and $\rho(1) = 1$ since $\mathbb{R}$ has no proper vector subspaces.

The following result is **Schur's lemma**, which provides an easy description of the elements of $\mathrm{End}_k(V)$ which commute with the representations of a group.

**Definition 3.4.** Let $\rho : G \longrightarrow \mathsf{GL}(V)$ be a representation of the Lie group $G$ over a vector space $V$. We shall write $\mathrm{End}_k(V)^G := \{h \in \mathrm{End}_k(V) \mid \rho(g)h = h\rho(g),\ \forall g \in G\}$.

**Theorem 3.1.1.3** (Schur's lemma)**.** Let $\mathbb{K}$ be an algebraically closed field and let $\rho : G \longrightarrow \mathsf{GL}(V)$ be a $\mathbb{K}$-linear representation of $G$. If $\rho$ is irreducible, then $\mathrm{End}_k(V)^G = k \cdot I$.

*Proof.* Let $h \in \mathrm{End}_k(V)^G$ and consider $v \neq 0$ an eigenvector of eigenvalue $\lambda \in k$. Let $W := \ker(h - \lambda I) \subset \mathrm{End}_k(V)^G$ and note that $W \neq 0$. Let $g \in G$, $x \in W$. Then

$$h(\rho(g)(x)) = \rho(g)(h(x)) = \rho(g)(\lambda x) = \lambda\rho(g)(x).$$

Therefore, $W$ is invariant by $G$ and since $\rho$ is simple, it follows that $W = \ker(h - \lambda I)$. $\square$

**Theorem 3.1.1.4** (Schur's lemma for $k = \mathbb{R}$)**.** Let $\rho : G \longrightarrow \mathsf{GL}(V)$ be an $\mathbb{R}$-linear representation of $G$. If $\rho$ is irreducible and $h \in \mathrm{End}(V)^G$, then the minimal polynomial of $h$ is either $m(t) = t - \lambda$, $\lambda \in \mathbb{R}$, or $m(t) = t^2 + a_1 t + a_2$, $a_1, a_2 \in \mathbb{R}$ with $a_1^2 - 4a_2 < 0$.

*Proof.* The first case follows applying the former lemma. Now, if $h$ has a non-real eigenvalue, then its minimal polynomial is the product of polynomials $p_1, \dots, p_r$ of second degree with $\Delta_{p_i} < 0$. Let $p_i(t)$ be one of them. Then $W = \{x \in V \mid p(h)x = 0\}$ is $G$-invariant, thus $W = V$ and the minimal polynomial of $h$ is $p_i(t)$. $\square$

### 3.1.2 Centers

In this part we recall some concepts of group theory and pay a visit to the center of a group, which we study in the case of Lie groups. We are interested in the centers of the classical Lie groups, which we will be describing here.

**Definition 3.5.** An **action** of the group $G$ over the set $X$ is a group homomorphism

$$\phi : G \longrightarrow \mathcal{F}(X),$$

$\mathcal{F}(X)$ denoting the group of bijective functions $f : X \longrightarrow X$. That is:

(i). For each $x \in X$, $\phi(1, x) = x$, where 1 is the unit in $G$.

(ii). For each $g, h \in G$ and $x \in X$, $\phi(gh, x) = \phi(g, \phi(h, x))$.

**Definition 3.6.** We say that an action of a group $G$ over a set $X$ is **transitive** (or that $G$ **acts transitively** on $X$) if for each $x, y \in X$, there exists $g \in G$ such that $\phi(g, x) = y$.

**Definition 3.7.** The **center** of a group $G$ is the set $Z(G) := \{g \in G \mid gh = hg,\ \forall h \in G\}$.

**Proposition 3.1.2.1.** Easy consequences follow from the definition:

(i). $1 \in Z(G)$.

(ii). If $g \in Z(G)$ then $g^{-1} \in Z(G)$.

(iii). If $g_1, g_2 \in Z(G)$ then $g_1 g_2 h = g_1 h g_2 = h g_1 g_2$, so $g_1 g_2 \in Z(G)$.

(iv). If $G$ is abelian, then $Z(G) = G$.

In particular, $Z(G)$ is a subgroup of $G$ and it is a proper subgroup if, and only if, $G$ is not abelian.

We want to describe the centers of some matrix Lie groups. For that purpose we need to state a lemma first.

**Lemma 3.1.2.2.** Let $\rho : G \longrightarrow \mathsf{GL}(V)$, $\dim V = n < \infty$, be a representation of the Lie group $G$. Assume that $V$ acts transitively (as a group) on $\mathbb{S}^{n-1} \subset V$. Then $\rho$ is simple.

*Proof.* Assume that this was not the case, then let $W \subset V$ be a $G$-invariant subspace. Let $x \in W \cap \mathbb{S}^{n-1}$, $y \in \mathbb{S}^{n-1} \setminus W$. Since $V$ acts transitively on $\mathbb{S}^{n-1}$ there exists $g \in G$ such that $\rho(g)(x) = y$, thus $W$ cannot be $G$-invariant and we get a contradiction, so $\rho$ has to be simple. □

Now, we begin the center computations with the unitary groups.

**Proposition 3.1.2.3.** $Z(\mathsf{U}(n)) = \left\{z \cdot I_n \mid z \in \mathbb{S}^1\right\}$.

*Proof.* Let $A \in \mathsf{U}(n)$. We want to describe the matrices $Z \in \mathsf{U}(n)$ such that $ZA = AZ$. Since we are in $\mathsf{U}(n)$, for each $v \in \mathbb{S}^{2n-1}$ there is $A \in \mathsf{U}(n)$ such that $Av = (1, 0, \ldots, 0) \in \mathbb{S}^{2n-1}$, thus $\mathsf{U}(n)$ acts transitively on $\mathbb{S}^{2n-1}$, then by lemma 3.1.2.2 the standard representation

$$\rho : \mathsf{U}(n) \hookrightarrow \mathsf{GL}(n, \mathbb{C}),$$

is simple, so applying Schur's lemma we get

$$\mathrm{End}(\mathbb{C}^n)^{\mathsf{U}(n)} = \{\lambda I_n \mid \lambda \in \mathbb{C}\},$$

which by definition means that, for each $A \in \mathsf{U}(n)$, $\rho(A)\lambda I_n = \lambda I_n \rho(A) \Rightarrow A\lambda I_n = \lambda I_n A$. Now, since this has to remain inside $\mathsf{U}(n)$, we have that $\|\lambda I_n A\| = \|\lambda\| = 1$, thus $\lambda \in \mathbb{S}^1$. Therefore,

$$Z(\mathsf{U}(n)) = \left\{z \cdot I_n \mid z \in \mathbb{S}^1\right\}.$$

□

**Corollary 3.1.2.4.** $Z(\mathsf{SU}(n)) = \{\zeta_i \cdot I_n \mid \zeta_i^n = 1\}$.

*Proof.* $Z(\mathsf{SU}(n)) = \mathbb{C} \cdot I_n \cap \mathsf{SU}(n) = \{\lambda \cdot I_n \mid \lambda^n = 1\}$. □

**Proposition 3.1.2.5.** Let $n > 2$. Then

(i). $Z(\mathsf{SO}(2n-1)) = \{I_{2n-1}\}$.

(ii). $Z(\mathsf{SO}(2n)) = \{\pm I_{2n}\}$

*Proof.* It is clear that $\mathsf{SO}(n)$ acts tansitively on $\mathbb{S}^{n-1}$ for each $n \geq 2$, so the standard representation $\rho : \mathsf{SO}(n) \hookrightarrow \mathsf{GL}(n, \mathbb{C})$ is simple. Now, for odd $n$, each endomorphism of $\mathbb{R}^n$ has a real eigenvalue, so applying the real version of Schur's lemma, we get $\mathrm{End}(\mathbb{R}^n)^{\mathsf{SO}(n)} = \mathbb{R} \cdot I_n$. Therefore,

$$Z(\mathsf{SO}(n)) = \mathsf{SO}(n) \cap \mathbb{R} \cdot I_n = \{I_n\}.$$

Assume now that $n$ is even. Let $\varphi \in \mathrm{End}(\mathbb{R}^n)^{\mathsf{SO}(n)}$. If $\varphi$ has a real eigenvalue then $\varphi = \pm I_n$. Now, if $\varphi$ has no real eigenvalues, then the minimal polynomial of $\varphi$ has two conjugated complex roots, $\lambda$ and $\bar{\lambda}$. Since $\varphi \in \mathsf{SO}(n)$, we have that $\|\lambda\| = 1$, that is, $\lambda = e^{i\theta}$, so there exists a matrix $P \in \mathsf{SO}(n)$ such that

$$P\varphi P^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & & & \\ \sin\theta & \cos\theta & & & \\ & & \ddots & & \\ & & & \cos\theta & -\sin\theta \\ & & & \sin\theta & \cos\theta \end{pmatrix},$$

which is true because $\varphi \in \mathrm{End}(\mathbb{R}^n)^{\mathsf{SO}(n)}$ implies that $P\varphi = \varphi P$. Let now $P$ be in the form

$$P = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & & \\ & \vdots & & & \vdots \\ & 0 & & \dots & I \end{pmatrix} \in \mathsf{SO}(n).$$

Then,

$$\varphi P = \begin{pmatrix} 0 & \sin\theta & \dots & \\ & & & \end{pmatrix},$$

while $P\varphi$ is the zero matrix, thus $\sin\theta = 0$ and therefore $\varphi = \pm I_n$. □

**Proposition 3.1.2.6.** $Z(\mathsf{SO}(2)) = \mathsf{SO}(2)$.

*Proof.* Just observe that $\mathsf{SO}(2) = \mathbb{S}^1$, which is an abelian group as a subgroup of $\mathbb{C}$. Therefore, $Z(\mathsf{SO}(2)) = \mathsf{SO}(2)$. □

**Proposition 3.1.2.7.** $Z(\mathsf{O}(n)) = \{I_n\}$.

*Proof.* Let $A \in \mathsf{O}(n)$. By general results of linear algebra, there exists an orthonormal basis $\{e_1, e_2\}$ of a 2-dimensional subspace such that if $v \in \langle e_1, e_2 \rangle$ then $Av \in \langle e_1, e_2 \rangle$. In particular, $A$ restricted to the subsapce $\langle e_1, e_2 \rangle$ is a matrix lying in $\mathsf{SO}(n)$, then by the Projection theorem $\mathbb{R}^n = \langle e_1, e_2 \rangle \oplus \langle e_1, e_2 \rangle^{\perp}$ and we have that $\langle e_1, e_2 \rangle^{\perp}$ is invariant by $A$. We may apply now lemma 3.1.2.2 to conclude that the standard representation is therefore simple, so since orthogonal matrices always have a real eigenvalue, Schur's lemma for $\mathbb{R}^n$ yields $\mathrm{End}^{\mathsf{O}(n)} = \mathbb{R} \cdot I_n$ and hence $Z(\mathsf{O}(n)) = \mathsf{O}(n) \cap \mathbb{R} \cdot I = \{I_n\}$. □

**Proposition 3.1.2.8.** $Z(\mathsf{Sp}(n)) = \{\pm I_n\}$.

*Proof.* We consider again the standard representation but this time we note that $\mathsf{Sp}(n)$ acts transitively on $\mathbb{S}^{4n-1} \subset \mathbb{C}^{2n} \setminus \mathbb{H}^n$, where $\mathbb{H}$ denotes the quaternion ring. Then, this representation is simple and hence by Schur's lemma

$$Z(\mathsf{Sp}(n)) = \mathsf{Sp}(n) \cap \mathbb{C} \cdot I_n.$$

But by the properties of the symplectic group, the only diagonal matrices in $\mathsf{Sp}(n)$ are indeed $\pm I_n$. □

### 3.1.3 The adjoint representation

Recall that back in section 2.1.2 we defined the adjoint map for a Lie algebra. A similar map can be defined for Lie groups, namely take $g \in G$, then the **conjugation with respecto to** $g$ is the map

$$\begin{aligned} C_g : G &\longrightarrow G \\ h &\longmapsto ghg^{-1} \end{aligned}$$

This is a Lie group homomorphism, thus differentiable. The differential at the identity element of $G$ is nothing but the adjoint map introduced in section 2.1.2

$$\mathrm{ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}$$

and since $\mathfrak{g}$ is a vector space it makes sense to consider the group $\mathsf{GL}(\mathfrak{g})$, which a Lie group itself. Then ad induces a map

$$\begin{aligned} \mathfrak{ad}: G &\longrightarrow \mathsf{GL}(\mathfrak{g}) \\ g &\longmapsto \mathrm{ad}_g \end{aligned}$$

**Proposition 3.1.3.1.** Ad is a Lie group morphism and therefore a representation of the group $G$, called the **adjoint representation**.

A proof for this has been already given in section 2.1.2. We are more interested in its inmediate applications.

**Proposition 3.1.3.2.** If $G$ is a connected Lie group then $Z(G) = \ker(\mathrm{Ad})$.

*Proof.* We shall check both inclusions:

$\subseteq$: If $g \in Z(G)$ then, for $h \in G$, $C_g(h) = ghg^{-1} = hgg^{-1} = h$ so $C_g = id_G$ and hence $\mathrm{ad}_g = id_{\mathfrak{g}}$.

$\supseteq$: If $g \in \ker(\mathrm{Ad})$ then $d_eC_g = \mathrm{Ad}(g) = id_{\mathfrak{g}} = d_e id_G$. Now, recall that for connected Lie groups the (restricted) Lie functor is faithful, thus $C_g = id_G$ and therefore $g \in Z(G)$.

Therefore the result follows. □

In section 4.2.3 we will delve into representations of Lie algebras, but let us give the definition for now in order to take a look at the adjoint representation of a Lie algebra.

**Definition 3.8.** Let $K_1$ and $K_2$ be fields (particularly $\mathbb{R}$ or $\mathbb{C}$) such that $K_1 \subset K_2$. Let $\mathfrak{g}$ be a $K_1$-Lie algebra and $V$ a $K_2$-vector space. A **$K_2$-linear Lie algebra representation** of $\mathfrak{g}$ is a Lie algebra morphism

$$f : \mathfrak{g} \longrightarrow \mathrm{End}_{K_2}(V),$$

with $\mathrm{End}_{K_2}(V)$ being a $K_2$-vector space and thus a $K_1$-vector space.

As an inmediate consequence of the definition and the definition of Lie algebra morphism we get that if $\rho : G \longrightarrow \mathsf{GL}(V)$ is a Lie group representation, then $d\rho : \mathrm{Lie}(G) \longrightarrow \mathfrak{gl}(V) \cong \mathrm{End}(V)$ is a Lie algebra representation. In particular, if $\rho = \mathrm{Ad}$ we have a representation

$$d_e\mathrm{Ad} = \mathfrak{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

which we call the **adjoint representation of the Lie algebra $\mathfrak{g}$**.

