ON THE CONNECTIVITY OF THE JULIA SETS OF MEROMORPHIC FUNCTIONS

KRZYSZTOF BARAŃSKI, NÚRIA FAGELLA, XAVIER JARQUE, AND BOGUSŁAWA KARPIŃSKA

Abstract. We prove that every transcendental meromorphic map f with disconnected Julia set has a weakly repelling fixed point. This implies that the Julia set of Newton's method for finding zeroes of an entire map is connected. Moreover, extending a result of Cowen for holomorphic self-maps of the disc, we show the existence of absorbing domains for holomorphic self-maps of hyperbolic regions, whose iterates tend to a boundary point. In particular, the results imply that periodic Baker domains of Newton's method for entire maps are simply connected, which solves a well-known open question.

1. Introduction

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant and non-Möbius holomorphic map from the complex plane $\mathbb{C}$ to the Riemann sphere $\widehat{\mathbb{C}}$. If the point at infinity is an essential singularity of f, then we call f a transcendental meromorphic map; otherwise f extends to the sphere as a rational map. We consider the dynamical system given by the iterates of f, which induces a dynamical partition of the complex sphere into two completely invariant sets: the *Fatou set* $F(f)$, which is the set of points $z \in \widehat{\mathbb{C}}$, where the family of iterates $\{f^n\}_{n\geq 0}$ is defined and normal in some neighborhood of z, and its complement, the *Julia set* $J(f) = \widehat{\mathbb{C}} \setminus F(f)$. The Fatou set is open and consists of points with, in some sense, stable dynamics, while the Julia set is closed and its points exhibit chaotic behavior. Moreover, $J(f)$ is the closure of the set of repelling periodic points of f (see [4]). If f is transcendental meromorphic, then the Julia set always contains the point at infinity and (unless f has a unique omitted pole), it is the closure of the set of all prepoles of f, while the Fatou set is unbounded or empty. For general background on the dynamics of rational and meromorphic maps we refer to [7, 13, 31].

Connected components of the Fatou set, known as *Fatou components*, are mapped by f among themselves. A Fatou component U is *periodic* of period p, or *p-periodic*, if $f^p(U) \subset U$; a component which is not eventually periodic is called *wandering*. Unlike the rational case [41], transcendental meromorphic maps may have wandering components. There is a complete classification of periodic Fatou components: such a component can either be a rotation domain (Siegel disc or Herman ring), the basin of attraction of an attracting or parabolic periodic point or a *Baker domain* (the latter possibility can occur only for transcendental maps). Recall that a p-periodic Fatou component $U \subset \mathbb{C}$ is a Baker domain, if f^{pn} on U tend to a point ζ in the boundary of U as $n \to \infty$, and $f^j(\zeta)$ is not defined for some $j \in \{0, \dots p-1\}$. This implies

Date: January 8, 2014.

2000 *Mathematics Subject Classification.* Primary 30D05, 37F10, 30D20.

The first author is partially supported by Polish NCN Grant N N201 607940. The second and third authors were partially supported by the Catalan grant 2009SGR-792, and by the Spanish grants MTM-2006-05849 and MTM-2008-01486 Consolider (including a FEDER contribution) and MTM2011-26995-C02-02. The fourth author is partially supported by Polish NCN Grant N N201 607940 and Polish PW Grant 504G 1120 0011 000.

the existence of an unbounded Fatou component U' in the same cycle, such that $f^{pn} \to \infty$ on U'. The first example of a Baker domain was given by Fatou [21], who considered the function $f(z) = z + 1 + e^{-z}$ and showed that the right half-plane is contained in an invariant Baker domain. If f is an entire function, then all its Baker domains (and other periodic Fatou components) must be simply connected [2]. In the case of meromorphic maps, Baker domains are, in general, multiply connected, as shown in examples by Dominguez [15] and König [25]. There are a number of papers studying dynamical properties of Baker domains, see e.g. [6, 17, 18] for the entire case and [9, 34, 35] for the meromorphic one.

In this paper we study the relation of the connectivity of the Julia set and the existence of weakly repelling fixed points for meromorphic maps. We say that a fixed point z_0 of a holomorphic map f is *weakly repelling*, if $|f'(z_0)| > 1$ or $f'(z_0) = 1$ (with the standard extension to $z_0 = \infty$ in the rational case). It was proved by Julia [24, pp. 84, 243] and Fatou [21, Ch. 1, p. 168] that a rational map of degree greater than one has at least one weakly repelling fixed point in $\widehat{\mathbb{C}}$. In 1990, Shishikura [40] proved a remarkable result, showing that if f is rational and its Julia is disconnected, then f has at least two weakly repelling fixed points in $\widehat{\mathbb{C}}$. For transcendental meromorphic maps the situation is more complicated, since they need not have fixed points at all. However, the point at infinity can be treated as an additional "fixed point".

In this paper we prove the following result.

Main Theorem. *Let f be a transcendental meromorphic function with disconnected Julia set. Then f has at least one weakly repelling fixed point.*

An important motivation for this theorem is the question of the connectivity of Julia sets of the celebrated Newton's method

$$N_g(z) = z - \frac{g(z)}{g'(z)}$$

for finding zeroes of an entire map $g : \mathbb{C} \to \mathbb{C}$. The dynamical properties of Newton's method, especially for polynomials g, were studied in a number of papers, see e.g. [23, 26, 28, 29, 30, 33, 37, 42]. Notice that the map N_g is meromorphic, its fixed points in $\mathbb{C}$ are, precisely, zeroes of g, and all of them are attracting. For a polynomial g, the map N_g is rational and the point at infinity is a repelling fixed point, while for transcendental entire g, its Newton's method is transcendental meromorphic (except the case $g = pe^q$ for polynomials p, q, when N_g is rational and the point at infinity is a parabolic fixed point of multiplier 1, see [22, Proposition 1] or [37, Proposition 2.11]). Hence, Shishikura's result shows that for polynomials g, the Julia set of N_g is connected. Our theorem immediately implies the following corollary, which solves a well-known open problem, formulated e.g. in [38, Question 8.6].

Corollary. *If g is an entire map and N_g is its Newton's method, then $J(N_g)$ is connected.*

Since the Julia set is closed, it is connected if and only if all the Fatou components are simply connected. Therefore, the proof of the Main Theorem splits into several cases – for each type of the Fatou component one should show that if it is multiply connected, then the map has a weakly repelling fixed point. However, Shishikura's proofs in the rational case cannot be directly extended to the transcendental one, because of the appearance of new phenomena such as lack of compactness, presence of asymptotic values and new types of Fatou components.

For transcendental meromorphic maps, the case of wandering domains was solved by Bergweiler and Terglane in [10], while the cases of attracting or parabolic cycles and preperiodic components were dealt with by Fagella, Jarque and Taixés in [19, 20]. Therefore, the remaining cases were Baker domains and Herman rings, which are the subject of the present work.

The known proofs for a p-periodic Fatou component U, such that $f^{pn} \to \zeta$ on U as $n \to \infty$ (i.e. when U is the basin of attraction of an attracting or parabolic periodic point), are based on the existence of a simply connected domain $W \subset U$, which is absorbing for $F = f^p$ and tends to ζ under iterations of F.

Definition (Absorbing domain). Let U be a domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map. A domain $W \subset U$ is *absorbing* in U for F, if $F(W) \subset W$ and for every compact set $K \subset U$ there exists $n = n(K) \geq 0$, such that $F^n(K) \subset W$.

The problem of existence of suitable absorbing domains has a long history. For attracting and parabolic basins it is a part of the classical problem of studying the local behavior of an analytic map near a fixed point. In particular, if U is the basin of a (super)attracting p-periodic point ζ, then $F = f^p$ is conformally conjugate to $z \mapsto F'(\zeta)z$ (if $F'(\zeta) \neq 0$) or $z \mapsto z^k$ for some integer $k \geq 2$ (if $F'(\zeta) = 0$) near $z = 0$. In this case, if we take W to be the preimage of a small disc centered at $z = 0$ under the conjugating map, then W is a simply connected absorbing domain for F and $\bigcap_{n\geq 0} F^n(\overline{W}) = \{\zeta\}$. Likewise, if U is a basin of a parabolic p-periodic point, an attracting petal in U would provide a similar example.

The existence of such absorbing regions in Baker domains was an open question, and one of the main obstacles for the completion of the proof of the Main Theorem. In this paper we prove that we can always construct suitable absorbing regions in Baker domains, if we drop the condition of simple connectedness. This is a corollary of the following more general theorem, which we prove in Section 3. We consider here holomorphic maps $F : U \to U$ on a hyperbolic domain $U \subset \mathbb{C}$, such that $F^n \to \zeta$ as $n \to \infty$ for some ζ in the boundary of U in $\widehat{\mathbb{C}}$. Changing coordinates by a Möbius transformation, we can assume $\zeta = \infty$. We denote by $\mathcal{D}_U(z, r)$ the disc of radius r centered at $z \in U$, with respect to the hyperbolic metric in U.

Theorem A (Existence of absorbing regions for holomorphic self-maps of hyperbolic domains). *Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map, such that $F^n \to \infty$ as $n \to \infty$. Then for every point $z \in U$ and every sequence of positive numbers r_n, $n \geq 0$ with $\lim_{n\to\infty} r_n = \infty$, there exists a domain $W \subset U$, such that:*

(*a*) $W \subset \bigcup_{n=0}^{\infty} \mathcal{D}_U(F^n(z), r_n)$,
(*b*) $\overline{W} \subset U$,
(*c*) $F^n(\overline{W}) = \overline{F^n(W)} \subset F^{n-1}(W)$ *for every* $n \geq 1$,
(*d*) $\bigcap_{n=0}^{\infty} F^n(\overline{W}) = \emptyset$,
(*e*) W *is absorbing in* U *for* F.

Moreover, F is locally univalent on W.

This theorem is an extension of the well-known Cowen's result [14] (see also Pommerenke [32] and Baker–Pommerenke [5]) on absorbing regions for holomorphic self-maps of simply connected domains. Recall that if G is a holomorphic self-map of the right half-plane $\mathbb{H}$ without fixed points, then Denjoy–Wolff's Theorem ensures that (after a possible change of coordinates) $G^n \to \infty$ uniformly on compact sets in $\mathbb{H}$. Cowen's result implies the existence of a simply connected absorbing domain $V \subset \mathbb{H}$, such that $\overline{V} \subset \mathbb{H}$, $G^n(\overline{V}) = \overline{G^n(V)} \subset G^{n-1}(V)$ for $n \geq 1$ and $\bigcap_{n\geq 0} G^n(\overline{V}) = \emptyset$. Moreover, there exists a univalent map $\varphi : V \to \mathbb{C}$ conjugating G to a map T of the form $T(\omega) = \omega + 1$, $T(\omega) = \omega \pm i$ or $T(\omega) = a\omega$, $a > 1$ on

$\Omega \in \{\mathbb{C}, \mathbb{H}\}$ and φ extends to a holomorphic map from $\mathbb{H}$ to Ω, which semi- conjugates G to T (see Theorem 2.6 for details). Using the Riemann Mapping Theorem, one can apply this result to a holomorphic self-map F of any simply connected region U, without fixed points.

Applied to the case of Baker domains, Theorem A has the following form.

Corollary A' (Existence of absorbing regions in Baker domains). *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic map and let U be a periodic Baker domain of period p such that $f^{pn} \to \infty$ as $n \to \infty$. Then there exists a domain $W \subset U$ with the properties listed in Theorem* A *for the map $F = f^p$.*

Note that if U is a simply connected Baker domain (which is always the case for entire maps), Cowen's Theorem immediately provides the existence of a suitable simply connected absorbing region in U. In the case of a multiply connected p-periodic Baker domain U of a meromorphic map f, one can consider a universal covering map $\pi : \mathbb{H} \to U$ and lift $F = f^p$ by π to a holomorphic map $G : \mathbb{H} \to \mathbb{H}$ without fixed points. König [25] showed that if f has finitely many poles, then the absorbing region $V \subset \mathbb{H}$ projects under π to a suitable simply connected absorbing region $W \subset U$ (see Theorem 2.7 for a precise statement). However, [25] contains examples showing that there are Baker domains which do not admit simply connected absorbing regions.

Hence, Corollary A' can be treated as a generalization of König's result, which weakens the assumptions on the map f and provides some estimates on the size of the absorbing region, but does not ensure simple connectivity of W.

Using Corollary A', we are able to prove:

Theorem B. *Let f be a transcendental meromorphic map with a multiply connected periodic Baker domain. Then f has at least one weakly repelling fixed point.*

In particular, Theorem B implies:

Corollary B'. *Periodic Baker domains of a Newton's method N_g for an entire map g are simply connected.*

This solves a well-known open question, raised e.g. by Bergweiler, Buff, Rückert, Mayer and Schleicher [8, 12, 28, 37]. In particular, Corollary B' implies that so-called virtual immediate basins for Newton maps (i.e. invariant simply connected unbounded domains in $\mathbb{C}$, where the iterates of the map converge locally uniformly to ∞), defined by Mayer and Schleicher [28], are equal to the entire invariant Baker domains.

Apart from Corollary A', the proof of Theorem B uses several general results on the existence of weakly repelling fixed points of meromorphic maps on some domains in the complex plane, under certain combinatorial assumptions. These tools, which are developed in Section 4, have some interest in themselves, since they generalize the results used by Shishikura, Bergweiler and Terglane [10, 40] and can be applied in a wider setup. In particular, we use them to prove the following result, which completes the proof of the Main Theorem.

Theorem C. *Let f be a transcendental meromorphic map with a cycle of Herman rings. Then f has at least one weakly repelling fixed point.*

The proof of Theorem C applies also to the rational setting and is an alternative to Shishikura's arguments for Herman rings of rational maps.

The paper is organized as follows. In Section 2 we state and reference some results we use in this paper. They include estimates of the hyperbolic metric, the theorems of Cowen and

König on the existence of absorbing domains and the results of Buff and Shishikura on the existence of weakly repelling fixed points for holomorphic maps. Section 3 contains the proof of Theorem A. The proofs of Theorems B and C are contained, respectively, in Sections 5 and 6, with an initial Section 4 which contains preliminary results on the existence of weakly repelling fixed points in various configurations of domains.

Acknowledgements. We wish to thank the Institut de Matemàtiques de la Universitat de Barcelona for its hospitality.

2. Background and tools

In this section we introduce notation and review the necessary background to prove the main results of the paper.

First, we present basic notation. The symbol $\operatorname{dist}(\cdot,\cdot)$ denotes the Euclidean distance on the complex plane $\mathbb{C}$. For a set $A \subset \mathbb{C}$, the symbols $\overline{A}$, ∂A denote, respectively, the closure and boundary in $\mathbb{C}$. The Euclidean disc of radius r centered at $z \in \mathbb{C}$ and the right half-plane are denoted, respectively, by $\mathbb{D}(z, r)$ and $\mathbb{H}$. The unit disc $\mathbb{D}(0, 1)$ is simply written as $\mathbb{D}$.

For clarity of exposition we divide this section into three parts. The first one contains standard estimates of hyperbolic metric. In the second and third one we present, respectively, some known results on the existence of absorbing domains and weakly repelling fixed points for holomorphic maps.

2.1. Hyperbolic metric and Schwarz–Pick's Lemma.

Let U be a domain in the Riemann sphere $\widehat{\mathbb{C}}$. We call U hyperbolic, if its boundary in $\widehat{\mathbb{C}}$ contain at least three points. By the Uniformization Theorem, in this case there exists a universal holomorphic covering π from $\mathbb{D}$ (or $\mathbb{H}$) onto U. Every holomorphic map $F : U \to U$ can be lifted by π to a holomorphic map $G : \mathbb{H} \to \mathbb{H}$, such that the diagram

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{G} & \mathbb{H} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{F} & U \end{array}$$

commutes. By $\varrho_U(\cdot)$ and $\varrho_U(\cdot,\cdot)$ we denote, respectively, the density of the hyperbolic metric and the hyperbolic distance in U. In particular, we will extensively use the hyperbolic metric in $\mathbb{D}$ and $\mathbb{H}$ of density

$$\varrho_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2} \quad \text{and} \quad \varrho_{\mathbb{H}}(z) = \frac{1}{\operatorname{Re}(z)},$$

respectively. In particular, we have

$$\varrho_{\mathbb{D}}(z, 0) = \ln \frac{1 + |z|}{1 - |z|} \tag{1}$$

for $z \in \mathbb{D}$.

By $\mathcal{D}_U(z, r)$ we denote the hyperbolic disc of radius r, centered at $z \in U$ (with respect to the hyperbolic metric in U). The following lemma contains well-known inequalities related to the hyperbolic metric.

