Contributions to the study of Cartier algebras and local cohomology modules

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CONTRIBUTIONS TO THE STUDY OF CARTIER ALGEBRAS AND LOCAL COHOMOLOGY MODULES

by

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Dedication

Respectfully devoted to the memory of my great grandmother Josefa Gómez.

To my own guardian angels, my mother Nuria Boix Cebrian and my grandmother Joaquina Cebrian Melguizo.

To my friend Cristóbal Canales.

Acknowledgements

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Secondly, I thank Mordechai Katzman the fact that share with me his ideas and insight. Actually, chapter 3 is mostly a modest attempt in order to understand his ideas concerning test ideals.

Thirdly, because I do not understand research without teaching and conversely, I want to express my gratitude to Julian Pfeifle. Julian, during the course on a seminar about toric varieties, taught me how to present mathematics to others emphasizing pictures rather than formal definition and results. It turns out that his advices altered completely my way of teaching mathematics, specially when I have to teach to undergraduate students.

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In October, 2012 I became reviewer from Mathematical Reviews. I want to express my gratitude to the anonymous person who suggested my name; I learn a lot of mathematics making my reviews.

Lastly (but not the least) are Pilar Bayer and Iván Blanco Chacón. Although our joint work (cf. [11]) has not been included as part of this dissertation, I want to express my gratitude towards them to share with me their ideas concerning Shimura curves and modular forms. I have learned with their help a lot of Number Theory in a small piece of time.

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Contents

Preface

Starting from the pioneering work by C. Peskine and L. Szpiro in their joint thesis (cf. [110]), the *Frobenius morphism*, which acts by raising each element of a ring of prime characteristic p to its pth power, has become in a fundamental tool in the study of Commutative Algebra in positive characteristic. A tremendous breakthrough in such study was the introduction of the so-called *tight closure theory*. This theory, originally due to M. Hochster and C. Huneke, has played a crucial role in many advances in the study of commutative Noetherian rings, even in characteristic zero. Indeed, seemingly unrelated results can be proved using tight closure. We mention here a few: the existence of big Cohen-Macaulay modules (cf. [131, 11.5.1 and Chapter 12, Theorem I]); rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay (cf. [69, Main Theorem]); the Briançon-Skoda Theorem (cf. [71, Chapter 12]); the monomial conjecture (cf. [30, 6.5.9]) and the syzygy Theorem (cf. [70, Chapter 10]). We refer to [70] and the references therein for more information about tight closure.

It occurs that inside tight closure theory grew out certain invariants, the so-called *test ideals*. These invariants are crucial in the study of higher Birational Geometry in positive characteristic because it turns out that, in many situations, are the characteristic p analogs of *multiplier ideals* in characteristic zero. This fact, as far as we know, was firstly pointed out by K. E. Smith in [128, Theorem 3.1]. Actually, it is a longstanding question in Algebraic Geometry whether test ideals are, for arbitrary algebraic varieties, the prime characteristic analogs of multiplier ideals in characteristic zero. In this way, as far as multiplier ideals in characteristic zero allow to define some special types of singularities of algebraic varieties (essentially, the singularities stemming from the so-called *Minimal Model Program*), the same type of singularity can be defined in the prime characteristic setting replacing the multiplier ideal by the test ideal. The main advantage of this approach is that one does not need to appeal to resolution of singularities of algebraic varieties, which nowadays is still an open problem in prime characteristic. Maybe at this point, it is convenient to point out that here we are using the phrase *test ideal* in order to refer to what in tight closure theory is called either the big or the non-finitistic test ideal.

In this way, a first problem arises in this context.

Problem 1. Define test ideals without appealing to any tight closure theory.

One of the main problems of tight closure is that it is an operation defined at the level of ideals and modules which, in general, does not commute with localization (cf. [28] or [104]). Nevertheless, as people working on Algebraic Geometry is more interested in test ideals rather than in tight closure, it was natural to ask whether test ideals can be defined without appealing to tight closure theory, expecting that this new definition would imply that these important invariants commutes with localization.

As far as we know, S. Takagi proposed in [132] a definition of test ideals which only involves p^{-e}*-linear maps* (that is, homogeneous maps of degree p^{-e}). Later on, K. Schwede, building on Takagi's approach, defines test ideals through algebras of p^{-e}-linear maps (cf. [117]). Finally, in [18], M. Blickle proposes a completely algebraic definition of test ideals through algebras of p^{-e}-linear maps, the so-called *Cartier algebras*. Blickle's definition of test ideals agrees with the previous ones in the F-finite case. Even more important than this is the fact that, using Blickle's approach, is straightforward to prove, among other things, that test ideals commute with localization. It is worth noting here that, at this point, the reader may think about Cartier algebras just as a certain ring on which we are packing all the possible homogeneous maps of degree p^{-e}, where e runs over $\mathbb{N}$.

It is worth mentioning here that, in the F-finite case, Cartier algebras correspond, roughly speaking (cf. Theorem 1.5.1 for the precise statement), to the so-called *Frobenius algebras*. These algebras were introduced and studied by G. Lyubeznik and K. E. Smith in [96]; the goal of the authors in such paper was to determine sufficient conditions in order to guarantee that test ideals commutes with localization and completion. It turns out that, whenever R is a complete local F-finite ring of prime characteristic, the Frobenius algebra attached to the injective hull of the residue field of R *corresponds through Matlis duality* (once more, we encourage the reader to look at Theorem 1.5.1 for the precise statement) to the Cartier algebra associated to R.

So, it is of some interest to know the structure of these Cartier algebras.

Problem 2. Determine whether Cartier algebras are finitely generated or not.

It is known (cf. [18]) that Cartier algebras of Gorenstein rings are principally generated. It turns out that Gorensteinness is enough to characterize principal Cartier algebras of normal rings. However, the situation in the non-normal case is unclear. So far, the only known result about non-normal Cartier algebras was provided by M. Katzman in [79]. Indeed, Katzman gave an example of a non-finitely generated Cartier algebra over a non-normal ring; more precisely, his example is

$$\frac{\mathbb{K}[\![x,y,z]\!]}{\langle xy, yz\rangle},$$

(where $\mathbb{K}$ is any field of prime characteristic) which may be regarded as a complete Stanley-Reisner ring. Motivated by such fact, we have:

Dissertation goal 1. Study Cartier algebras attached to complete Stanley-Reisner rings.

In Chapter 2, we are certainly to study the Cartier algebra $\mathcal{C}^R$ attached to the complete, Stanley-Reisner ring

$$R := \mathbb{K}[\![x_1, \dots, x_d]\!]/I,$$

where $\mathbb{K}$ is any ground field of prime characteristic and I is a squarefree monomial ideal inside $\mathbb{K}[x_1, \dots, x_d]$. In such case, we shall determine completely the generation of $\mathcal{C}^R$ as R-algebra. Actually, we shall see that, under our assumptions, $\mathcal{C}^R$ can only be either principally generated or infinitely generated as R-algebra, and that such issue only depends on the primary decomposition associated to the Stanley-Reisner ideal I.

So far, we have discussed issues regarding theoretical aspects of test ideals. But a very natural question arises in this context.

Question. How we can compute effectively test ideals?

As we have previously explained, test ideals grew out inside tight closure theory. So, one possible answer to this question might be: *compute tight closure of ideals and then compute test ideals*. Unfortunately, nowadays it is still an open problem to determine an effective method in order to compute the tight closure of an arbitrary ideal even in a polynomial ring. Therefore, our previous answer does not seem a good choice nowadays.

A different and more fruitful approach was provided by M. Katzman in [77]. Building from ideas originally established by J. Àlvarez Montaner, M. Blickle and G. Lyubeznik in [2] and later on generalized by M. Blickle, M. Mustaţă and K. E. Smith in [21], Katzman proposed an effective procedure (implemented in Macaulay2) to compute test ideals in Cohen-Macaulay rings. Bearing in mind that test ideals are the smallest ones which are fixed under the action of Cartier algebras, it is natural to ask for a method to determine all the ideals which are fixed under the action of such algebras.

Dissertation goal 2. Provide an algorithm in order to compute effectively all the ideals fixed with respect to a principally generated Cartier subalgebra of the Cartier algebra of a polynomial ring.

In Chapter 3, we shall give an effective method to calculate all the ideals fixed with respect to a principally generated subalgebra of $\mathcal{C}^S$, where $S := \mathbb{K}[x_1, \dots, x_d]$ and $\mathbb{K}$ is an *F-finite field* of prime characteristic p; in this case, $\mathcal{C}^S$ turns out to be the S-algebra of all the homogeneous maps of S with degree p^{-e}, where e runs through $\mathbb{N}$.

From now on, we have to make a change of topic, because Chapter 4 revolves around contents which are not directly connected with the ones discussed so far.

Indeed, in [6] J. Àlvarez Montaner, R. García López, and S. Zarzuela generalized the Mayer-Vietoris long exact sequence of local cohomology modules (cf. [30, 3.2.3]) to a spectral sequence, the so-called *Mayer-Vietoris spectral sequence*; such spectral sequence was only given in [6] in case of arrangements of linear varieties. Later on, G. Lyubeznik in [95, Theorem 2.1] provided the Mayer-Vietoris spectral sequence in full generality. Moreover, it was also pinpointed in [6] that the formalism used to construct the Mayer-Vietoris spectral sequence of local cohomology modules might be applied to other functors.

Coming back to [6], in op. cit. sufficient conditions were provided in order to ensure that the Mayer-Vietoris spectral sequence degenerates at the E_2-page; finally, also in [6] the authors determined the extension problems associated to the filtration produced by such degeneration.

On the other hand, in a quite technical article, M. Brun, W. Bruns and T. Römer (cf. [31]) provided a vast generalization of Hochster's decomposition of local cohomology modules of a Stanley-Reisner ring (cf. [31, Theorem 1.1 and Theorem 1.3]); in the final remark of such paper (cf. [31, Remark 8.8]), the authors pointed out that some of the results obtained there can be recovered using spectral sequences.

Regarding both observations, it seems natural to ask the next:

Question. Is there a general formalism to produce spectral sequences which recovers and generalizes the Mayer-Vietoris spectral sequence of local cohomology modules? If so, under what assumptions these spectral sequences degenerate at the E_2-page? In case of degeneration, what are the extension problems produced by the filtration attached to such degeneration? Finally, can we extend the results of [31] through this kind of formalism?

Motivated by all the foregoing issues, we have:

Dissertation goal 3. First of all, build homological spectral sequences which recover and extend the Mayer-Vietoris spectral sequence of local cohomology modules. Secondly, provide cohomological spectral sequences which recover and generalize some of the results obtained in [31]. Finally, give sufficient conditions to ensure their degeneration at the E_2-page and, following the spirit of [6], study the extension problems attached to the filtration produced by such degeneration.

Indeed, the purpose of Chapter 4 will be to construct, on one hand, spectral sequences which involve the left derived functors of the direct limit (for this reason, we often refer to them as *homological spectral sequences*); in particular, we recover and extend the Mayer-Vietoris spectral sequence of local cohomology modules firstly obtained in [6] and later on established in full generality in [95]. On the other hand, we also produce spectral sequences which involve the right derived functors of the inverse limit (this fact will make that we often refer to them as *cohomological spectral sequences*); in particular, these cohomological spectral sequences allow us to recover and extend some of the results of [31]. Finally, following the spirit of [6], we provide sufficient requirements to ensure the degeneration of all of these spectral sequences at the E_2-page and we study their extension problems.

Overview of contents

From now onward, we are to provide a more detailed outline of the contents and main results of this mimeograph.

In Chapter 1, we present the definitions and concepts that will be used throughout the subsequent chapters. It includes mild generalizations of the notions of Frobenius algebras and Cartier algebras. This chapter also includes a rough sketch of the theory of a distinguished class of non-commutative rings, the so-called *Ore extensions* and *skew polynomial rings*, which will play a key role in Chapter 2. It is worth mentioning that the material presented in Chapter 1 is known; the only new aspects here are the organization of the material and, overall, a mild generalization of Frobenius and Cartier algebras which we hope might be interesting for the reader.

Chapter 2 is devoted to the study of Cartier algebras of complete *Stanley-Reisner rings*; that is, rings of the form

$$R := \mathbb{K}[\![x_1, \dots, x_d]\!]/I_\Delta,$$

where I_Δ is a squarefree monomial ideal and $\mathbb{K}$ is an arbitrary ground field of prime characteristic. It turns out that such Cartier algebras can only be either principally generated or infinitely generated, and that this fact just depends on the minimal primary decomposition of the ideal I_Δ (cf. Theorem 2.3.5). As a main application, we are able to show the discreteness of F-jumping numbers of pairs of the form $(\operatorname{Spec}(R), \mathbf{V}(\mathfrak{a}))$, where $\mathfrak{a}$ is an arbitary ideal of R (cf. Theorem 2.5.3).

In Chapter 3, we are to provide an effective procedure in order to compute all the ideals which are fixed with respect to the action of a principally generated Cartier subalgebra of $\mathcal{C}^A$, where A is either the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$, the localization $\mathbb{K}[x_1, \dots, x_d]_{\mathfrak{m}}$, or the formal power series ring $\mathbb{K}[\![x_1, \dots, x_d]\!]$, and $\mathbb{K}$ is an F-finite field (cf. Theorem 3.2.5). We shall also give some theoretical evidence that this algorithm might be of some help in order to tackle the same issue when one drops the F-finiteness assumption on the ground field.

As we have essentially pinpointed before, Chapter 4 is likely the most technical part of this mimeograph. The main goal of such chapter is, on one hand, to work out a formalism in order to produce several homological spectral sequences which, in particular, recover and extend the Mayer-Vietoris one (cf. Theorem 4.2.17 and Example 4.2.18). On the other hand, following carefully a similar strategy, we produce some cohomological spectral sequences which, in particular, recover and generalize some of the results obtained by M. Brun, W. Bruns and T. Römer in [31] (cf. Theorem 4.3.5). Furthermore, following the philosophy carried out in [6, Section 3], we give sufficient conditions in order to ensure the degeneration of all these spectral sequences in their E_2-sheet (cf. Theorem 4.2.30 and Theorem 4.3.15) and, in this case, we study the extension problems associated to the filtration produced by such degeneration (cf. Subsection 4.3.4). We conclude providing, as application of these results, a generalization of the so-called *Gräbe formula* (cf. Theorem 4.3.30).

Moreover, we also want to point out that each chapter concludes with an unnumbered section called *Bibliographical Notes.* Our goal in such bibliographical notes is, on one hand, to provide further references about the research topics covered in each chapter which complement some aspects of them which are not explicitly pinpointed in the main text. On the other hand, these bibliographical notes are sometimes used in order to express our very particular point of view about the results obtained by our colleagues and how we think such results grew out.