**Proposition 3.1.3.3.** The representation $\mathfrak{ad}$ is given by

$$\begin{aligned} \mathfrak{ad} : \mathfrak{g} &\longrightarrow \mathfrak{gl}(\mathfrak{g}) \\ x &\longmapsto \mathfrak{ad}(x) : \mathfrak{g} \longrightarrow \mathfrak{g} \\ &\qquad\qquad\qquad y \longmapsto [x, y] \end{aligned}$$

*Proof.* Further formalizing what we saw just before proposition 2.1.2.1 for matrix groups, for general Lie groups we have

$$\begin{aligned} \mathrm{ad}_{x_1x_2}(y) &= x_1x_2y(x_1x_2)^{-1} = x_1x_2y{x_2}^{-1}{x_1}^{-1} \\ &= x_1\mathrm{ad}_{x_2}(y){x_1}^{-1} = \mathrm{ad}_{x_1}(\mathrm{ad}_{x_2}(y)) = (\mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2})(y), \end{aligned}$$

thus $\mathfrak{ad}(x_1x_2) = \mathrm{ad}_{x_1x_2} = \mathrm{ad}_{x_1} \circ \mathrm{ad}_{x_2} = \mathfrak{ad}(x_1) \circ \mathfrak{ad}(x_2)$; where $x_1, x_2, y \in \mathfrak{g}$. For the inversion, let $x, y \in \mathfrak{g}$, then

$$(\mathrm{ad}_{x^{-1}} \circ \mathrm{ad}_x)(y) = x^{-1}\mathrm{ad}_x(y)x = x^{-1}xyx^{-1}x = y,$$

so $\mathrm{ad}_{x^{-1}} = \mathrm{ad}_x^{-1}$ and hence $\mathfrak{ad}(x^{-1}) = (\mathfrak{ad}(x))^{-1}$. □

## 3.2 Maximal tori

In this section we study a particular class of subgroups of a Lie group called tori due to their fundamental property.

### 3.2.1 Tori and ranks

**Definition 3.9.** Let $G$ be a Lie group. A **torus** in $G$ is a Lie subgroup $\mathbb{T} \subseteq G$ such that $\mathbb{T} \cong \mathbb{S}^1 \times \dots \mathbb{S}^1$.

Note that a torus can be the group itself, as happens with $\mathsf{SO}(2) \cong \mathbb{S}^1$. There is a theorem stating that a torus is a compact, connected and abelian Lie subgroup of $G$.

**Definition 3.10.** A torus $\mathbb{T}$ in a Lie group $G$ is called **maximal** if $\mathbb{T} \subset G$ is a proper subgroup and there is no other torus $\mathbb{T}' \subseteq G$ such that $\mathbb{T} \subsetneq \mathbb{T}'$.

**Lemma 3.2.1.1.** All finite-dimensional Lie groups have maximal tori.

*Proof.* We shall begin by observing the following fact: if $G$ is a connected Lie group and $H \subset G$ is a subgroup then $\dim H < \dim G$. Indeed, if $H \subset G$, then $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$. Assume towards contradiction that $\dim H = \dim G$, so $\dim \mathrm{Lie}(H) = \dim \mathrm{Lie}(G)$. Then, $\mathrm{Lie}(H) = \mathrm{Lie}(G)$ and, in particular, $\langle \exp(\mathrm{Lie}(H)) \rangle = \langle \exp(\mathrm{Lie}(G)) \rangle$. Now, $\langle \exp(\mathrm{Lie}(H)) \rangle \subset H$ but since $G$ is connected, $\langle \exp(\mathrm{Lie}(G)) \rangle = G$, thus $G \subset H$, which cannot be. Now, if there were no maximal tori in a Lie group $G$, then an infinite sequence $\mathbb{T}_1 \subsetneq \dots \subsetneq \mathbb{T}_n \subsetneq \dots \subseteq G$ with $\dim \mathbb{T}_1 < \dots < \dim \mathbb{T}_n < \dots \leq \dim G$ exists, but this is impossible since $\dim G < \infty$. □

**Definition 3.11.** Let $G$ be a Lie group. We define the **rank** of $G$ as

$$\mathrm{rank}(G) := \max \{\dim \mathbb{T} \mid \mathbb{T} \subset G \text{ torus}\}.$$

That is to say that the rank of a Lie group is the dimension of its maximal tori. Observe that if $\dim G < \infty$ then $\mathrm{rank}(G) < \infty$ but not conversely.

**Example 3.2.1.2.** (i). $\mathrm{rank}(\mathsf{SO}(2)) = 1$.

(ii). $\mathrm{rank}(\mathsf{O}(2)) = 1$, as $\mathsf{O}(2) = \mathsf{SO}(2) \cup \mathsf{O}^-(2)$. The second component denotes the real matrices with determinant $-1$, which is not a Lie group, so any connected subgroup of $\mathsf{O}(2)$ has to be contained in $\mathsf{SO}(2)$, being this the maximal torus itself.

(iii). $\mathrm{rank}(\mathbb{R}^n) = 0$, since $\exp(\mathbb{R}^n) \cong \mathbb{R}^n$, thus if $\mathbb{T} \subset \mathbb{R}^n$ is a torus then $\mathbb{T} \cong \mathrm{Lie}(\mathbb{T})$, so $\mathbb{T} = \{(0, \dots, 0)\}$.

**Remark:** From (ii) it can be inferred that any maximal torus of a Lie group $G$ must be contained in $G_I$, thus $\mathrm{rank}(G) = \mathrm{rank}(G_I)$.

A notorious property of tori is that they are always conjugated within the father Lie group. This is what the following theorem tells us.

**Theorem 3.2.1.3** (Conjugacy)**.** Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two different tori in a Lie group $G$. Then there exists $g \in G$ such that $\mathbb{T}_1 = g\mathbb{T}_2 g^{-1}$.

This theorem requires heavy machinery on differential forms and map degree theory so we will assume it true without proof. A proof and a whole lot of theoretical background on this theorem can be found in [Wr].

The next step is to give a characterization of maximal tori for matrix Lie groups, so we can construct some of them in the next section. For that purpose, a number of results in linear algebra adapted to the scope of Lie groups follow now.

**Proposition 3.2.1.4.** Let $G$ be a compact matrix Lie group. Then all matrices in $G$ diagonalize over $\mathbb{C}$ and have all eigenvalues $\lambda \in \mathbb{S}^1$.

*Proof.* Let $A \in G$ and consider its characetersitic polynomial $p_A(\lambda)$. Consider matrices $J, P \in \mathsf{GL}(n, \mathbb{C})$ such that $A = P^{-1}JP$, with

$$J = D + N := \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} + \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \lambda_i \in \mathbb{C},$$

then

$$J^n = \sum_{k=1}^{n} \binom{n}{k} D^k N^{n-k} = D^n + nD^{n-1}N + \dots$$

Let $X_n := nD^{n-1}N$ and observe that $\|X_n\| \xrightarrow[n\to\infty]{} \infty$ unless $N = 0$, thus $J = D$ is a diagonal matrix. Now, assume $\|\lambda_i\| > 1$, for a certain $i$, then $D^n \xrightarrow[n\to\infty]{} \infty$, which cannot be because $A^n = P^{-1}D^nP \in G$, which is bounded. Assume now that $\|\lambda_i\| < 1$, then $\det D^n \xrightarrow[n\to\infty]{} 0$, thus $D^n \xrightarrow[n\to\infty]{} D \notin G$, which again cannot be because $G$ is closed. Therefore, $A \in G$ diagonalizes over $\mathbb{C}$ and has all its eigenvalues within $\mathbb{S}^1$. $\square$

**Corollary 3.2.1.5.** If $G$ is a compact matrix Lie group, then $\det A = \pm 1, \forall A \in G$. If, moreover, $G$ is connected, then $\det A = 1$.

*Proof.* For the first assertion, observe that $\|\det A\| = \|\det(P^{-1}DP)\| = \|\det D\| = 1$, as $\mathbb{S}^1$ is a group. Now, $\det A \in \mathbb{R}$, thus $\det A = \pm 1$. For the second, recall that det is a continuous map, thus if $G$ is connected and $\gamma$ is a path joining $A$ with $I$, then $\det \gamma(t)$ cannot jump from 1 to $-1$ at any $t$, thus $\det A = \det I = 1$. $\square$

**Proposition 3.2.1.6.** Let $\mathcal{D}$ be a set of matrices such that they commute and diagonalize. Then there exists a basis (in $\mathbb{R}^n$ or $\mathbb{C}^n$) such that all matrices in $\mathcal{D}$ diagonalize simultaneously.

*Proof.* We proceed by induction, the case $n = 1$ being trivial. If all matrices in $\mathcal{D}$ are homotecies then this is obvious, so let $A \in \mathcal{D}$ be a matrix such that it is not a homotecy and let $\lambda_1 \neq \lambda_2$ be eigenvalues of $A$. Let $V := \ker(A - \lambda_1 I) \neq 0$ and let $W := \bigoplus_{i\neq 1} \ker(A - \lambda_i I) \neq 0$, so $\mathbb{C}^n = V \oplus W$. Let $B \in \mathcal{D}$, then $B(V) \subset V$ and $B(W) \subset W$, since if $x \in \ker(A - \lambda_i I)$, then

$$A(Bx) = B(Ax) = B(\lambda_i x) = \lambda_i Bx.$$

Consider now the restrictions of $\mathcal{D}$ to $V$ and $W$, denoted $\mathcal{D}_V$ and $\mathcal{D}_W$ respectively. Matrices in both sets do commute, so by induction hypothesis, there exists a basis in $V$ such that all matrices in $\mathcal{D}_V$ diagonalize and the same goes for $\mathcal{D}_W$ as these were homotecies in the first place. Considering the union of these bases the result follows. $\square$

**Corollary 3.2.1.7.** Let $\mathbb{T} \subset \mathsf{GL}(n, \mathbb{C})$ be a torus. Then there exists a matrix $P \in \mathsf{GL}(n, \mathbb{C})$ such that

$$P\mathbb{T}P^{-1} \subseteq \mathbb{T}^n := \left\{ \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} \mid \theta_1, \dots, \theta_n \in \mathbb{R} \right\}.$$

In particular, $\dim \mathbb{T} \leq n$ and thus $\operatorname{rank}(\mathsf{GL}(n, \mathbb{C})) = n$. $\mathbb{T}^n$ is called **the standard torus** in $\mathsf{GL}(n, \mathbb{C})$.

*Proof.* Since $\mathbb{T}$ is a compact Lie group, all its matrices have its eigenvalues within $\mathbb{S}^1$, so should they diagonalize, they would belong to $\mathbb{T}^n$. Now, since $\mathbb{T}$ is compact, all its matrices diagonlise and since $\mathbb{T}$ is abelian, they do so under the same basis, which is the one generating the matrix $P$. $\square$

**Theorem 3.2.1.8.** Let $G$ be a Lie subgroup of $\mathsf{GL}(n,\mathbb{C})$. Consider the connected component of $G\cap\mathbb{T}^n$ containing the identity element, denoted by $T := (G\cap\mathbb{T}^n)_I$. If there exists $A\in T$ such that $A$ has pairwise different eigenvalues, then $T$ is a maximal torus in $G$.

*Proof.* Since $G$ is closed by the definition of matrix Lie group, $T$ is abelian, connected and compact, thus $T$ is a torus. Assume $T$ is not maximal, that is, there exists a torus $T'\subseteq G$ such that $T\subsetneq T'$. Then, $T'$ is a set of matrices that commute so there exists a basis such that all matrices in $T'$ diagonalize simultaneously. Since $A\in T'$ has its eigenvalues pairwise distinct, such basis is unique up to multiplication by scalars. Now, $A\in\mathbb{T}^n$, so $A$ is diagonal and thus there is no need for a change of basis, hence all matrices in $T'$ have eigenvalues with moduli 1. Therefore

$$\left.\begin{array}{l} T'\subset\mathbb{T}^n \\ T'\subset G \end{array}\right\} \Rightarrow T'\subset (G\cap\mathbb{T}^n)_I = T.$$

We have a contradtiction and then $T$ is a maximal torus in $G$. $\square$

### 3.2.2 Maximal tori of matrix Lie groups

In the previous section we have proved that the standard torus $\mathbb{T}^n$ is a maximal torus in $\mathsf{GL}(n,\mathbb{C})$. Now we are going to describe maximal tori for the classical matrix Lie groups.

**Proposition 3.2.2.1** (Unitary groups)**.** (i). $\mathbb{T}^n$ is a maximal torus in $\mathsf{U}(n)$.

(ii). The torus

$$T = \left\{ \begin{pmatrix} e^{i\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{i\theta_n} \end{pmatrix} \mid \theta_1,\dots,\theta_n\in\mathbb{R}, \sum_i \theta_i = 0 \right\}.$$

is maximal in $\mathsf{SU}(n)$.

*Proof.* For (i), apply theorem 3.2.1 by observing that all matrices in $\mathbb{T}^n$ are trivially orthonormal and that $(\mathsf{U}(n)\cap\mathbb{T}^n)_I = \mathbb{T}^n$ by the connectedness of $\mathsf{U}(n)$. Now it is clear that $\mathbb{T}^n$ contains matrices with pairwise distinct eigenvalues.

For (ii) we use the same idea, but first we shall note that $T$ is indeed $(\mathsf{SU}(n)\cap\mathbb{T}^n)_I$, which has to be because if $A\in\mathsf{SU}(n)$ then $\det A = 1$, so

$$\exp\left(\sum_{i=1}^n \theta_i\right) = 1 \Leftrightarrow \sum_{i=1}^n \theta_i = 0.$$

Note also that $\mathsf{SU}(n)\subset\mathsf{U}(n)$ hence $T\subset\mathbb{T}^n$. It remains to check that $T$ contains matrices with pairwise distinct eigenvalues, but again this is the case because if $\zeta^{n(n-1)/2}=1$, then the matrix

$$\begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{n-1} \end{pmatrix}$$

lies within $T$. $\square$

Next are the orthogonal groups, for whose maximal tori construction we first need a couple of lemmas.

**Lemma 3.2.2.2.** If $\mathcal{D} \subset \mathsf{GL}(n, \mathbb{C})$ is a set of pairwise commuting matrices, $A \in \mathcal{D}$ and $p(t) \in \mathbb{R}[t]$, then all elements of $A$ keep $\ker p(A)$ invariant.

*Proof.* Take $x \in \ker p(A)$ and set $p(t) = \sum_i a_i t^i$, then $p(A)(Ax) = (a_0 I + a_1 A + a_2 A^2 + \dots)(Ax) = (a_0 A + a_1 A^2 + a_2 A^3 + \dots)x$. Now, the hypothesis is that matrices in $\mathcal{D}$ commute, so we can now write this as $(A a_0 I + A a_1 A + A a_2 A^2 + \dots)x = A(p(A)x) = 0$. Thus, $Ax \in \ker p(A)$. □

**Lemma 3.2.2.3.** If $B \in \mathsf{O}(2)$ commutes with all matrices in $\mathsf{SO}(2)$ then $B \in \mathsf{SO}(2)$.

*Proof.* If $B \in \mathsf{O}(2) \setminus \mathsf{SO}(2)$ then it can be written in the form

$$B = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix},$$

but taking the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathsf{SO}(2)$$

we get that $AB \neq BA$, thus $B$ must lie in $\mathsf{SO}(2)$ in order to commute. □

**Proposition 3.2.2.4** (Special orthogonal groups)**.** (i) The torus

$$\mathbb{T}^n_{\mathsf{SO}(2n)} := \left\{ \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & \\ \sin\theta_1 & \cos\theta_1 & & & \\ & & \ddots & & \\ & & & \cos\theta_n & -\sin\theta_n \\ & & & \sin\theta_n & \cos\theta_n \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}$$

is maximal in $\mathsf{SO}(2n)$.

(ii) The torus

$$\mathbb{T}^n_{\mathsf{SO}(2n+1)} := \left\{ \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & & & & \\ \sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_n & -\sin\theta_n & \\ & & & \sin\theta_n & \cos\theta_n & \\ & & & & & 1 \end{pmatrix} \mid \theta_i \in \mathbb{R} \right\}$$

is maximal in $\mathsf{SO}(2n+1)$.