Lemma 2.1 (**Hyperbolic estimates I** [13, Theorem 4.3]). *Let $U \subset \mathbb{C}$ be a hyperbolic domain. Then*

$$\varrho_U(z) \le \frac{2}{\operatorname{dist}(z, \partial U)} \qquad \text{for } z \in U$$

and

$$\varrho_U(z) \ge \frac{1 + o(1)}{\operatorname{dist}(z, \partial U) \log(1/\operatorname{dist}(z, \partial U))} \qquad \text{as } z \to \partial U.$$

Moreover, if U is simply connected, then

$$\varrho_U(z) \ge \frac{1}{2\operatorname{dist}(z, \partial U)} \qquad \text{for } z \in U.$$

Every holomorphic map between hyperbolic domains does not increase the hyperbolic metric. This very useful result is known as the Schwarz–Pick Lemma.

Lemma 2.2 (**Schwarz–Pick's Lemma** [13, Theorem 4.1]). *Let $U, V \subset \mathbb{C}$ be hyperbolic domains and let $f : U \to V$ be a holomorphic map. Then*

$$\varrho_V(f(z_1), f(z_2)) \le \varrho_U(z_1, z_2)$$

for every $z_1, z_2 \in U$. In particular, if $U \subset V$, then

$$\varrho_V(z_1, z_2) \le \varrho_U(z_1, z_2),$$

with strict inequality unless $z_1 = z_2$ or f lifts to a Möbius transformation from $\mathbb{H}$ onto $\mathbb{H}$.

Using this lemma and properties of the hyperbolic metric in $\mathbb{C} \setminus \{0, 1\}$ we can easily deduce the following estimate, which will be useful in further parts of the paper. We sketch its proof for completeness.

Lemma 2.3 (**Hyperbolic estimates II**). *Let $U \subset \mathbb{C}$ be an unbounded hyperbolic domain. Then there exists $c > 0$ such that*

$$\varrho_U(z) > \frac{c}{|z| \log |z|}$$

if $z \in U$ and $|z|$ is sufficiently large.

Proof. Since U is hyperbolic, there exist two distinct points $z_0, z_1 \in \mathbb{C} \setminus U$, so U is a subset of $U' = \mathbb{C} \setminus \{z_0, z_1\}$. By Schwarz–Pick's Lemma 2.2, we have $\varrho_U(z) \ge \varrho_{U'}(z)$ for $z \in U$. At the same time, $\varrho_{U'}(z) = c\varrho_{U''}(w)$ for $U'' = \mathbb{C} \setminus \{0, 1\}$, where $w = (z_1 - z_0)z + z_0$ is the affine map transforming U'' onto U' and $c = 1/|z_0 - z_1|$. The standard estimates of the hyperbolic metric in U'' (see e.g. [1, 13]) give

$$\varrho_{U''}(w) = \frac{\mathcal{O}(1)}{|w| \log(1/|w|)}$$

as $|w| \to 0$. Transforming the metric under $1/w$, which leaves U'' invariant, we obtain

$$\varrho_{U''}(w) = \frac{\mathcal{O}(1)}{|w| \log |w|}$$

as $|w| \to \infty$, so

$$\varrho_U(z) \ge c\varrho_{U''}(w) = \frac{\mathcal{O}(1)}{|w| \log |w|} = \frac{\mathcal{O}(1)}{|z| \log |z|}$$

as $|z| \to \infty$, from which the estimate follows. ∎

The next result follows easily from the algebraic properties of universal coverings (see e.g. [27, Theorem 2] or [25, Lemma 4]). We include its proof for completeness.

Lemma 2.4. *Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map, such that for some ζ in the boundary of U in $\widehat{\mathbb{C}}$ we have $F^n(z) \to \zeta$ as $n \to \infty$ for $z \in U$. Let $\pi : \mathbb{H} \to U$ be a holomorphic universal covering and let $G : \mathbb{H} \to \mathbb{H}$ be a lift of F by π, i.e. $F \circ \pi = \pi \circ G$. Suppose that G is univalent. Then the induced endomorphism F^* of the fundamental group of U is an isomorphism. Moreover, if additionally, for every closed curve $\gamma \subset U$ there exists $n \ge 0$ such that $F^n(\gamma)$ is contractible in U, then U is simply connected and π is a Riemann map.*

Proof. The domain U is isomorphic (as a Riemann surface) to the quotient $\mathbb{H}/\Gamma$, where Γ is the group of cover transformations acting on $\mathbb{H}$. The group Γ is isomorphic to the fundamental group of U, denoted by $\pi_1(U)$. For $n \ge 0$ let $\theta_n : \Gamma \to \Gamma$ be an endomorphism induced by G^n (i.e. $G^n \circ g = \theta_n(g) \circ G^n$ for $g \in \Gamma$). The endomorphism θ_n corresponds to an endomorphism $\tilde{\theta}_n = (F^n)^* : \pi_1(U) \to \pi_1(U)$ induced by F^n (see [27]). Set $N = \bigcup_{n=0}^{\infty} \ker \theta_n$, $\tilde{N} = \bigcup_{n=0}^{\infty} \ker \tilde{\theta}_n$. Since G is univalent, we have $N = \{\mathrm{id}\} = \tilde{N}$, so $(F^n)^*$ is an isomorphism.

Suppose that for every closed curve $\gamma \subset U$ there exists $n \ge 0$ such that $F^n(\gamma)$ is contractible in U. Then $\pi_1(U) = \tilde{N} = \{\mathrm{id}\}$, so U is simply connected and π is a Riemann map. □

2.2. **Lifts of maps and absorbing domains.** Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map. Recall that a domain $W \subset U$ is absorbing in U for F, if $F(W) \subset W$ and for every compact set $K \subset U$ there exists $n > 0$, such that $F^n(K) \subset W$. The main goal of this subsection is to present results due to Cowen and König on the existence of absorbing domains.

Recall first the classical Denjoy–Wolff Theorem, which describes the dynamics of a holomorphic map G in $\mathbb{H}$.

Theorem 2.5 (**Denjoy–Wolff's Theorem** [13, Theorem 3.1])**.** *Let $G : \mathbb{H} \to \mathbb{H}$ be a non-constant holomorphic map, which is not an automorphism of $\mathbb{H}$. Then there exists a point $z_0 \in \overline{\mathbb{H}} \cup \{\infty\}$ (called the Denjoy–Wolff point of G), such that G^n tends to z_0 uniformly on compact subsets of $\mathbb{H}$ as $n \to \infty$.*

The following result, due to Cowen, gives the main tool for constructing absorbing domains.

Theorem 2.6 (**Cowen's Theorem** [14, Theorem 3.2], see also [25, Lemma 1])**.** *Let $G : \mathbb{H} \to \mathbb{H}$ be a holomorphic map such that $G^n \to \infty$ as $n \to \infty$. Then there exists a simply connected domain $V \subset \mathbb{H}$, a domain Ω equal to $\mathbb{H}$ or $\mathbb{C}$, a holomorphic map $\varphi : \mathbb{H} \to \Omega$, and a Möbius transformation T mapping Ω onto itself, such that:*

(*a*) *V is absorbing in $\mathbb{H}$ for G,*
(*b*) *$\varphi(V)$ is absorbing in Ω for T,*
(*c*) *$\varphi \circ G = T \circ \varphi$ on $\mathbb{H}$,*
(*d*) *φ is univalent on V.*

Moreover, φ, T depend only on G. In fact (up to a conjugation of T by a Möbius transformation preserving Ω), one of the following cases holds:

- *$\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$,*
- *$\Omega = \mathbb{H}$, $T(\omega) = \omega \pm i$,*
- *$\Omega = \mathbb{H}$, $T(\omega) = a\omega$ for some $a > 1$.*

Using Cowen's result, König proved the following theorem which provides the existence of simply connected absorbing domains in U for F under certain assumptions. In particular, these assumptions are trivially satisfied if U is simply connected.

Theorem 2.7 (**König's Theorem** [25]). *Let U be a hyperbolic domain in $\mathbb{C}$ and let $F : U \to U$ be a holomorphic map, such that $F^n \to \infty$ as $n \to \infty$. Suppose that for every closed curve $\gamma \subset U$ there exists $n > 0$ such that $F^n(\gamma)$ is contractible in U. Then there exists a simply connected domain $W \subset U$, a domain Ω and a transformation T as in Cowen's Theorem* 2.6*, and a holomorphic map $\psi : U \to \Omega$, such that:*

- (*a*) *W is absorbing in U for F,*
- (*b*) *$\psi(W)$ is absorbing in Ω for T,*
- (*c*) *$\psi \circ F = T \circ \psi$ on U,*
- (*d*) *ψ is univalent on W.*

In fact, if we take V and φ from Cowen's Theorem 2.6 *for G being a lift of F by a universal covering $\pi : \mathbb{H} \to U$, then π is univalent in V and one can take $W = \pi(V)$ and $\psi = \varphi \circ \pi^{-1}$, which is well defined in U.*

Moreover, if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a meromorphic map with finitely many poles, and U is a periodic Baker domain of period p, then the above assumptions are satisfied for $F = f^p$, and consequently, there exists $W \subset U$ with the properties (*a*)–(*d*) *for $F = f^p$.*

2.3. **Existence of weakly repelling fixed points.** We shall use several tools to establish the existence of weakly repelling fixed points in certain subsets of the plane. The results in this section will not be used until Section 5.

The first classical result in this direction is due to Julia and Fatou.

Theorem 2.8 ([21, Ch. 1, p. 168], [24, pp. 84, 243]). *Every rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with* $\deg f \geq 2$ *has at least one weakly repelling fixed point.*

In view of this, a map which locally behaves as a rational map should also have points of the same character. This is formalized in the following two propositions. By a proper map $f : D' \to D$ we mean a map from D' onto D, such that for every compact set $X \subset D$, the set $f^{-1}(X)$ is compact. Proper maps always have well defined finite degree.

Theorem 2.9 (**Polynomial-like maps** [16]). *Let D and D' be simply connected domains in $\mathbb{C}$ such that $\overline{D'} \subset D$ and let $f : D' \to D$ be a proper holomorphic map. Then f has a weakly repelling fixed point in D'.*

Indeed, if $\deg f|_{D'} = 1$, then f is invertible and, by Schwarz–Pick's Lemma 2.2 applied to f^{-1}, the map f has a repelling fixed point. Otherwise, (f, D', D) form a polynomial-like map. By the Straightening Theorem (see [16]), $f|_{D'}$ is conjugate to a polynomial and therefore has a weakly repelling fixed point.

Theorem 2.10 (**Rational-like maps** [11]). *Let D and D' be domains in $\mathbb{C}$ with finite Euler characteristic, such that $\overline{D'} \subset D$ and let $f : D' \to D$ be a proper holomorphic map. Then f has a weakly repelling fixed point in D'.*

Maps with this property are called rational-like (see [36]). The proof of the result above is due to Buff and can be found in [11], where he actually shows the existence of *virtually repelling fixed points*, which is a stronger statement. (Note that in [11] rational-like maps are assumed to have degree larger than one. However, the proof is valid also in the case of degree one.)

In the following result, also proved in [11], the hypothesis of compact containment is relaxed. In return, the image is assumed to be a disc.

Theorem 2.11 (**Rational-like maps with boundary contact** [11]). *Let D be an open Euclidean disc in $\mathbb{C}$ and $D' \subset D$ be a domain with finite Euler characteristic. Let $f : D' \to D$ be a proper map of degree greater than one, such that $|f(z) - z|$ is bounded away from zero as $z \to \partial D'$. Then f has a weakly repelling fixed point in D'.*

By a meromorphic map on a domain $D \subset \widehat{\mathbb{C}}$ we mean an analytic map from D to $\widehat{\mathbb{C}}$. The result above implies the following corollary.

Corollary 2.12 (**Rational-like maps with boundary contact**). *Let D be a simply connected domain in $\widehat{\mathbb{C}}$ with locally connected boundary and $D' \subset D$ a domain in $\widehat{\mathbb{C}}$ with finite Euler characteristic. Let f be a continuous map on the closure of D' in $\widehat{\mathbb{C}}$, meromorphic in D', such that $f : D' \to D$ is proper. If* $\deg f > 1$ *and f has no fixed points in $\partial D \cap \partial D'$, or* $\deg f = 1$ *and $D \neq D'$, then f has a weakly repelling fixed point in D'.*

Proof. Suppose $\deg f > 1$. Changing the coordinates in $\widehat{\mathbb{C}}$ by a Möbius transformation, we can assume $D \subset \mathbb{C}$. Let φ be a Riemann map from the unit disc $\mathbb{D}$ onto D. Since the boundary of D is locally connected, the map φ extends continuously to $\overline{\mathbb{D}}$. Let $g = \varphi^{-1} \circ f \circ \varphi$ on $\varphi^{-1}(D')$. Then $g : \varphi^{-1}(D') \to \mathbb{D}$ satisfies the assumptions of Theorem 2.11. Indeed, one should only check that $|g(z) - z|$ is bounded away from zero as $z \to \partial(\varphi^{-1}(D'))$. If it was not the case, then there would exist a sequence $z_n \in \varphi^{-1}(D')$ with $z_n \to \partial(\varphi^{-1}(D'))$ and $|z_n - g(z_n)| \to 0$. We can assume $z_n \to z \in \partial(\varphi^{-1}(D'))$. Then $g(z_n) \to z$, $\varphi(z_n) \to \varphi(z)$ and $\varphi(z)$ is in the boundary of D', so $f(\varphi(z_n)) = \varphi(g(z_n)) \to \varphi(z)$ and $f(\varphi(z)) = \varphi(z)$. Since $f : D' \to D$ is proper, $\varphi(z)$ is in the boundary of D, so $\varphi(z)$ is a fixed point of f in the intersection of the boundaries of D and D', which contradicts the assumptions of the corollary.

If $\deg f = 1$, then by the Riemann–Hurwitz Formula, D' is simply connected and f is invertible, so the existence of a repelling fixed point of f follows from Schwarz–Pick's Lemma 2.2 applied to f^{-1}. □

To apply this corollary we have to ensure the local connectedness of the boundary of the domain. We shall often use the following result due to Torhorst.

Theorem 2.13 (**Torhorst's Theorem** [43, p. 106, Theorem 2.2]). *If X is a locally connected continuum in $\widehat{\mathbb{C}}$, then the boundary of every component of $\widehat{\mathbb{C}} \setminus X$ is a locally connected continuum.*

We conclude this section stating a surgery result due to Shishikura, which will be generalized in Section 4 (see Proposition 4.7).

Theorem 2.14 (**Shishikura** [40, Theorem 2.1]). *Let V_0, V_1 be simply connected domains in $\widehat{\mathbb{C}}$ with $V_0 \neq \widehat{\mathbb{C}}$ and let f be a meromorphic map in a neighbourhood N of $\widehat{\mathbb{C}} \setminus V_0$, such that $f(\partial V_0) = \partial V_1$ and $f(V_0 \cap N) \subset V_1$. Suppose that for some $k \geq 1$, the map f^k is defined on V_1, such that*

$$f^j(V_1) \cap V_0 = \emptyset \text{ for } j = 0, \dots, k-1 \quad \text{and} \quad f^k(\overline{V_1}) \subset V_0.$$

Then f has a weakly repelling fixed point in $\widehat{\mathbb{C}} \setminus \overline{V_0}$.

See Figure 1.

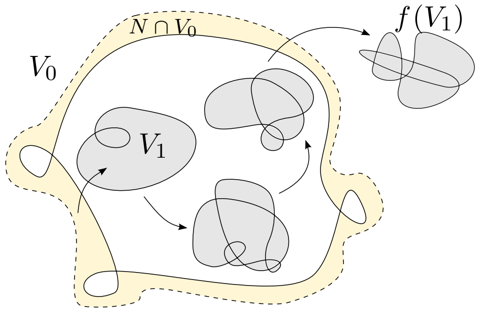

FIGURE 1. Setup of Theorem 2.14.