Finally, it is noteworthy that this mimeograph contains three appendices. On one hand, in Appendix A, our main purpose is to describe the main results obtained in Chapter 2 in an algorithmic way. More specifically, Appendix A provides the pseudo-code of the procedures which have been used in order to deduce some of the theoretical results obtained in Chapter 2 and, of course, to build examples. CoCoA has been used extensively in the implementation of such methods. The code is located in [24].

On the other hand, Appendix B is devoted to turn the main result obtained in Chapter 3 into an effective method in order to compute all the ideals of the polynomial ring $A := \mathbb{F}_p[x_1, \ldots, x_d]$ which are fixed with respect to a fixed principally generated Cartier subalgebra of $\mathcal{C}^A$. In this case, Macaulay2 (cf. [55]) has been used extensively in the implementation of such procedure. The code is located in [25].

Finally, Appendix C introduces a chain complex which, under reasonable assumptions, provides a free resolution which may be regarded as a Koszul resolution in a very specific non-commutative setting; we hope that such resolution is of some interest in its own right.

Note on references

Some parts of this mimeograph have already been published. More precisely, Chapter 2 is based in joint work with J. Àlvarez Montaner and S. Zarzuela and it has been published by Journal of Algebra (cf. [4]). On the other hand, Chapter 3 is based in a submitted joint project with M. Katzman (cf. [26]); the corresponding Macaulay2 package (cf. [25]) was also jointly written with professor Katzman. It is also worth noting that a report of [26], in the format of an extended abstract, have already been published in [27]. Finally, Chapter 4 is based is an ongoing joint project with J. Àlvarez Montaner and S. Zarzuela (cf. [3]).

Chapter 1

Preliminaries

The purpose of this chapter is to collect the basic notions and results which will play some role during this mimeograph; in what follows, we present a brief overview of its contents for the convenience of the reader. Before doing so, it is worth noting that all the material presented in this chapter is known; maybe, the only new aspects here are, on one hand, the organization of such material and, on the other hand, the slight generalization we produce about the notions of Frobenius and Cartier algebras (cf. Section 1.3).

Firstly, we shall introduce Cartier and Frobenius algebras. In our presentation, we follow the notation and terminology of the recent survey [23]. Actually, we present both notions in a slightly more general context (cf. Section 1.3) mostly in the spirit provided by A. K. Singh and U. Walther in [126, Section 2]; we hope that this extra generality may be useful for the reader.

Secondly, we are to review the notions of Ore extensions and skew polynomial rings, following the notations and terminology of [52]. We want to prevent the reader about the fact that we are to use the terminology *Frobenius Ore extension ring* to refer to what R. Y. Sharp called in several papers ([122] is such as a point) the *Frobenius skew polynomial ring*; this change of terminology is due to some historical remarks made by K. Goodearl and R. Warfield Jr. in [52]. It turns out that the basic (aka principally generated) example of Cartier algebra is the Frobenius skew polynomial ring and the basic (aka principally generated) example of Frobenius algebra is the Frobenius Ore extension ring.

Thirdly, we shall introduce a rough sketch of the theory of algebraic D-modules. It is customary in the literature ([39] is such as a point) to restrict such study to the case where the ground field has characteristic zero. Regardless, our sketch also includes some information about the case when the coefficient field has prime characteristic. This sketch is mainly introduced because modules over the Frobenius algebra (respectively, over the Cartier algebra) can be regarded as left (respectively, right) modules over the ring of differential operators in positive characteristic (cf. Section 1.6); this fact will play some role in Chapter 2 (cf. Section 2.5.2).

On the other hand, in Section 1.7 we introduce, borrowing from [21], test ideals on smooth ambient rings and we shall see how Cartier algebras allow to define test ideals even in non-smooth ambient rings. We shall also make a rough comparison between test ideals and their characteristic zero counterpart, the so-called *multiplier ideals*.

Finally, we conclude this chapter recalling basic facts about modules endowed with a Frobenius action; most of the material of this section is borrowed from [146].

1.1 Algebras attached to a ring endomorphism

Unless otherwise is specified, throughout this chapter A is to denote a commutative ring and $A\xrightarrow{\varphi}A$ will stand for a homomorphism of rings.

The purpose of this section is to associate to φ a non necessarily commutative algebra.

1.1.1 The pushforward and pullback functors of a fixed map

Our first aim is to recall how to attach to φ the pushforward functor φ_* and the pullback functor φ^*. Later on (cf. Section 1.4), we shall specialize such constructions in case A is of prime characteristic and $\varphi = F$ is the Frobenius map on A.

We start with the pushforward functor.

Construction 1.1.1. We denote by $\varphi_* A$ the abelian group A with the following structure as (A, A)-bimodule. Before establishing such structure, we underline that we denote $\varphi_* a$ ($a \in A$) an arbitrary element of $\varphi_* A$. Bearing in mind this convention, for any $(a, b, c) \in A \times A \times A$, set

$$b \cdot \varphi_* a \cdot c := \varphi_*(\varphi(b)ac).$$

This construction can be also applied to modules. Indeed, given a left A-module M we may define $\varphi_* M$ in similar terms; that is, if we denote by $\varphi_* m$ ($m \in M$) an arbitrary element of $\varphi_* M$ then, for any $a \in A$, set

$$a \cdot \varphi_* m := \varphi_*(\varphi(a)m).$$

Finally, given a map $M\xrightarrow{g}N$ of left A-modules we can define a map $\varphi_* M\xrightarrow{\varphi_* g}\varphi_* N$ of left A-modules by setting, for any $m \in M$,

$$\varphi_* g(\varphi_* m) := \varphi_* g(m).$$

In this way, the symbol φ_* defines a covariant functor from the category of left A-modules to the category of left A-modules.

Now, we review the construction of the pullback functor in this setup.

Construction 1.1.2. Let M be an A-module. Set $\varphi^* M := \varphi_* A \otimes_A M$. Moreover, given a map $g \in \mathrm{Hom}_A(M, N)$ set

$$\varphi^* g := \mathbb{1}_{\varphi_* A} \otimes g.$$

In this way, we have produced a covariant functor φ^* from the category of A-modules to the category of A-modules which is just $\varphi_* A \otimes_A (-)$.

We conclude this subsection with the following:

Remark 1.1.3. By a slight abuse of notation, we shall use the symbol $\varphi_* A$ to denote the target of the ring homomorphism $A \xrightarrow{\varphi} A$. Taking into account such convention, we shall regard φ as a ring homomorphism with source A and target $\varphi_* A$. In this way, φ_* may be viewed as a functor from the category of $\varphi_* A$-modules to the category of A-modules.

Moreover, φ^* may be regarded as a functor from the category of A-modules to the category of $\varphi_* A$-modules. In the same spirit, the so-called *extraordinary inverse image functor*

$$\varphi^! := \mathrm{Hom}_A(\varphi_* A, -)$$

may be considered too as a functor from the category of A-modules to the category of $\varphi_* A$-modules.

Remark 1.1.4. It is worth noting that we have chosen the language of Algebraic Geometry in order to define the previous functors; however, we want to compare, for the convenience of the Commutative Algebraist reader, these functors with other which such reader will quickly recognize.

(i) The pushforward functor φ_* is what is called in the context of Commutative Algebra the *restriction of scalars* functor; for instance, in [114, page 670] this functor is denoted by U.

(ii) The pullback functor φ^* is what is called in the context of Commutative Algebra the *extension of scalars* functor; in [114, page 671] such functor is denoted by ${}_{\varphi}(-)$.

(iii) The extraordinary inverse image functor $\varphi^!$ is denoted by $(-)^{\varphi}$ in [114, page 671].

Explicit adjunction

The following discussion establishes some duality isomorphisms between the previously defined functors. We want to underline that not all of such isomorphisms are canonical.

Discussion 1.1.5. Let M be a $\varphi_* A$-module and let N be an A-module.

(i) There is a canonical isomorphism

$$\begin{aligned} \mathrm{Hom}_{\varphi_* A}(\varphi^* N, M) &\xrightarrow{\sim} \mathrm{Hom}_A(N, \varphi_* M) \\ f &\longmapsto (n \longmapsto \varphi_*(f(\varphi_* 1 \otimes n))) \end{aligned}$$

in the category of $\varphi_* A$-modules; the reader should notice that this isomorphism is just a particular case of the well-known adjointness between Hom and tensor product, as described in either [114, Theorem 2.76] or [114, Lemma 10.68].

(ii) There is a canonical isomorphism

$$\begin{aligned} \mathrm{Hom}_A(\varphi_* M, N) &\xrightarrow{\sim} \mathrm{Hom}_{\varphi_* A}\big(M, \varphi^! N\big) \\ f &\longmapsto \big(m \longmapsto \mu_{f(\varphi_* m)}\big) \end{aligned}$$

in the category of $\varphi_* A$-modules, where $\mu_{f(\varphi_* m)} \in \varphi^! N$ denotes right multiplication by $f(\varphi_* m)$; in this case, this isomorphism turns out to be a particular case of [114, part (iv) of Lemma 10.70].

From now on, we assume that $\varphi_* A$ is a finitely generated free left A-module.

(iii) Since $\varphi_* A$ is finitely generated and free, there is a canonical isomorphism

$$\begin{aligned} \varphi^! A \otimes_A N &\xrightarrow{\sim} \varphi^! N \\ h \otimes n &\longmapsto (\varphi_* a \longmapsto h(\varphi_* a) n) \end{aligned}$$

in the category of $\varphi_* A$-modules.

(iv) Combining part (ii) and part (iii) it follows that there is a canonical isomorphism

$$\mathrm{Hom}_A(\varphi_* M, N) \xrightarrow{\sim} \mathrm{Hom}_{\varphi_* A}\big(M, N \otimes_A \varphi^! A\big)$$

in the category of $\varphi_* A$-modules.

In the sequel, we assume that $\varphi^! A$ is abstractly isomorphic to $\varphi_* A$ as $\varphi_* A$-module. In this way, we fix such abstract isomorphism

$$\varphi^! A \xrightarrow[\sim]{\phi} \varphi_* A.$$

(v) ϕ induces an abstract isomorphism

$$\varphi^! A \otimes_A N \xrightarrow[\sim]{\phi \otimes \mathbb{1}_N} \varphi_* A \otimes_A N = \varphi^* N$$

in the category of $\varphi_* A$-modules.

(vi) Combining part (iv) and (v) it follows that there is an abstract isomorphism

$$\mathrm{Hom}_A(\varphi_* M, N) \xrightarrow{\sim} \mathrm{Hom}_{\varphi_* A}(M, \varphi^* N)$$

in the category of $\varphi_* A$-modules.

We sum up the relevant adjunction isomorphisms obtained in the previous discussion as follows.

Proposition 1.1.6. *Let M be a left $\varphi_* A$-module and let N be an A-module. Then, the following statements hold.*

(a) There is a canonical isomorphism

$$\begin{aligned} \operatorname{Hom}_{\varphi_* A}(\varphi^* N, M) &\xrightarrow{\sim} \operatorname{Hom}_A(N, \varphi_* M) \\ f &\longmapsto (n \longmapsto \varphi_*(f(\varphi_* 1 \otimes n))) \end{aligned}$$

in the category of $\varphi_* A$-modules.

(b) If, in addition, $\varphi_* A$ is a finitely generated free A-module which is abstractly isomorphic to $\varphi^! A$ as $\varphi_* A$-module, then there is an abstract isomorphism

$$\operatorname{Hom}_A(\varphi_* M, N) \xrightarrow{\sim} \operatorname{Hom}_{\varphi_* A}(M, \varphi^* N)$$

in the category of $\varphi_* A$-modules.

1.1.2 φ-linear maps and φ^{-1}-linear maps

Carrying over notions defined by M. Blickle in [18, Section 2], we introduce:

Definition 1.1.7. Let M be a left A-module and $\psi \in \operatorname{End}_A(M)$.

(i) We say that ψ is *φ-linear* provided $\psi(am) = \varphi(a)\psi(m)$ for any $(a, m) \in A \times M$.

(ii) We say that ψ is *φ^{-1}-linear* provided $\psi(\varphi(a)m) = a\psi(m)$ for any $(a, m) \in A \times M$.

We denote by $\operatorname{End}_{\varphi}(M)$ (respectively, $\operatorname{End}_{\varphi^{-1}}(M)$) the A-endomorphisms of M which are φ-linear (respectively, φ^{-1}-linear).

We have to notice that these notions are straightforward generalizations of the following well-known concept.

Example 1.1.8. Let $k \in \mathbb{N}$ and set

$$\begin{aligned} A &\xrightarrow{(-)^k} A \\ a &\longmapsto a^k. \end{aligned}$$

Thus, the set of $(-)^k$-linear maps is clearly the set of homogeneous maps of degree k.

We can interpret both $\operatorname{End}_{\varphi}(M)$ and $\operatorname{End}_{\varphi^{-1}}(M)$ in terms of our previously introduced functors; the proof of the below result is very straightforward and therefore it is left to the interested reader.

Lemma 1.1.9. *The following statements hold.*

(a) The map

$$\begin{aligned} \mathrm{End}_{\varphi}(M) &\longrightarrow \mathrm{Hom}_A(M, \varphi_* M) \\ \psi &\longmapsto [m \longmapsto \varphi_*(\psi(m))] \end{aligned}$$

is bijective.

(b) The map

$$\begin{aligned} \mathrm{End}_{\varphi^{-1}}(M) &\longrightarrow \mathrm{Hom}_A(\varphi_* M, M) \\ \psi &\longmapsto [\varphi_* m \longmapsto \psi(m)] \end{aligned}$$

is bijective.

1.2 Classical non-commutative algebras attached to a single ring endomorphism

In this section, we are to recall a pair of ring constructions which stem from non-Commutative Algebra and which will play a key role in this mimeograph.

1.2.1 Ore extensions

The aim of this subsection is to collect the basic facts which we shall need later in this mimeograph concerning the so-called *Ore extensions*. The interested reader may like to consult [52, Chapter 2] for further details.

Definition 1.2.1. A left *φ-derivation* is any additive map $A \xrightarrow{\delta} A$ such that $\delta(ab) = \varphi(a)\delta(b) + \delta(a)b$ for all $(a, b) \in A \times A$. In case $\varphi = \mathbb{1}_A$ a φ-derivation is just called a *derivation*.

Now, we are ready for presenting Ore extensions.

Theorem/Definition 1.2.2. Let δ be a left φ-derivation on A. Then, there exists a ring B, containing A as a subring, such that B is a free left A-module with a basis of the form $\{\theta^i\}_{i \in \mathbb{N}_0}$ and $\theta \cdot a = \varphi(a)\theta + \delta(a)$ for all $a \in A$. In fact, B is denoted $A[\Theta; \varphi, \delta]$ and it is called the *Ore extension* of A *determined by φ and δ.*

(i) In case $\varphi = \mathbb{1}_A$ we shall write $A[\Theta; \delta]$ instead of $A[\Theta; \mathbb{1}_A, \delta]$ and we shall call this ring a *(formal) differential operator ring.*

(ii) In case $\delta = 0_A$ we shall write $A[\Theta; \varphi]$ rather than $A[\Theta; \varphi, 0_A]$.