*Proof.* Let $T := \mathbb{T}^n_{\mathsf{SO}(2n)}$ and assume it was not maximal, so there is some torus $T'$ such that $T \subsetneq T'$. Let $B \in T' \setminus T$. We shall choose a matrix $A \in T$ with pairwise distinct eigenvalues, then its characteristic polynomial is in the form

$$p(t) = \prod_{i=1}^{n} (t^2 - 2\cos\theta_i t + 1).$$

In that case, we have

$$\mathbb{R}^{2n} = \bigoplus_{i=1}^{n} \ker(A^2 - 2\cos\theta_i A + I) = \{v_1, v_2\} \oplus \{v_3, v_4\} \oplus \dots$$

Applying the first lemma, $B$ keeps those subspaces invariant, that is

$$B = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix}.$$

Now, since $B^T B = I$ then $B_i^T B_i = I_2$ thus $B_i \in \mathsf{O}(2)$ and since $B$ commutes with each matrix in $T$ then $B_i$ commutes with each matrix in $\mathsf{SO}(2)$, so the second lemma yields $B_i \in \mathsf{SO}(2)$ and therefore $B \in T$, triggering a contradiction. For the odd case the proof goes the same but adding the factor $(t-1)$ in the characteristic polynomial and noting that then $\mathbb{R}^{2n+1} = \{v_1, v_2\} \oplus \{v_3, v_4\} \oplus \cdots \oplus \{v_{2n+1}\}$. □

**Corollary 3.2.2.5** (Orthogonal groups)**.** $\mathbb{T}^n_{\mathsf{SO}(2n)}$ and $\mathbb{T}^n_{\mathsf{SO}(2n+1)}$ are maximal tori of $\mathsf{O}(2n)$ and $\mathsf{O}(2n+1)$.

*Proof.* Since maximal tori must lie in the connected component containing the identity element, which necessarily belong to $\mathsf{SO}(2n)$ and $\mathsf{SO}(2n+1)$ respectively, the result follows. □

**Proposition 3.2.2.6** (Symplectic groups)**.** The torus

$$\mathbb{T}^n_{\mathsf{Sp}(n)} := \left\{ \begin{pmatrix} e^{i\theta_1} & & & & & \\ & \ddots & & & 0 & \\ & & e^{i\theta_n} & & & \\ & & & e^{-i\theta_1} & & \\ & 0 & & & \ddots & \\ & & & & & e^{-i\theta_n} \end{pmatrix} \mid \theta_1, \dots, \theta_n \in \mathbb{R} \right\}$$

is maximal in $\mathsf{Sp}(n)$.

*Proof.* We begin by observing that since $\mathsf{Sp}(n)$ is connected, we have the following:

$$(\mathsf{Sp}(n) \cap \mathbb{T}^n_{\mathsf{Sp}(n)})_I = \mathsf{Sp}(n) \cap \mathbb{T}^n_{\mathsf{Sp}(n)} = \mathbb{T}^n_{\mathsf{Sp}(n)}.$$

Now it will suffice to prove that $\mathbb{T}^n_{\mathsf{Sp}(n)}$ contains matrices with pairwise distinct eigenvalues, but this is immediate observing that, as vector spaces, $\mathbb{T}^n_{\mathsf{Sp}(n)} \cong \mathbb{T}^n$, the standard torus, which already had this property. Therefore, the result follows. □

As a final comment in this section, observe that a maximal torus for $\mathsf{SL}(n)$ is the same as for $\mathsf{SU}(n)$ since $\mathsf{SL}(n) \subset \mathsf{GL}(n, \mathbb{C})$ and matrices in such torus have to have determinant one, so the sum of the arguments must be zero as seen in proposition 3.2.2.1.

# Chapter 4

# Compact Lie groups

## 4.1 Integration on Lie groups

The purpose of this section is to construct an integral for Lie groups such that it is **left-invariant** and **normalized**. We will see the approach of algebra and geometry, which suits best our background, but let's mention that such construction can be also achieved by means of measure theory in the more general setting of topological groups (which Lie groups are part of). See [Mu] for a brief yet nice overview on that. We are saving most proofs as they are all matter of different courses within the Master's degree. Details can be followed in [DC].

### 4.1.1 Forms and integration on manifolds

**Definition 4.1.** Let $X$ be a topological space and $\mathcal{C}_c^0(X)$ the $\mathbb{R}$-algebra of continuous functions with compact support on $X$ and values in $\mathbb{R}$. We call **integral on** $X$ to an $\mathbb{R}$-linear map

$$\begin{aligned} \int_X : \mathcal{C}_c^0(X) &\longrightarrow \mathbb{R} \\ f &\longmapsto \textstyle\int_X f \end{aligned}$$

which is monotonic, meaning that if $f(x) \leq g(x)$ then $\displaystyle\int_X f \leq \int_X g$.

**Definition 4.2.** Let $L_h$ denote the multiplication on the left by an element $h$. If $G$ is a Lie group and $\displaystyle\int_G$ is an integral on $G$, we say that $\displaystyle\int_G$ is **left-invariant** if for each $h \in G$

$$\int_G f = \int_G f \circ L_h,$$

that is,

$$\int_G f(h \cdot g) = \int_G f(g).$$

**Definition 4.3.** If $G$ is a compact Lie group, we say that $\displaystyle\int_G$ is **normalized** if

$$\int_G 1 = 1.$$

In the end we want an integral on $G$ with the above properties. Recall now the language of integration on manifolds ([DC]), specifically we want to deal with the space of multilinear alternated differential forms on a differential manifold $M$, which is denoted by $\Omega^k(M)$. We can set an operation between different alternated multilinear differential forms called **wedge product**

$$\wedge : \Omega^k(M) \times \Omega^l(M) \longrightarrow \Omega^{k+l}(M)$$

which is defined pointwise as

$$(\alpha_p \wedge \beta_p)(v_1, \ldots, v_{k+l}) = \frac{1}{k! \cdot l!} \sum_{\sigma} \varepsilon(\sigma) \cdot \alpha_p(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \beta_p(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}),$$

where $\sigma \in \mathcal{G}_{k+l}$, the group of permutations of $k + l$ elements, $\varepsilon$ is the signature map on $\mathcal{G}_{k+l}$ and $v_1, \ldots, v_{k+l} \in T_pM$. This construction works the same for vector spaces, where we denote by $\text{Alt}^k(V)$ the space of $\mathbb{K}$-linear alternated forms on the space $V$.

It is also worth noting that the assignation $M \longrightarrow \Omega^k(M)$ is a contravariant functor, meaning that if $\varphi : M \longrightarrow N$ is a smooth map, then it induces a map $\varphi^* : \Omega^k(N) \longrightarrow \Omega^k(M)$ defined as

$$(\varphi^*\alpha)(v_1, \ldots, v_k) = \alpha(d\varphi(v_1), \ldots, d\varphi(v_k)).$$

This functor satsifies that:

- $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.
- $(id_M)^* = id_{\Omega^k(M)}$.

With this vocabulary already set we may now define what a volume form is.

**Definition 4.4.** For a form $\alpha \in \Omega^k(M)$ we define the support of $\alpha$ as the set

$$\text{Supp}(\alpha) := \overline{\{x \in M \mid \alpha_x \neq 0\}}.$$

We denote by $\Omega_c^k(M)$ the $\mathbb{R}$-linear space of $k$-forms with compact support.

**Definition 4.5.** Let $M$ be a smooth manifold of dimension $n$. A form $\omega \in \Omega^n(M)$ is a **volume form** if $\omega_p \neq 0$ for each $p \in M$. If $M$ admits a volume form then we say that $M$ is **orientable**.

We now go on to left-invariance and normalization.

**Definition 4.6.** Let $M$ be an orientable manifold. Let $\omega$ be a volume form in $M$. If $f \in \mathcal{C}_c^0(M)$, we define

$$\int_M f := \int_M f\omega.$$

**Definition 4.7.** Let $G$ be a Lie group and let $\alpha \in \Omega^k(M)$. We say that $\alpha$ is **left-invariant** if $L_h{}^*(\alpha) = \alpha$ for all $h \in G$. The space of such forms is denoted by $\Omega^k_{\text{Li}}(G)$.

**Proposition 4.1.1.1.** The correspondence

$$\begin{aligned} \Omega^k_{\text{Li}}(G) &\longrightarrow \text{Alt}^k(T_eG) \\ \alpha &\longmapsto \alpha_e \end{aligned}$$

is bijective.

*Proof.* **Exhaustive**: Let $\xi \in \mathrm{Alt}^k(T_eG)$, we shall construct a pre-image of $\xi$ as follows: if $g \in G$ and $v_1, \ldots, v_k \in T_eG$, then

$$\alpha_g(v_1, \ldots, v_k) = \xi(d_g L_{g^{-1}}(v_1), \ldots, d_g L_{g^{-1}}(v_k)).$$

**Injective**: If $\alpha_e = \beta_e$,

$$\alpha_e(v_1, \ldots, v_k) = \left(L_{g^{-1}}^*\right)_g (v_1, \ldots, v_k) = \alpha_e(d_g L_{g^{-1}}(v_1), \ldots, d_g L_{g^{-1}}(v_k)).$$

□

**Corollary 4.1.1.2.** All Lie groups are orientable.

### 4.1.2 The Haar integral

Let $G$ denote a Lie group for all this section.

**Definition 4.8.** Let G be a compact Lie group with $\dim G = n$. Let $dg \in \Omega^n(G)$ a left-invariant volume form such that

$$\int_G dg = 1,$$

which can be obtained as

$$dg = \frac{\omega}{\displaystyle\int_G 1_G \omega},$$

for a given left-invariant volume form $\omega \in \Omega^n(G)$. The integral associated to $dg$ is called the **Haar integral**.

**Proposition 4.1.2.1.** The Haar integral is left-invariant, that is, for each $f \in \mathcal{C}_C^0(G)$ and each $h \in G$,

$$\int_G f(h \cdot g) dg = \int_G f(g) dg.$$

*Proof.* Using the $L_h$ map we have

$$\int_G f(h \cdot g) dg = \int_G \left(f \circ L_h\right)(g) dg.$$

Now, $dg$ is a left-invariant volume form, thus $dg = L_h^*(dg)$ and then

$$\int_G \left(f \circ L_h\right)(g) dg = \int_G \left(f \circ L_h\right)(g) L_h^*(dg).$$

Using the change of variable formula we obtain

$$\int_G \left(f \circ L_h\right)(g) L_h^*(dg) = \int_G f\left(L_h(g)\right) L_h^*(dg) = \int_G f(g) dg.$$

□

**Proposition 4.1.2.2.** Let $G$ be a compact Lie group and let $\mathcal{C}(G)$ be the $\mathbb{R}$-vector space of all continuous functions in $G$. The Haar integral

$$\int_G dg : \mathcal{C}(G) \longrightarrow \mathbb{R}$$

is the only normalized left-invariant integral on $G$.

*Proof.* Start by observing that a normalized right-invariant integral can be constructed in the same fashion we had for left-invariant ones; that is, using the right-multiplication map $R_g(h) = g \cdot h$ in order to obtain a right-invariant volume form $\delta h = R_g^*(\delta h)$, so we have

$$\int_G f(h \cdot g)\delta h = \int_G f(h)\delta h.$$

**Fubini's theorem** for manifolds states that if $F : G \times G \longrightarrow \mathbb{R}$ is a continuous function, then the integration order can be swapped, allowing us to write

$$\int_G \left( \int_G F(g,h)dg \right) \delta h = \int_G \left( \int_G F(g,h)\delta h \right) dg.$$

Consider now the function $F : G \times G \longrightarrow \mathbb{R}$ defined by $F(g,h) = f(g \cdot h)$ for a given $f \in \mathcal{C}(\mathbb{R})$. This is a continous function, so Fubini's theorem holds. Then, we can use the fact that $\int \delta g$ is normalized to write

$$\begin{aligned} \int_G f(g)dg = \int_G \left( \int_G f(g)dg \right) \delta h = \int_G \left( \int_g f(g \cdot h)dg \right) \delta h \\ = \int_G \left( \int_G f(g \cdot h)\delta h \right) dg = \int_G f(h)\delta h. \end{aligned}$$

$\square$

**Corollary 4.1.2.3.** Let $G$ be a compact Lie group and $\varphi : G \longrightarrow G$ a Lie group isomorphism. Then,

$$\int_G f(g)dg = \int_G (f \circ \varphi)(g)dg, \ \forall f \in \mathcal{C}^0(G).$$

*Proof.* As a map between smooth manifolds, $\varphi$ is a diffeomorphism, hence the map

$$f \longmapsto \int_G f \circ \varphi dg$$

is $\mathbb{R}$-linear, monotonic and normalized. If we manage to check that it is also left-invariant, proposition 4.1.2.2 will yield that it has to be $\int_g f dg$ by uniqueness. Let $h \in G$ and consider $k = \varphi^{-1}(h)$, then

$$\int_G f(h \cdot \varphi(g))dg = \int_G (f \circ \varphi)(k \cdot g)dg = \int_G (f \circ \varphi)(g)dg.$$

Thus,

$$\int_G f \circ L_h \circ \varphi dg = \int_G f \circ \varphi dg,$$

so $\int_G f \circ \varphi dg$ is left-invariant and therefore coincides with $\int_g f dg$. $\square$

## 4.2 More on representations

The Haar integral plays an important role on compact Lie group representations and classification. In this section and the forecoming ones we will be digging deeper in the theory of Lie group representations.

### 4.2.1 Morphisms of representations

**Definition 4.9.** Let $G$ be a group and $\mathbb{K}$ a field. Let $V$ and $W$ be $k\mathbb{K}$-vector spaces. Let $\rho_1 : G \longrightarrow \mathrm{Aut}(V)$ and $\rho_2 : G \longrightarrow \mathrm{Aut}(W)$ be representations of $G$. A **morphism of representations** is a $\mathbb{K}$-linear map $f : V \longrightarrow W$ which is **equivariant**, that is, such that $f(\rho_1(g)v) = \rho_2(g)(f(v))$.

We can view $\rho_1$ and $\rho_2$ as $G$-modules, so $f$ is a $G$-module morphism. We will denote the $\mathbb{K}$-vector space of morphisms between $V$ and $W$ as $\mathrm{Hom}_G(V, W)$. Note that with this definition, group representations form a category.

**Definition 4.10.** Let $G$ be a group and consider $V$ to be a complex $G$-module ($k = \mathbb{C}$). We say that a hermitic product

$$\begin{aligned} V \times V &\longrightarrow \mathbb{C} \\ (u, v) &\longmapsto \langle u, v\rangle \end{aligned}$$

is $G$**-invariant** if $\langle gu, gv\rangle = \langle u, v\rangle$ for each $u, v \in V$ and $g \in G$. A representation of $G$ endowed with a $G$-invariant hermitic product is called a **unitary representation**.

**Remark:** If we choose an orthonormal basis of $V$ for $\langle\ ,\ \rangle$ and we make the identification $V \simeq \mathbb{C}^n$, then

$$\rho : G \longrightarrow \mathsf{U}(n),$$

where $\rho$ is a representation of $G$.