3. Proof of Theorem A

The general setup for this section is the following. Let U be a hyperbolic domain in $\mathbb{C}$. Then there exists a holomorphic universal covering π from $\mathbb{H}$ onto U. Take a holomorphic map $F : U \to U$ as in Theorem A. Then F can be lifted to a holomorphic map $G : \mathbb{H} \to \mathbb{H}$, such that

$$F \circ \pi = \pi \circ G.$$

Since F has no fixed points, the map G has no fixed points either, so by the Denjoy–Wolff's Theorem 2.5, conjugating G by a suitable Möbius transformation preserving $\mathbb{H}$, we can assume that $G^n \to \infty$ as $n \to \infty$. Hence, by Cowen's Theorem 2.6, G is semi-conjugated to a Möbius transformation $T : \Omega \to \Omega$, where $\Omega \in \{\mathbb{C}, \mathbb{H}\}$, by a holomorphic map φ, which is univalent on a simply connected absorbing domain $V \subset \mathbb{H}$. In other words, we have the following commutative diagram.

$$\begin{array}{ccccc}
\varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\
\Big\downarrow{\scriptstyle \varphi^{-1}} & & \Big\uparrow{\scriptstyle \varphi} & & \Big\uparrow{\scriptstyle \varphi} \\
V & \subset & \mathbb{H} & \xrightarrow{G} & \mathbb{H} \\
 & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
 & & U & \xrightarrow{F} & U
\end{array}$$

We use the above notation throughout the proof.

Since the proof of Theorem A is rather technical, we first briefly discuss its geometric ideas. We will define the absorbing set W as the projection $W = \pi(\varphi^{-1}(A))$ of a suitable domain $A \subset \varphi(V)$, which is absorbing for T. Then one can easily show that W is absorbing for F. However, we should be careful to define A sufficiently "thin", so that $\overline{W} \subset U$ and $\bigcap_{n=1}^{\infty} F^n(\overline{W}) = \emptyset$ (a priori, we could have e.g. $W = U$).

Notice that the map T is an isometry with respect to the hyperbolic metric in $\mathbb{H}$ (in the case $\Omega = \mathbb{H}$) or the Euclidean metric in $\mathbb{C}$ (in the case $\Omega = \mathbb{C}$). Hence, the idea is to define A (in the case $\Omega = \mathbb{H}$) in the form

$$A = \bigcup_{n \geq m} \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)$$

for a point $\omega \in \Omega$ and a suitable sequence c_n which increases to ∞ sufficiently slowly (in the case $\Omega = \mathbb{C}$ we take Euclidean discs instead of hyperbolic ones). Then we show that

$\overline{A} \subset \varphi^{-1}(V)$, A is absorbing for T and (by Schwarz–Pick's Lemma), $T(\overline{A}) \subset A$. Moreover, taking a suitable sequence c_n, we can achieve

$$\overline{A} \subset \bigcup_{n \geq m} \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n)$$

for any given sequence b_n with $b_n \to \infty$. (Notice that since $V \subset \mathbb{H}$ is simply connected and φ is univalent, the set $\varphi(V)$ is simply connected and $\varphi(V) \subsetneq \mathbb{C}$, so $\varphi(V)$ is hyperbolic.) The precise construction of the suitable domain A will be done in Proposition 3.1.

Then, using Schwarz–Pick's Lemma, for any $z_0 \in U$ and any sequence r_n with $r_n \to \infty$ we will choose ω and b_n such that

$$W = \pi(\varphi^{-1}(A)) \subset \bigcup_{n \geq 0} \mathcal{D}_U(F^n(z_0), r_n).$$

Taking r_n converging to ∞ slowly enough, depending on the speed of escaping of $F^n(z_0)$ to ∞, we will show that W is sufficiently "thin" to satisfy the assertions of Theorem A. Notice that although we construct A to be simply connected, the set W will not be in general simply connected, unless U is simply connected.

The construction of the absorbing domain A is done in the following proposition.

Proposition 3.1 (Absorbing domains in Ω). *Under the notation of Cowen's Theorem* 2.6*, for every $\omega \in \Omega$ and every sequence of positive numbers b_n, $n \geq 0$ with $\lim_{n\to\infty} b_n = \infty$, there exist $m \in \mathbb{N}$ and a simply connected domain $A \subset \Omega$ with the following properties:*

(*a*) $\overline{A} \subset \bigcup_{n=m}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n) \subset \varphi(V)$*,*

(*b*) $T(\overline{A}) \subset A$*,*

(*c*) *A is absorbing for T in Ω.*

Moreover, if $\Omega = \mathbb{C}$, $T(\omega) = \omega + 1$, then for every $\omega \in \Omega$ and $b > 0$ there exist a sequence b_n, $n \geq 0$ with $b_n < b$ and $\lim_{n\to\infty} b_n = 0$, a number $m \in \mathbb{N}$ and a simply connected domain $A \subset \Omega$, such that the conditions (*a*)*–*(*c*) *are satisfied.*

Proof. The proof splits in two cases, according to $\Omega = \mathbb{H}$ or $\Omega = \mathbb{C}$ in Cowen's Theorem 2.6.

Case 1. $\Omega = \mathbb{H}$. Then $T(\omega) = a\omega$, $a > 1$ or $T(\omega) = \omega \pm i$. Notice that in this case T is an isometry with respect to the hyperbolic metric in $\mathbb{H}$. Take $\omega \in \mathbb{H}$ and a sequence b_n, $n \geq 0$ of positive numbers with $b_n \to \infty$ as $n \to \infty$.

To define the domain A, first we show that there is $m \in \mathbb{N}$ and a sequence of positive numbers d_n, $n \geq 0$ with $d_n \to \infty$ as $n \to \infty$, such that

$$\overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), d_n)} \subset \varphi(V) \qquad \text{for every } n \geq m. \tag{2}$$

To see the claim, suppose it is not true. Then there exists $d > 0$ such that $\overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), d)} \not\subset \varphi(V)$ for infinitely many n, which contradicts the assertion (b) of Cowen's Theorem for the compact set $K = \overline{\mathcal{D}_{\mathbb{H}}(\omega, d)}$. Hence, we can take a sequence d_n satisfying (2).

Now we define the absorbing set A as

$$A = \bigcup_{n=m}^{\infty} \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n),$$

where

$$c_n = \frac{1}{2} \min \left(\inf_{k \geq n} \ln \frac{1 + B_k D_k}{1 - B_k D_k},\ \varrho_{\mathbb{H}}(T^n(\omega), \omega) \right)$$

for

$$B_n = \frac{e^{b_n} - 1}{e^{b_n} + 1}, \qquad D_n = \frac{e^{d_n} - 1}{e^{d_n} + 1}.$$

Since, by definition, $b_n, d_n > 0$ and $b_n \to \infty$, $d_n \to \infty$ as $n \to \infty$, it follows that $0 < B_n < 1$, $0 < D_n < 1$ and $B_n \to 1$, $D_n \to 1$ as $n \to \infty$. In fact, we have

$$b_n = \varrho_{\mathbb{D}}(B_n, 0), \qquad d_n = \varrho_{\mathbb{D}}(D_n, 0) \tag{3}$$

(see (1)). The definition of c_n implies (notice that $\varrho_{\mathbb{H}}(T^n(\omega), \omega) \nearrow \infty$ as $n \to \infty$) that the sequence c_n, $n \geq 0$ is positive, increasing, tends to infinity and satisfies

$$c_n < \ln \frac{1 + B_n D_n}{1 - B_n D_n} = \varrho_{\mathbb{D}}(B_n D_n, 0).$$

To ensure that A is a domain we enlarge m if necessary, so that $c_n > \varrho_{\mathbb{H}}(\omega, T(\omega)) = \varrho_{\mathbb{H}}(T^{n+1}(\omega), T^n(\omega))$ for all $n \geq m$. Hyperbolic discs in $\mathbb{H}$ are Euclidean discs, so they are convex. Consequently, A is simply connected, because it is a union of convex sets, all of them intersecting the straight line containing the trajectory of $T^n(\omega)$ under T. Notice also that defining

$$C_n = \frac{e^{c_n} - 1}{e^{c_n} + 1},$$

we have $C_n > 0$ and $c_n = \ln((1 + C_n)/(1 - C_n)) = \varrho_{\mathbb{D}}(C_n, 0)$, so

$$C_n < B_n D_n < D_n \quad \text{and} \quad c_n < d_n. \tag{4}$$

The main ingredient to end the proof of the proposition is to show that the closure of A equals the union of the closures of the respective discs, i.e.

$$\overline{A} = \bigcup_{n=m}^{\infty} \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)}. \tag{5}$$

Before proving (5) we show how it implies the particular statements of the proposition. To prove the statement (b), it is enough to use (5) and notice that

$$T(\overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)}) = \overline{\mathcal{D}_{\mathbb{H}}(T^{n+1}(\omega), c_n)} \subset \mathcal{D}_{\mathbb{H}}(T^{n+1}(\omega), c_{n+1}),$$

because $c_{n+1} > c_n$. To show the assertion (c), take a compact set $K \subset \mathbb{H}$. Then $K \subset \mathcal{D}_{\mathbb{H}}(\omega, r)$ for some $r > 0$, so

$$T^n(K) \subset T^n(\mathcal{D}_{\mathbb{H}}(\omega, r)) = \mathcal{D}_{\mathbb{H}}(T^n(\omega), r) \subset \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n) \subset A$$

for sufficiently large n, because $c_n \to \infty$.

Now we prove the statement (a) of the proposition. By (5), it suffices to show that

$$\overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)} \subset \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n). \tag{6}$$

Note that by (2) and Schwarz–Pick's Lemma 2.2 for the inclusion map, we have

$$\overline{\mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(\omega), d_n)}(T^n(\omega), b_n)} \subset \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n),$$

and so, to show (6) it is enough to prove

$$\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n) \subset \mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(\omega), d_n)}(T^n(\omega), b_n). \tag{7}$$

To show (7), let h_1 be a Möbius transformation of $\widehat{\mathbb{C}}$ mapping $\mathbb{H}$ onto $\mathbb{D}$ with $h_1(T^n(\omega)) = 0$. Then

$$h_1(\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)) = \mathcal{D}_{\mathbb{D}}(0, c_n),$$
$$h_1(\mathcal{D}_{\mathcal{D}_{\mathbb{H}}(T^n(\omega), d_n)}(T^n(\omega), b_n)) = \mathcal{D}_{\mathcal{D}_{\mathbb{D}}(0, d_n)}(0, b_n) = \mathcal{D}_{\mathbb{D}(0, D_n)}(0, b_n),$$

where the latter equality follows from (3). Hence, to prove (7), it suffices to check that

$$\mathcal{D}_{\mathbb{D}}(0, c_n) \subset \mathcal{D}_{\mathbb{D}(0, D_n)}(0, b_n). \tag{8}$$

Let $h_2(v) = v/D_n$ be the Möbius transformation which maps univalently $\mathbb{D}(0, D_n)$ onto $\mathbb{D}$. Similarly as before, we have

$$h_2(\mathcal{D}_{\mathbb{D}}(0, c_n)) = h_2(\mathbb{D}(0, C_n)) = \mathbb{D}\left(0, \frac{C_n}{D_n}\right),$$
$$h_2(\mathcal{D}_{\mathbb{D}(0, D_n)}(0, b_n)) = \mathcal{D}_{\mathbb{D}}(0, b_n) = \mathbb{D}(0, B_n).$$

Therefore, to prove (8) (and consequently (6) and the statement (a)), it is enough to show

$$\mathbb{D}\left(0, \frac{C_n}{D_n}\right) \subset \mathbb{D}(0, B_n),$$

which holds by (4).

To end the proof of the proposition, it remains to prove (5). Obviously, it suffices to show the inclusion $\overline{A} \subset \bigcup_{n=m}^{\infty} \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)}$. Take $v \in \overline{A}$ and a sequence $v_k \in A$ such that $v_k \to v$ as $k \to \infty$. By the definition of A, there exists a sequence $n_k \geq m$, such that

$$v_k \in \mathcal{D}_{\mathbb{H}}(T^{n_k}(\omega), c_{n_k}).$$

Since, by definition, $c_{n_k} \leq \varrho_{\mathbb{H}}(T^{n_k}(\omega), \omega)/2$, we have

$$\frac{\varrho_{\mathbb{H}}(T^{n_k}(\omega), \omega)}{2} \geq c_{n_k} > \varrho_{\mathbb{H}}(T^{n_k}(\omega), v_k) \geq \varrho_{\mathbb{H}}(T^{n_k}(\omega), \omega) - \varrho_{\mathbb{H}}(v_k, \omega),$$

so

$$\varrho_{\mathbb{H}}(v_k, \omega) > \frac{\varrho_{\mathbb{H}}(T^{n_k}(\omega), \omega)}{2}.$$

On the other hand, the sequence $\varrho_{\mathbb{H}}(v_k, \omega)$ is bounded, because $v_k \to v$. Hence, the sequence $\varrho_{\mathbb{H}}(T^{n_k}(\omega), \omega)$ must be bounded, so n_k is bounded. Therefore, taking a subsequence, we can assume that there exists $n \geq m$ such that $n_k = n$ for every k, so

$$v_k \in \mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n).$$

This implies

$$v \in \overline{\mathcal{D}_{\mathbb{H}}(T^n(\omega), c_n)},$$

which finishes the proof of (5).

Case 2: $\mathbf{\Omega} = \mathbb{C}$. In this case $T(\omega) = \omega + 1$, so T is an isometry with respect to the Euclidean metric in $\mathbb{C}$. Since most of the arguments here are similar to the previous case (with the Euclidean metric instead of the hyperbolic one), we skip some details.

Similarly as before, we claim that the absorbing region $\varphi(V)$ must contain a union of appropriate discs of increasing radii. More precisely, for a given $\omega \in \mathbb{C}$ there exists $m \in \mathbb{N}$ and a sequence d_n, $n \geq 0$ of positive numbers with $d_n \to \infty$ as $n \to \infty$ such that

$$\overline{\mathbb{D}(T^n(\omega), d_n)} \subset \varphi(V) \qquad \text{for every } n \geq m. \tag{9}$$

(If the claim was not true, then for the compact set $K = \overline{\mathbb{D}(\omega, d)}$ we would have a contradiction with the assertion (b) of Cowen's Theorem.) Hence, in what follows we will assume that the sequence d_n satisfies (9).

Take $b > 0$ and let $b_n = 1/\sqrt{d_n} \to 0$. Enlarging m if necessary, we may assume $b_n < b$ for all $n \geq m$. We define the absorbing set A as

$$A = \bigcup_{n=m}^{\infty} \mathbb{D}(T^n(\omega), c_n)$$

for

$$c_n = \frac{1}{2} \min \left(\inf_{k \geq n} \frac{e^{b_k} - 1}{e^{b_k} + 1} d_k, n \right).$$

Clearly, c_n, $n \geq 0$ is an increasing sequence of positive numbers. Moreover, we have

$$c_n < \frac{e^{b_n} - 1}{e^{b_n} + 1} d_n < d_n \quad \text{and} \quad \frac{e^{b_n} - 1}{e^{b_n} + 1} d_n = \frac{e^{1/\sqrt{d_n}} - 1}{e^{1/\sqrt{d_n}} + 1} d_n \to \infty \tag{10}$$

as $n \to \infty$. Hence, $c_n \to \infty$.

As in the previous case, enlarging m if necessary, we can assume A is a domain. Moreover, A is simply connected, since it is a union of Euclidean discs intersecting the straight line containing the T-trajectory of ω.

The main ingredient of the proof is to prove

$$\overline{A} = \bigcup_{n=m}^{\infty} \overline{\mathbb{D}(T^n(\omega), c_n)}. \tag{11}$$

As in Case 1, first we show how (11) implies the particular statements of the proposition. To show the statement (b), we use (11) and notice that

$$T(\overline{\mathbb{D}(T^n(\omega), c_n)}) = \overline{\mathbb{D}(T^{n+1}(\omega), c_n)} \subset \mathbb{D}(T^{n+1}(\omega), c_{n+1}),$$

because $c_{n+1} > c_n$. To prove the assertion (c), take a compact set $K \subset \mathbb{C}$. Then $K \subset \mathbb{D}(\omega, r)$ for some $r > 0$, so

$$T^n(K) \subset T^n(\mathbb{D}(\omega, r)) = \mathbb{D}(T^n(\omega), r) \subset \mathbb{D}(T^n(\omega), c_n) \subset A$$

for sufficiently large n, because $c_n \to \infty$.