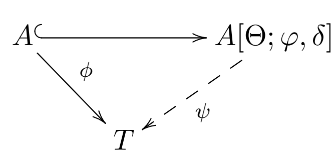
Sketch of proof. In fact, $A[\Theta;\varphi,\delta]$ is just

$$\frac{A\langle\Theta\rangle}{\langle \Theta a - \varphi(a)\Theta - \delta(a) \mid \quad a \in A\rangle}.$$

The other statements of this result can be verified easily using this equality. □

We shall refer to the next result as the *universal property for Ore extensions.*

Proposition 1.2.3. *Let $B = A[\Theta;\varphi,\delta]$ be the Ore extension of A determined by φ and δ. Suppose that we have a ring T, a ring homomorphism $A \xrightarrow{\phi} T$ and an element $y \in T$ such that, for each $a \in A$, $y\phi(a) = \phi(\varphi(a))y + \phi(\delta(a))$. Then, one has that there is a unique ring homomorphism $B \xrightarrow{\psi} T$ such that $\psi(\Theta) = y$ which makes the triangle*



commutative.

From now on, we shall restrict our attention to Ore extensions with $\delta = 0_A$.

Our next aim is to produce Ore extensions through elementary ring homomorphisms. This is the content of the following:

Example 1.2.4. Let $u \in A$. As $u\Theta$ is an element of $A[\Theta;\varphi]$ such that, for any $a \in A$,

$$(u\Theta)a = u\Theta a = u\varphi(a)\Theta = \varphi(a)(u\Theta),$$

it follows from the universal property for Ore extensions that there is a unique ring homomorphism $A[\Theta';\varphi] \longrightarrow A[\Theta;\varphi]$ such that it maps Θ' to $u\Theta$ and which fixes the elements of A. We shall denote the image of this map by $A[u\Theta;\varphi]$. The previous argument shows that $A[u\Theta;\varphi]$ can be regarded as an Ore extension too.

We end this subsection relating the approach used here with the one employed in Section 1.1.

Proposition 1.2.5. *There is a bijective correspondence between* $\mathrm{End}_\varphi(M)$ *and the left $A[\Theta;\varphi]$-module structures which can be attached to M.*

Sketch of proof. Any φ-linear map ψ induces on M a structure as left $A[\Theta;\varphi]$-module given by the following rule:

$$\Theta \cdot m := \psi(m).$$

Conversely, any left $A[\Theta;\varphi]$-module structure on M produces a φ-linear map; namely,

$$\begin{aligned} M &\xrightarrow{\psi} M \\ m &\longmapsto \Theta m. \end{aligned}$$

We omit the routine verifications, which are left to the interested reader. □

1.2.2 Skew polynomial rings

In this subsection, we are to review the basic facts which we shall need later on concerning the so-called *skew polynomial rings*. The interested reader may like to consult [101, Chapter 1] for additional details.

Definition 1.2.6. A right φ-*derivation* is any additive map $A \xrightarrow{\delta} A$ such that $\delta(ab) = \delta(a)\varphi(b) + a\delta(b)$ for all $(a, b) \in A \times A$. In case $\varphi = \mathbb{1}_A$ a right φ-derivation is just called a *derivation.*

Theorem/Definition 1.2.7. Let δ be a right φ-derivation of A. Then, there exists a ring B, containing A as a subring, such that B is a free right A-module with a basis of the form $\{\varepsilon^i\}_{i\in\mathbb{N}_0}$ and $a \cdot \varepsilon = \varepsilon\varphi(a) + \delta(a)$ for all $a \in A$. In fact, B is denoted $A[\varepsilon; \varphi, \delta]$ and it is called the *skew polynomial ring* of A *determined by* φ *and* δ.

(i) In case $\varphi = \mathbb{1}_A$ we shall write $A[\varepsilon; \delta]$ rather than $A[\varepsilon; \mathbb{1}_A, \delta]$ and call this ring a *(formal) differential operator ring.*

(ii) In case $\delta = 0_A$ we are to write $A[\varepsilon; \varphi]$ instead of $A[\varepsilon; \varphi, 0_A]$.

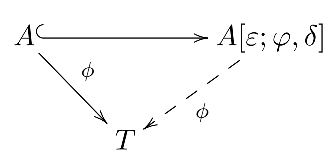
Sketch of proof. We should only check the statements taking $A[\varepsilon; \varphi, \delta]$ as

$$\frac{A\langle\varepsilon\rangle}{\langle a\varepsilon - \varepsilon\varphi(a) - \delta(a) \mid \quad a \in A\rangle}.$$

The rest of the details are omitted. □

We shall refer to the next result as the *universal property for skew polynomial rings.*

Proposition 1.2.8. *Let $B = A[\varepsilon; \varphi, \delta]$ be the skew polynomial ring of A determined by φ and δ. Suppose that we have a ring T, a ring homomorphism $A \xrightarrow{\phi} T$ and an element $y \in T$ such that, for each $a \in A$, $\phi(a)y = y\phi(\varphi(a)) + \phi(\delta(a))$. Then, one has that there is a unique ring homomorphism $B \xrightarrow{\psi} T$ such that $\phi(\varepsilon) = y$ which makes the triangle*



commutative.

We provide in the below lines the basic example of skew polynomial rings which we shall consider later on.

Example 1.2.9. Let $u \in A$. As εu is an element of $A[\varepsilon;\varphi]$ such that, for any $a \in A$,

$$a(\varepsilon u) = \varepsilon\varphi(a)u = (\varepsilon u)\varphi(a),$$

it follows from the universal property for skew polynomial rings that there is a unique ring homomorphism $A[\varepsilon';\varphi]\longrightarrow A[\varepsilon;\varphi]$ such that it maps ε' to εu and which fixes the elements of A. We shall denote the image of this map by $A[\varepsilon u;\varphi]$. The previous argument shows that $A[\varepsilon u;\varphi]$ can be regarded as a skew polynomial ring too.

The following result can be proved along the same lines of Proposition 1.2.5. We omit the details.

Proposition 1.2.10. *There is a bijective correspondence between* $\operatorname{End}_{\varphi^{-1}}(M)$ *and the left* $A[\varepsilon;\varphi]$*-module structures which can be attached to* M.

1.3 Algebras associated to a family of maps

So far, we have fixed a ring endomorphism φ of A and we have studied the action of φ on modules. In this section, we collect in suitable algebras all the maps which we are interested on. From this point of view, the current section may be regarded as a generalization of the previous ones.

1.3.1 Rings of φ^{-1}-linear operators

Our first purpose is to pack all the φ^{-e}-linear maps in an algebra. It leads us to introduce the so-called *Cartier algebras* with respect to φ; as the reader can easily see, next definition may be regarded as a generalization of the notion of Cartier algebra introduced by M. Blickle in [18].

Definition 1.3.1. An *A-Cartier algebra with respect to φ* is an $\mathbb{N}$-graded A-algebra

$$\mathcal{C}^{\varphi} := \bigoplus_{e\geq 0}\mathcal{C}_e^{\varphi}$$

such that, for any $(a,\phi_e)\in A\times\mathcal{C}_e^{\varphi}$, we have that $a\cdot\phi_e = \phi_e\cdot\varphi^e(a)$. The A-algebra structure of $\mathcal{C}^{\varphi}$ is given by the natural map from A to $\mathcal{C}_0^{\varphi}$. Moreover, set

$$\mathcal{C}_+^{\varphi} := \bigoplus_{e\geq 1}\mathcal{C}_e^{\varphi}.$$

We also assume that the structural map $A\longrightarrow\mathcal{C}_0^{\varphi}$ is surjective.

The reader should notice that in $\mathcal{C}^{\varphi}$ we are collecting all the φ^{-e}-linear maps, where e runs through $\mathbb{N}$. Our next goal is to propose a possible (and useful) way to produce Cartier algebras.

Construction 1.3.2. Let M be a left A-module. We have to point out that the elements of $\mathrm{End}_{\varphi^{-e}}(M)$ are abelian group homomorphisms $M \xrightarrow{\phi_e} M$ such that $\phi_e(\varphi^e(a)m) = a\phi(m)$ for all $(a, m) \in a \times M$. We can now turn

$$\mathcal{C}^{M,\varphi} := \bigoplus_{e \geq 0} \mathrm{Hom}_A(\varphi^e_* M, M) = \bigoplus_{e \geq 0} \mathrm{End}_{\varphi^{-e}}(M)$$

into an A-algebra by defining the product of a $\phi_e \in \mathcal{C}^{M,\varphi}_e$ and a $\phi_{e'} \in \mathcal{C}^{M,\varphi}_{e'}$ as the element of $\mathcal{C}^{M,\varphi}_{e+e'}$ given by

$$\varphi^{e+e'}_* M \xrightarrow{\phi_{e'} \circ \varphi^{e'}_* \phi_e} M.$$

We point out that $\mathcal{C}^{M,\varphi}$ is generally NOT an A-Cartier algebra with respect to φ, since $\mathcal{C}^{M,\varphi}_0 = \mathrm{End}_A(M)$ and therefore the natural map $R \longrightarrow \mathrm{End}_A(M)$ is, in general, not surjective. Nevertheless, if $M = A/I$ (where I is any ideal of A) then $\mathrm{End}_A(M) = A/I$ and therefore it follows that $\mathcal{C}^{A/I,\varphi}$ is an A-Cartier algebra.

1.3.2 Rings of φ-linear operators

In the same spirit, we may collect all the φ^e-linear maps in a suitable algebra. In this case, it leads us to introduce the so-called *Frobenius algebras* with respect to φ; indeed, next definition provides a generalization of the notion of Frobenius algebra introduced by G. Lyubeznik and K. E. Smith in [96, Definition 3.5].

Construction 1.3.3. Let $A[\Theta; \varphi^e]$ be the Ore extension of A with respect to φ^e. Since $A \subset A[\Theta; \varphi^e]$, any $A[\Theta; \varphi^e]$-module is an A-module by restriction of scalars. Conversely, an $A[\Theta; \varphi^e]$-module is simply an A-module together with a suitable action of Θ on M; that is, to define an $A[\Theta; \varphi^e]$-module structure on an A-module M, one only needs to define an additive map $M \xrightarrow{\psi_e} M$ such that, for any $(a, m) \in A \times M$, $\psi_e(am) = \varphi^e(a)\psi_e(m)$. We can quickly note that $\psi_e \in \mathrm{Hom}_A(M, \varphi^e_* M)$. In this way, set

$$\mathcal{F}^{M,\varphi}_e := \mathrm{Hom}_A(M, \varphi^e_* M) = \mathrm{End}_{\varphi^e}(M).$$

We underline as well that we may define a product among these pieces; that is, if $\psi_e \in \mathcal{F}^{M,\varphi}_e$ and $\psi_{e'} \in \mathcal{F}^{M,\varphi}_{e'}$ then one sets

$$\psi_{e'} \cdot \psi_e := \varphi^e_* (\psi_{e'}) \circ \psi_e \in \mathcal{F}^{M,\varphi}_{e+e'}.$$

Therefore, we are ready for introducing the following notion.

Definition 1.3.4. The *ring of Frobenius operators with respect to* φ on M is the associative, not necessarily commutative ring

$$\mathcal{F}^{M,\varphi} := \bigoplus_{e \geq 0} \mathcal{F}^{M,\varphi}_e.$$

Notice that we are packing in such Frobenius algebra all the possible φ^e-linear maps, where e runs through $\mathbb{N}$.

In the below example, we provide the basic instance of Frobenius algebra with respect to a ring homomorphism; it is worth noting that the below calculation is a mild generalization of a computation carried out by G. Lyubeznik and K. E. Smith in [96, Example 3.6].

Example 1.3.5. We claim that $\mathcal{F}^{A,\varphi} \cong A[\Theta;\varphi]$. Indeed, fix $e \in \mathbb{N}$ and let $\psi_e \in \mathcal{F}_e^{A,\varphi}$. We point out that, for any $a \in A$,

$$\psi_e(a) = \psi_e(a \cdot 1) = \varphi^e(a)\psi_e(1) = \psi_e(1)\varphi^e(a).$$

In this way, set

$$\begin{aligned} \mathcal{F}_e^{A,\varphi} &\xrightarrow{b_e} A\Theta^e \\ \psi_e &\longmapsto \psi_e(1)\Theta^e. \end{aligned}$$

The previous straightforward calculation shows the injectivity of this map. In fact, it is a bijective map with inverse

$$\begin{aligned} A\Theta^e &\longrightarrow \mathcal{F}_e^{A,\varphi} \\ a\Theta^e &\longmapsto a\varphi^e. \end{aligned}$$

In this way, setting $\mathcal{F}^{A,\varphi} \xrightarrow{b} A[\Theta;\varphi]$ as the unique map of rings given in degree e by b_e it follows that b is an isomorphism of graded algebras.

1.4 Algebras attached to the Frobenius endomorphism

So far, we have considered several abstract constructions which actually stem from considerations about a particular endomorphism (namely, φ) on a general commutative ring A.

In this section, we shall specialize all these abstract constructions in the case we are really interested. Namely, hereafter A will stand for a commutative Noetherian ring of prime characteristic p and $A \xrightarrow{F} A$ will denote the Frobenius map on A; that is, F raises an element $a \in A$ to its pth power $a^p \in A$.

1.4.1 Frobenius pushforward, Frobenius pullback, and adjunction

In Section 1.1.1, we reviewed the constructions of the pushforward and pullback functors with respect to a given map φ. In case $\varphi = F^e$, we fix once and for all the following notation.

(i) The e-fold of the Frobenius pushforward functor will be denoted F_*^e.

(ii) The e-fold of the Frobenius pullback functor will be denoted

$$F^{*e} = F_*^e A \otimes_A (-).$$

Notice that $F^* := F^{*1}$ is the classical Peskine-Szpiro Frobenius functor (cf. [110, Définition (I.2)]). As the reader can easily see, when discussing the case $e = 1$ we drop the e from the notation.

It is well-known when F^* is exact. We shall refer to the following classical result as *Kunz's Theorem* (cf. [89, Theorem 1.2 and Corollary 2.7]).

Theorem 1.4.1 (Kunz)**.** *Let A be a commutative Noetherian reduced ring of prime characteristic p. Then, the following statements are equivalent.*

(i) A is regular.

(ii) F_*A is a flat A-module.

(iii) F^* is exact.

We also state in this case the adjunction established before in the general case (cf. Proposition 1.1.6); it is worth noting that the below result was originally proved by G. Lyubeznik, W. Zhang and Y. Zhang in [97].