**Theorem 4.2.1.1.** Let $G$ be a compact Lie group. Then, each complex representation of $G$ is unitary.

*Proof.* Let $\rho : G \longrightarrow \mathrm{Aut}(V)$ a representation of the Lie group $G$, with $V$ a $\mathbb{C}$-vector space. Let $b : V \times V \longrightarrow \mathbb{C}$ an arbitrary hermitic product. If $u, v \in V$, we shall define

$$c(u, v) := \int_G b(gu, gv)dg,$$

where $\int$ is the right-invariant Haar integral on $G$. Then:

- Let $u_1, u_2, v \in V$, $g \in G$ and $\lambda \in \mathbb{C}$. Since $b$ is hermitic,

$$\begin{aligned} c(\lambda u_1 + u_2, gv) &= \int_G b(g\left(\lambda u_1 + u_2\right), v)dg = \int_G b(g\lambda u_1 + gu_2, gv)dg \\ &= \int_G \left(\lambda b(gu_1, gv) + b(gu_2, gv)\right)dg = \lambda \int_G b(gu_1, gv)dg + \int_G b(gu_2, gv)dg \\ &= \lambda c(u_1, v) + c(u_2, v). \end{aligned}$$

- Analogously, let $u, v_1, v_2 \in V$ and $\mu \in \mathbb{C}$, so

$$c(u, \mu v_1 + v_2) = \overline{\mu} c(u, v_1) + c(u, v_2).$$

- For the very same reasons we may write

$$c(u, v) = \int_G b(gu, gv)dg = \int_G \overline{b(gv, gu)}dg = \overline{\int_G b(gv, gu)dg} = \overline{c(v, u)}.$$

Now, $c$ is $G$-invariant: let $h \in G$, then by right-invariance

$$c(hu, hv) = \int_G b(ghu, ghv)dg = \int_G b\left(R_h(g)u, R_h(g)v\right)dg = \int_G b(gu, gv)dg = c(u, v).$$

Finally, $c$ is positively-defined: let $u \neq 0$, then

$$c(u, v) = \int_G b(gu, gv)dg,$$

but $b(gu, gv) > 0$ by the compactness of $G$, as there exists $\varepsilon$ such that $b(gu, gv) \geq \varepsilon > 0$. Now, by the monotonicity and normalization of the Haar integral, $c(u, v) > 0$. $\square$

**Corollary 4.2.1.2.** Let $G$ be a compact Lie group and $U$ a $G$-module. Let $V \subset U$ a sub-module, then there exists a $G$-sub-module $W \subset U$ such that $U = V \oplus W$ (this can be viewed as a group-like version of the Projection Theorem [Rud]).

*Proof.* It is enough to choose a $G$-invariant hermitic product, which will be unitary by the theorem above, and then take $W := V^{\perp}$. □

**Corollary 4.2.1.3.** If $G$ is a compact Lie group, then each complex representation of $G$ is a direct sum of irreducible representations.

Let's take a look at an easy example of such irreducible representations.

**Proposition 4.2.1.4.** If $G$ is abelian, then an irreducible representation has dimension 1.

*Proof.* Let $V$ a $G$-module and $g \in G$. Since $G$ is abelian,

$$\rho(g) : V \longrightarrow V$$

is a morphism of $G$-modules, that is, $\rho(G) \in \mathrm{End}^G(V)$. By Schur's lemma, $\rho(g) = \lambda g$ for some $\lambda \in \mathbb{C}$. Therefore, each sub-space of $V$ is $G$-invariant and hence $\dim V = 1$. □

### 4.2.2 Trace and characters

**Definition 4.11.** Let $V$ be a finite dimensional $\mathbb{K}$-vector space. There is an isomorphism of $\mathbb{K}$-vector spaces

$$\begin{array}{ccccc} V^* \otimes V & \longrightarrow & \mathrm{Hom}_k(V,V) & & \\ \sum v_i^* \otimes u_i & \longmapsto & V & \longrightarrow & V \\ & & x & \longmapsto & \sum v_i^*(x) u_i \end{array}$$

Let now $f \in \mathrm{Hom}_k(V,V)$. The **trace** of $f$ is the scalar $\mathrm{tr}(f) \in \mathbb{K}$ which is the image of $f$ by the map composition

$$\begin{array}{ccccc} \mathrm{Hom}_k(V,V) & \cong & V^* \otimes V & \longrightarrow & k \\ & & v^* \otimes u & \longmapsto & v^*(u) \end{array}$$

If $A$ is the matrix associated to the linear map $f$ in a fixed basis, then $\mathrm{tr}(f) = \mathrm{tr}(A)$.

**Proposition 4.2.2.1.** The trace has the following properties:

(i). $\mathrm{tr} : \mathrm{Hom}_k(V,V) \longrightarrow k$ is a $\mathbb{K}$-linear map.

(ii). For each $\varphi \in \mathrm{Aut}_k(V)$, $\mathrm{tr}(\varphi \circ f \circ \varphi^{-1}) = \mathrm{tr}(f)$.

(iii). If $f : V \longrightarrow W$ and $g : W \longrightarrow V$, then $\mathrm{tr}(f \circ h) = \mathrm{tr}(h \circ f)$.

(iv). $\mathrm{tr}(f \oplus g) = \mathrm{tr}(f) + \mathrm{tr}(g)$.

(v). $\mathrm{tr}(f \otimes g) = \mathrm{tr}(f) \cdot \mathrm{tr}(g)$.

(vi). $\mathrm{tr}(f^*) = \mathrm{tr}(f)$.

(vii). If $f^2 = f$ then $\mathrm{tr}(f) = \dim(\mathrm{Im} f)$.

(viii). If $k = \mathbb{C}$ and $f \in \mathrm{Hom}_{\mathbb{C}}(V,V)$, then $f$ induces $\overline{f} \in \mathrm{Hom}_{\mathbb{C}}(\overline{V}, \overline{V})$ and $\mathrm{tr}(\overline{f}) = \overline{\mathrm{tr}(f)}$.

**Definition 4.12.** Let $G$ be a compact Lie group, let $V$ be a complex $G$-module. The **character** of $V$ is defined to be the function

$$\begin{array}{cccccccc} \chi_V : & G & \longrightarrow & \mathbb{C} & & & & \\ & g & \longrightarrow & \mathrm{tr}(L_g : & V & \longrightarrow & V & ) \\ & & & & v & \longrightarrow & g \cdot v & \end{array}$$

If the representation $V$ is irreducible then we say that $\chi_V$ is an **irreducible character**.

**Proposition 4.2.2.2.** Just like with the trace, we have the eight following properties for characters:

(i). $\chi_V$ is a $\mathcal{C}^\infty$ function.

(ii). If $V \cong W$ then $\chi_V = \chi_W$.

(iii). $\chi_V(ghg^{-1}) = \chi_V(g)$.

(iv). $\chi_{V\oplus W} = \chi_V + \chi_W$.

(v). $\chi_{V\otimes W} = \chi_V \cdot \chi_W$.

(vi). $\chi_{V^*}(f) = \chi_V(f^{-1})$.

(vii). $\chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$.

(viii). $\chi_V(e) = \dim_{\mathbb{C}} V$, $e$ the unit element in $G$.

*Proof.* (i). If we fix a basis on $V$, the representation will be given by matrices

$$g \longrightarrow (r_{ij}(g))$$

with the $r_{ij} \in \mathcal{C}^\infty$. Now, $\chi_V(g) = \sum r_{ij}(g) \in \mathcal{C}^\infty$.

(ii). Let $f : V \longrightarrow W$ be a $G$-module isomorphism. Then

$$L_g^W \circ f = f \circ L_g^V,$$

thus

$$\chi_V(g) = \mathrm{tr}(L_g^V) = \mathrm{tr}(f^{-1} \circ L_g^W \circ f) = \mathrm{tr}(L_g^W) = \chi_W(g).$$

(iii). By property (iii) of the trace,

$$\chi_V(ghg^{-1}) = \mathrm{tr}(L_{ghg^{-1}}^V) = \mathrm{tr}(L_g^V \circ L_h^V \circ L_{g^{-1}}^V) = \mathrm{tr}(L_g^V \circ L_{g^{-1}}^V \circ L_h^V) = \mathrm{tr}(L_h^V) = \chi_V(h).$$

(iv). By the fourth property of the trace,

$$\chi_{V\oplus W}(g) = \mathrm{tr}(L_g^{V\oplus W}) = \mathrm{tr}(L_g^V \oplus L_g^W) = \mathrm{tr}(L_g^V) + \mathrm{tr}(L_g^W) = \chi_V(g) + \chi_W(g).$$

(v). Again, by property (v) of the trace,

$$\chi_{V\otimes W}(g) = \mathrm{tr}(L_g^{V\otimes W}) = \mathrm{tr}(L_g^V \otimes L_g^W) = \mathrm{tr}(L_g^V) \cdot \mathrm{tr}(L_g^W) = \chi_V(g) \cdot \chi_W(g).$$

(vi). If the representation $\rho_V(g)$ is given by the matrix $A$, then $\rho_{V^*}(g)$ is given by $(A^T)^{-1}$, thus $\chi_V^*(g) = \chi_V(g^{-1})$.

(vii). If $V$ is a representation of a compact group $G$ then we already know that $\overline{V} \cong V^*$.

(viii). $\chi_V(e) = \mathrm{tr}(L_g^V) = \mathrm{tr}(\mathrm{id_V}) = \dim_{\mathbb{C}}(V)$.

$\square$

The purpose of this whole section is to be able to prove the following:

**Theorem 4.2.2.3** (Orthogonality relations between characters)**.** Let $G$ be a compact Lie group, $V$ and $W$ complex $G$-modules. Then,

(i). $\displaystyle\int_G \chi_V(g)dg = \dim_{\mathbb{C}}(V^G)$.

(ii). If we define

$$\langle \chi_W, \chi_V \rangle := \int_G \overline{\chi_V(g)}\chi_W(g)dg,$$

then we have

$$\langle \chi_W, \chi_V \rangle = \dim_{\mathbb{C}} (\mathrm{Hom}_G(V, W)) .$$

Moreover, if $V$ and $W$ are irreducible, then

$$\langle \chi_W, \chi_V \rangle = \begin{cases} 1, & \text{if } W \cong V, \\ 0, & \text{else.} \end{cases}$$

For the proof we need to prove a lemma first.

**Lemma 4.2.2.4.** Let $W_1$ and $W_2$ be complex irreducible representations of a group $G$. Then

$$\dim (\mathrm{Hom}_G(W_1, W_2)) = \begin{cases} 1, & \text{if } W_1 \cong W_2, \\ 0, & \text{else.} \end{cases}$$

*Proof.* Let $f \in \mathrm{Hom}_G(W_1, W_2)$. Note that $\ker f$ and $\mathrm{Im} f$ are $G$-modules. If $f \neq 0$, since $W_1$ and $W_2$ are irreducible, then $\ker f = \{0\}$ and $\mathrm{Im} f = W_2$, then $f$ is an isomorphism. Now, if $f_1, f_2 \in \mathrm{Hom}_G(W_1, W_2)$ are isomorphisms then $f_1 \circ f_2^{-1} \in \mathrm{Aut}(W_2)^G = \mathbb{C} \cdot \mathrm{id}$ and therefore $f_1 = kf_2$, $k \in \mathbb{C}$, so the result follows. □

Now for the proof of the theorem.

*Proof.* (Theorem 4.2.2.3) Let's begin proving (i) by defining the map

$$\begin{array}{rccccl} p: & V & \longrightarrow & V & & \\ & v & \longmapsto & p(v) & = & \displaystyle\int_g gvdg \end{array}$$

Now, observe that if $h \in G$, then $h$ acts like a constant in the integration sense, so we have

$$h \cdot p(v) = h \int_G gvdg = \int_G hgvdg = \int_G L_h(gv)dg = \int_G gvdg = p(v).$$

Hence $p(v) \in V^G$ and if $v \in V^G$,

$$p(v) = \int_G gvdg = \int_G vdg = v.$$

We've got $p : V \longrightarrow V^G$ such that $p^2 = p$ and $p\mid_{V^G} = \mathrm{id}_{V^G}$. From that we infer that $\mathrm{tr}(p) = \dim V^G$, and furthermore

$$\mathrm{tr}(p) = \mathrm{tr}\left[v \longmapsto \int_G gvdg\right] = \mathrm{tr}\left[\int_G L_g dg\right] = \int_G \mathrm{tr}(L_g)dg = \int_G \chi_V(g)dg.$$

For (ii), since $\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$,

$$\dim (\mathrm{Hom}_G(V, W)) = \int_G \chi_{\mathrm{Hom}(V,W)}(g)dg,$$

but using the properties of the character map

$$\chi_{\mathrm{Hom}(V,W)} = \chi_{V^* \otimes W} = \overline{\chi_V} \cdot \chi_W.$$

Let now $G$ be a compact Lie group, $V$ a complex representation of $G$. We know that there exist complex representations $\{V_j\}_j$ of $G$ which are not mutually isomorphic and such that

$$V \cong \bigoplus_j (V_j^{\oplus n_j}).$$

Such decomposition is unique, as by the lemma above, if $W$ is an irreducible representation of $G$, then

$$\dim\left(\mathrm{Hom}_G(W,V)\right) = \sum \dim\left(\mathrm{Hom}_G(W, V_j^{\oplus n_j})\right) = \begin{cases} n_j, & \text{if } W \cong V_j, \\ 0, & \text{else.} \end{cases}$$

□

**Definition 4.13.** If $n_j \neq 0$, we say that $v_j$ is **contained** in $V$ and $n_j$ is called the **multiplicity** of $V_j$ in $V$.

From the orthogonality relations theorem we obtain:

**Corollary 4.2.2.5.** A complex representation of a compact Lie group is determined, up to isomorphims, by its character.

*Proof.* If $V \cong \bigoplus_j n_j V_j$, then $n_j = \langle \chi_V, \chi_{V_j} \rangle$. □

**Corollary 4.2.2.6.** Under the same assumptions, $V$ is irreducible if, and only if, $\langle \chi_V, \chi_v \rangle = 1$.

*Proof.* If $V \cong \bigoplus_j n_j V_j$, then $\langle \chi_V, \chi_V \rangle = \sum n_j^2$, which is one if $V$ only decomposes into itself. □

### 4.2.3 Representations of Lie algebras

Much like with Lie groups, Lie algebras have also representations of their own. This is particularly convenient, as Ado's theorem tells us that all Lie algebras are isomorphic to some matrix algebra.

**Definition 4.14.** Let $K_1$ be a field, $L$ a $K_1$-Lie algebra. Let $K_2 \supset K_1$ be another field and let $V$ be a $K_2$-vector space. A $K_2$**-representation of** $L$ is a Lie algebra morphism

$$L \longrightarrow \mathrm{End}_{K_2}(V).$$

We will denote by $\mathrm{Rep}_{K_2}(L_{K_1})$ the set of these representations. It is easy to see that they form a category.

**Definition 4.15.** Let $K_1 \subset K_2$ be fields.

(i). Let $\mathfrak{g}_1$ be a $K_1$-Lie algebra. Within the $K_2$-vector space $\mathfrak{g}_1 \otimes_{K_1} K_2$ we define

$$[v \otimes \lambda, w \otimes \mu] := [v,w] \otimes \lambda\mu.$$

$\mathfrak{g}_1 \otimes_{K_1} K_2$ with this bracket is a Lie algebra called a **scalar extension of** $\mathfrak{g}_1$ or, in the case $K_1 = \mathbb{R}$ and $K_2 = \mathbb{C}$, the **complexification** of $\mathfrak{g}_1$.

(ii). Let $\mathfrak{g}_\mathbb{C}$ be a complex Lie algebra. A **real form** of $\mathfrak{g}_\mathbb{C}$ is a real Lie algebra $\mathfrak{g}_\mathbb{R}$ such that $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$.