To prove the statement (a), in view of (11), it suffices to show

$$\overline{\mathbb{D}(T^n(\omega), c_n)} \subset \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n). \tag{12}$$

Note that by (9) and Schwarz–Pick's Lemma 2.2 we have

$$\overline{\mathcal{D}_{\mathbb{D}(T^n(\omega), d_n)}(T^n(\omega), b_n)} \subset \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n),$$

so, to show (12), it is enough to prove

$$\mathbb{D}(T^n(\omega), c_n) \subset \mathcal{D}_{\mathbb{D}(T^n(\omega), d_n)}(T^n(\omega), b_n). \tag{13}$$

To see this is true we apply the univalent function $h(v) = (v - T^n(\omega))/d_n$, which maps $\mathbb{D}(T^n(\omega), d_n)$ onto $\mathbb{D}$. We have

$$h(\mathbb{D}(T^n(\omega), c_n)) = \mathbb{D}\left(0, \frac{c_n}{d_n}\right),$$

$$h(\mathcal{D}_{\mathbb{D}(T^n(\omega), d_n)}(T^n(\omega), b_n)) = \mathcal{D}_{\mathbb{D}}(0, b_n) = \mathbb{D}\left(0, \frac{e^{b_n} - 1}{e^{b_n} + 1}\right).$$

Therefore, to prove (13) (and consequently the statement (a)), it is sufficient to check

$$\mathbb{D}\left(0, \frac{c_n}{d_n}\right) \subset \mathbb{D}\left(0, \frac{e^{b_n}-1}{e^{b_n}+1}\right),$$

which follows from (10).

Finally, we prove (11). As in Case 1, it suffices to show $\overline{A} \subset \bigcup_{n=m}^{\infty} \overline{\mathbb{D}(T^n(\omega), c_n)}$. Take $v \in \overline{A}$ and a sequence $v_k \in A$ such that $v_k \to v$ as $k \to \infty$. Then there exists a sequence $n_k \geq m$, such that

$$v_k \in \mathbb{D}(T^{n_k}(\omega), c_{n_k}).$$

Since, by definition, $c_{n_k} \leq n_k/2$, we have

$$\frac{n_k}{2} \geq c_{n_k} > |T^{n_k}(\omega) - v_k| = |n_k + \omega - v_k| \geq n_k - |\omega| - |v_k|,$$

so

$$|v_k| > \frac{n_k}{2} - |\omega|.$$

On the other hand, the sequence v_k is bounded, because $v_k \to v$. Hence, the sequence n_k must be bounded, so taking a subsequence, we can assume that $n_k = n$ for every k and some $n \geq m$, so

$$v_k \in \mathbb{D}(T^n(\omega), c_n) \text{ for every } k > 0 \quad \text{and} \quad v \in \overline{\mathbb{D}(T^n(\omega), c_n)}.$$

Hence, (11) follows. □

With Proposition 3.1 in hand, we are ready to prove Theorem A. We construct the absorbing region W by projecting A into the domain U.

Proof of Theorem A. Note that by Lemma 2.3, there exist $c > 0$ and a large $r > 0$ such that

$$\varrho_U(u) > \frac{c}{|u| \log |u|} \qquad \text{for } u \in U,\ |u| \geq r. \tag{14}$$

Fix some $v_0 \in \varphi(V)$ and let $z_0 = \pi(\varphi^{-1}(v_0))$. Since $F^n(z_0) \to \infty$, replacing v_0 by $T^j(v_0)$ for sufficiently large j, we can assume

$$|F^n(z_0)| > r^{\log r} > r \quad \text{for every } n \geq 0. \tag{15}$$

Take $z \in U$ and a sequence of positive numbers $\{r_n\}_{n \geq 0}$ with $r_n \to \infty$. Fix a number $n_0 \in \mathbb{N}$ such that

$$r_n > 2\varrho_U(z, z_0) \quad \text{for every } n \geq n_0. \tag{16}$$

We define the sequence

$$a_n = \frac{1}{2} \min\left(r_n, \frac{c}{2} \inf_{k \geq n} \log\log |F^k(z_0)|\right). \tag{17}$$

Clearly, $a_n \to \infty$ as $n \to \infty$. Let $A \subset \Omega$ be the domain from Proposition 3.1 defined for $\omega = T^{n_0}(v_0)$ and $b_n = a_{n+n_0}$. Finally, let

$$W = \pi(\varphi^{-1}(A)).$$

By construction, we have the following commutative diagram.

$$\begin{array}{ccccccc}
A & \subset & \varphi(V) & \subset & \Omega & \xrightarrow{T} & \Omega \\
\downarrow{\scriptstyle \varphi^{-1}} & & \downarrow{\scriptstyle \varphi^{-1}} & & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi} \\
\varphi^{-1}(A) & \subset & V & \subset & \mathbb{H} & \xrightarrow{G} & \mathbb{H} \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
W & \subset & \pi(V) & \subset & U & \xrightarrow{F} & U
\end{array}$$

In the remaining part of the proof we show that W satisfies the conditions listed in Theorem A.

First, we prove the statement (a). By Proposition 3.1 we know that, for some $m \in \mathbb{N}$,

$$A \subset \bigcup_{n=m}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(\omega), b_n) = \bigcup_{n=m+n_0}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(v_0), a_n) \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_{\varphi(V)}(T^n(v_0), a_n). \tag{18}$$

Hence, by Schwarz–Pick's Lemma 2.2 for φ^{-1} and the inclusion map, we obtain

$$\varphi^{-1}(A) \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_V(\varphi^{-1}(T^n(v_0)), a_n) = \bigcup_{n=n_0}^{\infty} \mathcal{D}_V(G^n(\varphi^{-1}(v_0)), a_n) \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_{\mathbb{H}}(G^n(\varphi^{-1}(v_0)), a_n)$$

and

$$W \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_U(\pi(G^n(\varphi^{-1}(v_0))), a_n) = \bigcup_{n=n_0}^{\infty} \mathcal{D}_U(F^n(z_0), a_n).$$

Using this together with (16), (17) and Schwarz–Pick's Lemma 2.2, we get

$$\begin{aligned} W \subset \bigcup_{n=n_0}^{\infty} & \mathcal{D}_U(F^n(z), a_n + \varrho_U(F^n(z), F^n(z_0))) \\ & \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_U(F^n(z), a_n + \varrho_U(z, z_0)) \subset \bigcup_{n=n_0}^{\infty} \mathcal{D}_U(F^n(z), r_n), \end{aligned}$$

which ends the proof of the statement (a).

Now we prove the assertions (b)–(d). Fix $j \geq 0$ and consider an arbitrary $u \in \overline{F^j(W)}$. Let w_k, $k \geq 1$ be a sequence of points in W, such that for $u_k = F^j(w_k)$ we have $u_k \to u$ as $k \to \infty$. Since $W = \pi(\varphi^{-1}(A))$, there exists a sequence of points $v_k \in A$ with $w_k = \pi(\varphi^{-1}(v_k))$. By (18), for every k there exists $n_k \geq n_0$, such that

$$v_k \in \mathcal{D}_{\varphi(V)}(T^{n_k}(v_0), a_{n_k}). \tag{19}$$

Thus, by Schwarz–Pick's Lemma 2.2, we have

$$w_k \in \mathcal{D}_U(F^{n_k}(z_0), a_{n_k}), \quad u_k \in \mathcal{D}_U(F^{n_k+j}(z_0), a_{n_k}). \tag{20}$$

The key ingredient in the proof of the assertions (b)–(d) is to show

$$|u_k| > e^{\sqrt{\log |F^{n_k+j}(z_0)|}}. \tag{21}$$

To prove (21), take $\gamma_k : [0,1] \to U$ to be a curve in U such that $\gamma_k(0) = F^{n_k+j}(z_0)$, $\gamma_k(1) = u_k$,

$$\int_{\gamma_k} \varrho_U(\xi)|d\xi| < 2\varrho_U(F^{n_k+j}(z_0), u_k) \tag{22}$$

and let

$$t_k = \sup\{t \in [0,1] : |\gamma_k(t')| \geq r \text{ for all } 0 < t' < t\}.$$

By (15), $|\gamma_k(0)| > r$, so the supremum is well defined. Moreover, we have $|\gamma_k(t)| \geq r$ for $t \in [0, t_k]$ and $|\gamma_k(t_k)| \in \{r, |\gamma_k(1)|\}$. Notice that if $|\gamma_k(0)| < |\gamma_k(1)|$, then (21) follows from (15). Hence, we may assume $|\gamma_k(0)| \geq |\gamma_k(1)|$, which implies $|\gamma_k(0)| \geq |\gamma_k(t_k)|$. Using this together with (14), (17), (20) and (22), we obtain

$$\begin{aligned}\frac{c}{4}\log\log|F^{n_k+j}(z_0)| &\geq a_{n_k} > \varrho_U(F^{n_k+j}(z_0), u_k)\\ &> \frac{1}{2}\int_{\gamma_k}\varrho_U(\xi)|d\xi| \geq \frac{1}{2}\int_{\gamma_k|_{[0,t_k]}}\varrho_U(\xi)|d\xi| \geq \frac{c}{2}\int_{\gamma_k|_{[0,t_k]}}\frac{|d\xi|}{|\xi|\log|\xi|} \geq \frac{c}{2}\int_{|\gamma_k(t_k)|}^{|\gamma_k(0)|}\frac{ds}{s\log s}\\ &= \frac{c}{2}(\log\log|F^{n_k+j}(z_0)| - \log\log|\gamma_k(t_k)|),\end{aligned}$$

where the latter inequality follows from the definition of the Riemann integral. We conclude that $\log\log|\gamma_k(t_k)| > (\log\log|F^{n_k+j}(z_0)|)/2$, which means

$$|\gamma_k(t_k)| > e^{\sqrt{\log|F^{n_k+j}(z_0)|}}. \tag{23}$$

In particular, this implies that $|\gamma_k(t_k)| \neq r$, because otherwise we have a contradiction with (15). Hence, $|\gamma_k(t_k)| = |\gamma_k(1)| = |u_k|$, so (23) shows (21).

Having (21), we now prove the assertions (b)–(d) of Theorem A. First, notice that since $u_k \to u$ as $k \to \infty$ and $F^n(z_0) \to \infty$ as $n \to \infty$, (21) implies that the sequence n_k is bounded. Hence, (19) shows that the sequence v_k is bounded, so taking a subsequence, we can assume that

$$v_k \to v \in \overline{A},$$

and, by Proposition 3.1, $v \in \varphi(V)$. Therefore, by continuity,

$$w_k \to w = \pi(\varphi^{-1}(v)) \in \overline{W} \cap U \quad \text{and} \quad F^j(w) = u. \tag{24}$$

Recall that u was taken as an arbitrary point in $\overline{F^j(W)}$. Hence, for $j = 0$, (24) implies $u = w \in U$, which proves the statement (b) and shows that $F^j(\overline{W})$ is well defined for $j \geq 1$. To prove the assertion (c), notice that (24) gives $u = F^j(w) \in F^j(\overline{W})$, which shows $\overline{F^j(W)} \subset F^j(\overline{W})$. On the other hand, the inclusion $F^j(\overline{W}) \subset \overline{F^j(W)}$ is obvious by the continuity of F^j, so $F^j(\overline{W}) = \overline{F^j(W)}$ for $j \geq 1$. To end the proof of the assertion (c), it is sufficient to show $F^j(\overline{W}) \subset F^{j-1}(W)$ for $j \geq 1$. To do it, notice that Proposition 3.1 implies $T(v) \in T(\overline{A}) \subset A$, so for $j = 1$ (24) gives $u = F(w) = F(\pi(\varphi^{-1}(v))) = \pi(\varphi^{-1}(T(v))) \in W$. Hence,

$$F(\overline{W}) = \overline{F(W)} \subset W.$$

This and induction on j proves $F^j(\overline{W}) \subset F^{j-1}(W)$ for $j \geq 1$, which ends the proof of the assertion (c).

To show the statement (d), notice that (21) implies $|u| \geq \inf_{n \geq j+n_0} e^{\sqrt{\log|F^n(z_0)|}}$, so

$$F^j(\overline{W}) = \overline{F^j(W)} \subset \mathbb{C} \setminus \mathbb{D}\left(0, \inf_{n \geq j+n_0} e^{\sqrt{\log|F^n(z_0)|}}\right).$$

This proves (d), because $|F^n(z_0)| \to \infty$ as $n \to \infty$.

Now we show the statement (e). Take a compact set $K \subset U$ and a point $u \in K$. Let $w \in \mathbb{H}$ be such that $\pi(w) = u$ and take $N(w)$ to be an open neighbourhood of w, such that $\overline{N(w)} \subset \mathbb{H}$. Then $\pi(N(w))$ is an open neighbourhood of u, so by the compactness of

K, we can choose a finite number of points $u_1, \ldots, u_k \in K$, such that $K \subset \bigcup_{j=1}^k \pi(N(w_j))$. Since $L = \bigcup_{j=1}^k \varphi(\overline{N(w_j)})$ is a compact set in Ω, by Proposition 3.1, there exists n such that $T^n(L) \subset A$. This implies

$$\bigcup_{j=1}^k G^n(N(w_j)) \subset \varphi^{-1}\left(\bigcup_{j=1}^k T^n(\varphi(N(w_j)))\right) = \varphi^{-1}\left(\bigcup_{j=1}^k \varphi(G^n(N(w_j)))\right) \subset \varphi^{-1}(A),$$

so

$$F^n(K) \subset \bigcup_{j=1}^k F^n(\pi(N(w_j))) = \bigcup_{j=1}^k \pi(G^n(N(w_j))) \subset W,$$

which ends the proof of the statement (e).

To show that F is locally univalent on W, take $z \in W$. Then $z = \pi(\varphi^{-1}(\omega))$ for some $\omega \in A$, so F near z can be expressed as $F = \pi \circ \varphi^{-1} \circ T \circ \varphi \circ \pi^{-1}$, where π^{-1} is the inverse branch of π mapping z onto $\varphi^{-1}(\omega)$. Since $\varphi|_V$ and T are univalent, F is locally univalent. This ends the proof of Theorem A. □

4. Configurations of domains and their images

In this section we present preliminary lemmas which we use repeatedly throughout the proofs of Theorems B and C. They provide the existence of weakly repelling fixed points for meromorphic maps in some domains under certain combinatorial conditions related to the configuration of the domain and its subsequent images. These lemmas are formulated in a general setup and may have further applications apart from the ones used in this paper.

The first lemma shows that a meromorphic map is proper on bounded components of the preimage of a domain with finite Euler characteristic.

Lemma 4.1 (**Proper restrictions of meromorphic maps).** *Let $D \subset \widehat{\mathbb{C}}$ be a domain with finite Euler characteristic and let f be a map, which is non-constant and meromorphic on a neighbourhood of $\overline{D'}$, where D' is a bounded component of $f^{-1}(D)$. Then D' has finite Euler characteristic and the restriction $f : D' \to D$ is proper.*

Proof. Clearly, we have $f(D') = D$. Since D has finite Euler characteristic, its boundary has a finite number of connected components, and each component of $\partial D'$ is mapped by f onto a component of ∂D. Hence, the boundary of D' has finitely many components, because otherwise we could find w_0 in the boundary of D, such that f takes the value w_0 on a set with an accumulation point in $\overline{D'}$, so $f \equiv w_0$. This implies that D' has finite Euler characteristic and $f : D' \to D$ is proper. □

Definition (**Exterior of a compact set).** For a compact set $X \subset \mathbb{C}$ we denote by $\operatorname{ext}(X)$ the connected component of $\widehat{\mathbb{C}} \setminus X$ containing infinity. We set $K(X) = \widehat{\mathbb{C}} \setminus \operatorname{ext}(X)$. For a Jordan curve $\gamma \subset \mathbb{C}$ we denote by $\operatorname{int}(\gamma)$ the bounded component of $\mathbb{C} \setminus \gamma$.

The following facts are immediate consequences of some standard topological facts and the maximum principle. We will use them repeatedly without explicit quotation.

Lemma 4.2 (**Properties of $K(X)$ and $\operatorname{ext}(X)$).** *Let $X \subset \mathbb{C}$ be a compact set. Then:*

(*a*) *if X is connected, then* $\operatorname{ext}(X)$ *is a simply connected subset of $\widehat{\mathbb{C}}$ and $K(X)$ is a connected subset of $\mathbb{C}$,*

(*b*) *if X has a finite number of components, then* $\operatorname{ext}(X)$ *has finite Euler characteristic,*

(c) *$K(X)$ is a compact set in $\mathbb{C}$ and $\mathbb{C} \setminus K(X)$ is connected,*
(d) *if $Y \subset X$ is a compact set, then $\operatorname{ext}(Y) \supset \operatorname{ext}(X)$ and $K(Y) \subset K(X)$,*
(e) *if f is meromorphic map in a neighbourhood of $K(X)$ and $K(X)$ does not contain poles of f, then $f(K(X)) \subset K(f(X))$.*

The next lemma shows that the multiple connectivity of a Fatou component U implies the existence of a pole of f in a bounded component of the complement of some image of U. This will be an important property used in the proofs of the main theorems.