Proposition 1.4.2 (Lyubeznik, Zhang, Zhang)**.** *Let M be a left F_*A-module and let N be an A-module. Then, the following statements hold.*

(a) There is a canonical isomorphism

$$\operatorname{Hom}_{F_*A}(F^*N, M) \xrightarrow{\sim} \operatorname{Hom}_A(N, F_*M)$$
$$f \longmapsto (n \longmapsto F_*(f(F_*1 \otimes m)))$$

in the category of F_*A-modules.

(b) If, in addition, F_*A is a finitely generated free A-module which is abstractly isomorphic to $F^!A$ as F_*A-module, then there is an abstract isomorphism

$$\operatorname{Hom}_A(F_*M, N) \xrightarrow{\sim} \operatorname{Hom}_{F_*A}(M, F^*N)$$

in the category of F_*A-modules.

1.4.2 p^e-linear maps and p^{-e}-linear maps

Taking into account Example 1.1.8, the set of F^e-linear maps is just the set of homogeneous maps of degree p^e. Inspired by this fact, M. Blickle in [18, Definition 2.1] introduced the following notions.

Definition 1.4.3 (Blickle)**.** Let $\psi \in \mathrm{End}_A(M)$.

(i) We say that ψ is p^e-*linear* provided $\psi(am) = a^{p^e}\psi(m)$ for any $(a,m) \in A \times M$. Equivalently, $\psi \in \mathrm{Hom}_A(M, F_*^e M)$.

(ii) We say that ψ is p^{-e}-*linear* provided $\psi(a^{p^e} m) = a\psi(m)$ for any $(a,m) \in A \times M$. Equivalently, $\psi \in \mathrm{Hom}_A(F_*^e M, M)$.

Remark 1.4.4. A note on terminology. In [8, pp. 293], G. W. Anderson used a slightly different terminology. Namely, Anderson used the phrase *Frobenius linear* (respectively, *Cartier linear*) to refer to any p^e-linear map (respectively, p^{-e}-linear map). It is worth mentioning that the notion of p^e-linear map is classical (cf. [131, page 218]).

1.4.3 Frobenius-Ore extensions and Frobenius skew polynomial rings

In Section 1.2, we have established with complete generality the notions of Ore extensions and skew polynomial rings. The aim of this subsection is to specialize such constructions in case φ is the Frobenius map.

(i) The *Frobenius-Ore extension ring* of A is the non-commutative graded ring $A[\Theta; F]$; that is, the free left A-module with basis $\{\Theta^e\}_{e\in\mathbb{N}}$ and right multiplication given by

$$\Theta \cdot a = a^p \Theta.$$

(ii) The *Frobenius skew polynomial ring* of A is the non-commutative graded ring $A[\varepsilon; F]$; that is, the free right A-module with basis $\{\varepsilon^e\}_{e\in\mathbb{N}}$ and left multiplication given by

$$a \cdot \varepsilon := \varepsilon a^p.$$

Remark 1.4.5. As it was pointed out in the introduction of this chapter, our terminology concerning the previous non-commutative ring differs from the one adopted by R. Y. Sharp in recent papers (e. g. [122]). Namely, Sharp refers to $A[\Theta; F]$ as the Frobenius skew polynomial ring. On the other hand, it is also worth noting that $A[\Theta; F]$ is what G. Lyubeznik denoted by $A\{F\}$ in [94, Section 4].

1.4.4 Frobenius algebras and Cartier algebras

In Section 1.3, we have introduced Frobenius and Cartier algebras with respect to a given map φ. Due to the fact that both play a key role in this dissertation, we review their definition in this specific setting for the convenience of the reader.

Definition 1.4.6. Let A be a commutative Noetherian ring of prime characteristic p and let M be an A-module.

(i) The *Frobenius algebra* attached to M is the associative, $\mathbb{N}$-graded, not necessarily commutative ring
$$\mathcal{F}^M := \bigoplus_{e \geq 0} \operatorname{Hom}_A(M, F_*^e M).$$

(ii) The *Cartier algebra* attached to M is the associative, $\mathbb{N}$-graded, not necessarily commutative ring
$$\mathcal{C}^M := \bigoplus_{e \geq 0} \operatorname{Hom}_A(F_*^e M, M).$$

Remark 1.4.7. Slightly loosely speaking, whereas in $\mathcal{F}^M$ we are collecting all the homogeneous maps on M of degree p^e (where e runs through $\mathbb{N}$), in $\mathcal{C}^M$ we are packing all the homogeneous maps on M of degree p^{-e} (where again e runs through $\mathbb{N}$).

1.4.5 The trace map

In this subsection, all the fields are assumed to have prime characteristic p. The aim of this subsection is to justify why the name of P. Cartier has appeared in the definition of the previously mentioned Cartier algebras.

The Cartier operator

Firstly, we introduce some preliminary notions which will be useful not only in this subsection, but also in other places of this dissertation.

Definition 1.4.8 (Infinity norm)**.** Let $\mathbb{K}$ be any field, S will stand for $\mathbb{K}[x_1, \ldots, x_d]$, let $g \in S$, and write
$$g = \sum_{\alpha \in \mathbb{N}^d} g_\alpha \mathbf{x}^\alpha,$$
with $g_\alpha \in \mathbb{K}$ and $g_\alpha = 0$ up to a finite number of terms.

(i) We define the *support* of g (which will be denoted $\operatorname{supp}(g)$) as
$$\operatorname{supp}(g) := \left\{ \alpha \in \mathbb{N}^d \mid \quad g_\alpha \neq 0 \right\}.$$

(ii) We define the *infinity norm* of g (which will be denoted $||g||_\infty$) as

$$||g||_\infty := \max_{\alpha \in \mathrm{supp}(g)} ||\alpha||_\infty,$$

where $||\alpha||_\infty := \max\{a_1, \ldots, a_d\}$ and $\alpha := (a_1, \ldots, a_d)$.

The other well-known concept that we are to introduce below is the notion of *p-basis*. The reader may like to consult [44, A.1.3] and the references therein for additional information.

Definition 1.4.9. Let $\mathbb{L}$ be an arbitrary ground field of prime characteristic p.

(i) Finitely many elements $\lambda_1, \ldots, \lambda_s$ are called *p-independent* if the following three equivalent conditions are satisfied.

(a) $[\mathbb{L}^p(\lambda_1, \ldots, \lambda_s) : \mathbb{L}^p] = p^s$.

(b) The chain

$$\mathbb{L}^p \subseteq \mathbb{L}^p(\lambda_1) \subseteq \mathbb{L}^p(\lambda_1, \lambda_2) \subseteq \ldots \subseteq \mathbb{L}^p(\lambda_1, \ldots, \lambda_s)$$

is an strictly increasing tower of fields.

(c) The p^s monomials $\lambda := \lambda_1^{a_1} \cdots \lambda_s^{a_s}$ with $0 \le ||\alpha||_\infty \le p-1$ are an $\mathbb{L}^p$-vector space basis for $\mathbb{L}$.

(ii) An infinite subset of $\mathbb{L}$ is called *p-independent* provided any finite subset of it is p-independent in the previous sense.

(iii) A maximal p-independent subset of $\mathbb{L}$ is called a *p-basis* for $\mathbb{L}$. It is worth mentioning that Zorn's Lemma guarantees the existence of p-basis; indeed, it stems from the fact that the union of a chain of p-independent sets is also p-independent.

Now, we proceed recalling the following classical construction (cf. [34, pp. 200]).

Construction 1.4.10 (Cartier). Let $\mathbb{K}^p \subseteq \mathbb{L} \subseteq \mathbb{K}$ be a tower of fields such that $[\mathbb{K} : \mathbb{L}] < \infty$, let $\mathrm{Der}_{\mathbb{L}}(\mathbb{K})$ the ring of $\mathbb{L}$-derivations over $\mathbb{K}$ and let

$$\Omega := \bigoplus_{r \ge 0} \Omega^r_{\mathbb{K}|\mathbb{L}}$$

be the ring of differential forms of $\mathbb{K}$ over $\mathbb{L}$. In this way, we define the operator

$$\Omega^1 \otimes_{\mathbb{L}} \mathrm{Der}_{\mathbb{L}}(\mathbb{K}) \xrightarrow{C} \mathbb{K}^{1/p}$$

given by the assignment $\omega \otimes \partial \longmapsto (C\omega)(\partial)$, where

$$[(C\omega)(\partial)]^p = \omega(\partial^p) - \partial^{p-1}(\omega(\partial)).$$

We have to underline that $\partial^p \in \mathrm{Der}_{\mathbb{L}}(\mathbb{K})$ because all the involved fields have prime characteristic p.

We are interested in extending such construction. For simplicity, hereafter in this section we shall assume that $\mathbb{K}$ is F-finite

Construction 1.4.11 (Cartier). Set $\Omega := \Omega_{\mathbb{K}|\mathbb{K}^p}$, let $\Omega \xrightarrow{d} \Omega$ be the differential of Ω, set $B := d(\Omega)$, $Z := \ker(d)$, and let $x_1, \dots, x_t$ be a p-basis of $\mathbb{K}$. In this way, set

$$Z \xrightarrow{C} \Omega$$

which sends an element

$$\omega = d\varphi + \sum_{1 \le i_1 < \ldots < i_r \le t} a^p_{i_1,\dots,i_r} (x^{p-1}_{i_1} dx_{i_1}) \wedge \ldots \wedge (x^{p-1}_{i_r} dx_{i_r}) \tag{1.1}$$

to

$$C\omega := \sum_{1 \le i_1 < \ldots < i_r \le t} a_{i_1,\dots,i_r} dx_{i_1} \wedge \ldots \wedge dx_{i_r}.$$

We omit the proof of the following result. We refer to [35, pp. 195–204] for a detailed treatment.

Theorem/Definition 1.4.12 (Cartier)**.** The following statements hold.

(i) The monomials $\{x^{p-1}_{i_1} dx_{i_1} \wedge \ldots \wedge x^{p-1}_{i_r} dx_{i_r}\}_{1 \le i_1 < \ldots < i_r \le t}$ form a basis of Z modulo B. Actually, this fact justifies why we can write an arbitrary element of $\ker(d)$ as in (1.1). In this way, the operator C is well-defined.

(ii) C is a ring homomorphism from Z to Ω.

(iii) For any $x \in \mathbb{K}$, one has that $C(x^p) = x$, $C(x^{p-1}dx) = dx$ and $C(dx) = 0$.

(iv) C is surjective and induces a ring isomorphism

$$H_{\mathrm{dR}}(\mathbb{K} \mid \mathbb{K}^p) \xrightarrow[\sim]{C} \Omega.$$

In this way, the de Rham cohomology ring $H_{\mathrm{dR}}(\mathbb{K} \mid \mathbb{K}^p)$ is abstractly isomorphic to Ω; indeed, it depends on the choice of a p-basis on $\mathbb{K}$. For such reason, this is not a bijection of *differential algebras* (cf. [90] for unexplained terminology).

The operator C is called the *Cartier operator.*

Remark 1.4.13. In fact, it is straightforward to check that C is the unique ring homomorphism from Z to Ω verifying part (iii) of the previous theorem.

The trace map on a polynomial ring

From now on in this subsection, let $\mathbb{K}$ be an F-finite field and set $S := \mathbb{K}[x_1, \dots, x_d]$.

The following technical tool will be used later on in this mimeograph; albeit this result is well known, we provide a detailed proof for the reader's benefit.

Proposition 1.4.14. *Let $\mathcal{B}_e$ be a basis of $\mathbb{K}$ as $\mathbb{K}^{p^e}$-vector space. Then, the set*

$$\{b\mathbf{x}^\alpha \mid \quad b \in \mathcal{B}_e, \quad 0 \leq ||\alpha||_\infty \leq p^e - 1\}$$

is a finite basis of $F_*^e S$ as left S-module.

Proof. Let $g \in S$ and write

$$g = \sum_{\substack{b \in \mathcal{B}_e \\ 0 \leq ||\alpha||_\infty \leq p^e - 1}} g_{\alpha b}^{p^e} b\mathbf{x}^\alpha,$$

with $g_{\alpha b} \in S$ and $g_{\alpha b} = 0$ up to a finite number of terms (indeed, this is always possible after performing the euclidean quotient with p^e as denominator in the exponent set of g). In this way, if follows that

$$F_*^e g = \sum_{\substack{b \in \mathcal{B}_e \\ 0 \leq ||\alpha||_\infty \leq p^e - 1}} g_{\alpha b} F_*^e(b\mathbf{x}^\alpha),$$

just what we would want to check. □

Now, we introduce the main result of this subsection. We skip the proof and refer to [29, Chapter 1] for details.

Theorem/Definition 1.4.15. Let $\mathbb{K}$ be an F-finite field, let $d \in \mathbb{N}$, let $S := \mathbb{K}[x_1, \dots, x_d]$ and set Φ_e as the only S-linear map such that

$$\begin{aligned} &F_*^e S \xrightarrow{\Phi_e} S \\ &F_*^e(b\mathbf{x}^\alpha) \longmapsto b x_1^{\frac{a_1 - (p^e - 1)}{p^e}} \cdots x_d^{\frac{a_d - (p^e - 1)}{p^e}} \end{aligned}$$

(where b and α are as in Proposition 1.4.14) with the convention that if $\frac{a_i - (p^e - 1)}{p^e} \notin \mathbb{Z}$ for some $1 \leq i \leq d$ then $\Phi_e(F_*^e(b\mathbf{x}^\alpha)) = 0$. Then, the map

$$\begin{aligned} &F_*^e S \longrightarrow \operatorname{Hom}_S(F_*^e S, S) \\ &F_*^e s \longmapsto [F_*^e t \longmapsto \Phi_e(F_*^e(st))] \end{aligned}$$

is bijective. Therefore, $\operatorname{Hom}_S(F_*^e S, S)$ is the cyclic free $F_*^e S$-module generated by Φ_e. Hereafter, we shall refer to Φ_e as the (Grothendieck) *trace map*.

Remark 1.4.16. Whenever $\mathbb{K}$ is F-finite, the set

$$\{b\mathbf{x}^\alpha \mid \quad b \in \mathcal{B}_e, \quad 0 \leq ||\alpha||_\infty \leq p^e - 1\}$$

is a finite basis of $F^e_* S$ as left S-module, where $S = \mathbb{K}[x_1, \dots, x_d]$ and $\mathcal{B}_e$ is a basis of $\mathbb{K}$ as $\mathbb{K}^{p^e}$-vector space. In this way, we may also regard the trace map Φ_e as the unique p^{-e}-linear map which is the projection onto the direct summand $Sx_1^{p^e-1} \cdots x_d^{p^e-1}$.