**Example 4.2.3.1.** Observe that $\mathfrak{su}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \otimes \mathbb{C}$. Indeed, any matrix $X \in \mathfrak{su}(2,\mathbb{C})$ can be decomposed as follows

$$X = \frac{1}{2}(X - X^*) + \frac{i}{2i}(X + X^*),$$

where the summands are traceless skew matrices; that is, $X = X_1 + iX_2$ with $X_1, X_2 \in \mathfrak{sl}(2,\mathbb{R})$. Therefore, $\mathfrak{su}(2,\mathbb{C})$ has $\mathfrak{sl}(2,\mathbb{R})$ as a real form.

**Proposition 4.2.3.2.** Let $\mathfrak{g}$ be a real form of a complex Lie algebra $\tilde{\mathfrak{g}}$. The map

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{C}}(\tilde{\mathfrak{g}}_{\mathbb{C}}) & \longrightarrow & \mathrm{Rep}_{\mathbb{C}}(\mathfrak{g}_{\mathbb{R}}) \\ p & \longmapsto & p\mid_{\mathfrak{g}} \end{array}$$

is bijective.

*Proof.* We have to check injectivity and exhaustivity.

- **Injectivity**: Let $\rho_1$ and $\rho_2$ be $\mathbb{C}$-linear representations of $\tilde{\mathfrak{g}}$, let us consider, for $i = 1, 2$,

$$\begin{array}{rcccccc} \rho_i\mid_{\mathfrak{g}}: & \mathfrak{g} & \hookrightarrow & \tilde{\mathfrak{g}} & \xrightarrow{\rho_i} & \mathfrak{gl}(n,\mathbb{C}) \\ & v & \longrightarrow & v \otimes 1 & & \end{array}$$

  Now, if $\rho_1\mid_{\mathfrak{g}} = \rho_2\mid_{\mathfrak{g}}$, since $\mathfrak{g}$ generates $\tilde{\mathfrak{g}}$ as a $\mathbb{C}$-algebra and the $\rho_i$ are $\mathbb{C}$-linear, we have that $\rho_1 = \rho_2$.

- **Exhaustivity**: Given $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(n,\mathbb{R})$, take $\rho \otimes \mathrm{id}_{\mathbb{C}}$ as a pre-image.

$\square$

**Example 4.2.3.3.** We have, as an example,

$$\begin{array}{ccccc} \mathrm{Rep}_{\mathbb{C}}(\mathsf{SU}(2)) & \xrightarrow{d\rho} & \mathrm{Rep}_{\mathbb{C}}(\mathfrak{su}(2,\mathbb{R})) & \longleftrightarrow & \mathrm{Rep}_{\mathbb{C}}(\mathfrak{sl}(2,\mathbb{C})) \\ & & & & \updownarrow \\ & & \mathrm{Rep}_{\mathbb{C}}(\mathsf{SL}(2,\mathbb{R})) & \xrightarrow{d\rho} & \mathrm{Rep}_{\mathbb{C}}(\mathfrak{sl}(2,\mathbb{R})) \end{array}$$

### 4.2.4 Representations of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(3,\mathbb{C})$

The remaining matters on Lie algebra representations will be studied under two specific examples, the special linear Lie algebras of dimension 2 and 3.

**Representations of $\mathfrak{sl}(2,\mathbb{C})$** Consider the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Taking commutators as Lie brackets we have the following:

$$[X,Y] = XY - YX = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H,\ [H,X] = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2X,\ [H,Y] = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 2Y.$$

**Definition 4.16.** Let $\rho : \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(V)$ be a $\mathbb{C}$-representation. We say that a non-zero vector $v \in V$ **has weight** $\lambda$ if $\rho(H)v = \lambda v$. We say that $v$ is **primitive** (or maximal) of weight $\lambda$ if $\rho(H)v = \lambda v$ and, moreover, $\rho(X)v = 0$.

**Lemma 4.2.4.1.** Let $v \in V$ with weight $\lambda$. Then:

(i). If $Xv \neq 0$ then $Xv$ has weight $\lambda + 2$.

(ii). If $Yv \neq 0$ then $Yv$ has weight $\lambda - 2$.

*Proof.* $HXv = XHv + 2Xv = (\lambda+2)Xv$ and $HYv = YHv - 2Yv = (\lambda-2)Yv$. □

**Lemma 4.2.4.2.** If $\dim V < \infty$, then each representation $V$ of $\mathfrak{sl}(2,\mathbb{C})$ has a primitive vector.

*Proof.* Let $v_0 \in V \setminus \{0\}$ an eigenvector of $H$. Let us consider

$$\begin{array}{ccccc} & v_0 & Xv_0 & X^2v_0 & \dots \\ \text{weights} & \lambda & \lambda+2 & \lambda+4 & \dots \end{array}$$

In particular, the $X^k v_0$, $k \geq 0$ are linearly independent, hence there exists $k \geq 0$ such that $X^k v_0 = 0$ and $X^{k-1}v_0 \neq 0$, so it suffices to take $v = X^{k-1}v_0$. In a similar fashion, it can be proved that there is $m \geq 0$ such that $Y^m v_0 = 0$. □

**Theorem 4.2.4.3.** Let $V$ be a representation of $\mathfrak{sl}(2,\mathbb{C})$. Let $v_0 \in V$ be a primitive vector of weight $\lambda$. Let us define

$$v_{-1} = 0, \quad v_n = \frac{1}{n!}Y^n v_0.$$

Then:

(i). $Yv_n = (n+1)v_{n+1}$.

(ii). $Hv_n = (\lambda - 2n)v_n$.

(iii). $Xv_n = (\lambda - n + 1)v_{n-1}$.

(iv). If $m$ is maximal in the sense that $Y^m v_0 \neq 0$, then $\lambda = m$.

(v). If $V$ is irreducible, then $V = \langle v_0, \dots, v_m \rangle$.

*Proof.* (i). $Yv_n = Y\left(\frac{1}{n!}Y^n v_0\right) = \frac{1}{n!}Y^{n+1}v_0 = \frac{n+1}{(n+1)!}Y^{n+1}v_0 = (n+1)v_{n+1}$.

(ii). By induction on $n$:

$$\begin{aligned} Hv_{n+1} &= \frac{1}{n+1}HYv_n = \frac{1}{n+1}(YH - 2Y)v_n = \frac{1}{n+1}Y(\lambda - 2n)v_n + \frac{1}{n+1}2Yv_n \\ &= (\lambda - 2n)v_{n+1} + 2v_{n+1} = [\lambda - 2(n+1)]\, v_{n+1}. \end{aligned}$$

(iii). Completely analogous to (ii).

(iv). Let $m$ be the maximal element such that $Y^m v_0 \neq 0$. Then, $v_{n+1} = 0$, so

$$0 = Xv_{m+1} = (\lambda - (n+1) + 1)v_m = (\lambda - m)v_m,$$

now, since $v_m \neq 0$, we get $\lambda = m$.

(v). As a consequence of the above, if $V$ is irreducible, then $V = \langle v_0, \dots, v_m \rangle$.

□

In this case, the matrices of $H$, $X$ and $Y$ in the basis $v_0, \dots, v_m$ are

$$Y = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \\ & & m & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & m & & \\ & \ddots & m-1 & \\ 0 & \ddots & \ddots & 1 \\ & & & 0 \end{pmatrix}, H = \begin{pmatrix} m & & \\ & m-2 & & 0 \\ 0 & & \ddots & \\ & & & -m \end{pmatrix}.$$

We will denote by $V(m)$ the representation

$$\mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}(\mathbb{C}^{m+1})$$

given by these matrices.

**Lemma 4.2.4.4.** The representations $V(m)$ are irreducible.

*Proof.* Let $W \subset V(m)$ be a representation. Let $v \in W$ be a primitive vector, that is, an eigenvector of $H$ such that $Xv = 0$. Since $\ker X = \langle e_1 \rangle$, we have that $e_1 \in W$ and since $W$ is stable under the action of $X, Y, H$, we get that $W = V(m)$. □

**Theorem 4.2.4.5.** For each $m \geq 0$ there exists a unique complex irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $m+1$ defined by the matrices above. Moreover, each complex irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ is a direct sum of irreducible representations.

**Corollary 4.2.4.6.** As a consequence, for each representation $\rho$ of $\mathfrak{sl}(2,\mathbb{C})$, the eigenvectors of $\rho(H)$ are always integer.

**Representations of $\mathfrak{sl}(3,\mathbb{C})$** We go now for the representations of $\mathfrak{sl}(3,\mathbb{C})$. We will use the following basis for this algebra:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

**Definition 4.17.** Let $V$ be a representation of $\mathfrak{sl}(3,\mathbb{C})$. We say that $\mu = (m_1, m_2) \in \mathbb{C}^2$ is a **weight** of $V$ if there is $v \in V \setminus \{0\}$ such that

$$H_1 v = m_1 v, \qquad H_2 v = m_2 v,$$

that is, $\rho(H_1)v = m_1 v$, $\rho(H_2)v = m_2 v$. We say that $v$ is an **eigenvector of weight** $\mu$. The dimension of the subspace $V_\mu$ of eigenvectors of weight $\mu$ is called the **multiplicity** of $\mu$.

**Proposition 4.2.4.7.** Each representation of $\mathfrak{sl}(3,\mathbb{C})$ has at least a weight.

*Proof.* Let $m_1 \in \mathbb{C}$ be an eigenvalue of $\rho(H_1)$. Let $W \subset V$ be the subspace of eigenvectors of eigenvalue $m_1$. Since $[H_1, H_2] = 0$, $[\rho(H_1), \rho(H_2)]$ so $\rho(H_2)(W) \subset W$. Let $w \in W$ be an eigenvector of $\rho(H_2)\mid_W$ with eigenvalue $m_2 \in \mathbb{C}$. Then, $w$ is an eigenvector of weight $\mu = (m_1, m_2)$. □

**Proposition 4.2.4.8.** If $\rho$ is a representation of $\mathfrak{sl}(3,\mathbb{C})$ and $\mu = (m_1, m_2)$ is a weight of $\rho$, then $m_1, m_2 \in \mathbb{Z}$

*Proof.* The subalgebra $\langle H_1, X_1, Y_1 \rangle \subset \mathfrak{sl}(3,\mathbb{C})$ is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Thus, by restriction, we get a representation of $\langle H_1, X_1, Y_1 \rangle \cong \mathfrak{sl}(2,\mathbb{C})$ where $m_1$ is an eigenvalue of $\rho(H)$ and thus $m_1 \in \mathbb{Z}$. We can check that $m_2 \in \mathbb{Z}$ by an analogous argument with $\langle X_1, Y_1, H_2 \rangle \subset \mathfrak{sl}(3,\mathbb{C})$ □

**Definition 4.18.** A pair $\alpha := (a_1, a_2) \in \mathbb{C}^2$ is a **root** of $\mathfrak{sl}(3,\mathbb{C})$ if (i). $a_1a_2 \neq 0$.

(ii). There exists $Z \in \mathfrak{sl}(3, \mathbb{C})$ such that

1. $[H_1, Z] = a_1 Z$.
2. $[H_2, Z] = a_2 Z$.

We say that $Z$ is a **root vector** corresponding to $\alpha$.

**Remark:** Condition number (ii) tells us that $Z$ is an eigenvector of $\mathrm{ad}(H_1)$ and $\mathrm{ad}(H_2)$, thus $Z$ is an eigenvector of the adjoint representation of weight $(a_1, a_2)$ and hence $(a_1, a_2) \in \mathbb{Z}^2$.

$\mathfrak{sl}(3, \mathbb{C})$ has six roots, namely

| $\alpha$ | $Z$ |
|---|---|
| $(2,-1)$ | $X_1$ |
| $(-1,2)$ | $X_2$ |
| $(1,1)$ | $X_3$ |
| $(-2,1)$ | $Y_1$ |
| $(1,-2)$ | $Y_2$ |
| $(-1,-1)$ | $Y_3$ |

$H_1$ and $H_2$ are not in the list because they have zero eigenvalues.

We will denote $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$, and we will call these the positive simple roots. They possess the following property.

**Proposition 4.2.4.9.** All roots are linear combinations of $\alpha_1$ and $\alpha_2$ with integer coefficients, and such coefficients are either both positive or both negative.

*Proof.* We just have to perform the computations

$$\begin{aligned}(2,-1) &= \alpha_1\\ (-1,2) &= \alpha_2\\ (1,1) &= \alpha_1+\alpha_2\\ (-2,1) &= -\alpha_1\\ (1,-2) &= -\alpha_2\\ (-1,-1) &= -\alpha_1-\alpha_2.\end{aligned}$$

□

Now, this is not the only possible election. The part played by roots in the representations of $\mathfrak{sl}(3, \mathbb{C})$ is given by the following lemma:

**Lemma 4.2.4.10.** Let $\alpha = (a_1, a_2)$ be a root of $\mathfrak{sl}(3, \mathbb{C})$ and let $Z_\alpha$ be a root vector corresponding to $\alpha$. Let $\rho$ be a representation of $\mathfrak{sl}(3, \mathbb{C})$, $\mu = (m_1, m_2)$ a weight of $\rho$ and $v \neq 0$ an eigenvector of weight $\mu$. Then:

(i). $\rho(H_1)\rho(Z_\alpha)v = (m_1 + a_1)\rho(Z_\alpha)v$.

(ii). $\rho(H_2)\rho(Z_\alpha)v = (m_2 + a_2)\rho(Z_\alpha)v$.

*Proof.* By the definition of a root, $[H_1, Z_\alpha] = a_1 Z_\alpha$. Then

$$H_1Z_\alpha v = (Z_\alpha H_1 + a_1 Z_\alpha)v = Z_\alpha(m_1 v) + a_1 Z_\alpha v = (m_1 + a_1)Z_\alpha v.$$

□

That is to say, if $\mu = (m_1, m_2)$ is a weight of a representation of $\mathfrak{sl}(3,\mathbb{C})$ and $v$ is an eigenvector of weight $\mu$, then applying the root vectors $Z_\alpha = X_1,\ X_2,\ X_3,\ Y_1,\ Y_2,\ Y_3$ we obtain new weights $\mu + \alpha$ (except if $Z_\alpha v = 0$).

If we want to keep getting analogies with the case $\mathfrak{sl}(2,\mathbb{C})$ we need to state a notion of maximal weight. The following definition tells us of an adequate way of describing such concept.

**Definition 4.19.** Let $\alpha_1$ and $\alpha_2$ be the simple positive roots of $\mathfrak{sl}(3,\mathbb{C})$. Let $\rho$ be a representation. We will say that $\mu_1$ is **greater** than $\mu_2$ if $\mu_1 - \mu_2$ can be written as

$$\mu_1 - \mu_2 = a\alpha_1 + b\alpha_2$$

where $a \geq 0$ and $b \geq 0$. We will denote $\mu_1 \geq \mu_2$. We say that $\mu_0$ is a **maximal weight** if for any other weight $\mu$ we have $\mu_0 \geq \mu$.

**Remark:**

(i). The relationship $\geq$ is a partial ordering, as it may happen that $\mu_1 \not\geq \mu_2$ and $\mu_2 \not\geq \mu_1$. In particular, a set of weights may not have a maximal weight.

(ii). The coefficients $a$ and $b$ are not necessarily integer. For instance, $(1,0) \geq (0,0)$ as $(1,0) = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2$.