Lemma 4.3 (Poles in loops). *Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental non-entire meromorphic map and let $\gamma \subset \mathbb{C}$ be a closed curve in a Fatou component U of f, such that $K(\gamma) \cap J(f) \neq \emptyset$. Then there exists $n \geq 0$, such that $K(f^n(\gamma))$ contains a pole of f. Consequently, if U is multiply connected then there exists a bounded component of $\widehat{\mathbb{C}} \setminus f^n(U)$, which contains a pole.*

Proof. If f has exactly one pole which is an omitted value, then f is a self-map of a punctured plane and the claim follows easily from [3, Theorem 1]. Hence, we can assume that f has at least two poles or exactly one pole, which is not an omitted value. Then prepoles are dense in $J(f)$, so there is a prepole in $K(\gamma)$. Suppose $K(f^n(\gamma))$ does not contain poles of f for every $n \geq 0$. Then f^n is holomorphic in a neighbourhood of $K(\gamma)$, so by Lemma 4.2, $f^n(K(\gamma)) \subset K(f^n(\gamma))$ for every $n \geq 0$. Hence, $K(\gamma)$ cannot contain any prepoles of f, which gives a contradiction. □

The next lemma is a consequence of Buff's results on the existence of weakly repelling fixed points for rational-like maps (Theorem 2.10 and Corollary 2.12).

Lemma 4.4 (Boundary maps out). *Let $\Omega \subset \mathbb{C}$ be a bounded domain with finite Euler characteristic and let f be a meromorphic map in a neighbourhood of $\overline{\Omega}$. Assume that there exists a component D of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$, such that:*
(a) *$\overline{\Omega} \subset D$,*
(b) *there exists $z_0 \in \Omega$ such that $f(z_0) \in D$.*

Then f has a weakly repelling fixed point in Ω. Moreover, if additionally Ω is simply connected with locally connected boundary, then the assumption (a) can be replaced by:
(a′) *$\Omega \subsetneq D$ and f has no fixed points in $\partial\Omega \cap f(\partial\Omega)$.*

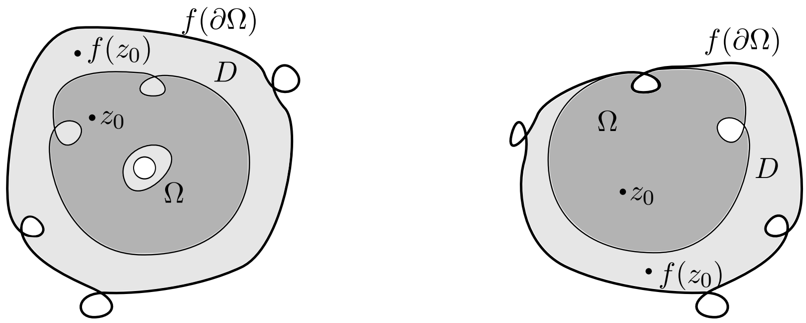

FIGURE 2. Setup of Lemma 4.4 with the assumption (a) (left) and (a') (right).

Remark. Observe that if Ω is simply connected with locally connected boundary, then $f(\partial\Omega)$ is allowed to have common points with $\partial\Omega$ (see Figure 2). A version of this lemma requiring $f(\partial\Omega)$ to be disjoint from $\partial\Omega$ and $f(z_0) = \infty$ appeared in [10, Lemma 1].

Proof of Lemma 4.4. By the assumption (b), there exists a component D' of $f^{-1}(D)$ containing z_0. Observe that

$$D' \subset \Omega.$$

To see this, suppose that D' is not contained in Ω. Then there exists $z \in D' \cap \partial\Omega$. Consequently, $f(z) \in D \cap f(\partial\Omega)$. This is a contradiction since, by definition, $D \cap f(\partial\Omega) = \emptyset$.

As a consequence, D' is bounded. Moreover, since Ω has finite Euler characteristic, $\partial\Omega$ (and hence $f(\partial\Omega)$ and ∂D) has a finite number of components, so D has finite Euler characteristic. Therefore, by Lemma 4.1, D' has finite Euler characteristic and the restriction $f : D' \to D$ is proper. Moreover, the assumption (a) implies $\overline{D'} \subset D$. Hence (possibly after a change of coordinates in $\widehat{\mathbb{C}}$ by a Möbius transformation), $f : D' \to D$ is a rational-like map, so by Theorem 2.10, the map f has a weakly repelling fixed point in $D' \subset \Omega$.

Finally, assume that Ω is simply connected with locally connected boundary, and the assumption (a) is replaced by (a'). Then $\partial\Omega$ (and hence $f(\partial\Omega)$) is a locally connected continuum in $\widehat{\mathbb{C}}$, so D is simply connected and, by the Torhorst Theorem 2.13, has locally connected boundary. Moreover, since $D' \subset \Omega \subset D$ and the boundary of D is contained in $f(\partial\Omega)$, the intersection of the boundaries of D and D' is either empty or is contained in $\partial\Omega \cap f(\partial\Omega)$. This together with the condition (a') implies that the restriction $f : D' \to D$ satisfies the assumptions of Corollary 2.12, which ends the proof. □

In particular, Lemma 4.4 implies the following two corollaries (see Figures 3–4).

Corollary 4.5 (Continuum surrounds a pole and maps out). *Let $X \subset \mathbb{C}$ be a continuum and let f be a meromorphic map in a neighbourhood of $K(X)$. Suppose that:*

(*a*) *f has no poles in X,*
(*b*) *$K(X)$ contains a pole of f,*
(*c*) *$K(X) \subset \mathrm{ext}(f(X))$.*

Then f has a weakly repelling fixed point in the interior of $K(X)$.

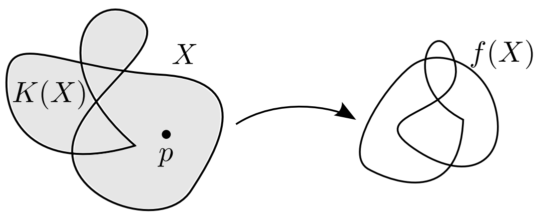

FIGURE 3. Setup of Corollary 4.5.

Proof. Let $p \in K(X)$ be a pole of f. Observe that by the assumption (a), the set $f(X)$ (and hence $K(f(X))$) is a continuum in $\mathbb{C}$. Moreover, (a) implies

$$p \in \Omega \subset \overline{\Omega} \subset K(X)$$

for a bounded simply connected component Ω of $\widehat{\mathbb{C}} \setminus X$. We have $\partial\Omega \subset X$, which gives $f(\partial\Omega) \subset f(X)$, so by the assumption (c),

$$K(X) \subset \text{ext}(f(\partial\Omega)),$$

which implies $\overline{\Omega} \subset \text{ext}(f(\partial\Omega))$.

Let $D = \text{ext}(f(\partial\Omega))$. We have $\overline{\Omega} \subset D$, $p \in \Omega$ and $f(p) = \infty \in D$. Hence, the assumptions of Lemma 4.4 are satisfied for Ω, D, p, so f has a weakly repelling fixed point in Ω, which is a subset of the interior of $K(X)$. □

Corollary 4.6 (**Continuum maps out twice).** *Let $X \subset \mathbb{C}$ be a continuum and let f be a meromorphic map in a neighbourhood of $X \cup K(f(X))$. Suppose that:*

(*a*) *f has no poles in X,*
(*b*) *$X \subset K(f(X))$,*
(*c*) *$f^2(X) \subset \text{ext}(f(X))$.*

Then f has a weakly repelling fixed point in the interior of $K(f(X))$.

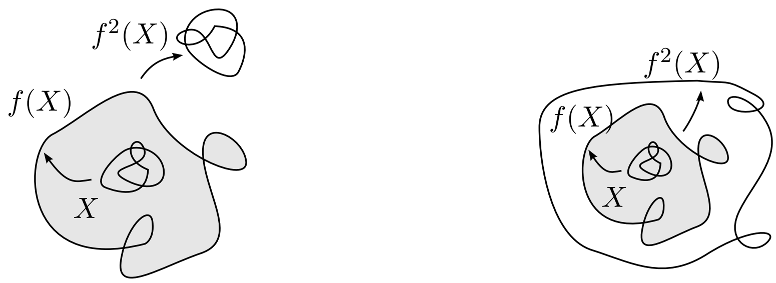

FIGURE 4. Two possible setups of Corollary 4.6.

Proof. By the assumption (a), the set $f(X)$ (and hence $K(f(X))$) is a continuum in $\mathbb{C}$ and $f^2(X)$ is a continuum in $\widehat{\mathbb{C}}$. Moreover, $X \cap f(X) = \emptyset$ (otherwise $f(X) \cap f^2(X) \neq \emptyset$, which contradicts the assumption (c)). Hence, by (b),

$$X \subset \Omega \subset \overline{\Omega} \subset K(f(X))$$

for some bounded simply connected component Ω of $\widehat{\mathbb{C}} \setminus f(X)$. We have $\partial\Omega \subset f(X)$, so $f(\partial\Omega) \subset f^2(X)$ and by the assumption (c),

$$K(f(X)) \subset \widehat{\mathbb{C}} \setminus f^2(X) \subset \widehat{\mathbb{C}} \setminus f(\partial\Omega),$$

which gives $K(f(X)) \subset D$ for some component D of $\widehat{\mathbb{C}} \setminus f(\partial\Omega)$. Consequently, $\overline{\Omega} \subset K(f(X)) \subset D$. Moreover, for any $z_0 \in X$ we have $z_0 \in \Omega$ and $f(z_0) \in f(X) \subset D$. Hence, the assumptions of Lemma 4.4 are satisfied for Ω, D, z_0, so f has a weakly repelling fixed point in Ω, which is contained in the interior of $K(f(X))$. □

The previous results give some conditions for the existence of a weakly repelling fixed point in the case when a closed curve is mapped by f into its exterior. The following proposition, which is a considerable generalization of Shishikura's Theorem 2.14, gives conditions for the existence of a weakly repelling fixed point in the case when a closed curve before mapping out is mapped by f several times into its interior (see Figure 5).

Proposition 4.7 (**Boundary maps in**)**.** *Let $\Omega \subset \mathbb{C}$ be a bounded simply connected domain and let f be a meromorphic map in a neighbourhood of $\overline{\Omega}$. Suppose that:*

(*a*) *there exists $m \geq 2$, such that f^m is defined on $\partial\Omega$,*
(*b*) *$f^j(\partial\Omega) \subset \overline{\Omega}$ for $j = 1, \ldots, m-1$,*
(*c*) *$f^m(\partial\Omega) \cap \overline{\Omega} = \emptyset$.*

Then f has a weakly repelling fixed point in Ω.

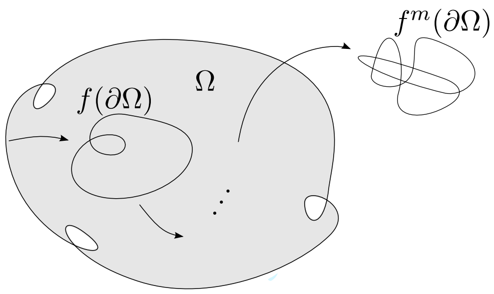

FIGURE 5. Possible setup for Proposition 4.7.

Proof. We proceed by contradiction, i.e. we assume that f has no weakly repelling fixed points in Ω. The proof is split into a number of steps.

Step 1. First, note that the simple connectedness of Ω implies that $\partial\Omega$ (and hence $f^j(\partial\Omega)$ for $j = 1, \ldots, m$) is connected. Moreover, the following conditions are satisfied:

$$\partial\Omega, f(\partial\Omega), \ldots, f^m(\partial\Omega) \text{ are pairwise disjoint,} \tag{25}$$

$$K(f^j(\partial\Omega)) \subset \Omega \quad \text{for } j = 1, \ldots, m-1. \tag{26}$$

To see (25), notice that if $z \in f^{j'}(\partial\Omega) \cap f^{j''}(\partial\Omega)$ for some $0 \leq j' < j'' \leq m$, then $f^{m-j''}(z) \in f^{m+j'-j''}(\partial\Omega) \cap f^m(\partial\Omega)$ and $0 \leq m + j' - j'' < m$, which contradicts the assumptions (b)–(c). Hence, (25) follows. Now (25) together with (b) implies (26).

Step 2. We show that we can reduce the proof to the case

$$f^{j+1}(\partial\Omega) \subset \text{ext}(f^j(\partial\Omega)) \quad \text{for } j = 1, \ldots, m-1. \tag{27}$$

To see this, suppose that there exists $j_0 \in \{1, \ldots, m-1\}$ such that $f^{j_0+1}(\partial\Omega) \subset K(f^{j_0}(\partial\Omega))$ and take the maximal number j_0 with this property. Then by the assumption (c) and (26), $j_0 \neq m-1$ and $f^{j_0+1}(\partial\Omega) \subset \Omega_0 \subset \Omega$ for some bounded simply connected component Ω_0 of $\widehat{\mathbb{C}} \setminus f^{j_0}(\partial\Omega)$. We have $f^k(\partial\Omega_0) \subset f^{k+j_0}(\partial\Omega)$ for $k \geq 0$. Hence, it follows from (26) and (c) that there exists $m_0 \geq 2$ such that $f^j(\partial\Omega_0) \subset \Omega_0$ for $j = 1, \ldots, m_0 - 1$ and $f^{m_0}(\partial\Omega_0) \cap \overline{\Omega_0} = \emptyset$. Thus, the assumptions (a)–(c) are satisfied for Ω_0, m_0. Since j_0 was maximal, this implies that replacing, respectively, Ω and m by Ω_0 and m_0, we can assume $f^{j+1}(\partial\Omega) \not\subset K(f^j(\partial\Omega))$ for $j = 1, \ldots, m-1$. Since by (25), there is no intersection between the images of $\partial\Omega$, we have proven that we can reduce the proof to the case (27).

Step 3. We claim that there exists a Jordan curve $\sigma_1 \subset \mathbb{C}$ close to $f(\partial\Omega)$ such that:

(28) $\quad \mathrm{int}(\sigma_1) \supset K(f(\partial\Omega))$,

$\quad \sigma_1$ contains no images of critical points of f in $\overline{\Omega}$,

(29) $\quad \sigma_1, f(\sigma_1), \ldots, f^{m-2}(\sigma_1)$ are pairwise disjoint subsets of Ω and $f^{m-1}(\sigma_1) \cap \overline{\Omega} = \emptyset$,

(30) $\quad f^{j+1}(\sigma_1) \subset \mathrm{ext}(f^j(\sigma_1))$ for $j = 0, \ldots, m-2$.

The existence of a curve satisfying these four conditions follows easily from (25), (26), (27), the assumption (c) and the fact that the set of critical points in $\overline{\Omega}$ is finite.

We then consider the set

$$D = \mathrm{ext}(\sigma_1).$$

By the assumption (c) and (29), we have $f^m(\partial\Omega) \subset D$. Hence, there exists a component D' of $f^{-1}(D)$ containing $f^{m-1}(\partial\Omega)$. By definition, D' intersects Ω and contains a pole of f. Consequently,

$$D' \subset \Omega,$$

because otherwise $D' \cap \partial\Omega \neq \emptyset$, so $D \cap f(\partial\Omega) \neq \emptyset$, which is impossible by (28). Therefore, D' is bounded and by Lemma 4.1, it has finite Euler characteristic and the restriction $f : D' \to D$ is proper. In fact, since ∂D contains no values of critical points of f in $\partial D'$, the boundary of D' consists of finitely many disjoint Jordan curves, f is a finite degree covering in a neighbourhood of every component of $\partial D'$ and maps this component onto σ_1.

We now define σ_0 to be the Jordan curve, which is the boundary of the unbounded component of $\widehat{\mathbb{C}} \setminus D'$. Notice that $D' \subset \mathrm{int}(\sigma_0) \subset \Omega$, moreover $\mathrm{int}(\sigma_0)$ contains a pole of f and $f(\sigma_0) = \sigma_1$. We will use the notation

$$\sigma_j = f^j(\sigma_0).$$

By (29), we have $\sigma_0 \cap \sigma_j = \emptyset$ for $j = 1, \ldots, m$, which means that

$$K(\sigma_j) \subset \mathrm{int}(\sigma_0) \text{ or } \sigma_j \subset \mathrm{ext}(\sigma_0). \tag{31}$$

(see Figure 6). Finally, we note that σ_0 and σ_1 are, by construction, Jordan curves, while σ_j for $j = 2, \ldots m$ are closed curves, which are not necessarily Jordan.