We conclude this subsubsection with the following:

Remark 1.4.17. Albeit we do not exploit it in what follows, it is worth noting that in [126, Example 2.2] was presented a generalization of the trace map Φ_e; indeed, given $\mathbb{K}$ an arbitrary field and $S := \mathbb{K}[x_1, \dots, x_d]$ there is a $\mathbb{K}$-linear endomorphism $S \xrightarrow{\varphi_t} S$ such that $\varphi_t(x_i) = x_i^t$ for $1 \leq i \leq d$ (where $t \in \mathbb{N}$). We have to point out that the inclussion $\varphi_t(S) \subseteq S$ splits because S is a free $\varphi_t(S)$-module with basis $\mathbf{x}^\alpha$ with $||\alpha||_\infty \leq t-1$. Moreover, given a squarefree monomial ideal I, it is clear that $\varphi_t(I) \subseteq I$, whence φ_t induces an endomorphism $S/I \xrightarrow{\overline{\varphi_t}} S/I$. The reader should notice that $\overline{\varphi_t}(S/I)$ is the $\mathbb{K}$-vector space spanned by those monomials in $x_1^t, \dots, x_d^t$ which are not in I. In this way, setting $S/I \xrightarrow{\pi_t} S/I$ as the unique $\mathbb{K}$-linear endomorphism which is the identity on the previous monomials, and acts as zero on the remainder ones, we obtain a splitting of $\overline{\varphi_t}$. In particular, if $t = p^e$ and $\mathbb{K}$ is of prime characteristic, then $\varphi_t = F^e$ and therefore π_t turns out to be the trace map Φ_e.

1.4.6 Some examples of Frobenius algebras

Now, we shall exhibit several examples of principally generated Frobenius algebras. The first one turns out to be a calculation which was carried out by G. Lyubeznik and K. E. Smith in [96, Example 3.7].

Example 1.4.18 (Lyubeznik, Smith). Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring of characteristic p. Then

$$\mathcal{F}^{H^{\dim(R)}_{\mathfrak{m}}(R)} \cong S[\Theta; F],$$

where S denotes the S_2-ification of $\widehat{R}$ and F denotes the standard Frobenius action on $H^{\dim(R)}_{\mathfrak{m}}(R)$. Indeed, as $S := \operatorname{Hom}_{\widehat{R}}\left(H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R}), H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R})\right)$ is the S_2-ification of $\widehat{R}$ (cf. [30, 12.2.9]) and

$$\operatorname{Hom}_R\left(F^{*e}_R H^{\dim(R)}_{\mathfrak{m}}(R), H^{\dim(R)}_{\mathfrak{m}}(R)\right) = \operatorname{Hom}_{\widehat{R}}\left(F^{*e}_{\widehat{R}} H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R}), H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R})\right)$$

(indeed, it follows combining flat base change of local cohomology [30, 4.3.2] joint with the fact that $H^{\dim(R)}_{\mathfrak{m}}(R)$ is Artinian [30, 7.1.6]) we finally obtain, bearing in mind that $F^{*e}_{\widehat{R}} = \widehat{R}\Theta^e \otimes_{\widehat{R}} (-)$ and $H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R}) \otimes_{\widehat{R}} (-)$ commutes,

$$\mathcal{F}^{H^{\dim(R)}_{\mathfrak{m}}(R)} \cong \bigoplus_{e \geq 0} \operatorname{Hom}_{\widehat{R}}\left(H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R}), H^{\dim(\widehat{R})}_{\widehat{\mathfrak{m}}}(\widehat{R})\right)\Theta^e \cong \bigoplus_{e \geq 0} S\Theta^e \cong S[\Theta; F],$$

and therefore we get the desired conclusion.

Another source of examples is given by the following result.

Proposition 1.4.19. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a complete F-finite local ring of characteristic p and E_R will stand for a choice of injective hull of $\mathbb{K}$ over R. Then, the following statements hold.*

(i) If R is quasi Gorenstein then $\mathcal{F}^{E_R}$ is principal.

(ii) If R is normal then $\mathcal{F}^{E_R}$ is principal if and only if R is Gorenstein.

(iii) If R is a normal domain which, in addition, is $\mathbb{Q}$-Gorenstein (cf. Theorem/Definition 2.4.1), then $\mathcal{F}^{E_R}$ is a finitely generated R-algebra if and only if p is relatively prime with the index of R.

(iv) If R is a normal domain which, in addition, is $\mathbb{Q}$-Gorenstein, then $\mathcal{F}^{E_R}$ is principal if and only if the index of R divides $p-1$.

Proof. If R is quasi Gorenstein then $E_R \cong H_{\mathfrak{m}}^{\dim(R)}(R)$. But we have seen in Example 1.4.18 that, under our assumptions, $\mathcal{F}^{H_{\mathfrak{m}}^{\dim(R)}(R)} \cong R[\Theta; F]$; indeed, R is complete and any quasi Gorenstein ring is, in particular, S_2. The second part is proved in [18, Example 2.7]. On the other hand, part (iii) follows combining [84, Proposition 4.3] and [47, Theorem 4.5]; finally, part (iv) also follows from [84, Proposition 4.3]. □

Next result computes explicitly $\mathcal{F}^{E_R}$. We shall refer to this result in what follows as *Fedder's Theorem* (cf. [49, pp. 465]). This is the main result of this subsection.

Theorem 1.4.20 (Fedder)**.** *Let $\mathbb{K}$ be a field of prime characteristic p, T will stand for $\mathbb{K}[\![x_1, \ldots, x_d]\!]$, and I will denote an arbitrary ideal of T. Then, one has that*

$$\mathcal{F}^{E_R} \cong \bigoplus_{e \geq 0} \left\{ (I^{[p^e]} :_T I)/I^{[p^e]} \right\} \Theta^e,$$

where E denotes a choice of injective hull of $\mathbb{K}$ over T, $R := T/I$, Θ is the standard Frobenius action on E and $E_R := (0 :_E I)$.

Proof. Compute explicitly:

$$\begin{aligned} &\operatorname{Hom}_T((T/I)^\vee, F_*^e((T/I)^\vee)) \cong \operatorname{Hom}_T(F^{*e}((T/I)^\vee), (T/I)^\vee) \cong \operatorname{Hom}_T((T/I^{[p^e]})^\vee, (T/I)^\vee) \\ &\cong \operatorname{Hom}_T(T/I, T/I^{[p^e]}) \cong (I^{[p^e]} :_T I)/I^{[p^e]}. \end{aligned}$$

□

1.5 Duality between Cartier algebras and Frobenius algebras

From now on, let $T = \mathbb{K}[\![x_1, \dots, x_d]\!]$ be a formal power series ring with d indeterminates over an F-finite field $\mathbb{K}$ of characteristic p, I an ideal of T and $R := T/I$. The purpose of this section is to review the correspondence between Frobenius and Cartier algebras obtained through Matlis duality; before doing so, we want to stress that such correspondence is known (cf. [19] and [125]). However, we produce here a simplified proof which we hope may be useful for the reader.

Next result establishes the explicit correspondence between $\mathcal{C}^R$ and $\mathcal{F}^{E_R}$ given by Matlis duality. It is the main result of this section.

Theorem 1.5.1. *We have that*

$$\operatorname{Hom}_T(F_*^e R, R)^\vee \cong \operatorname{Hom}_T(E_R, F_*^e E_R) \quad \textit{and} \quad \operatorname{Hom}_T(E_R, F_*^e E_R)^\vee \cong \operatorname{Hom}_T(F_*^e R, R).$$

Before proving this theorem we have to show a previous statement which we shall need during its proof; albeit the below result was obtained by F. Enescu and M. Hochster in [46, Discussion (3.4)], we review here their proof for the convenience of the reader.

Lemma 1.5.2. *Let $(A, \mathfrak{m}, \mathbb{K}) \longrightarrow (B, \mathfrak{n}, \mathbb{L})$ be a local homomorphism of local rings, and suppose that $\mathfrak{m}B$ is $\mathfrak{n}$-primary and that $\mathbb{L}$ is finite algebraic over $\mathbb{K}$ (both these conditions hold if B is module-finite over A). Let $E := E_A(\mathbb{K})$ and $E_B(\mathbb{L})$ denote choices of injective hulls for $\mathbb{K}$ over A and for $\mathbb{L}$ over B, respectively. Then, the functor $\operatorname{Hom}_A(-, E)$, on B-modules, is isomorphic with the functor $\operatorname{Hom}_B(-, E_B(\mathbb{L}))$.*

Proof. First of all, we underline that $\operatorname{Hom}_A(-, E)$, on B-modules, can be identified via adjunction with

$$\operatorname{Hom}_A((-) \otimes_A B, E) \cong \operatorname{Hom}_B(-, \operatorname{Hom}_A(B, E))$$

and therefore $\operatorname{Hom}_A(B, E)$ is injective as B-module. Moreover, as $\mathfrak{m}B$ is $\mathfrak{n}$-primary any element of $\operatorname{Hom}_A(B, E)$ is killed by a power of $\mathfrak{n}$ and therefore

$$\operatorname{Hom}_A(B, E) \cong E_B(\mathbb{L})^{\oplus l}.$$

In this way, it only remains to check that $l = 1$. Indeed, we note that

$$\operatorname{Hom}_A(\mathbb{L}, E) \cong \operatorname{Hom}_A(\mathbb{L}, \mathbb{K}).$$

However, as A-module, $\operatorname{Hom}_A(\mathbb{L}, \mathbb{K})$ is abstractly isomorphic to $\mathbb{L}$ (here we are using the assumption that $\mathbb{L}$ is finite algebraic over $\mathbb{K}$). Thus, all these foregoing facts imply that

$$E_B(\mathbb{L}) \cong \operatorname{Hom}_A(B, E),$$

hence $\operatorname{Hom}_A((-) \otimes_A B, E) \cong \operatorname{Hom}_B(-, E_B(\mathbb{L}))$ and we get the desired conclusion. □

Proof of Theorem 1.5.1. First of all, we underline that

$$\operatorname{Hom}_T(F_*^e R, R)^\vee \cong \operatorname{Hom}_T(E_R, F_*^e(R)^\vee).$$

Now, let E_* be the injective hull of the residue field of $F_*^e R$. In this way, from Lemma 1.5.2 we deduce that $\operatorname{Hom}_T(-, E) \cong \operatorname{Hom}_{F_*^e T}(-, E_*)$ as functors of $F_*^e T$-modules. Therefore, combining all these facts joint with the exactness of F_*^e it follows that

$$F_*^e(R)^\vee \cong \operatorname{Hom}_T(F_*^e R, E) \cong \operatorname{Hom}_{F_*^e T}(F_*^e R, E_*) \cong F_*^e \operatorname{Hom}_T(R, E) \cong F_*^e E_R.$$

Thus, taking into account this last chain of isomorphisms one obtains the first desired conclusion.

On the other hand, using once more Lemma 1.5.2 it turns out that

$$F_*^e(E_R)^\vee \cong \operatorname{Hom}_T(F_*^e E_R, E) \cong \operatorname{Hom}_{F_*^e T}(F_*^e E_R, F_*^e E) \cong F_*^e(E_R^\vee) \cong F_*^e R.$$

Thus, bearing in mind this last chain of isomorphisms it follows that

$$\operatorname{Hom}_T(E_R, F_*^e E_R)^\vee \cong \operatorname{Hom}_T(F_*^e(E_R)^\vee, E_R^\vee) \cong \operatorname{Hom}_T(F_*^e R, R),$$

just what we finally wanted to show. □

We end this part with the following:

Remark 1.5.3. The reader should point out that Theorem 1.5.1 only guarantees a correspondence between Frobenius and Cartier algebras at the level of homogeneous elements.

1.6 Cartier operators, Frobenius operators and differential operators

The purpose of this section is to remind the close connection between Cartier algebras (respectively, Frobenius algebras) and the ring of differential operators in prime characteristic; in fact, it turns out that modules over $\mathcal{C}^A$ (where A is either $\mathbb{K}[x_1, \dots, x_d]$ or $\mathbb{K}[\![x_1, \dots, x_d]\!]$ and $\mathbb{K}$ is a field of positive characteristic) can be regarded as right modules over D_A and that modules over $\mathcal{F}^A$ can also be considered as left D_A-modules. This connection will be of some interest for our purposes in the final part of Chapter 2 (cf. Section 2.5.2).

Before doing so, we remind Grothendieck's construction of the ring of differential operators of a general commutative ring.

Definition 1.6.1 (Grothendieck)**.** Let A be a commutative ring and let A_0 be a subring of A. The ring of *algebraic differential operators* $D_{A|A_0}$ is the subring of $\operatorname{Hom}_{A_0}(A, A)$ whose elements are defined inductively on the so-called *order*. Namely, a differential operator of order 0 is the multiplication by an element of A, and a differential operator of order n is a $\delta \in \operatorname{Hom}_{A_0}(A, A)$ such that, for any $a \in A$, the commutator $[\delta, a] := \delta \circ a - a \circ \delta$ is a differential operator of order less or equal than $n-1$. In case $A_0 = \mathbb{Z}$, we shall simply write D_A instead of $D_{A|\mathbb{Z}}$ for the sake of simplicity.

Example 1.6.2. Let $\mathbb{F}$ be any field of characteristic 0 and let A be either the polynomial ring $\mathbb{F}[x_1,\dots,x_d]$, the convergent power series ring $\mathbb{F}\{x_1,\dots,x_d\}$, or the formal power series ring $\mathbb{F}[\![x_1,\dots,x_d]\!]$. In this case,

$$D_{A|\mathbb{F}} = A\left[y_1,\dots,y_d;\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_d}\right],$$

where $\frac{\partial}{\partial x_i}$ $(i\in\{1,\dots,n\})$ denotes the partial derivative with respect to x_i. In this case, $D_{A|\mathbb{F}}$ is a left Noetherian (non-commutative) ring. In this case, $D_{A|\mathbb{F}}$ is the so-called *Weyl algebra* and it is often denoted $\mathbb{A}_d(\mathbb{F})$.

The reader should remind that the fact that $\mathbb{A}_d(\mathbb{F})$ is left Noetherian turns out to be a non-trivial result; indeed, it is a consequence of a non-commutative version of the Hilbert Basis Theorem. The interested reader may like to consult [52, Theorem 2.6 and Corollary 2.7] for additional details.

The situation is quite different in prime characteristic.

Example 1.6.3. Let $\mathbb{K}$ be any field of characteristic p and let A be a finitely generated $\mathbb{K}$-algebra (or a localization or a completion of it). In this case, D_A is far from being left or right Noetherian. However, we can describe explicitly $D_{A|\mathbb{K}}$ under mild assumptions. Indeed, if one sets $D_A^{(e)} := \operatorname{Hom}_A(F_*^e A, F_*^e A)$ then it follows that

$$D_{A|\mathbb{K}} \subseteq D_A \subseteq \bigcup_{e\geq 0} D_A^{(e)}.$$

Furthermore, if A is an F-finite ring (in case A is local this is equivalent to say that $\mathbb{K}$ is an F-finite field) then the previous rightmost inclusion becomes in equality because of Yekutieli's Theorem (cf. [145, 1.4.8a]). In any case, we may think that the elements of $D_{A|\mathbb{K}}^{(e)}$ are nothing but differential operators that are linear over A^{p^e}. It is also well known that the inclusion $D_{A|\mathbb{K}} \subseteq D_A$ is an equality in case the ground field $\mathbb{K}$ is perfect.