(iii). $(m_1, m_2) \geq (n_1, n_2)$ does not imply that $m_1 \geq n_1$ and $m_2 \geq n_2$. For instance, $(1,1) \geq (-1,2)$.

We end this part with the maximal weight theorem, whose proof will be omitted in this work.

**Theorem 4.2.4.11** (Maximal weight). (i). A representation of $\mathfrak{sl}(3,\mathbb{C})$ is the direct sum of its weight subspaces:

$$V_\mu = \{v \in V \mid v \text{ has maximal weight or } v = 0\}.$$

(ii). An irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ has a unique maximal weight $\mu_0$. Two irreducible representations are isomorphic if, and only if, their maximal weights coincide.

(iii). If $\mu_0 = (m_1, m_2)$ is the maximal weight of an irreducible representation, then $m_1, m_2 \geq 0$. Conversely, if $m_1, m_2 \in \mathbb{N}$, there exists an irreducible representation of $\mathfrak{sl}(3,\mathbb{C})$ with maximal weight $\mu_0 = (m_1, m_2)$.

## 4.3 Second review on $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$

This final section of the chapter deals with the classifcation of the Lie groups $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ as a completion for the study strated at the end of Chapter 2. We study the representations of $\mathsf{SU}(3)$ as well as both the 2-dimensional and 3-dimensional special unitary groups will be regarded in Chapter 5 as the link between algebraic Lie theory and Quantum Physics.

### 4.3.1 Representations of $\mathsf{SU}(2)$

Let us begin with some notation. Let $V_0$ be the trivial representation of $\mathsf{SU}(2)$ within $\mathsf{GL}(1,\mathbb{C})$ and let $V_1$ denote the standard representation within $\mathsf{GL}(2,\mathbb{C})$. Let us denote $V_n := \mathbb{C}^n[z_1, z_2]$, the $\mathbb{C}$-vector space of homoegenous polynomials of degree $n$ in two variables, which has dimension $n+1$. We shall define the following action of $\mathsf{SU}(2)$ over $V_n$: Let $g \in \mathsf{SU}(2)$ and $p \in V_n$, then

$$(g \cdot P)(z_1, z_2) = p\left((z_1, z_2) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = p(az_1 + cz_2, bz_1 + dz_2).$$

**Proposition 4.3.1.1.** (i). If $n = 0, 1$ we get the representations above.

(ii). For each $n \geq 0$ we obtain a representation

$$\rho : \mathsf{SU}(2) \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_n).$$

*Proof.* (i). If $n = 0$, then $V_0 = \mathbb{C}^*$ (we are considering that the polynomial 0 has degree $-\infty$), thus the action of $\mathsf{SU}(2)$ is given by $(g \cdot \omega)(z_1, z_2) = \omega g,\ g \in \mathsf{SU}(2)$, which clearly defines a group morphism $\rho_\omega$ into $\mathsf{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$ by setting $\rho_\omega(g) = \omega \det g$. If $n = 1$, then the action is

$$\begin{aligned}(g \cdot p)(z_1, z_2) &= p(az_1 + cz_2, bz_1 + cz_2) = (az_1 + cz_2) + k_2(bz_1 + dz_2)\\ &= (ak_1 + bk_2)z_1 + (ck_1 + dk_2)z_2.\end{aligned}$$

This defines an action on $\mathsf{GL}(2, \mathbb{C})$ by the following rule:

$$\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ak_1 & bk_2 \\ dk_2 & ck_1 \end{pmatrix}.$$

This is well defined as $k_1k_2 \neq 0$ (otherwise $p \notin V_1$), so $\rho(g) \in \mathsf{GL}(2, \mathbb{C})$ for $g \in \mathsf{SU}(2)$.

(ii). The general case follows from observing that $\mathrm{Aut}_{\mathbb{C}}(V_n) \cong M_{n+1}(\mathbb{C})^* = \mathsf{GL}(n + 1, \mathbb{C})$.

□

**Proposition 4.3.1.2.** The representations $V_n$ are irreducible.

*Proof.* It is enough to show that if $A : V_n \longrightarrow V_n$ is $\mathsf{SU}(2)$-invariant then $A = \lambda I_{n+1}$, $\lambda \in \mathbb{C}$, as were not to be irreducible, then $V_n = W_1 \oplus W_2$ with the $W_i$ representing homoteces with different ratios.

We shall denote:

(i). If $0 \leq k \leq n$, then $p_k(z_1, z_2) = z_1^k z_2^{n-k}$, and it is clear that the $p_k$ form a basis of $V_n$.

(ii). If $a \in \mathbb{S}^1$, then $g_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in \mathsf{SU}(2)$.

Observe that we have that $g_a \cdot p_k = a^{2k-n} \cdot p_k$ and hence

$$g_a \cdot Ap_k = Ag_a \cdot p_k = Aa^{2k-n} \cdot p_k = a^{2k-n} A \cdot p_k.$$

We shall choose $a \in \mathbb{C}$ such that $\|a\| = 1$ and all the powers $a^{2k-n}$ are different. Then

$$\left\{\text{Subspaces of } V_n \text{ of eigenvectors of eigenvalue } a^{2k-n}\right\} = \langle p_k \rangle.$$

This is implies that $A \cdot p_k = c_k p_k$ for a certain $c_k \in \mathbb{C}$. Let now

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathsf{SU}(2),\ t \in \mathbb{R}.$$

Then,

$$\begin{aligned}AR(t) \cdot p_n = A(z_1 \sin t + z_2 \cos t)^n &= \sum_{k=0}^{n} \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot Ap_k\\ &= \sum_{k=0}^{n} \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot c_k p_k.\end{aligned}$$

Analogously,

$$R(T)Ap_n = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot c_n p_k,$$

so $R(t)Ap_k = AR(t)p_k$. Since the $p_k$ are linearly independent, comparing coefficients yields $c_n = c_k$ and therefore $A = c_n I_{n+1}$. □

We shall see that the $V_n$ are actually all the irreducible representations of $\mathsf{SU}(2)$ up to isomorphisms.

**Definition 4.20.** Let $G$ be a Lie group. We say that $f : G \longrightarrow \mathbb{C}$ is a **class function** if

$$f(ghg^{-1}) = f(g),\ \forall g, h \in G.$$

This definition allows us to present the characters of the representations of $\mathsf{SU}(2)$. We know that any matrix in $\mathsf{SU}(2)$ is conjugated to one of the form

$$e(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

Note that $e(s)$ and $e(t)$ are conjugated to each other if, and only if, $s = \pm t \mod 2\pi$. Thus, if

$$f : \mathsf{SU}(2) \longrightarrow \mathbb{C}$$

is a class function, then

$$\begin{array}{rccc} f_e : & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & f(e(t)) \end{array}$$

is a continuous, even function with period $2\pi$ and the assignation $f \longmapsto f \circ e$ is injective over the class functions in $\mathsf{SU}(2)$.

Let $\chi_n$ be the character of the representation $V_n$. Using the matrices of $V_n$ in the basis $\{p_k\}_{0\le k\le n}$, we obtain $\tilde{\chi}_n(t) := \chi_n \circ e(t) = \sum_{k=0}^{n} e^{i(2n-k)t}$. If $t \neq \mathbb{Z}\pi$, then

$$\tilde{\chi}_n = \frac{\sin\left[(n+1)t\right]}{\sin t}.$$

Using the formula of the sum of the sine function, we get

$$\tilde{\chi}_n = \cos nt + \tilde{\chi}_{n-1}(t)\cos t$$

and from here

$$\langle \tilde{\chi_0}, \ldots, \tilde{\chi_n} \rangle = \langle 1, \cos t, \ldots, \cos nt \rangle.$$

Recall the following theorem from analysis:

**Theorem 4.3.1.3.** The $\mathbb{C}$-vector space of continuous, even functions of period $2\pi$ can be endowed with a topology such that the subset

$$\{\cos nt \mid n \in \mathbb{N}\}$$

is dense and $\langle , \rangle$ is continuous over the subsapce of characters.

Now for the main result of this section:

**Theorem 4.3.1.4.** Any representation of $\mathsf{SU}(2)$ is isomorphic to $V_n$, for some $n \geq 0$.

*Proof.* Let $W$ be an irreducible representation of $\mathsf{SU}(2)$ and let $\chi_W$ be its character. If $W \not\cong V_n$ then $\langle \chi_W, \chi_n \rangle = 0$. Since the subspace $\langle \{\tilde{\chi}_n\}_n \rangle$ is dense, we may write

$$\tilde{\chi}_W = \lim_{k\to\infty} \sum c_n^k \tilde{\chi}_n.$$

If we now define $\langle \tilde{\chi}_W, \tilde{\chi}_W \rangle := \langle \chi_W, \chi_n \rangle$, then by continuity we may write

$$\langle \tilde{\chi}_W, \tilde{\chi}_W \rangle = \lim_{k\to\infty} \sum c_n^k \langle \tilde{\chi}_W, \tilde{\chi}_n \rangle = 0$$

and therefore $\langle \chi_W, \chi_W \rangle = 0$, but if $W$ is irreducible then this product should be one, hence a contradiction. $\square$

### 4.3.2 Representations of $\mathsf{SO}(3)$

The representations of $\mathsf{SO}(3)$ are easily drawn from those of $\mathsf{SU}(2)$ using that the last is the universal cover of the former, as we saw back in chapter 2. Recall from 2.4.2 that there exists a Lie group morphism $\mathsf{SU}(2) \longrightarrow \mathsf{SO}(3)$ such that it is exhaustive and has kernel $\{I, -I\}$.

Let now $W$ be an irreducible $\mathsf{SO}(3)$-module. The composition

$$\mathsf{SU}(2) \longrightarrow \mathsf{SO}(3) \longrightarrow \mathrm{Aut}(W)$$

is an irreducible representation of $\mathsf{SU}(2)$ on which $-I$ acts trivially. Conversely, if we have an irreducible representation $\rho : \mathsf{SU}(2) \longrightarrow \mathrm{Aut}(W)$ such that $\rho(-I) = I$, then $\rho$ induces a representation of $\mathsf{SO}(3)$ which is also irreducible. The action of $-I$ on $V_n$ is nothing else than the homotecy of ratio $(-1)^n$, thus:

**Theorem 4.3.2.1.** The complex irreducible representations of $\mathsf{SO}(3)$ are

$$\mathsf{SO}(3) \longrightarrow \mathsf{SU}(2)/\{I, -I\} \longrightarrow \mathrm{Aut}(V_{2n}),$$

with $n \geq 0$.

The idea involved on this theorem comes directly from the discussions made back at the end of chapter 2, where we saw that $\mathsf{SU}(2)$ is a two-to-one cover of $\mathsf{SO}(3)$.

## 4.4 Representations of $\mathsf{SU}(3)$

Once we know how to construct the representations of $\mathsf{SU}(2)$ it is useful to know how to get those of $\mathsf{SU}(3)$ too, as with these in hand it is easy to derive the general setting for the representations of all special unitary groups $\mathsf{SU}(n)$. This section is dedicated to that matter.

### 4.4.1 Theoretical insight

A theorem we are to state inmediately tells us that we have already set the theory for the representations of $\mathsf{SU}(3)$, so this subsection will deal with the construction of specific examples.

**Theorem 4.4.1.1.** There is a one-to-one correspondence between the representations of the Lie group $\mathsf{SU}(3)$ and the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$.

*Proof.* It is enough observing that $\mathfrak{su}(3)$ is a subalgebra of $\mathfrak{sl}(3, \mathbb{C})$. Then, if $\pi$ is a representation of $\mathfrak{sl}(3, \mathbb{C})$ and $X \in \mathfrak{su}(3) \subset \mathfrak{sl}(3, \mathbb{C})$, we have the representation $\rho(Y) := \rho(\exp(X)) = \exp(\pi(X))$ (definition 3.8), where we know that $Y \in \mathsf{SU}(3)$ and the result follows. $\square$

In particular, $\rho$ is irreducible if, and only if, $\pi$ is irreducible. Observe also that, in light of corollary 4.2.1.3, all representations of $\mathsf{SU}(3)$ are a direct sum of irreducible representations.

However, there are some facts we might point before starting with the representations, especially those concerning to what we will be calling the $L_i$ functionals. First of all, refer back to subsection 4.2.3, where we set a basis for the representaions of $\mathfrak{sl}(3, \mathbb{C})$ consisting of some $H_1, H_2, X_1, X_2, X_3, Y_1, Y_2, Y_3$ matrices. The first thing we shall observe is that, here, the $H$ matrix of $\mathfrak{sl}(2, \mathbb{C})$ is replaced by $H_1$ and $H_2$. These matrices generate a subspace $\mathfrak{h} \subset \mathfrak{sl}(3, \mathbb{C})$ which is the space of (traceless) diagonal matrices, namely

$$\mathfrak{h} = \left\{ \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} | \alpha_1 + \alpha_2 + \alpha_3 = 0 \right\}.$$

The thing with this subspace is that any finite-dimensional representation $V$ of $\mathfrak{sl}(3,\mathbb{C})$ admits a decomposition

$$V = \bigoplus V_\alpha,$$

where each vector $v \in V_\alpha$ is an eigenvector for each element $H \in \mathfrak{h}$. This leads to some sort of particular terminology, as we are dealing not with the action of a matrix $H$ like with $\mathfrak{sl}(2,\mathbb{C})$ but with the action of a whole subspace $\mathfrak{h}$. Thus, an **eigenvalue** for the action of $\mathfrak{h}$ will mean an element $\alpha \in \mathfrak{h}^*$ such that there exists a non-zero element $v \in V$ satisfying that $H(v) = \alpha H \cdot v$, where $v$ is an eigenvector for each element of $\mathfrak{h}$. Such vectors will span an **eigenspace** of the action of $\mathfrak{h}$ and hence we can state the following:

**Proposition 4.4.1.2.** Any finite-dimensional representation $V$ of $\mathfrak{sl}(3,\mathbb{C})$ has a decomposition

$$V = \bigoplus V_\alpha,$$

where $V_\alpha$ is an eigenspace for $\mathfrak{h}$ and $\alpha$ ranges over a finite subset of $\mathfrak{h}^*$.

This is, on its turn, a particular case of a more general statement, namely that for any complex semisimple Lie algebra $\mathfrak{g}$ ([Se]) we can find a non-trivial abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the action of $\mathfrak{h}$ on any $\mathfrak{g}$-module $V$ will be simultaneously diagonalizable for a decomposition of $V$ on eigenspaces $V_\alpha$.