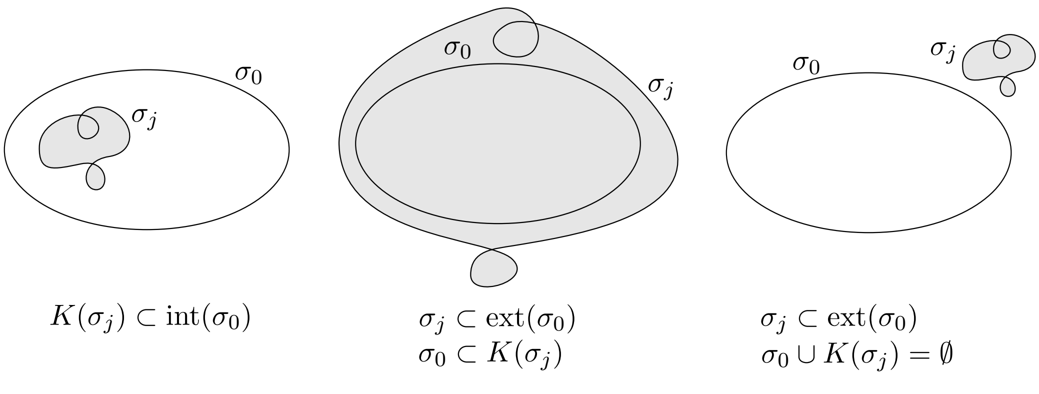

FIGURE 6. Possible relative distribution of the curve σ_j for some $j = 1, \ldots m$ and the curve σ_0.

Step 4. We show that the following conditions hold:

$$K(\sigma_1) \subset \mathrm{int}(\sigma_0), \tag{32}$$

$$K(\sigma_j) \subset \mathrm{ext}(\sigma_{j+1}) \text{ for } j = 1, \ldots, m-2, \tag{33}$$

$$f \text{ has no poles in } K(\sigma_j) \quad \text{for } j = 1, \ldots, m-2. \tag{34}$$

To prove it, note first that if $\sigma_j \subset K(\sigma_{j+1})$ for some $j \in \{0, \ldots, m-2\}$, then for $X_1 = \sigma_j$ we have $X_1 \subset K(f(X_1))$, f has no poles in X_1 and, by (30), $f^2(X_1) \subset \mathrm{ext}(f(X_1))$, so the assumptions of Corollary 4.6 are satisfied for X_1. Hence, f has a weakly repelling fixed point in $K(f(X_1)) = K(f^{j+1}(\sigma_0))$, which is contained in Ω by (29). This makes a contradiction. Hence, we have $\sigma_j \not\subset K(\sigma_{j+1})$ for $j = 0, \ldots, m-2$, which together with (30) and (31) shows $K(\sigma_1) \cap \sigma_0 = \emptyset$ and (33).

To end the proof of (32), it remains to exclude the case $K(\sigma_0) \subset \mathrm{ext}(\sigma_1)$. If it holds, then (since $\mathrm{int}(\sigma_0)$ contains a pole of f), the assumptions of Corollary 4.5 are satisfied for $X = \sigma_0$. Hence, f has a weakly repelling fixed point in $K(\sigma_0) \subset \Omega$, which is a contradiction. In this way we have proved (32).

Finally, to show (34), suppose that f has a pole in $K(f^j(\sigma_0))$ for some $j \in \{1, \ldots, m-2\}$ and take $X_2 = f^j(\sigma_0)$. Then by (33), we have $K(X_2) \subset \mathrm{ext}(f(X_2))$, moreover f has no poles in X_2 and $K(X_2)$ contains a pole of f, so by Corollary 4.5 for X_2, the map f has a weakly repelling fixed point in $K(X_2) = K(f^j(\sigma_0))$, which is contained in Ω by (29). This is a contradiction. Hence, the assertion (34) is proved.

Notice that by (29), (31) and (32), there exists $1 \le k \le m-1$, such that

$$K(\sigma_j) \subset \mathrm{int}(\sigma_0) \text{ for } j = 1, \ldots, k \quad \text{and} \quad \sigma_{k+1} \subset \mathrm{ext}(\sigma_0) \tag{35}$$

(see Figure 7).

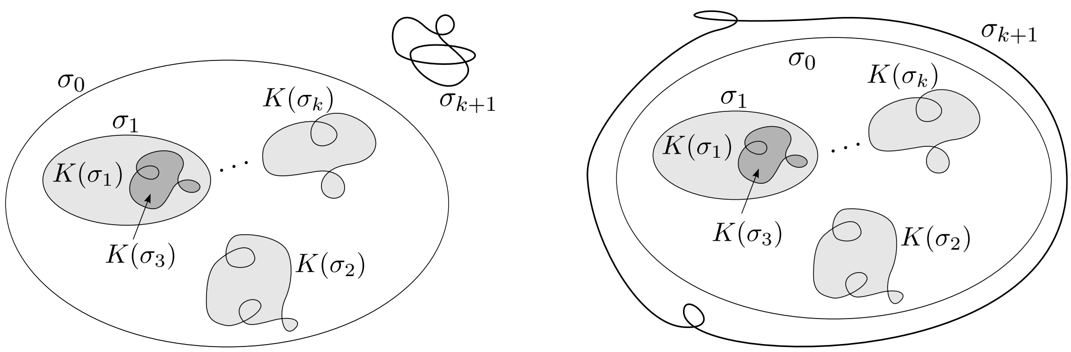

FIGURE 7. Two possible relative positions of $\sigma_k = f^k(\sigma_0)$ and σ_0 under the condition (35). In both cases, $\sigma_{k+1} \subset \mathrm{ext}(\sigma_0)$.

Step 5. We show

$$f(K(\sigma_k)) \subset \mathrm{ext}(\sigma_0). \tag{36}$$

To see it, suppose otherwise, i.e. $f(K(\sigma_k)) \not\subset \mathrm{ext}(\sigma_0)$ (see Figure 8). Then there exists $z_0 \in K(\sigma_k)$ such that $f(z_0) \in K(\sigma_0)$. By (35), we have

$$z_0 \in \Omega_1 \subset \overline{\Omega_1} \subset K(\sigma_k)$$

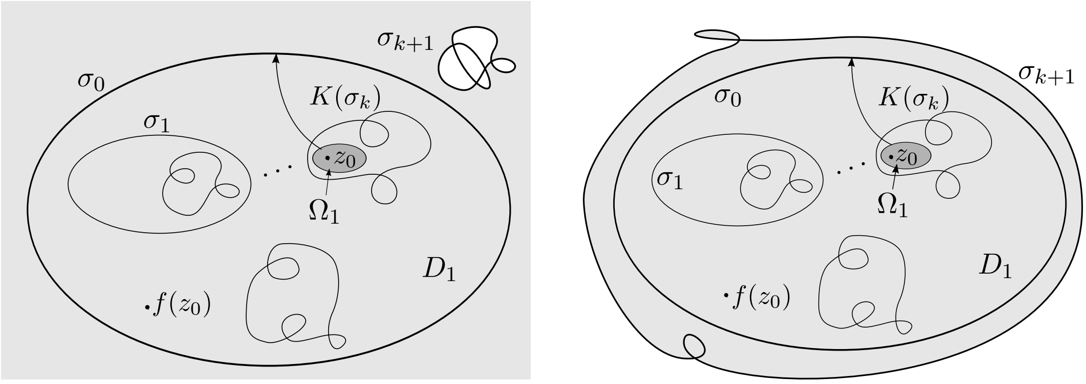

FIGURE 8. Sketch of Step 5.

for some bounded simply connected component Ω_1 of $\widehat{\mathbb{C}} \setminus \sigma_k$. We have $\partial\Omega_1 \subset \sigma_k$, so $f(\partial\Omega_1) \subset \sigma_{k+1}$, which together with (35) implies $\overline{\Omega_1} \subset K(\sigma_0) \subset D_1$ for some component D_1 of $\widehat{\mathbb{C}} \setminus f(\partial\Omega_1)$. Moreover, $z_0 \in \Omega_1$ and $f(z_0) \in K(\sigma_0) \subset D_1$. Hence, the assumptions of Lemma 4.4 are satisfied for Ω_1, D_1, z_0, so f has a weakly repelling fixed point in Ω_1, which is contained in Ω by (29). This makes a contradiction. Therefore, (36) is satisfied.

Step 6. We check that we are under the assumptions of Shishikura's Theorem 2.14. Let

$$V_0 = \operatorname{ext}(\sigma_0), \qquad V_1 = \operatorname{int}(\sigma_1),$$

and let us check that V_0, V_1 satisfy the required assumptions. By definition, V_0, V_1 are simply connected and $f(\partial V_0) = \partial V_1$. Since f is a covering in some neighbourhood N of $\sigma_0 = \partial V_0$, we have

$$f(V_0 \cap N) = f(N \setminus \overline{D'}) \subset \mathbb{C} \setminus \overline{D} = V_1.$$

By (35), $K(f^j(\partial V_1)) \subset \mathbb{C} \setminus \overline{V_0}$ for $j = 0, \dots, k-1$ and $f^k(\partial V_1) \subset V_0$. Moreover, by (34) and (36), the map f^k is defined on $\overline{V_1}$ and

$$f^j(\overline{V_1}) \subset K(f^j(\partial V_1)) \subset \mathbb{C} \setminus \overline{V_0} \text{ for } j = 0, \dots, k-1 \quad \text{and} \quad f^k(\overline{V_1}) \subset V_0.$$

See Figure 9. Hence, Shishikura's Theorem 2.14 concludes that f has a weakly repelling point

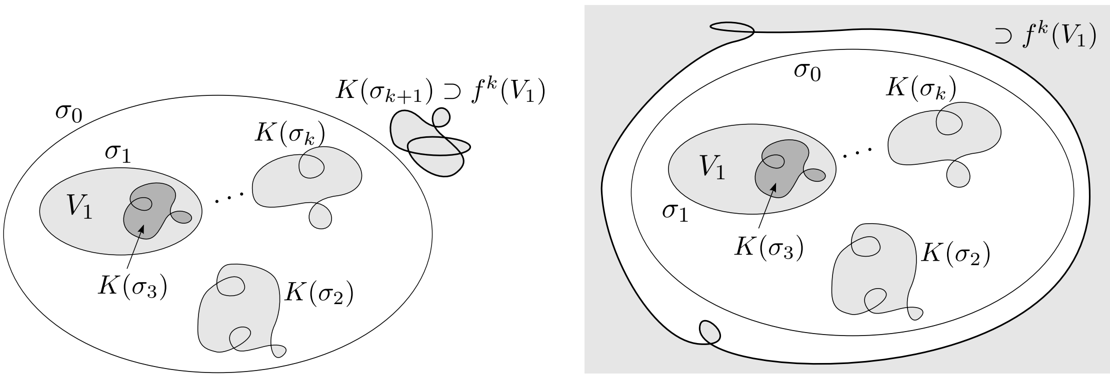

FIGURE 9. Sketch of Step 6.

in $\widehat{\mathbb{C}} \setminus \overline{V_0} = \operatorname{int}(\sigma_0) \subset \Omega$, which finishes the proof. □

5. Proof of Theorem B

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a transcendental meromorphic map and let $U_0, \ldots, U_{p-1}$ be a periodic cycle of Baker domains of f of (minimal) period $p \geq 1$. Recall that for $j = 0, \ldots, p-1$ we have $f^{pn} \to \zeta_j$ locally uniformly on U_j as $n \to \infty$ for some $\zeta_j \in \widehat{\mathbb{C}}$ such that $\zeta_j = \infty$ for at least one j. Renumbering the Baker domains, we may assume $\zeta_0 = \infty$, i.e. the domain U_0 is unbounded and

$$f^{pn}(z) \to \infty \quad \text{for } z \in U_0 \quad \text{as } n \to \infty.$$

As the first step in the proof of Theorem B we show a technical lemma which allows us to discard some of the possible configurations of the U_j's. More precisely, we show that under certain relative positions of the U_j's the existence of a weakly repelling fixed point follows directly from the results in Section 4.

Lemma 5.1 (Configurations of Baker domains). *Suppose there exist a simply connected bounded domain $\Omega \subset \mathbb{C}$ and a pole p_0 of f, such that*

$$p_0 \in \Omega \quad \textit{and} \quad \partial\Omega \subset U_j$$

for some $j = 0, \ldots, p-1$. Then f has a weakly repelling fixed point or there exist $n \geq 0$ and a bounded simply connected domain $\Omega_0 \subset \mathbb{C}$, such that

$$p_0 \in \Omega_0 \quad \textit{and} \quad \partial\Omega_0 \subset f^n(\partial\Omega) \subset U_0.$$

Proof. If $p = 1$ then we can take $n = 0$ and $\Omega_0 = \Omega$. Hence, in what follows we assume $p > 1$.

Since $p > 1$, it is clear that $\partial\Omega, f(\partial\Omega), \ldots, f^{p-1}(\partial\Omega)$ are pairwise disjoint and we cannot have

$$K(\partial\Omega) \subset K(f(\partial\Omega)) \subset \cdots \subset K(f^p(\partial\Omega)),$$

because it would contradict the connectedness of U_j. Thus, there is a minimal $n \geq 0$ such that

$$K(f^n(\partial\Omega)) \not\subset K(f^{n+1}(\partial\Omega)). \tag{37}$$

Note that we have $p_0 \in K(f^n(\partial\Omega)) \setminus f^n(\partial\Omega)$. Hence, there exists a bounded component Ω_0 of $\widehat{\mathbb{C}} \setminus f^n(\partial\Omega)$, such that $p_0 \in \Omega_0$. Since Ω is simply connected, Ω_0 is also simply connected.

As $\partial\Omega_0 \cap f(\partial\Omega_0) = \emptyset$, one of the three possibilities holds: $\overline{\Omega_0} \subset K(f(\partial\Omega_0))$, $\overline{\Omega_0} \subset \operatorname{ext}(f(\partial\Omega_0))$ or $f(\partial\Omega_0) \subset \Omega_0$. Since $\partial\Omega_0 \subset f^n(\partial\Omega)$ and $f(\partial\Omega_0) \cap f^n(\partial\Omega) = \emptyset$, the first possibility does not occur by (37). If the second possibility holds, then the assumptions of Corollary 4.5 are satisfied for $X = \partial\Omega_0$, so f has a weakly repelling fixed point. Hence, we are left with the third possibility, i.e. $f(\partial\Omega_0) \subset \Omega_0$.

Note that $\partial\Omega_0, f(\partial\Omega_0), \ldots, f^{p-1}(\partial\Omega_0)$ are pairwise disjoint. Therefore, if there exists a (minimal) number $2 \leq m \leq p-1$ such that $f^m(\partial\Omega_0) \not\subset \Omega_0$, then $f(\partial\Omega_0), \ldots, f^{m-1}(\partial\Omega_0) \subset \Omega_0$ and $f^m(\partial\Omega_0) \cap \overline{\Omega_0} = \emptyset$, so the assumptions of Proposition 4.7 are fulfilled for Ω_0 and we conclude that f has a weakly repelling fixed point in that case. Thus, we can assume $f(\partial\Omega_0), \ldots, f^{p-1}(\partial\Omega_0) \subset \Omega_0$. This implies $\partial\Omega_0 \subset U_0$, because otherwise $\partial\Omega_0 \cap U_0 = \emptyset$ and one of the sets $f(\partial\Omega_0), \ldots, f^{p-1}(\partial\Omega_0)$ is contained in U_0, which contradicts the fact that U_0 is connected and unbounded. Hence, Ω_0 satisfies the assertion of the lemma. □

Let $W \subset U_0$ be an absorbing domain which exists according to Corollary A' (for the map $F = f^p$). Note that W is unbounded and does not contain poles of f. The proof of Theorem B splits into two cases depending on the simple connectivity of W.

Case 1. *W* is not simply connected.

Under this assumption we can take a closed curve

$$\gamma \subset W,$$

such that $K(\gamma) \cap J(f) \neq \emptyset$. Notice that, because of Corollary A', $f^{p\ell}(\gamma) \subset W$ for all $\ell \geq 0$.