From now on, we denote by $\mathbb{K}$ a perfect field of prime characteristic.

Example 1.6.4. Let A be either the polynomial ring $\mathbb{K}[x_1,\dots,x_d]$, or the formal power series ring $\mathbb{K}[\![x_1,\dots,x_d]\!]$. In this case, $D_{A|\mathbb{K}}$ is the ring extension of A generated by the differential operators

$$\partial_i^t := \frac{1}{t!}\frac{\partial^t}{\partial x_i^t}\quad (i\in\{1,\dots,d\}),$$

where $\frac{\partial^t}{\partial x_i^t}$ is the tth partial derivative with respect to x_i. More precisely, using the multi-graded notation

$$\partial^\alpha := \frac{1}{a_1!}\frac{\partial^{a_1}}{\partial x_1^{a_1}}\cdots\frac{1}{a_d!}\frac{\partial^{a_d}}{\partial x_d^{a_d}},$$

we have that $D_{A|\mathbb{K}}^{(e)}$ is the ring extension of A generated by the differential operators ∂^α with $||\alpha||_\infty \leq p^e-1$.

On the other hand, we recall as well that the ring of differential operators of a quotient ring $B := A/I$ is $D_{B|\mathbb{K}} = D_{A|\mathbb{K}}(I)/ID_{A|\mathbb{K}}$, where

$$D_{A|\mathbb{K}}(I) := \{\delta \in D_{A|\mathbb{K}} \mid \quad \delta(I) \subseteq I\}$$

is often called the ring of I-logarithmic differential operators of A.

Now, we introduce the main ingredient of this section.

Let A be a essentially of finite type $\mathbb{K}$-algebra. We observe that we have a natural map

$$\mathcal{F}_e^A \otimes_A \mathcal{C}_e^A \xrightarrow{\langle -,- \rangle_e} D_A^{(e)}.$$

given by the assignment $\phi_e \otimes \psi_e \longmapsto \phi_e \circ \psi_e$. However, we have to take into account that $\mathcal{F}_e^A = \operatorname{Hom}_A(A, F_*^e A)$ and that $\mathcal{C}_e^A = \operatorname{Hom}_A(F_*^e A, A)$; in this way, regarding $\operatorname{Hom}_A(A, F_*^e A)$ as an $(\operatorname{Hom}_A(F_*^e A, F_*^e A), A)$-bimodule and $\operatorname{Hom}_A(F_*^e A, A)$ as an $(A, \operatorname{Hom}_A(F_*^e A, F_*^e A))$-bimodule, the previous tensor product is a $(D_A^{(e)}, D_A^{(e)})$-bimodule, whence $\langle -,- \rangle_e$ is a natural morphism of $(D_A^{(e)}, D_A^{(e)})$-bimodules.

Next result tells that, when the ambient ring is smooth, the previous pairing turns out to be bijective; although the below result was proved in [2, Proposition 2.1], we provide a proof for the sake of completeness.

Theorem 1.6.5 (Àlvarez, Blickle, Lyubeznik)**.** *Suppose that, in addition, A is regular. Then $\langle -,- \rangle_e$ is an isomorphism.*

Proof. As being isomorphism is a local property we may assume, without loss of generality, that A is a local regular ring. In this situation, Kunz's Theorem says us that $F_*^e A$ is a free left A-module with finite rank; namely, r. Once a basis of it is fixed, we can identify $F_*^e A$ with $\mathcal{M}_{r\times 1}(A)$; we may as well identify $\operatorname{Hom}_A(F_*^e A, A)$ with $\mathcal{M}_{1\times r}(A)$ and, finally, we identify $D_{A|\mathbb{K}}^{(e)}$ with $\mathcal{M}_{r\times r}(A)$. In this way, $\langle -,- \rangle_e$ can be identified with the map

$$\mathcal{M}_{r\times 1}(A) \otimes_A \mathcal{M}_{1\times r}(A) \longrightarrow \mathcal{M}_{r\times r}(A)$$

given by the assignment $B_1 \otimes B_2 \longmapsto B_1 \cdot B_2$; but this is clearly bijective. □

Later on (cf. Section 2.5.2 of Chapter 2) we shall check that, if one drops regularity, then one loses bijectivity.

On the other hand, it is worth mentioning that Theorem 1.6.5 has been generalized by Y. Toda and T. Yasuda. We omit their proof and refer to [135, Proposition 2.1] for further details.

Proposition 1.6.6 (Toda, Yasuda)**.** *Let R be a commutative ring and let L, M, N denote R-modules. We regard L as an $(\operatorname{End}_R(L), R)$-bimodule, and analogously for M and N.*

Suppose that L is a direct summand of $M^{\oplus r}$ for some $r \geq 1$. Then, the natural morphism of $(\mathrm{End}_R(N), \mathrm{End}_R(L))$*-bimodules*

$$\mathrm{Hom}_R(M,N) \otimes_{\mathrm{End}_R(M)} \mathrm{Hom}_R(L,M) \longrightarrow \mathrm{Hom}_R(L,N)$$

given by the assignment $f \otimes g \longmapsto f \circ g$ is an isomorphism.

The reader should notice that, if one takes $M = A$ and $N = L = F_*^e A$ in the previous result, then one recovers Proposition 1.6.5 as an immediate corollary.

1.6.1 Basic examples of D-modules

We are to conclude this section presenting some basic examples of algebraic D-modules. For the purposes of this subsection, $\mathbb{K}$ is to denote either a field of characteristic zero or a perfect field of prime characteristic, and A will denote the polynomial ring $\mathbb{K}[x_1, \ldots, x_d]$. As we have noticed in this section, $D_A = D_{A|\mathbb{K}}$ is the ring extension of A generated by the differential operators

$$\partial_i^t := \frac{1}{t!} \frac{\partial^t}{\partial x_i^t},$$

where $(i,t) \in \{1, \ldots, d\} \times \mathbb{N}$ and $\partial^t / \partial x_i^t$ denotes the tth partial derivative with respect to x_i.

Example 1.6.7. A has a natural structure as left D_A-module given by

$$\partial_i^t \cdot f := \frac{1}{t!} \frac{\partial^t f}{\partial x_i^t},$$

where $f \in A$. In this case, there is an isomorphism $A \cong D_A/(D_A)_+$ in the category of left D_A-modules, where $(D_A)_+$ is the left ideal of D_A generated by ∂_i^t, and (i,t) runs through $\{1, \ldots, d\} \times \mathbb{N}^*$.

Next example is also basic, but it is non-trivial. Actually, it requires to introduce the following notion (cf. [39, Chapter 10, Theorem 3.3]).

Theorem/Definition 1.6.8 (Bernšteĭn)**.** Let $\mathbb{K}$ be a field of characteristic zero, set $A := \mathbb{K}[x_1, \ldots, x_d]$ and $f \in A$ a non-zero polyomial. Then, there exists a monic polynomial $b_f \in \mathbb{K}[s]$ and a differential operator $Q \in D_A[s]$ such that

$$Q(s) \cdot f^{s+1} = b_f(s) \cdot f^s$$

for any s. The polynomial $b_f(s)$ is called the *Bernšteĭn-Sato polynomial* of f.

Example 1.6.9. Let $f \in A$. Then, A_f has a natural structure as left D_A-module. In this case, there is an isomorphism

$$A_f \cong D_A \cdot \frac{1}{f^l}$$

in the category of left D_A-modules, where $l = 1$ if $\mathbb{K}$ has prime characteristic (cf. [2, Theorem 3.7 and Corollary 3.8]) or $-l$ is the negative integer root of greatest absolute value of the Bernšteĭn-Sato polynomial of f if $\mathbb{K}$ has characteristic zero (cf. [12]). In the characteristic zero case, it is known as well that A_f can not be isomorphic to $D_A \cdot 1/f^i$ for $i < l$ (cf. [138, Lemma 1.3]).

We end this subsection with the last basic example of D-module which will appear later on: the local cohomology module $H_I^*(A)$.

Example 1.6.10. Let I be an arbitrary ideal of A (at this point, it is very important to stress that $A = \mathbb{K}[x_1, \dots, x_d]$). We have seen before that A_f has a natural structure as D-module. Using this fact, it is straightforward to check, with the help of the Čech complex with respect to a minimal generating set for I, that the cohomology of such complex has a natural structure as D-module. But the cohomology of this Čech complex is just the local cohomology modules $H_I^*(A)$. This last fact is proved carefully in [30, Chapter 5].

We give two concrete (and enlightening) examples. When $I = \mathfrak{m} = \langle x_1, \dots, x_d \rangle$, one has (cf. [93, Proposition 2.3]) that

$$H_{\mathfrak{m}}^d(A) \cong \frac{D_A}{D_A \cdot \mathfrak{m}}$$

More generally, one can check that if $I = \langle x_1, \dots, x_t \rangle$ (where $t \leq d$) then

$$H_I^t(A) \cong \frac{D_A}{D_A \cdot \langle x_1, \dots, x_t, \partial_{t+1} \dots, \partial_d \rangle}.$$

1.7 F-jumping numbers of pairs

It is well-known (cf. [92, Chapter 9]) that the mere existence of resolution of singularities in characteristic zero implies the fact that the jumping numbers of the multiplier ideal are discrete and rational. On the other hand, although the existence of resolution of singularities in prime characteristic is still an open problem, it is well-known too that the multiplier ideal has an analogue in the characteristic p setting. In this section, we shall introduce such analog.

Before so, we present the organization of this section for the reader's benefit. Firstly, we recall the notions of multiplier ideals and jumping numbers, following the presentation due to R. Lazarsfeld in [92, Chapter 9]. Second, following Blickle, Mustaţă and Smith's approach, we introduce Hara-Yoshida's (generalized) test ideals (through p^{-e}-linear maps) and their corresponding jumping coefficients, the so-called *F-jumping numbers.* Third, we see how Cartier algebras allow to extend test ideals and F-jumping numbers to non-smooth (even non-normal) varieties. Finally, we see how the so-called *gauge boundedness* allows to show the discreteness of F-jumping numbers under some finiteness restrictions on a certain Cartier algebra.

1.7.1 Multiplier ideals and jumping numbers

As we have explained before, the material of this section is borrowed from [92, Chapter 9]; we only introduce the notions and results that we really need regarding our purposes.

We start with some well-known preliminaries.

Definition 1.7.1. Let X be an irreducible, normal algebraic variety over $\mathbb{C}$.

(i) A $\mathbb{Q}$-*divisor* on X is a finite formal linear combination

$$D = \sum_i a_i D_i$$

of codimension-one irreducible subvarieties $D_i \subseteq X$ with rational coefficients $a_i \in \mathbb{Q}$. Moreover, it is said that D is *effective* provided $a_i \geq 0$ for all i.

(ii) Let $D = \sum_i a_i D_i$ be a $\mathbb{Q}$-divisor on X. The *round down* (or *integral part*) $\lfloor D \rfloor$ is the integral divisor

$$\lfloor D \rfloor := \sum_i \lfloor a_i \rfloor D_i,$$

where, as usual, for $x \in \mathbb{Q}$ one denotes by $\lfloor x \rfloor$ the greatest integer less or equal than x.

From now on, assume that X is, in addition, smooth.

(iii) A divisor $D = \sum_i D_i$ has *simple normal crossings* (and, in this case, it is said that D is a *simple normal crossings divisor*) if each D_i is smooth, and if D is defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdots z_k = 0$$

for some $k \leq \dim(X)$. A $\mathbb{Q}$-divisor $\sum_i a_i D_i$ has *simple normal crossing support* provided $\sum_i D_i$ is a simple normal crossings divisor in the previous sense.

(iv) Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X. A *log resolution* of $\mathfrak{a}$ is a projective birational map $X' \xrightarrow{\mu} X$, where X' smooth, such that $\mu^{-1}\mathfrak{a} = \mathcal{O}_{X'}(-F)$, where F is an effective divisor on X such that $F + \text{except}(\mu)$ has simple normal crossing support; here, $\text{except}(\mu)$ denotes the sum of the exceptional divisors of μ.

In this way, given any such resolution, we denote by

$$K_{X'|X} := K_{X'} - \mu^* K_X$$

the so-called *relative canonical divisor* of X' over X.

After all the foregoing preliminaries, we are now ready for introducing multiplier ideals.

Definition 1.7.2. Let X be a smooth algebraic variety over $\mathbb{C}$, let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X, and $t > 0$ a real number. Fix a log resolution of $\mathfrak{a}$ with $\mu^{-1}\mathfrak{a} = \mathcal{O}_{X'}(-F)$. The *multiplier ideal* $\mathcal{J}(X, \mathfrak{a}^c)$ *attached to* c *and* $\mathfrak{a}$ is defined as follows:

$$\mathcal{J}(X, \mathfrak{a}^c) := \mu_* \mathcal{O}_{X'} \left(K_{X'|X} - \lfloor cF \rfloor \right).$$

Our next aim is to define the so-called *jumping numbers* attached to multiplier ideals; before doing so, we introduce the smallest non-zero of such digits:

Definition 1.7.3. Let X be a smooth algebraic variety over $\mathbb{C}$, and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X. The *log-canonical threshold* of $\mathfrak{a}$ at $x \in X$ is

$$\operatorname{lct}(\mathfrak{a}; x) := \inf\{c \in \mathbb{Q} \mid \quad \mathcal{J}(X, \mathfrak{a}^c)_x \subseteq \mathfrak{m}_x\}.$$

It is of some interest to characterize the intervals on which multiplier ideals are constant; it leads to the introduction of the so-called *jumping numbers*. More precisely:

Theorem/Definition 1.7.4. Let X be a smooth algebraic variety over $\mathbb{C}$, and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a non-zero ideal sheaf on X. Then, for each $x \in X$, there is an increasing sequence

$$0 = \xi_0(\mathfrak{a}; x) < \xi_1(\mathfrak{a}; x) < \xi_2(\mathfrak{a}; x) < \dots$$

of rational numbers $\xi_i = \xi_i(\mathfrak{a}; x)$ characterized by the properties that

$$\mathcal{J}(X, \mathfrak{a}^c)_x = \mathcal{J}(X, \mathfrak{a}^{\xi_i})_x \text{ for } c \in [\xi_i, \xi_{i+1}),$$

whereas $\mathcal{J}(X, \mathfrak{a}^{\xi_{i+1}})_x \subsetneq \mathcal{J}(X, \mathfrak{a}^{\xi_i})_x$ for every i. Here, one agrees the convention that $\mathcal{J}(X, \mathfrak{a}^0) = \mathcal{O}_X$; so, $\xi_1(\mathfrak{a}; x)$ is the log-canonical threshold of $\mathfrak{a}$ at x.