The next step is to check the role played by the $X_i$ and $Y_j$ matrices instead of $X$ and $Y$ in $\mathfrak{sl}(2,\mathbb{C})$. We have to look at the commutations relations

$$[H, X] = 2X, \quad [H, Y] = -2Y.$$

That is, $X$ and $Y$ are eigenvectors for the adjoint action of $H$ on $\mathfrak{sl}(2,\mathbb{C})$, so in our new *eigen*-setting we want to look for the eigenvectors of the adjoint action of $\mathfrak{h}$ on $\mathfrak{sl}(3,\mathbb{C})$. Applying the proposition above we obtain a decomposition of the adjoint representation

$$\mathfrak{sl}(3,\mathbb{C}) = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right), \tag{4.1}$$

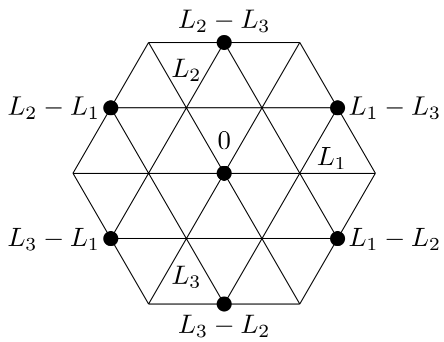
where $\alpha$ ranges over a finite subset of $\mathfrak{h}^*$ and $\mathfrak{h}$ acts on each space $\mathfrak{g}_\alpha$ by scalar multplication, that is, for each $H \in \mathfrak{h}$ and $Y \in \mathfrak{g}_\alpha$,

$$[H, Y] = \mathrm{ad}_H(Y) = \alpha H \cdot Y.$$

It can be argued that if $E_{i,j}$ are the $3 \times 3$ matrices with 1 in position $i, j$ and 0 elsewhere then $E_{i,j}$ generate the eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha$. We can now state that $\mathfrak{h}^*$ is generated by some functionals $L_1, L_2, L_3$ such that

$$L_i \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} = \alpha_i$$

then, the linear functionals appearing on the decomposition 4.1 are the six functionals $L_i - L_j$ and the spaces $\mathfrak{g}_{i-j}$ will be generated by the matrices $E_{i,j}$. This sets the following picture, which will be the main graphic idea for the $\mathsf{SU}(3)$ representations:



The interest of such functionals is to be able to state the following theorem, whose proof can be found in [Ful].

**Theorem 4.4.1.3.** For any pair of numbers $a, b \in \mathbb{N}$ there exists a unique irreducible, finite-dimensional representation $\Gamma_{a,b}$ of $\mathfrak{sl}(3, \mathbb{C})$ with highest weight $aL_1 - bL_3$.

So, with all this, the theory of $\mathsf{SU}(3)$ representations is analogous to that of $\mathfrak{sl}(3, \mathbb{C})$ and thus we can apply its machinery in order to analyze the proposed examples. We will use the notations introduced in 4.2.3.

### 4.4.2 Standard and dual representations

We start by looking at the most elementary representations. The standard and adjoint representations correspond, respectively, to those of maximal weight $(1, 0)$ and $(0, 1)$.

**Standard representation** Of course, here $V \cong \mathbb{C}^3$, the eigenvectors for the action of $\mathfrak{h}$ are those of the canonical basis $e_1, e_2, e_3$ with eigenvalues $L_1$, $L_2$ and $L_3$ respectively. Thus the diagram for $V$ is:

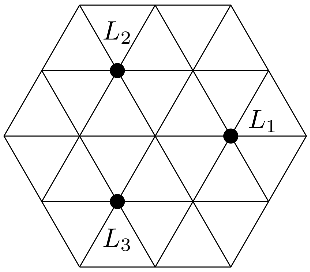
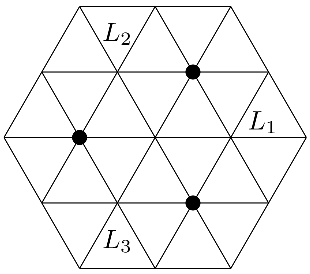



Figure 4.1: Standard representation of $\mathfrak{sl}(3, \mathbb{C})$.

**Dual representation** Here, we take $V^* \cong \mathbb{C}^3$ but with the eigenvalues being the opposites to those of the standard representation, so the diagram for $V^*$ is




In the case of $\mathsf{SU}(2)$ the dual representation would be isomorphic to the standard one, but this is no longer true for $\mathsf{SU}(3)$. However, there is still symmetry in the diagrams. This is due to the fact that the automorphism $\alpha : \mathfrak{sl}(3) \longrightarrow \mathfrak{sl}(3)$ given by $\alpha(X) = -X^T$ carries one representaion into the other.

### 4.4.3 Tensor product representations

Our discussions on representations ends with an overview on other representations of $\mathsf{SU}(3)$ which are obtained by means of performing tensor products with the standard and dual representations.

**Definition 4.21.** Let $V$ be a representation. The **symmetric square representation**, denoted by $\mathrm{Sym}^2 V$, is that whose weights are obtained by applying a symmetry to the weights of $V$ followed by a homotecy of ratio 2.

**Example 4.4.3.1.** In terms of $\mathsf{SU}(2)$ representations, $\mathrm{Sym}^2 V$ is given by the basis $\left\{x^2, xy, y^2\right\}$, where $V$ is the standard representation.

Now, let $V$ and $V^*$ denote the standard and dual representations of $\mathsf{SU}(3)$ as in the subsection above. The representation $\mathrm{Sym}^2 V$ will have weights $\{2L_i, L_i + L_j\}$, rendering the diagram

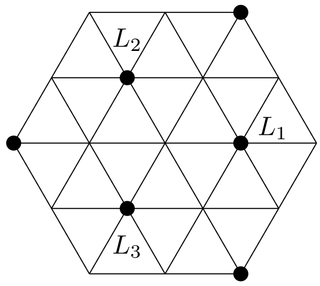
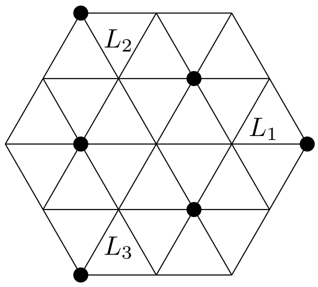



Figure 4.2: $\mathrm{Sym}^2 V$ representation.

Applying the automorphism $\alpha$ we get the dual square symmetric representation $\mathrm{Sym}^2 V^*$:




Let's now consider the tensor product $V \otimes V^*$. The weights are just the sums of the weights $L_i$ of the $V$ representation with those $-L_j$ of the dual representation. This spans the functionals $L_i - L_j$ and the 0 functional occuring twice, as the tensor product is indexing the 0 in $V$ by the 0 in $V^*$ and conversely. The diagram we get is the follwing:

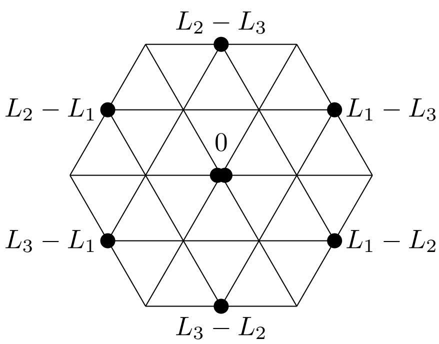



Figure 4.3: $V \otimes V^*$ representation.

The weight space $V_0$ is three dimensional, thus the double-pointed representation. Now, this is not an irreducible representation, as there is a linear map

$$\begin{array}{ccc} V \otimes V^* & \longrightarrow & \mathbb{C} \\ v \otimes u^* & \longmapsto & u^*(v) \end{array}$$

which can be viewed as the trace map of $\mathrm{Hom}(V, V)$. Therefore, the kernel of this map is the subspace of $V \otimes V^*$ of traceless matrices which is on its turn the adjoint representation of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$, this one being irreducible. In the chapter dedicated to particle physics we will see the relation of this representation to some important particle classification.

We finish with the $V \otimes V \otimes V$ representation. This one takes the weights of $V \otimes V$ an indexes them by the weights of $V$, obtaining a 10-dimensional representation with the following diagram:

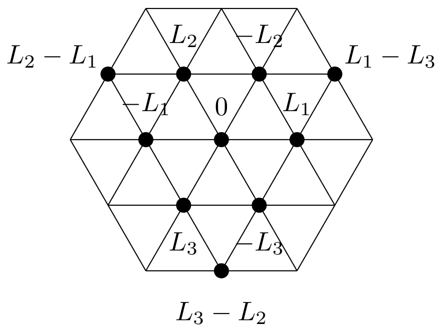
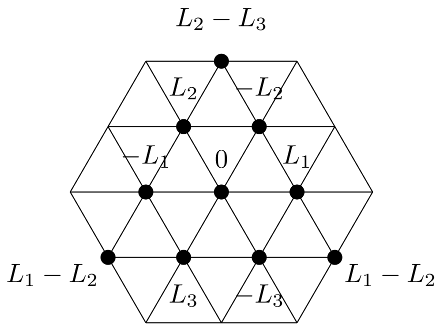



Figure 4.4: $V \otimes V \otimes V$ representation.

Likewise, we get the dual $V^* \otimes V^* \otimes V^*$ representation by applying the $\alpha$ automorphism. The diagram is, of course, symmetric to that of $V \otimes V \otimes V$:




Other diagrams may be obtained by combinations of truplets of $V$ of $V^*$, yielding $\pi/3$-rotations of the two diagrams above. Note that, if that is the case, then some representations are isomorphic to each other, namely $V \otimes V \otimes V^* \cong V^* \otimes V \otimes V$ and $V \otimes V^* \otimes V \cong V^* \otimes V \otimes V^*$.

# Chapter 5

# Lie groups and Quantum Physics

Our last chapter explores the application of Lie groups and algebras, especially those of the special unitary and special orthogonal groups, in the field of particle physics. The purpose here is not to give a deep description of the theory but to have a look at how an abstract mathematical concept such as Lie algebra has its interpretations in the natural world. The source material for this chapter is [Gut], which can be used as an introductory guide into particle physics for mathematicians.

## 5.1 Physicist's notation

One important thing to note prior to our discussions is that physicists often use different notations from that of mathematicians for certain objects. In order to be faithful with the source material, we are introducing some notations here which in the end represent objects we have already studied under different names and/or symbols.

### 5.1.1 Pauli matrices

Pauli matrices are used as a quick way to check that $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. Start by recalling from section 2.4.2 that all matrices within the Lie algebra of $\mathsf{SO}(3)$ come in the form

$$X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

which can be expressed as linear combinations of the following three matrices:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

**Notation:** We define the symbol $\epsilon_{abc}$ in the following way

$$\epsilon_{abc} = \begin{cases} 1, & \text{if } (a,b,c) \in \{(1,2,3),(2,3,1),(3,1,2)\}, \\ -1, & \text{if } (a,b,c) \in \{(1,3,2),(3,2,1),(2,1,3)\}, \\ 0, & \text{else.} \end{cases}$$

**Notation:** $\delta_{ij}$ denotes the usual Krönecker's delta.

With these notations it is easy to check that $[T_a, T_b] = \epsilon_{abc}T_c$.

**Definition 5.1.** The **Pauli matrices** $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{M}_2(\mathbb{C})$ are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

**Proposition 5.1.1.1.** Pauli matrices satisfy the relation $\sigma_a\sigma_b = \delta_{ab}I_2 + i\epsilon_{abc}\delta_c$.

The proof is a simple matter of computation. Let us define now the matrices

$$T_a = -\frac{i}{2}\sigma_a.$$

Observe that these matrices form a basis for the $2 \times 2$ traceless antihermitian matrices, which are those of the Lie algebra $\mathfrak{su}(2)$, and that $[T_a, T_b] = \epsilon_{abc}T_c$, giving a new insight on the fact that $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ share the same Lie algebra.

### 5.1.2 Bra-ket notation

The bra-ket notation is possibly the most famous one among physicists. It is used to denote vectors and, in general, all kinds of elements within a Hilbert or Banach space in a different way from that of mathematicians.

**Notation:** Given a vector $v \in V$, $V$ a vector space; we will denote $v := |v\rangle$. Shall the vector come with specific components $v_1, \dots, v_n$, then we will denote it by $|v_1, \dots, v_n\rangle$.

This notation proves handy when dur to certain particle properties we want to somehow integrate inner products in our descriptions.

**Notation:** If $V$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $u, v \in V$, then we denote $\langle u, v\rangle := \langle u|v\rangle$. In particular, $\|v\|^2 = \langle v|v\rangle$.

### 5.1.3 Eigenstates

This subsection introduces a notation used by physicits in order to deal with the most possibly general form of eigenobjects without any regard of the nature of the objects themselves, namely that they will use the same notation for the eigenvectors of an endomorphism or for the eigenfunctions of the Laplace operator.

**Definition 5.2.** Let $V$ be a vector space over a field $\mathbb{K}$. Let $T : V \longrightarrow V$ be a linear operator of the space onto istelf. An **eigenstate** of $T$, denoted $|\phi\rangle$, is an element $\phi \in V$ such that $T\phi = \lambda\phi$, for a certain eigenvalue $\lambda \in k$.

In the context of Lie groups, we can already infer that the concept of eigenstates is attached to that of weight of a representation of a Lie group (as these are eigenvalues of certain eigenvectors, i.e. eigenstates) and, when put into the languaje of particle physics, they lead to our first target, the electronic spin, which we will be discussing in the next section. For the pure mathematical setting, consider the basis of $\mathfrak{su}(2)$ formed by the matrices $T_a$ introduced at the end of 5.1.1. Let $V$ be a $\mathbb{C}$-vector space and suppose that $\rho$ is a finite-dimensional representation of $\mathfrak{su}(2)$ on $V$. Define the operators

$$J_3 := i\rho(T_3), \qquad J_\pm := \frac{i}{\sqrt{2}}(\rho(T_1) \pm i\rho(T_2)).$$

**Proposition 5.1.3.1.** $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = J_3$.

Again, this is a simple matter of computation. Now, since $V$ is a vector space, there exists an eigenstate $|\phi\rangle$ with a certain eigenvalue $\lambda$, so from the proposition it follows that

$$J_3 J_\pm = |\phi\rangle = (\lambda \pm 1) J_\pm |\phi\rangle.$$

By induction we can easily see that

$$J_3 (J_\pm)^n = |\phi\rangle = (\lambda \pm n)(J_\pm)^n |\phi\rangle.$$

That is to say, the $(J_\pm)^n|\phi\rangle$ either vanish or are eigenstates of $J_3$ of eigenvalue $(\lambda \pm n)$, $n \geq 0$ which translated into the lenaguaje of Lie algebra means that they are the weights of the representation $\rho$.

If the $(J_+)^n|\phi\rangle$ are non-vanishing then they are linealry independent, as they have different $J_3$-eigenvalues and, since $V$ is finite-dimensional, there exists a $J_3$-eigenstate $(J_+)^n|\phi\rangle$ denoted $|j,j\rangle$ with eigenvalue $j$ such that $J_+|j,j\rangle = 0$. That is, $|j,j\rangle$ is the maximal weight of the representation $\rho$.

**Definition 5.3.** The eigenvalue $j$ of the maximal weight $|j,j\rangle$ is called the **electronic spin**.

## 5.2 SU(2) and particle physics

It is now the time to explore the applications of the Lie group $\mathsf{SU}(2)$ into particle physics. All the discussions follow from the language introduced in the section above regarding weights and the electronic spin.