By Lemma 4.3, there exists $n_0 \geq 0$ and a pole p_0 of f, such that $p_0 \in K(f^{n_0}(\gamma))$. Then p_0 is in a bounded simply connected component Ω of $\widehat{\mathbb{C}} \setminus f^{n_0}(\gamma)$, such that $\partial\Omega \subset f^{n_0}(W)$. By Lemma 5.1, we may reduce the proof to the case when there exists a bounded simply connected domain Ω_0 with

$$\partial\Omega_0 \subset f^{n_1}(\partial\Omega) \subset f^{n_1}(\gamma) \subset U_0 \cap f^{n_0+n_1}(W)$$

for some $n_1 \geq 0$, such that $p_0 \in \Omega_0$. In particular, this implies that $n_0 + n_1 = \ell p$ for some $\ell \geq 0$, so by Corollary A' we have $f^{n_0+n_1}(W) \subset W$, which implies $\partial\Omega_0 \subset W$. We conclude that there exists a bounded component Ω_1 of $\mathbb{C} \setminus \overline{W}$, such that $p_0 \in \Omega_1$. Since W is connected we know that Ω_1 is simply connected. We claim that

$$\partial\Omega_1, f(\partial\Omega_1), \dots, f^p(\partial\Omega_1) \quad \text{are pairwise disjoint.} \tag{38}$$

To see the claim it is enough to notice that $\partial\Omega_1, f(\partial\Omega_1), \dots, f^{p-1}(\partial\Omega_1)$ are in different Fatou components. Moreover, $\partial\Omega_1 \subset \overline{W} \subset U_0$, so by Corollary A' we get

$$f^p(\partial\Omega_1) \subset f^p(\overline{W}) \subset f^p(W) \subset \mathbb{C} \setminus \overline{\Omega_1}. \tag{39}$$

Now we proceed like in the proof of Lemma 5.1. By (38), we have $f(\partial\Omega_1) \subset \Omega_1$, $\overline{\Omega_1} \subset \operatorname{ext}(f(\partial\Omega_1))$ or $\overline{\Omega_1} \subset K(f(\partial\Omega_1))$. In the first case, by (39) we have $p > 1$ and there exists $m \in \{2, \dots, p\}$ such that $f(\partial\Omega_1), \dots, f^{m-1}(\partial\Omega_1) \subset \Omega_1$ and $f^m(\partial\Omega_1) \cap \overline{\Omega_1} = \emptyset$. Hence, f has a weakly repelling fixed point by Proposition 4.7 applied to Ω_1. In the second case we use Corollary 4.5 for $X = \partial\Omega_1$. Thus, we can assume that the third possibility takes place, i.e.

$$\overline{\Omega_1} \subset K(f(\partial\Omega_1))$$

Note that this implies

$$p = 1,$$

because if $p > 1$, then $\Omega_1 \subset U_0$ and $f(\partial\Omega_1) \cap U_0 = \emptyset$, which contradicts the fact that U_0 is connected and unbounded.

Let

$$\mathcal{N} = \{n \geq 0 : p_0 \text{ is contained in a bounded component of } \mathbb{C} \setminus \overline{f^n(W)}\}.$$

Note that $0 \in \mathcal{N}$, so $\sup \mathcal{N}$ is well defined. We consider two further subcases.

Case (i): $\sup \mathcal{N} = N < \infty$.

Then p_0 is contained in a bounded component Ω_2 of $\mathbb{C} \setminus \overline{f^N(W)}$ but is not contained in any bounded component of $\mathbb{C} \setminus \overline{f^{N+1}(W)}$. Moreover, by Corollary A' we have

$$f(\partial\Omega_2) \subset f(\overline{f^N(W)}) = f^{N+1}(\overline{W}) \subset f^N(W) \subset \mathbb{C} \setminus \overline{\Omega_2}.$$

This implies $\overline{\Omega_2} \subset \operatorname{ext}(f(\partial\Omega_2))$. Consequently, the assumptions of Corollary 4.5 are satisfied for $X = \partial\Omega_2$, and so f has a weakly repelling fixed point.

Case (ii): $\sup \mathcal{N} = \infty$.

Fix some point $z_0 \in \mathbb{C}$, which is not a pole of f. By assumption and Corollary A', for sufficiently large n there exists a bounded component Ω_3 of $\mathbb{C} \setminus \overline{f^n(W)}$ containing $p_0, z_0, f(z_0)$, such that

$$f(\partial\Omega_3) \subset f(\overline{f^n(W)}) = f^{n+1}(\overline{W}) \subset f^n(W) \subset \mathbb{C} \setminus \overline{\Omega_3}.$$

Hence,

$$\overline{\Omega_3} \subset D,$$

where D is a component of $\widehat{\mathbb{C}} \setminus f(\partial\Omega_3)$. We have $z_0, f(z_0) \in \Omega_3 \subset D$. Hence, Ω_3, D, z_0 satisfy the assumptions of Lemma 4.4, so f has a weakly repelling fixed point. This ends the proof of Theorem B in Case 1 (W is multiply connected).

Case 2. *W* is simply connected.

By assumption, one of the domain U_j is multiply connected, so like in the proof in Case 1, using Lemmas 4.3 and 5.1 we can assume that there exists a curve

$$\gamma \subset U_0$$

and a pole p_0 of f, such that $p_0 \in K(\gamma)$ (the difference with respect to the previous case is that the curve γ was taken in W). Let

$$\Gamma = \bigcup_{n=0}^{\infty} f^n(\gamma).$$

Note that $p_0 \notin \Gamma$ and $f(\Gamma) \subset \bigcup_{n=1}^{\infty} f^n(\gamma) \subset \Gamma$. Moreover, Γ is the union of p disjoint sets

$$\Gamma_j = \bigcup_{n=0}^{\infty} f^{pn+j}(\gamma) \subset U_j,$$

for $j \in \{0, \dots p-1\}$, such that $f(\Gamma_j) \subset \Gamma_{j+1 \bmod p}$ and $f^{p\ell} \to \zeta_j$ uniformly on Γ_j as $\ell \to \infty$. In particular, this implies that Γ_0 is a closed subset of $\mathbb{C}$.

Define

$$\mathcal{N} = \{n \geq 0 : p_0 \text{ is contained in a bounded simply connected domain with boundary in } f^n(\Gamma_0)\}.$$

Since $p_0 \in K(\gamma) \setminus \gamma$ and $\gamma \subset \Gamma_0$, we have $0 \in \mathcal{N}$, so $\sup \mathcal{N}$ is well defined. By Lemma 5.1, we can reduce the proof to the case, when the following holds:

$$\text{for every } n \in \mathcal{N} \text{ there exists } N \in \mathcal{N} \text{ such that } N \geq n \text{ and } f^N(\Gamma_0) \subset U_0. \tag{40}$$

Suppose $\sup \mathcal{N} = \infty$. Then (40) implies that there are arbitrarily large N such that p_0 is contained in a bounded simply connected domain with boundary in $f^N(\Gamma_0) \cap U_0$. By Corollary A', this boundary is contained in W for large enough values of N. This is a contradiction since W is simply connected by assumption.

Hence, $\sup \mathcal{N} = N_0 < \infty$ and, again by (40), there exists a bounded simply connected domain V with

$$\partial V \subset f^{N_0}(\Gamma_0) \subset \Gamma_0 \subset U_0, \tag{41}$$

such that $p_0 \in V$ and p_0 is not contained in any bounded simply connected domain with boundary in $f^{N_0+1}(\Gamma_0)$. Define E to be the bounded component of $\mathbb{C} \setminus f^{N_0}(\Gamma_0)$, such that $p_0 \in E$. Note that by (41), the set $f^{N_0}(\Gamma_0)$ is closed in $\mathbb{C}$ and so

$$\partial E \subset f^{N_0}(\Gamma_0).$$

Let

$$\Omega = \bigcup \{K(\sigma) : \sigma \text{ is a closed curve in } E\}.$$

By definition, Ω is a bounded simply connected domain in $\mathbb{C}$, such that $E \subset \Omega$, $p_0 \in \Omega$ and

$$\partial\Omega \subset \partial E \subset f^{N_0}(\Gamma_0) \subset U_0.$$

We claim that for any given $n > 0$, one of the following must be satisfied:

$$f^n(\partial\Omega) \cap \Omega = \emptyset \quad \text{or} \quad f^n(\partial\Omega) \subset \Omega. \tag{42}$$

To see this observe that if $n \neq \ell p$ for all $\ell > 0$, then $f^n(\partial\Omega) \cap \partial\Omega = \emptyset$, so (42) holds due to the connectedness of $\partial\Omega$ and $f^n(\partial\Omega)$. If $n = \ell p$ for some $\ell > 0$, then $f^n(\partial\Omega) \subset f^{N_0}(\Gamma_0)$, so $f^n(\partial\Omega)$ is disjoint from E. Hence, if $f^n(\partial\Omega)$ intersects Ω, then $f^n(\partial\Omega) \cap K(\sigma) \neq \emptyset$ for a closed curve $\sigma \subset E$, so in fact $f^n(\partial\Omega) \subset K(\sigma) \subset \Omega$. This shows (42).

Using (42), we conclude that one of the following three cases holds: $\Omega \subset K(f(\partial\Omega))$, $\Omega \subset \operatorname{ext}(f(\partial\Omega))$ or $f(\partial\Omega) \subset \Omega$. The first case is not possible since it would imply that p_0 is in a bounded simply connected domain with boundary in $f^{N_0+1}(\Gamma_0)$, which contradicts the definition of N_0. The second case implies that the assumptions of Corollary 4.5 are satisfied for $X = \partial\Omega$ (by Torhorst's Theorem 2.13, $\partial\Omega$ is locally connected; moreover, f has no fixed points in $\partial\Omega$), so f has a weakly repelling fixed point. Hence, the remaining case is

$$f(\partial\Omega) \subset \Omega.$$

By (42) and the fact that $f^{pn} \to \infty$ as $n \to \infty$ uniformly on $\partial\Omega$, there exists a (minimal) number $m \geq 2$ such that

$$f(\partial\Omega), \ldots, f^{m-1}(\partial\Omega) \subset \Omega \quad \text{and} \quad f^m(\partial\Omega) \cap \Omega = \emptyset. \tag{43}$$

If $f^m(\partial\Omega) \cap \overline{\Omega} = \emptyset$, the domain Ω satisfies the assumptions of Proposition 4.7, so f has a weakly repelling fixed point. Hence, we are left with the case $f^m(\partial\Omega) \cap \partial\Omega \neq \emptyset$, which implies $m = \ell p$ for a certain $\ell > 0$ and, consequently, $f^m(\partial\Omega) \subset U_0$.

In this case we will see that we can slightly modify the domain Ω to a new domain Ω', so that Ω' satisfies the condition (43) and $f^m(\partial\Omega') \cap \overline{\Omega'} = \emptyset$. Then Proposition 4.7 applies to Ω' and f has a weakly repelling fixed point.

To define the set Ω' with the desired conditions, let

$$D_\varepsilon = \{z \in U_0 : \varrho_{U_0}(z, \partial\Omega) \leq \varepsilon\}$$

for a small $\varepsilon > 0$. Then D_ε is a compact subset of U_0. It is immediate by (43), that if ε is small enough, then all sets $f(\partial\Omega), \ldots, f^{m-1}(\partial\Omega)$ are contained in the same bounded component Ω' of $\Omega \setminus D_\varepsilon$, such that $\overline{\Omega'} \subset \Omega$. Since $\partial\Omega$ is connected, the set D_ε is also connected and, consequently, Ω' is simply connected. Moreover,

$$\partial\Omega' \subset \{z \in U_0 : \varrho_{U_0}(z, \partial\Omega) = \varepsilon\} \tag{44}$$

and, since $\overline{\Omega'} \subset \Omega$ and $f^m(\partial\Omega) \cap \Omega = \emptyset$, we have

$$\varrho_{U_0}(z, w) \geq \varepsilon \quad \text{for every } z \in \overline{\Omega'} \cap U_0 \text{ and } w \in f^m(\partial\Omega) \tag{45}$$

(otherwise, connecting z to w in U_0 by a curve κ of hyperbolic length smaller than ε, we would find $z' \in \partial\Omega' \cap \kappa$ and $w' \in \partial\Omega \cap \kappa$ such that $\varrho_{U_0}(z', w') < \varepsilon$, which contradicts (44)).

As f^m maps U_0 into itself, Schwarz–Pick's Lemma 2.2 implies that for $z \in \partial\Omega'$ and $w \in \partial\Omega$ we have

$$\varrho_{U_0}(f^m(z), f^m(w)) \le \varrho_{U_0}(z, w), \tag{46}$$

with strict inequality unless a lift of f^m to a universal cover of U_0 is a Möbius transformation. Suppose the inequality in (46) is not strict. Then the first assumption of Lemma 2.4 is satisfied for $U = U_0$ and $F = f^m$, while the additional assumption of this lemma is also fulfilled since W is simply connected. Hence, by Lemma 2.4 we conclude that U_0 is simply connected, a contradiction with $p_0 \in \Omega$ and $\partial\Omega \subset U_0$.

Therefore, the inequality in (46) is strict, and by the compactness of $\partial\Omega$ we have

$$\varrho_{U_0}(f^m(z), f^m(\partial\Omega)) < \varrho_{U_0}(z, \partial\Omega) = \varepsilon \quad \text{for every } z \in \partial\Omega'. \tag{47}$$

This together with (45) implies

$$f^m(\partial\Omega') \cap \overline{\Omega'} = \emptyset.$$

Note also that if ε is sufficiently small, then by (43),

$$f(\partial\Omega'), \dots, f^{m-1}(\partial\Omega') \subset \Omega'.$$

Hence, the assumptions of Proposition 4.7 are satisfied for Ω', and f has a weakly repelling fixed point. This concludes the proof in Case 2 (W is simply connected) and, in fact, the proof of Theorem B.

6. Proof of Theorem C

In what follows we assume that $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is a meromorphic map with a cycle of Herman rings $U_0, \dots, U_{p-1}$ for some $p > 0$. Then there exists a biholomorphic map

$$\psi : U_0 \to \{z : 1/r < |z| < r\}$$

for some $r > 1$, such that $\psi \circ f^p \circ \psi^{-1} = R_\alpha$, where $R_\alpha(z) = e^{2\pi i\alpha}z$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Herman rings are multiply connected by definition. The goal is to show that in this setup, f must have a weakly repelling fixed point. Let

$$\gamma = \psi^{-1}(\{z : |z| = 1\}).$$

Then γ is a Jordan curve in U_0. If $p = 1$, then Lemma 4.4 applies to $\Omega = \mathrm{int}(\gamma)$, and f has a weakly repelling fixed point. Hence, in what follows we assume $p > 1$ and, consequently, γ is a Jordan curve in U_0 such that $\gamma, f(\gamma), \dots, f^{p-1}(\gamma)$ are pairwise disjoint, $f^p(\gamma) = \gamma$ and $\mathrm{int}(\gamma) \cap J(f) \ne \emptyset$. By Lemma 4.3, the map f has a pole p_0 in $\mathrm{int}(f^j(\gamma))$ for some $0 \le j \le p-1$. Without loss of generality we assume that $j = 0$, i.e. $p_0 \in \mathrm{int}(\gamma)$.

Next we discuss different relative positions of the above curves to see that the results in Section 4 imply that f has a weakly repelling fixed point unless one situation occurs. In this case, to show the existence of a weakly repelling fixed point we will use a surgery argument, like in Shishikura's Theorem 2.14.

Observe that for all $j \ge 0$, we have $f^j(\gamma) \subset \mathrm{int}(f^{j+1}(\gamma))$ or $f^j(\gamma) \subset \mathrm{ext}(f^{j+1}(\gamma))$. Since $f^p(\gamma) = \gamma$, we cannot have $f^j(\gamma) \subset \mathrm{int}(f^{j+1}(\gamma))$ for all $j = 0, \dots, p-1$. Hence, there exists a minimal number $j_0 \in \{0, \dots, p-1\}$ such that $f^{j_0}(\gamma) \subset \mathrm{ext}(f^{j_0+1}(\gamma))$. Set

$$\sigma_0 = f^{j_0}(\gamma) \quad \text{and} \quad \sigma_j = f^j(\sigma_0), \; j \ge 1$$

By definition, $\sigma_0, \sigma_1, \ldots, \sigma_{p-1}$ are pairwise disjoint and $\sigma_p = \sigma_0$. Moreover, $\sigma_0 \subset \operatorname{ext}(\sigma_1)$ and $p_0 \in \operatorname{int}(\sigma_0)$, by the minimality of j_0.