The rational numbers $\xi_i(\mathfrak{a}; x)$ are called the *jumping numbers* (or *jumping coefficients*) of $\mathfrak{a}$ at x.

The reader should notice that, by the very definition of jumping numbers, there is an strictly decreasing chain of multiplier ideals

$$\mathcal{O}_{X,x} \supsetneq \mathcal{J}(X, \mathfrak{a}^{\xi_1})_x \supsetneq \mathcal{J}(X, \mathfrak{a}^{\xi_2})_x \supsetneq \dots \supsetneq \mathcal{J}(X, \mathfrak{a}^{\xi_i})_x \supsetneq \mathcal{J}(X, \mathfrak{a}^{\xi_{i+1}})_x \supsetneq \dots$$

We end this subsection with the following:

Remark 1.7.5. In fact, one can define multiplier ideal sheaves in non-smooth (but normal) algebraic varieties; the problem turns out to be that, in this setting, it is unclear whether the corresponding jumping numbers are discrete and rational. The interested reader in this issue may like to consult [92, Chapter 9] and overall [41] for further information about these general multiplier ideals.

1.7.2 Test ideals and F-jumping numbers

Test ideals have grown inside the so-called *tight closure theory* introduced by M. Hochster and C. Huneke thirty years ago (one may like to consult [70] for a gentle introduction on the study of tight closure). Starting from the pioneering work by K. E. Smith in [128], people working in Algebraic Geometry have tried to define test ideals without appealing to tight closure; as we have previously explained, their motivation stems from the fact that test ideals are in many situations the characteristic p analogs of multiplier ideals. It is worth mentioning here that what we call *test ideal* here is what is defined in tight closure theory as the *big test ideal*. The interested reader may like to consult [119] and the references therein for more information concerning test ideals.

Test ideals and F-jumping numbers on smooth habitats

In [59], N. Hara and K.-i. Yoshida defined the test ideals for pairs; regardless, their definition stills depends on a generalization of tight closure theory, the so-called *generalized tight closure*; sometimes, it is as well called *ideal-adic tight closure* (e .g . [139]). Finally, recent work due to S. Takagi (cf. [132]) (among others) have conducted M. Blickle, M. Mustaţă and K. E. Smith in [21] to define test ideals using p^{-e}-linear maps.

We start this section introducing test ideals through p^{-e}-linear maps following [21].

Theorem/Definition 1.7.6. Let A be an F-finite regular ring of prime characteristic p, let $c \in \mathbb{R}$ be non-negative, let $\mathfrak{a}$ be an ideal of A, and set

$$I_e(\mathfrak{a}^c) := \operatorname{Hom}_A(F^e_* A, A) \cdot F^e_*(\mathfrak{a}^{\lceil c(p^e-1)\rceil}).$$

Then, the following statements hold.

(i) For any $e \in \mathbb{N}$, $I_e(\mathfrak{a}^c) \subseteq I_{e+1}(\mathfrak{a}^{c+1})$. In this way, one obtains an increasing chain of ideals

$$\ldots \subseteq I_e(\mathfrak{a}^c) \subseteq I_{e+1}(\mathfrak{a}^{c+1}) \subseteq \ldots$$

(ii) The previous ascending chain of ideals eventually stabilizes.

In this way, we define the *test ideal* with respect to $\mathfrak{a}^c$ as

$$\tau(A; \mathfrak{a}^c) := I_e(\mathfrak{a}^c)$$

for $e \gg 0$.

After these preliminaries, we introduce F-jumping numbers of pairs; the reader is encouraged to compare the below result with 1.7.4.

Theorem/Definition 1.7.7. Let A be an F-finite regular ring of prime characteristic, let $\mathfrak{a}$ be an ideal of A and let c be a non-negative real number. Then, for all $\varepsilon > 0$, we have

$$\tau(A; \mathfrak{a}^c) \supseteq \tau(A; \mathfrak{a}^{c+\varepsilon})$$

with equality for sufficiently small $\varepsilon > 0$. In this way, a real number $c > 0$ is called an *F-jumping number* of the pair $(\mathrm{Spec}(A), \mathbf{V}(\mathfrak{a}))$ if

$$\tau(A; \mathfrak{a}^{c-\varepsilon}) \supsetneq \tau(A; \mathfrak{a}^c)$$

for all $\varepsilon > 0$.

As in the characteristic zero setting, it is clear, by the very definition of F-jumping numbers, that there is an strictly decreasing chain of test ideals:

$$A = \tau(A; \mathfrak{a}^0) = \tau(A; \mathfrak{a}^{\nu_0}) \supsetneq \tau(A; \mathfrak{a}^{\nu_1}) \supsetneq \ldots \supsetneq \tau(A; \mathfrak{a}^{\nu_i}) \supsetneq \tau(A; \mathfrak{a}^{\nu_{i+1}}) \supsetneq \ldots$$

The first non-zero of such digits (namely, ν_1 in our previous notation) is the so-called *F-pure threshold* of the pair $(\mathrm{Spec}(A), \mathfrak{a})$.

Test ideals and F-jumping numbers in singular habitats through Cartier algebras

Our next aim is to extend the definition of the test ideal (and also of F-jumping coefficients) to a non-necessarily regular ambient ring; it turns out that Cartier algebras are the key notion in order to attain this goal. More precisely, following [23, Definition 9.3.7] we introduce:

Theorem/Definition 1.7.8 (Blickle)**.** Let M be a finitely generated R-module that is also a $\mathcal{C}$-module for some R-Cartier algebra $\mathcal{C}$. Then there is a unique $\mathcal{C}$-submodule $\sigma(M) \subseteq M$ such that the following statements hold.

(a) $M/\sigma(M)$ is nilpotent; that is, $(\mathcal{C}_+)^n\,(M/\sigma(M)) = 0$ for some $n \in \mathbb{N}$.

(b) $\mathcal{C}_+\sigma(M) = \sigma(M)$.

In this way, we define the *test submodule* $\tau(M, \mathcal{C})$ to be the unique smallest submodule N of M (IF IT EXISTS) which verifies the following requirements.

(i) N is a $\mathcal{C}$-submodule of M.

(ii) $\sigma(M)_{\mathfrak{p}} = N_{\mathfrak{p}}$ for any minimal prime $\mathfrak{p}$ of R.

In this way, M. Blickle introduced (cf. [18, Definition 3.21]) Hara-Yoshida's generalized test ideals using Cartier algebras in the following manner.

Definition 1.7.9. Let $\mathfrak{a} \subseteq R$ be a non-zero ideal and let t be a non-negative real number. Set

$$\mathcal{C}^{\mathfrak{a}^t} := \bigoplus_{e \geq 0} \left(\operatorname{Hom}_R \left(F^e_* R, R \right) \cdot \left(F^e_* \mathfrak{a}^{\lceil t(p^e - 1) \rceil} \right) \right),$$

which is clearly a Cartier subalgebra of $\mathcal{C}^R$. Thus, the *generalized test ideal* $\tau(R; \mathfrak{a}^t)$ is defined as $\tau(R, \mathcal{C}^{\mathfrak{a}^t})$. This definition makes sense because the existence of $\tau(R; \mathfrak{a}^t)$ under our assumptions is well-known.

In this way, as far as Cartier algebras allow to define test ideals in arbitrary algebraic varieties, one can also define its corresponding F-jumping numbers as in 1.7.7. So, it is natural to ask in this general context the following:

Question 1.7.10. Are F-jumping numbers discrete and rational?

Gauge boundedness

Now, we present the fundamental technical tool of this section. It is worth mentioning that this technique is mainly introduced in order to deduce discreteness of jumping numbers of test ideals where the ambient ring is not necessarily normal.

From now on, set $S := \mathbb{K}[x_1, \ldots, x_d]$. It is worth mentioning that the below notion, introduced by G. W. Anderson in [8, pp. 291–292] and used by M. Blickle in [18, Section 4] in the context of test ideals, plays an important role in this mimeograph

Definition 1.7.11 (Anderson, Blickle)**.** Set S_d as the $\mathbb{K}$-span generated by monomials

$$\{\mathbf{x}^{\alpha} \mid \quad ||\alpha||_{\infty} \leq d\}$$

and $S_{-\infty} := 0$. Let M be a finitely generated S-module and let $m_1, \ldots, m_k$ be generators of M. Thus, the just introduced filtration on S together with this choice of generators of M induces an increasing filtration, indexed by $\{-\infty\} \cup \mathbb{N}_0$, on M given by

$$M_d := \begin{cases} 0, & \text{if } d = -\infty, \\ S_d \cdot \langle m_1, \ldots, m_k \rangle, & \text{otherwise.} \end{cases}$$

In this way, set

$$\begin{aligned} M &\xrightarrow{\delta_M} \{-\infty\} \cup \mathbb{N}_0 \\ m &\longmapsto \min\{d \mid \quad m \in M_d - M_{d+1}\}. \end{aligned}$$

We call δ_M a *gauge* for M.

The main example of gauge we shall consider in this dissertation has been already introduced in this mimeograph (cf. Definition 1.4.8). However, we remind it once more for the convenience of the reader.

Definition 1.7.12. Let $g \in S$ and write

$$g = \sum_{\alpha \in \mathbb{N}^d} g_\alpha \mathbf{x}^\alpha,$$

with $g_\alpha \in \mathbb{K}$ and $g_\alpha = 0$ up to a finite number of terms.

(i) We define the *support* of g (which will be denoted $\operatorname{supp}(g)$) as

$$\operatorname{supp}(g) := \left\{ \alpha \in \mathbb{N}^d \mid \quad g_\alpha \neq 0 \right\}.$$

(ii) We define the *infinity norm* of g (which will be denoted $||g||_\infty$) as

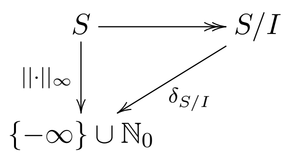
$$||g||_\infty := \max_{\alpha \in \operatorname{supp}(g)} ||\alpha||_\infty,$$

where $||\alpha||_\infty := \max\{a_1, \ldots, a_d\}$ and $\alpha := (a_1, \ldots, a_d)$.

Example 1.7.13. There are two examples of gauge which are interesting for our later purposes.

(i) If $M = S$ then $\delta_S = || \cdot ||_\infty$.

(ii) If $M = S/I$, where I is some ideal of S, then δ_M is just the map which makes the following triangle commutative.



In what follows, we make the abuse of identifying $\delta_{S/I}$ with $|| \cdot ||_\infty$.

Definition 1.7.14 (Blickle)**.** Let I be an ideal of S and set $R := S/I$. We say that $\mathcal{C}^R$ is *gauge bounded* if there exists a set $\{\psi_i \mid \quad \psi_i \in \mathcal{C}^R_{e_i}\}$, which generates $\mathcal{C}^R_+$ as right R-module, and a constant K such that, for any i and for any $r \in R$,

$$||\psi_i(r)||_\infty \quad \leq \frac{||r||_\infty}{p^{e_i}} + K.$$

The following result, proved by Blickle in [18], justifies all the previously introduced notions in this section. We skip the proof and refer to [18, Corollary 4.16] for details.

Proposition 1.7.15 (Blickle)**.** *Suppose that $\mathcal{C}^R$ is gauge bounded. Then, whenever C is a positive real number, the set $\{\tau(R; \mathfrak{a}^c) \mid \quad 0 \leq c \leq C\}$ is a finite set.*

We shall show that Proposition 1.7.15, in conjunction with the results obtained in Chapter 2, imply that the F-jumping numbers of $(\operatorname{Spec}(R), \mathbf{V}(\mathfrak{a}))$ forms a discrete subset inside the non-negative real numbers, where $R = \mathbb{K}[\![x_1, \ldots, x_d]\!]/I$ is a complete Stanley-Reisner ring, $\mathbb{K}$ is a perfect field of characteristic p, and $\mathfrak{a}$ is any ideal of R.

1.8 Modules with a Frobenius action

In this section, we shall focus on the study of modules M equipped with a homomorphism $\psi \in \mathrm{Hom}(M, F_* M)$. Equivalently, we want to analize some objects in the category of left $A[\Theta; F]$-modules, where A is a commutative Noetherian ring of prime characteristic p and $A \xrightarrow{F} A$ is the Frobenius map.

1.8.1 Artinian modules over the Frobenius-Ore extension ring

One of the main problems of $A[\Theta; F]$ is that it is rarely left or right Noetherian, as it is explictly established in the following result. We omit its proof and we refer to [146, Theorem (1.3)] for details.

Theorem 1.8.1 (Yoshino)**.** *Let A be a commutative Noetherian ring of characteristic p. Then, the following statements hold.*

(i) *$A[\Theta; F]$ is left Noetherian if and only if A is a direct product of a finite number of fields.*

(ii) *$A[\Theta; F]$ is right Noetherian if and only if A is Artinian and $A/\mathfrak{m}$ is a perfect field for any $\mathfrak{m} \in \mathrm{Max}(A)$.*

Now, we provide effective descriptions of basic examples of left modules over $A[\Theta; F]$. We refer to [146, Lemma (5.1)] for details.

Proposition 1.8.2 (Yoshino)**.** *Let A be a commutative Noetherian ring of characteristic p, let $a \in A$ be any element and let $J(a)$ denote the left ideal of $A[\Theta; F]$ generated by the infinite set*

$$\{a^{p^e-1}\Theta^e - 1 \mid \quad e \in \mathbb{N}\}.$$

Then, the following statements hold.

(i) *A has a natural structure as left $A[\Theta; F]$-module given by*

$$\Theta \cdot x := x^p \quad \textit{for any} \quad x \in A.$$

In such case, there is an isomorphism

$$A \cong A[\Theta; F]/A[\Theta; F]\langle \Theta - 1 \rangle$$

in the category of left $A[\Theta; F]$-modules.

(ii) *The localization A_a has a natural structure as left $A[\Theta; F]$-module given by*

$$\Theta \cdot \left(\frac{x}{a^t}\right) := \frac{x^p}{a^{tp}} \quad \textit{for any} \quad x \in A.$$

In such case, there is an isomorphism

$$\psi_a : \quad A[\Theta; F]/J(a) \cong A_a.$$

in the category of left $A[\Theta; F]$-modules.

Let I be any ideal of A. Our next aim is to show that local cohomology modules $H^i_I(A)$ have an abstract structure as finitely generated left $A[\Theta; F]$-modules. This is the main result of this subsection. We refer to [146, pp. 2490–2491] for further details.