### 5.2.1 Angular momentum

For this brief section, which shows a direct interpretation of the representations of $\mathsf{SU}(2)$, recall the notation for the differential of a map with respect to local cooradinates $(x_1, \ldots, x_n)$ (see [Cur]), namely:

$$df = \sum_{j \geq 1} g^j \frac{\partial}{\partial x^j}.$$

**Definition 5.4.** The **orbital angular momentum operators** are defined as

$$L_a := -i\epsilon_{abc} x^b \frac{\partial}{\partial x^c}.$$

These operators act on wavefunctions and they satisfy that

**Proposition 5.2.1.1.** $[L_a, L_b] = i\epsilon_{abc} L_c$.

*Proof.* Let $f$ be a wavefunction and recall that $\epsilon_{abc} = \epsilon_{bca} = \epsilon_{cab} = 1$. Then,

$$\begin{aligned}
[L_a, L_b](f) &= L_a\left(-i\epsilon_{bca} x^c \frac{\partial f}{\partial x^a}\right) - L_b\left(-i\epsilon_{abc} x^b \frac{\partial f}{\partial x^c}\right) \\
&= -i\epsilon_{abc} x^b \frac{\partial}{\partial x^c}\left(-i\epsilon_{bca} x^c \frac{\partial f}{\partial x^a}\right) + i\epsilon_{bca} x^c \frac{\partial}{\partial x^a}\left(-i\epsilon_{abc} x^b \frac{\partial f}{\partial x^c}\right) \\
&= -\epsilon_{cab} x^b \left(\frac{\partial}{\partial x^a} + x^c \frac{\partial^2}{\partial x^c \partial x^a}\right) f + \epsilon_{cab} x^c x^b \frac{\partial^2}{\partial x^a \partial x^c} f
\end{aligned}$$

Applying Schwarz's theorem, the second derivatives vanish, so making $-1 = i^2$ we get

$$\begin{aligned}
[L_a, L_b](f) &= \epsilon_{cab} x^b \left(-\frac{\partial}{\partial x^a}\right) f = i\epsilon_{abc}\left(i\epsilon_{cab} x^b \frac{\partial}{\partial x^a}\right) f \\
&= i\epsilon_{abc}\left(-i\epsilon_{cba} x^b \frac{\partial}{\partial x^a}\right) f = i\epsilon_{abc} L_c f.
\end{aligned}$$

Since this holds for any $f$, we get that $[L_a, L_b] = i\epsilon_{abc} L_c$. $\square$

**Corollary 5.2.1.2.** The orbital angular momentum operators form a complexified representation of $\mathsf{SU}(2)$.

Particles also carry a **spin angular momentum** $S$ which satisfies $[L, S] = 0$ (that is, it commutes with orbital angular momentum). With that, the **total angular momentum** is defined as

$$J = L + S.$$

### 5.2.2 Isospin symmetry

The $\mathsf{SU}(2)$ isospin symmetry was introduced by Heisenberg in order to give some mathematical meaning to some similarities between different particles. For example, neutrons and protons have similar mass, strong nuclear forces between nucleons being similar too. This way, particles are grouped into multiplets of the same isospin value $j$ and labelled by the weights, which we recall that are eigenvalues of the $J_3$ operator.

**Definition 5.5.** A **particle configuration** is any family of elementary particles. If $P$ is a particle, we denote its **anti-particle** by $\overline{P}$.

Physicists use a notation based on eigenstates and its eigenvalues to denote particle configurations. Some examples follow in the next lines.

**Example 5.2.2.1.** In this example $I$ and $I_3$ denote certain eigenvalues (isospin values, previously denoted $j$, which are weights) of the $J_3$ operator. $I_3$ is called the third component of isospin.

(i). Nucleons have isospin $I = \frac{1}{2}$. The proton has $I_3 = \frac{1}{2}$ and the neutron has $I_3 = -\frac{1}{2}$.

$$\begin{aligned} n &= |\frac{1}{2}, -\frac{1}{2}\rangle \\ p &= |\frac{1}{2}, \frac{1}{2}\rangle \end{aligned}$$

(ii). Other baryons (called strange baryons) have $I = 0$ and $I = 1$.

$$\begin{aligned} \Sigma^- &= |1, -1\rangle \\ \Sigma^0 &= |1, 0\rangle \\ \Sigma^+ &= |1, 1\rangle \\ \Lambda^0 &= |0, 0\rangle \end{aligned}$$

(iii). The light quarks have $I = \frac{1}{2}$.

$$\begin{aligned} d &= |\frac{1}{2}, -\frac{1}{2}\rangle \\ u &= |\frac{1}{2}, \frac{1}{2}\rangle \end{aligned}$$

There are other particles (pions and mesons) we are not showing here. See [Gut] for a wider description.

### 5.2.3 Pauli's Exclusion Principle

For that section we need to consider a tensor product representation for $\mathsf{SU}(2)$. Back in 5.1.3 we have defined the isospin to be the maximal weight of a representation; let now $\rho_1$ and $\rho_2$ be two representations of $\mathsf{SU}(2)$ such that they have isospin $j_1 = j_2 = \frac{1}{2}$. We shall consider the composite system $\rho_1 \otimes \rho_2$, which is an irreducible tensor product representation. Then, the composite spin values can be $j = 0$ and $j = 1$, spanning a 4-dimensional tensor product space.

The $j = 1$ states are
$$|1,1\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle.$$
Applying $J_-$ in both sides we get
$$|1,0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle + |\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle \right),$$
and applying $J_-$ once more we find
$$|1,-1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |-\frac{1}{2}, \frac{1}{2}\rangle.$$
Now there can only be one possible state for $j = 0$, which is
$$|0,0\rangle = \frac{1}{\sqrt{2}} \left( |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle - |\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle \right).$$

These states can be expressed in terms of what we call NN nucleon-nucleon bound states. Following with the notation started in last subsection, where $n$ is neutron and $p$ is proton we have, for $j = 1$
$$|1,1\rangle = pp, \quad |1,0\rangle = \frac{1}{\sqrt{2}}(np + pn), \quad |1,-1\rangle = nn.$$
These states are symmetric under a certain condition called **exchange of isospin degrees of freedom** (See [Gut]). If we turn to the $j = 0$ state, recall that there is only possibility, namely
$$|0,0\rangle = \frac{1}{\sqrt{2}}(np - pn),$$
which is anti-symmetric under exchange of isospin degrees of freedom.

**Pauli's Exclusion Principle** states that this always needs to be the case, that is, that not all four quantum numbers can be preserved at the same time within a bound quantum state, being always one that is "excluded".

## 5.3 SU(3) and the Quark Model

### 5.3.1 Quantum numbers

One of the main pruposes of particle physics is to be able to explain and classify the interactions that occur in Nature. Back before the 20th Century, it was believed that electrons, protons and neutrons were the elementary particles of Nature. by means of collisions inside of particle accelerators, other particles were observed, thus leading to the idea that these three particles might not be completely elementary.

The Standard Model postulates the existence of four forces in the Universe, namely the electromagnetic, the strong nuclear force, the weak nuclear force and the gravitational force; and tries to explain their interactions. The idea is that these interactions appear when a pair of particles exchange a third one. This exchange particles are the following:

- Photons for the electromagnetic force.
- Gluons for the strong interaction (from "glue", as they are responsible of keeping particles together within the atomic core).

- $W$ and $Z$ bosons for the weak nuclear interaction.
- It is postulated that a particle called graviton is responsible for the gravitational interaction.

Particles can be classified in many ways depending on the followed criteria. For instance, classifying according to the mass, we heve the following:

- Baryons (or Fermions), which have strong interactions. Examples are the proton and the neutron.
- Mesons. They also interact strongly. Examples are pions and kaons.
- Leptons. They do not have strong interactions and are considered to be truly elementary. Photons and neutrinos are examples of this.

Baryons and Mesons are often called Hadrons due to their strong interactions. To the above particles one shall add their antiparticles, postulated by Paul Dirac in 1930.

In order to determine the possible interactions occuring between particles, physicists often asign numerical invariants which obey some conservation laws. This numbers are called quantum numbers and depending on which ones are fixed and which ones are let to vary freely one gets different classifications. This is where the Lie group $\mathsf{SU}(3)$ comes into play.

### 5.3.2 Strangeness and representations of $\mathsf{SU}(3)$

Symmetries from $\mathsf{SU}(3)$ appear when dealing with Hadrons. In a heuristic sense, certain quantum numbers are conserved when strong interactions happen; however, some interactions that would conserve all of these numbers were never observed in the experimentation. This lead Murray Gell-Mann and Nishima to independently postulate an inital quantity called strangeness which should be conserved under strong interactions. Physicists usually consider a magnitude call **hypercharge**, $Y = B + S$, which is the sum of the baryonic number $B$ and the strangeness $S$.

Gell-Mann and Nishima observed that fixing two other magnitudes, the spin and the parity, Hadrons can be grouped in either groups of 8, called **octets**, or groups of 10, called **decuplets**; which, when represented in the plane $Y - I_3$, appear to be the similar to the weight diagrams of different representations of $\mathsf{SU}(3)$:

- The octet configuration corresponds to the tensor product representation $V \otimes V^*$.
- Decuplets correspond to the tensor representation $V \otimes V \otimes V$, where $V$ is the trivial representaion.

The representations included a spot for a still hidden particle, which was postulated to have spin $\frac{3}{2}$, parity $+$, hypercharge $-2$ and $I_3 = 0$. This particle was discovered in 1964, proving that the use of $\mathsf{SU}(3)$ symmetry could predict the existence of unknown particles.

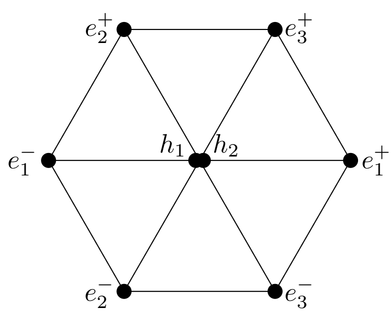



Figure 5.1: Octuplet representation of $\mathsf{SU}(3)$.

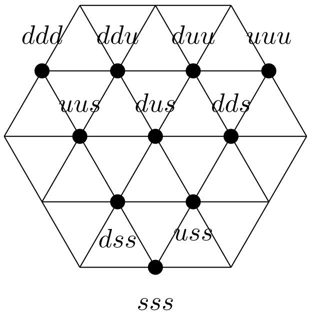



Figure 5.2: Decuplet representation of $\mathsf{SU}(3)$ showing some quark combinations.

When looking at the representations above, one question arises naturally: what happens when applying the trivial representation so the corresponding quantum numbers are set to be fixed? The answer is that we get three particles that sit on the positions of the third roots of unity. These particles were called "up", "down" and "charming"; and together with their antiparticles sum to what we know as **quarks**.

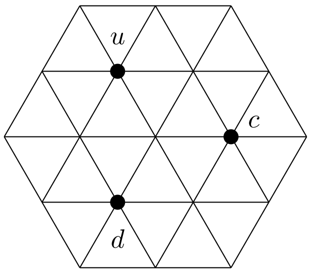



Figure 5.3: The standard representation showing the quarks

Taking the dual representation, one gets the set of anti-quarks: top, bottom and strange.

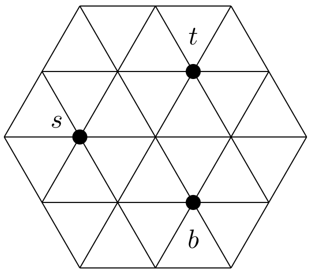



Figure 5.4: The dual representation with the quarks strange, top and bottom

# Final conclusions

We end this master thesis with a summary of the most important ideas we have been gathering during the whole work.

## On Lie groups and algebras

Regarding the formal definition, a Lie group is smooth manifold which is also a group, meaning that its product operation and inversion map are smooth maps under the corresponding differential structure. Nonetheless, we have approached Lie groups mainly as matrix Lie groups, which are those matrix groups such that any convergent sequence has a limit either inside the group itself or outside the General Linear group. It is worth noting that not all Lie groups can be realised as matrix Lie groups.

Lie algebras are vector spaces with an additional operation called Lie bracket. Unlike Lie groups, all Lie algebras are isomorphic to some matrix Lie algebra, where the Lie bracket operation translates into what we call the commutator of the algebra. In the differential setting, a Lie algebra of a Lie group can be defined as the tangent space to the group by its identity element, bringing the idea that Lie groups can be attached to Lie algebras.

The link between these two structures is the exponential map,

$$\exp : \mathrm{Lie}(G) \longrightarrow G$$

defined over the elements of the Lie algebra $\mathrm{Lie}(G)$ of the group $G$ as

$$\exp(A) = \sum_{k\geq 0} \frac{1}{k!} A^k,$$

$A \in \mathrm{Lie}(G)$. However, the relation Lie algebra versus Lie group is not one-to-one, meaning that two different Lie groups may have the same Lie algebra. Because of this, regarding Lie groups and Lie algebras as categories one finds that a functor between them is not fully faithful in general, though when restrictied to simply connected Lie groups a one-to-one relation can be achieved.

From a topological point of view, connectedness, simple connectedness and compactness are the most relevant aspects of a Lie group. An important fact is that any Lie group admits a universal cover which is a simple connected Lie group. However, because of the Lie functor not being fully faithful in general, two different Lie groups may have the same (simply connected) covering Lie group.

All this theory can be illustrated with the case of $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$, which are not isomorphic as Lie groups but have isomorphic Lie algebras, and relate themselves by the fact that the former is the 2-leaved universal cover of the last.

# On Lie group representations

Aside from the general results of groups representations such as Schur's lemma, the key points in Lie group representation theory are those concerning compact Lie groups representations. These representations can be accessed through the Haar integral, which is a normalized and left-invariant integral (in the sense that multyplying by the left does not change the integral). The Haar integral gives rise to the concepts of wieghts and roots, which play an important role in said representations.

We have studied the adjoint representation as an example, finding that if $\mathfrak{g}$ is the Lie algebra of the Lie group $G$, then the adjoint map lets us view $\mathsf{GL}(\mathfrak{g})$ as a representation of $G$.

As important results, we have seen that any compact Lie group is orientable, that representations of compact Lie groups are always unitary and that, moreover, any complex representation of a compact Lie group can we written as a direct sum of irreducible representations.

# On $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$

Aside from the fact that $\mathsf{SU}(2)$ is the universal cover of $\mathsf{SO}(3)$, the main features of these groups arise from their topological properties.

$\mathsf{SU}(2)$ is a compact, connected and simply connected Lie group so, when regarded as a smooth manifold, implies that it is isomorphic (as a Lie group) and diffeomorphic (as a manifold) to the real 3-sphere. In the other hand, and despite sharing the same Lie algebra and thus having the same dimension, $\mathsf{SO}(3)$ is compact and connected but not simply connected, as $\pi_1(\mathsf{SO}(3)) \cong \mathbb{Z}/2$. If we look at the fact that $\mathsf{SU}(2) \cong \mathbb{S}^3$ is the universal cover of $\mathsf{SO}(3)$, we can deduce that $\mathsf{SO}(3)$ is isomorphic to the 3-dimensional real projective space as a manifold. In the end, all this can be summarized with the following nice formula

$$\mathsf{SO}(3) \cong \mathsf{SU}(2)/\{I, -I\}.$$

On the representations side, we have seen that all the $n$-dimensional representations of $\mathsf{SU}(2)$ are isomorphic to the vector space

$$V_n := \{p \in \mathbb{C}[z_1, z_2] \mid \deg p = n,\ p \text{ homogeneous}\}$$

and that such representations are irreducible. Then, using the above relation between $\mathsf{SU}(2)$ and $\mathsf{SO}(3)$ we have concluded that $V_{2n}$ are the representations of $\mathsf{SO}(3)$.

Additionally, we have taken a peek on the representations of $\mathsf{SU}(3)$ by proving that they are isomorphic to the representations of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. Such representations can always be factorized as the direct sum of an abelian subalgebra plus an idexed set of other subalgebras, namely

$$\mathfrak{sl}(3, \mathbb{C}) = \mathfrak{h} \oplus \left(\bigoplus \mathfrak{g}_\alpha\right),$$

where $\alpha$ ranges over a finite subset of $\mathfrak{h}^*$ (regarded as the space of forms) and $\mathfrak{h}$ acts on $\mathfrak{g}_\alpha$ by scalar multiplication.

Both the representations of $\mathsf{SU}(2)$ and $\mathsf{SU}(3)$ have been applied to study certain aspects of particle physics, the first for the isospin and the second for a number of classifications of elementary particles.

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