Suppose first that $\sigma_1 \subset \operatorname{ext}(\sigma_0)$. Then $\operatorname{int}(\sigma_0) \subset \operatorname{ext}(\sigma_1)$, so by Corollary 4.5 for $X = \sigma_0$, the map f has a weakly repelling fixed point. Hence, we can assume

$$\sigma_1 \subset \operatorname{int}(\sigma_0). \tag{48}$$

If there exists $j \in \{2, \ldots, p-1\}$ such that $\sigma_j \subset \operatorname{ext}(\sigma_0)$, then the assumptions of Proposition 4.7 are satisfied for $\Omega = \operatorname{int}(\sigma_0)$, so f has a weakly repelling fixed point. Therefore, from now on we suppose that

$$\sigma_j \subset \operatorname{int}(\sigma_0) \quad \text{for } j = 1, \ldots, p-1. \tag{49}$$

Suppose now that there exists $j \in \{1, \ldots, p-1\}$ such that $\sigma_{j+1} \subset \operatorname{int}(\sigma_j)$. Then the assumptions of Proposition 4.7 are satisfied for $\Omega = \operatorname{int}(\sigma_j)$, so f has a weakly repelling fixed point. Thus, we can assume that

$$\sigma_{j+1} \subset \operatorname{ext}(\sigma_j) \quad \text{for } j = 1, \ldots, p-1. \tag{50}$$

If there exists $j \in \{1, \ldots, p-2\}$ such that $\sigma_j \subset \operatorname{int}(\sigma_{j+1})$ then, by (50), the assumptions of Corollary 4.6 are satisfied for $X = \sigma_j$, so f has a weakly repelling fixed point. Hence, we may also suppose that $\sigma_j \not\subset \operatorname{int}(\sigma_{j+1})$, so

$$\operatorname{int}(\sigma_j) \subset \operatorname{ext}(\sigma_{j+1}) \quad \text{for } j = 1, \ldots, p-2. \tag{51}$$

By Corollary 4.5 for $X = \sigma_j$, and using (51) we may assume that f has no poles in $\operatorname{int}(\sigma_j)$ for $j = 1, \ldots, p-2$. Consequently,

$$f(\operatorname{int}(\sigma_j)) = \operatorname{int}(\sigma_{j+1}) \quad \text{for } j = 1, \ldots, p-2. \tag{52}$$

We claim that we can also reduce the proof to the case where

$$\operatorname{int}(\sigma_1), \ldots, \operatorname{int}(\sigma_{p-1}) \quad \text{are pairwise disjoint subsets of } \operatorname{int}(\sigma_0). \tag{53}$$

To see this suppose otherwise, i.e. there exist $k > 0$ and $m > 1$ with $k + m \leq p - 1$, such that $\sigma_{k+m} \subset \operatorname{int}(\sigma_k)$ or $\sigma_k \subset \operatorname{int}(\sigma_{k+m})$. Observe that $m = 1$ is not possible by (51).

In the first case, observe that by (52), $f^{p-k-m}(\operatorname{int}(\sigma_k)) = \operatorname{int}(\sigma_{p-m}) \subsetneq \operatorname{int}(\sigma_0)$. Since $\sigma_{k+m} \subset \operatorname{int}(\sigma_k)$, we have $\sigma_0 = f^{p-k-m}(\sigma_{k+m}) \subset \operatorname{int}(\sigma_{p-m})$, which again is not possible.

In the second case, again by (52), $f^{p-k-m-1}(\operatorname{int}(\sigma_{k+m})) = \operatorname{int}(\sigma_{p-1})$. Since $\sigma_k \subset \operatorname{int}(\sigma_{k+m})$ we have $\sigma_{p-m-1} = f^{p-k-m-1}(\sigma_k) \subset \operatorname{int}(\sigma_{p-1})$. Hence, there exists $z_0 \in \operatorname{int}(\sigma_{p-1})$ such that $f(z_0) \in \operatorname{int}(\sigma_0)$. Then, Lemma 4.4 with $\Omega = \operatorname{int}(\sigma_{p-1})$ and $D = \operatorname{int}(\sigma_0)$ provides the existence of a weakly repelling fixed point of f.

Hence, we may assume (53). By Lemma 4.4 applied exactly as above we may also suppose that

$$f(\operatorname{int}(\sigma_{p-1})) = \overline{\operatorname{ext}(\sigma_0)}. \tag{54}$$

Finally, suppose that $f(\operatorname{int}(\sigma_0)) \supset \operatorname{int}(\sigma_1)$, which together with (52) implies that $f(\operatorname{int}(\sigma_0)) = \widehat{\mathbb{C}}$. By considering a preimage of $D = \operatorname{int}(\sigma_0)$ compactly contained inside D, and applying Lemma 4.4, it follows again that f has a weakly repelling fixed point. Hence, from now on we also suppose that

$$\operatorname{int}(\sigma_1) \not\subset f(\operatorname{int}(\sigma_0)), \tag{55}$$

which implies that there exists a neighborhood N of $\operatorname{int}(\sigma_0)$ such that $f(N \cap \operatorname{ext}(\sigma_0)) \subset \operatorname{int}(\sigma_1)$.

At this point we work under the assumptions (48)–(55), as shown in Figure 10.

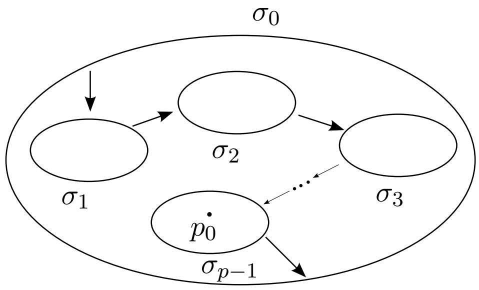

FIGURE 10. The final setup in the proof of Theorem C.

Observe that the situation is reminiscent of the setup of Shishikura's Theorem 2.14 for $V_0 = \mathrm{ext}(\sigma_0)$, $V_1 = \mathrm{int}(\sigma_1)$ and $k = p-1$, except for one hypothesis, namely $f^k(\overline{V_1}) \subset V_0$, which instead reads as $f^k(\overline{V_1}) = \overline{V_0}$.

We shall conclude the proof with an alternative surgery argument, which is a particular case of Shishikura's surgery in [39, Theorem 6]. The idea is to convert the p-cycle of Herman rings into a p-cycle of Siegel discs, by gluing a rigid rotation in $\mathrm{ext}(\sigma_0)$ (for the p-th iterate). This will provide the existence of a weakly repelling fixed point in $\mathrm{int}(\sigma_0) \setminus \bigcup_{j=1}^{p-1} \mathrm{int}(\sigma_j)$.

We sketch the details for completeness. Redefine the cycle of Herman rings so that $\sigma_0 \subset U_0$. Then $\sigma_0 = \psi^{-1}(\{z : |z| = 1\})$. Since $\psi|_{\sigma_0}$ is real analytic, there exists a quasiconformal homeomorphism

$$\Psi : \overline{\mathrm{ext}(\sigma_0)} \to \widehat{\mathbb{C}} \setminus \mathbb{D}$$

such that $\Psi = \psi$ on σ_0. We now define $h : \overline{\mathrm{ext}(\sigma_0)} \to \overline{\mathrm{ext}(\sigma_0)}$ as

$$h = \Psi^{-1} \circ R_\alpha \circ \Psi.$$

Note that $h^n = \Psi^{-1} \circ R_\alpha^n \circ \Psi$ and therefore h^n is uniformly quasiregular for all $n > 0$.

Since f^p is conjugate to R_α on σ_0, it follows that f has degree one on σ_j for all $j = 1, \ldots, p$. Together with (52) and (54), this implies that for all $j = 1, \ldots, p-1$, the map $f|_{\mathrm{int}(\sigma_j)}$ is univalent and hence it has a univalent inverse. We now define a new map on the Riemann sphere as follows:

$$F = \begin{cases} f & \text{on } \overline{\mathrm{int}(\sigma_0)} \\ \left(f\,|_{\mathrm{int}(\sigma_1)}\right)^{-1} \circ \cdots \circ \left(f\,|_{\mathrm{int}(\sigma_{p-1})}\right)^{-1} \circ h & \text{on } \mathrm{ext}(\sigma_0). \end{cases}$$

Note that $F^p|_{\mathrm{ext}(\sigma_0)} = h$ and F is holomorphic everywhere except on $\mathrm{ext}(\sigma_0)$, where it is quasiconformal. Now we define a conformal structure μ on $\widehat{\mathbb{C}}$ setting

$$\mu = \begin{cases} (\Psi^{-1})^* \mu_0 & \text{on } \mathrm{ext}(\sigma_0) \\ \left(\left(f\,|_{\mathrm{int}(\sigma_j)}\right)^{-1} \circ \cdots \circ \left(f\,|_{\mathrm{int}(\sigma_{p-1})}\right)^{-1}\right)^* \mu & \text{on } \mathrm{int}(\sigma_j) \text{ for } j = 1, \ldots, p-1 \\ \mu_0 & \text{elsewhere,} \end{cases}$$

where μ_0 is the standard structure. Then μ is bounded and F-invariant, so by the Measurable Riemann Mapping Theorem, F is quasiconformally conjugate to a rational map g, under a quasiconformal homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

One can check that on some neighborhood of $\phi(\overline{\text{ext}(\sigma_0)})$ the map $\Psi \circ \phi^{-1}$ is conformal and conjugates g^p to R_α. Hence, g has a p-cycle of Siegel discs containing $\phi(\overline{\text{ext}(\sigma_0)}) \cup \phi(\text{int}(\sigma_1)) \cup \cdots \cup \phi(\text{int}(\sigma_{p-1}))$. Since g is rational, it has a weakly repelling fixed point, which cannot lie in the Siegel cycle. But g is conformally conjugate to f everywhere else. Hence, f has a weakly repelling fixed point. This concludes the proof of Theorem C.

References

[1] Lars V. Ahlfors. *Conformal invariants.* AMS Chelsea Publishing, Providence, RI, 2010. Topics in geometric function theory, Reprint of the 1973 original, With a foreword by Peter Duren, Frederick W. Gehring and Brad Osgood.

[2] Irvine N. Baker. The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):277–283, 1975.

[3] Irvine N. Baker and Patricia Domínguez. Analytic self-maps of the punctured plane. *Complex Variables Theory Appl.*, 37(1-4):67–91, 1998.

[4] Irvine N. Baker, Janina Kotus, and Lü Yinian. Iterates of meromorphic functions. I. *Ergodic Theory Dynam. Systems*, 11(2):241–248, 1991.

[5] Irvine N. Baker and Christian Pommerenke. On the iteration of analytic functions in a halfplane. II. *J. London Math. Soc. (2)*, 20(2):255–258, 1979.

[6] Krzysztof Barański and Núria Fagella. Univalent Baker domains. *Nonlinearity*, 14(3):411–429, 2001.

[7] Walter Bergweiler. Iteration of meromorphic functions. *Bull. Amer. Math. Soc. (N.S.)*, 29(2):151–188, 1993.

[8] Walter Bergweiler. Newton's method and Baker domains. *J. Difference Equ. Appl.*, 16(5-6):427–432, 2010.

[9] Walter Bergweiler, David Drasin, and James K. Langley. Baker domains for Newton's method. *Ann. Inst. Fourier (Grenoble)*, 57(3):803–814, 2007.

[10] Walter Bergweiler and Norbert Terglane. Weakly repelling fixpoints and the connectivity of wandering domains. *Trans. Amer. Math. Soc.*, 348(1):1–12, 1996.

[11] Xavier Buff. Virtually repelling fixed points. *Publ. Mat.*, 47(1):195–209, 2003.

[12] Xavier Buff and Johannes Rückert. Virtual immediate basins of Newton maps and asymptotic values. *Int. Math. Res. Not.*, pages Art. ID 65498, 18, 2006.

[13] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.

[14] Carl C. Cowen. Iteration and the solution of functional equations for functions analytic in the unit disk. *Trans. Amer. Math. Soc.*, 265(1):69–95, 1981.

[15] Patricia Domínguez. Dynamics of transcendental meromorphic functions. *Ann. Acad. Sci. Fenn. Math.*, 23(1):225–250, 1998.

[16] Adrien Douady and John H. Hubbard. On the dynamics of polynomial-like mappings. *Ann. Sci. École Norm. Sup. (4)*, 18(2):287–343, 1985.

[17] Núria Fagella and Christian Henriksen. Deformation of entire functions with Baker domains. *Discrete Contin. Dyn. Syst.*, 15(2):379–394, 2006.

[18] Núria Fagella and Christian Henriksen. The Teichmüller space of an entire function. In *Complex dynamics*, pages 297–330. A K Peters, Wellesley, MA, 2009.

[19] Núria Fagella, Xavier Jarque, and Jordi Taixés. On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points. I. *Proc. Lond. Math. Soc. (3)*, 97(3):599–622, 2008.

[20] Núria Fagella, Xavier Jarque, and Jordi Taixés. On connectivity of Julia sets of transcendental meromorphic maps and weakly repelling fixed points II. *Fund. Math.*, 215(2):177–202, 2011.

[21] Pierre Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47:161–271, 1919.

[22] Mako E. Haruta. Newton's method on the complex exponential function. *Trans. Amer. Math. Soc.*, 351(6):2499–2513, 1999.

[23] John H. Hubbard, Dierk Schleicher, and Scott Sutherland. How to find all roots of complex polynomials by Newton's method. *Invent. Math.*, 146(1):1–33, 2001.

[24] Gaston Julia. Mémoire sur l'itération des fonctions rationnelles. *J. Math. Pures Appl. (8)*, 1:47–245, 1918.

[25] Harald König. Conformal conjugacies in Baker domains. *J. London Math. Soc. (2)*, 59(1):153–170, 1999.

[26] Anthony Manning. How to be sure of finding a root of a complex polynomial using Newton's method. *Bol. Soc. Brasil. Mat. (N.S.)*, 22(2):157–177, 1992.
[27] Albert Marden and Christian Pommerenke. Analytic self-mappings of infinite order of Riemann surfaces. *J. Analyse Math.*, 37:186–207, 1980.
[28] Sebastian Mayer and Dierk Schleicher. Immediate and virtual basins of Newton's method for entire functions. *Ann. Inst. Fourier (Grenoble)*, 56(2):325–336, 2006.
[29] Curtis T. McMullen. Families of rational maps and iterative root-finding algorithms. *Ann. of Math. (2)*, 125(3):467–493, 1987.
[30] Curtis T. McMullen. Braiding of the attractor and the failure of iterative algorithms. *Invent. Math.*, 91(2):259–272, 1988.
[31] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.
[32] Christian Pommerenke. On the iteration of analytic functions in a halfplane. *J. London Math. Soc. (2)*, 19(3):439–447, 1979.
[33] Feliks Przytycki. Remarks on the simple connectedness of basins of sinks for iterations of rational maps. In *Dynamical systems and ergodic theory (Warsaw, 1986)*, volume 23 of *Banach Center Publ.*, pages 229–235. PWN, Warsaw, 1989.
[34] Philip J. Rippon. Baker domains of meromorphic functions. *Ergodic Theory Dynam. Systems*, 26(4):1225–1233, 2006.
[35] Philip J. Rippon and Gwyneth M. Stallard. Singularities of meromorphic functions with Baker domains. *Math. Proc. Cambridge Philos. Soc.*, 141(2):371–382, 2006.
[36] Pascale Roesch. Puzzles de Yoccoz pour les applications à allure rationnelle. *Enseign. Math. (2)*, 45(1-2):133–168, 1999.
[37] Johannes Rückert and Dierk Schleicher. On Newton's method for entire functions. *J. Lond. Math. Soc. (2)*, 75(3):659–676, 2007.
[38] Dierk Schleicher. Dynamics of entire functions. In *Holomorphic dynamical systems*, volume 1998 of *Lecture Notes in Math.*, pages 295–339. Springer, Berlin, 2010.
[39] Mitsuhiro Shishikura. On the quasiconformal surgery of rational functions. *Ann. Sci. École Norm. Sup. (4)*, 20(1):1–29, 1987.
[40] Mitsuhiro Shishikura. The connectivity of the Julia set and fixed points. In *Complex dynamics*, pages 257–276. A K Peters, Wellesley, MA, 2009.
[41] Dennis Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math. (2)*, 122(3):401–418, 1985.
[42] Lei Tan. Branched coverings and cubic Newton maps. *Fund. Math.*, 154(3):207–260, 1997.
[43] Gordon Thomas Whyburn. *Analytic topology*. American Mathematical Society Colloquium Publications, Vol. XXVIII. American Mathematical Society, Providence, R.I., 1963.

Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland
E-mail address: baranski@mimuw.edu.pl

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, 08007 Barcelona, Catalonia, Spain
E-mail address: fagella@maia.ub.es

Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, 08007 Barcelona, Catalonia, Spain
E-mail address: xavier.jarque@ub.edu

Faculty of Mathematics and Information Science, Warsaw University of Technology, Pl. Politechniki 1, 00-661 Warszawa, Poland
E-mail address: bkarpin@mini.pw.edu.pl