Theorem 1.8.3 (Yoshino). *Let A be a commutative Noetherian ring of characteristic p, let $a, b \in A$ and let I be any ideal of A such that $I = \langle a_1, \ldots, a_t \rangle$. Then, the following statements hold.*

(i) There is a commutative diagram

$$\begin{array}{ccc} A[\Theta; F]/J(a) & \xrightarrow{\overline{\mu}_b} & A[\Theta; F]/J(ab) \\ \downarrow{\scriptstyle \psi_a} & & \downarrow{\scriptstyle \psi_{ab}} \\ A_a & \xrightarrow{\mu_b} & A_{ab} \end{array}$$

Here, the vertical arrows are the isomorphisms described in Proposition 1.8.2 and both $\overline{\mu}_b$ and μ_b are the natural maps induced by the right multiplication by b.

(ii) The Čech complex of A with respect to $a_1, \ldots, a_t$

$$0 \longrightarrow A \longrightarrow \bigoplus_{i=1}^{t} A_{a_i} \longrightarrow \bigoplus_{1 \leq i < j \leq t} A_{a_i a_j} \longrightarrow \ldots \longrightarrow \bigoplus_{i=1}^{t} A_{a_1 \cdots \widehat{a_i} \cdots a_t} \longrightarrow A_{a_1 \cdots a_t} \longrightarrow 0$$

is a complex of left $A[\Theta; F]$-modules which is isomorphic to the following complex:

$$0 \longrightarrow A[\Theta; F]/J(1) \longrightarrow \bigoplus_{i=1}^{t} A[\Theta; F]/J(a_i) \longrightarrow \bigoplus_{1 \leq i < j \leq t} A[\Theta; F]/J(a_i a_j) \longrightarrow \ldots$$

$$\ldots \longrightarrow \bigoplus_{i=1}^{t} A[\Theta; F]/J(a_1 \cdots \widehat{a_i} \cdots a_t) \longrightarrow A[\Theta; F]/J(a_1 \cdots a_t) \longrightarrow 0.$$

In this complex, for each $1 \leq i_1 < \ldots < i_l \leq t$, any map

$$A[\Theta; F]/J(a_{i_1} \cdots \widehat{a_{i_j}} \cdots a_{i_l}) \xrightarrow{\overline{\mu_{a_{i_j}}}} A[\Theta; F]/J(a_{i_1} \cdots a_{i_l})$$

is the one induced on any direct summand from the right multiplication by some a_{i_j}.

(iii) Any local cohomology module $H^i_I(A)$ has an abstract structure as a finitely generated left $A[\Theta; F]$-module.

Remark 1.8.4. It is worth mentioning here that part (iii) of the previous result is proved carefully in [30, 5.3.4 and 5.3.6] in a more functorial way.

We have to emphasize that Theorem 1.8.3 has the following interesting:

Corollary 1.8.5. *Let $(A, \mathfrak{m}, \mathbb{K})$ be a Cohen-Macaulay local ring of characteristic p of dimension $d \geq 1$, and let $x_1, \dots, x_d$ be a system of parameters for A. Then, there is an isomorphism*

$$H^d_{\mathfrak{m}}(A) \cong A[\Theta; F] / \left(J(x_1 \cdots x_d) + A[\Theta; F]\langle x_1, \dots, x_d \rangle \right).$$

in the category of left $A[\Theta; F]$-modules.

In particular, if A is Gorenstein then there is an isomorphism

$$E_A \cong A[\Theta; F] / \left(J(x_1 \cdots x_d) + A[\Theta; F]\langle x_1, \dots, x_d \rangle \right)$$

in the category of left $A[\Theta; F]$-modules.

1.8.2 Matlis duality

In this subsection, we are to establish an equivalence of categories which involves a slight modification of the well-known Matlis duality. We must notice that this construction was introduced by M. Katzman in [77, Section 2]. Actually, we shall follow his treatment. Regardless, we want to point out that the most of the ideas used here were already established by M. Blickle in his thesis; the interested reader may like to consult [15, Chapter 4] for further details.

Throughout this section, $\mathbb{K}$ is to denote an arbitrary field of characteristic p, T will be $\mathbb{K}[\![x_1, \dots, x_d]\!]$, I will stand for an arbitrary ideal of T and $R := T/I$. Moreover, we shall denote by $E = E_T$ a choice of injective hull of $\mathbb{K}$ over T and by $(-)^\vee$ the Matlis duality functor $\operatorname{Hom}_T(-, E)$.

Definition 1.8.6. Let M be a left $R[\Theta; F]$-module.

(a) Following Lyubeznik's terminology in [94, Section 4], we say that M is *cofinite* provided it it Artinian as R-module. We shall denote by ${}_{R[\Theta;F]}\mathcal{A}$ the category of cofinite $R[\Theta; F]$-modules.

(b) Following Blickle's terminology in [17, Definition 2.1], a γ-*sheaf* on T is a pair (N, γ_N) consisting of a left T-module N and a T-linear map $N \xrightarrow{\gamma_N} F_T^* N$. Moreover, we say that it is *coherent* provided its underlying T-module N is finitely generated. In addition, we say that a coherent γ-sheaf is *supported on I* provided its underlying module N has a structure as left R-module. It is worth mentioning here that Blickle's original definition of γ-sheaf is more general than ours.

On the other hand, *homomorphisms* of γ-sheaves are maps $N_1 \xrightarrow{\psi} N_2$ between the corresponding underlying T-modules such that the square

$$\begin{array}{ccc} N_1 & \xrightarrow{\psi} & N_2 \\ \downarrow{\scriptstyle \gamma_{N_1}} & & \downarrow{\scriptstyle \gamma_{N_2}} \\ F_T^* N_1 & \xrightarrow{F_T^*\psi} & F_T^* N_2 \end{array}$$

commutes. In the sequel, we shall denote by $\mathrm{Coh}_\gamma(T)$ (respectively, $\mathrm{Coh}_\gamma(R)$) the category of coherent γ-sheaves on T (respectively, supported on I).

Remark 1.8.7. A note on terminology. The notion of γ-sheaf previously introduced agrees with the concept of *quasi-$\mathcal{F}$-module* which was proposed by D. Tobisch in [134, Definition 3.1].

On the other hand, in [77, Section 2] our previously defined category ${}_{R[\Theta;F]}\mathcal{A}$ was denoted by $\mathcal{C}$ and the author says that $\mathcal{C}$ is the category of Artinian $R[\Theta; F]$-modules. We have avoided this terminology because the top local cohomology module $H_{\mathfrak{m}}^{\dim(R)}(R)$ belongs to ${}_{R[\Theta;F]}\mathcal{A}$; however, we have seen in Theorem 1.8.3 that it has a structure as finitely generated left $R[\Theta; F]$-module. So, it is not clear for us whether $H_{\mathfrak{m}}^{\dim(R)}(R)$ belongs to $\mathcal{C}$.

The following technical statement will be a key ingredient in the proof of the main result of this section. It was established by Lyubeznik in [94, Lemma 4.1] providing a generalization of an earlier result (cf. [62, Lemma 1.8]) proved by R. Hartshorne and R. Speiser. We provide here Lyubeznik's proof for the sake of completeness.

Proposition 1.8.8 (Lyubeznik). *There is a natural equivalence of functors*

$$(-)^\vee \circ F_T^* \overset{\tau}{\underset{\sim}{\Longrightarrow}} F_T^* \circ (-)^\vee$$

regarding these compositions as functors from the category of Artinian left T-modules to the category of finitely generated left T-modules.

Proof. Let M be an Artinian left T-module. So, (cf. [30, 10.2.8]) there are $(a, b) \in \mathbb{N} \times \mathbb{N}$ and an exact sequence

$$0 \longrightarrow M \longrightarrow E^{\oplus a} \xrightarrow{\psi} E^{\oplus b}.$$

Moreover, as $E^\vee$ is canonically isomorphic to T one has that ψ can be abstractly identified with the right multiplication by $A^t \in \mathcal{M}_{b\times a}(T)$. In this way, applying to the previous exact sequence the exact functors $(-)^\vee \circ F_T^*$ and $F_T^* \circ (-)^\vee$ respectively, we obtain the following exact sequences:

$$(F_T^*(E)^\vee)^{\oplus b} \xrightarrow{F_T^*(\psi)^\vee} (F_T^*(E)^\vee)^{\oplus a} \longrightarrow F_T^*(M)^\vee \longrightarrow 0$$

$$F_T^*(E^\vee)^{\oplus b} \xrightarrow{F_T^*(\psi^\vee)} F_T^*(E^\vee)^{\oplus a} \longrightarrow F_T^*(M^\vee) \longrightarrow 0.$$

It turns out that the following statements hold.

(i) Both $F_T^*(\psi)^\vee$ and $F_T^*(\psi^\vee)$ are abstractly isomorphic to the left multiplication by $A^{[p]}$, where $A^{[p]}$ is obtained from A by raising all its entries to the p-th power.

(ii) Since T is regular it is, in particular, quasi Gorenstein and therefore E is isomorphic to $H^d_{\mathfrak{m}}(T)$.

(iii) $F_T^*(H^d_{\mathfrak{m}}(T))$ is canonically isomorphic to $H^d_{\mathfrak{m}}(T)$. Indeed, set $\mathbf{x} := x_1 \cdots x_d$. Since

$$H^d_{\mathfrak{m}}(T) \cong \varinjlim \left(T \xrightarrow{\mathbf{x}\cdot} T/\langle x_1, \ldots, x_d\rangle \xrightarrow{\mathbf{x}\cdot} T/\langle x_1^2, \ldots, x_d^2\rangle \xrightarrow{\mathbf{x}\cdot} \ldots \right)$$

it follows, using that tensor products commute with filtered colimits, that

$$F_T^* H^d_{\mathfrak{m}}(T) \cong \varinjlim \left(T \xrightarrow{\mathbf{x}^p\cdot} T/\langle x_1^p, \ldots, x_d^p\rangle \xrightarrow{\mathbf{x}^p\cdot} T/\langle x_1^{2p}, \ldots, x_d^{2p}\rangle \xrightarrow{\mathbf{x}^p\cdot} \ldots \right)$$

Now, we fix $t \in \mathbb{N}$ and let $T/\langle x_1^t, \ldots, x_d^t\rangle \xrightarrow{F_t} T/\langle x_1^{pt}, \ldots, x_d^{pt}\rangle$ be the map induced by the Frobenius map. We have to note that the following square commutes:

$$\begin{array}{ccc} T/\langle x_1^t, \ldots, x_d^t\rangle & \xrightarrow{\mathbf{x}\cdot} & T/\langle x_1^{t+1}, \ldots, x_d^{t+1}\rangle \\ \downarrow{\scriptstyle F_t} & & \downarrow{\scriptstyle F_{t+1}} \\ T/\langle x_1^{pt}, \ldots, x_d^{pt}\rangle & \xrightarrow{\mathbf{x}^p\cdot} & T/\langle x_1^{p(t+1)}, \ldots, x_d^{p(t+1)}\rangle \end{array}$$

In this way, the universal property of the colimit implies that there is a natural homomorphism

$$H^d_{\mathfrak{m}}(T) \longrightarrow F_T^*(H^d_{\mathfrak{m}}(T)).$$

Finally, using that $\{\langle x_1^{pt}, \ldots, x_d^{pt}\rangle\}_{t\in\mathbb{N}}$ is an inverse system of ideals which is cofinal with respect to the inverse system of ideals given by $\{\langle x_1^t, \ldots, x_d^t\rangle\}_{t\in\mathbb{N}}$ it follows that the previous map between colimits yields the promised canonical isomorphism between $F_T^*(H^d_{\mathfrak{m}}(T))$ and $H^d_{\mathfrak{m}}(T)$.

In this way, combining the previous facts it turns out that we obtain a canonical isomorphism $F_T^*(E)^\vee \xrightarrow{\sim} F_T^*(E^\vee)$ which induces a canonical isomorphism

$$F_T^*(M)^\vee \xrightarrow[\sim]{\tau_M} F_T^*(M^\vee)$$

between the cokernels of $F_T^*(\psi)^\vee$ and $F_T^*(\psi^\vee)$. The comparison theorem between minimal free resolutions (cf. [114, Theorem 6.16]) imply that this isomorphism does not depend on the presentation of $M^\vee$ as cokernel of a matrix between free T-modules. □

Now, we are to introduce the following terminology.

Definition 1.8.9. Let M be a left R-module.

(i) The *evaluation map on M*, which we shall denote by e_M, is defined in the following way:

$$\begin{aligned} M &\xrightarrow{e_M} M^{\vee\vee} \\ m &\longmapsto e_m \end{aligned}$$

where

$$\begin{aligned} M^{\vee} &\xrightarrow{e_m} E \\ \omega &\longmapsto \omega(m). \end{aligned}$$

We note that e defines a natural transformation from the identity functor $(-)$ to the double dual functor $(-)^{\vee\vee}$.

(ii) We say that M is *Matlis-reflexive* provided e_M is an isomorphism of left R-modules.

Enochs-Zink Theorem describes explicitly the class of Matlis-reflexive modules. We shall omit the proof and refer to [131, 3.4.13] for details.

Theorem 1.8.10 (Enochs, Zink)**.** *A left R-module M is Matlis-reflexive if and only if it can be embedded into a short exact sequence*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

in which M' is a Noetherian left R-module and M'' is an Artinian left R-module. Therefore, the full subcategory of left R-modules consisting of all Matlis-reflexive modules is actually the smallest Serre subcategory of $R-\mathrm{Mod}$ that contains all Noetherian and all Artinian modules.

Now, we are to introduce two functors which are mild modifications of the standard Matlis duality.

Construction 1.8.11 (Katzman). Let M be a cofinite left $R[\Theta;F]$-module. We define $\Delta(M)$ as the γ-sheaf supported on I (N,γ_N), where $N := M^{\vee}$ and $N \xrightarrow{\gamma_N} F_T^*N$ will be defined as follows. Firstly, we set $F_T^*M \xrightarrow{\beta_M} M$ given by the assignment $F_*t \otimes m \longmapsto t\Theta m$. In this way, we define γ_N as the composition

$$M^{\vee} \xrightarrow{\beta_M^{\vee}} F_T^*(M)^{\vee} \xrightarrow[\sim]{\tau_M} F_T^*(M^{\vee}).$$

Therefore, the symbol Δ defines a contravariant functor from ${}_{R[\Theta;F]}\mathcal{A}$ to $\mathrm{Coh}_{\gamma}(R)$.

On the other hand, let (N, γ_N) be a coherent γ-sheaf supported on I. We define $\Psi(N)$ as the Artinian left R-module $N^\vee$. In addition, we want to attach to $\Psi(N)$ a structure as cofinite left $R[\Theta; F]$-module. Indeed, we denote by ν_N the following chain of isomorphisms:

$$F_T^*(N^\vee) \xrightarrow[\sim]{e_{F_T^*(N^\vee)}} F_T^*(N^\vee)^{\vee\vee} \xrightarrow[\sim]{(\tau_{N^\vee}^\vee)^{-1}} F_T^*(N^{\vee\vee})^\vee \xrightarrow[\sim]{(F_T^*(e_N^{-1})^\vee)^{-1}} F_T^*(N)^\vee.$$

In this way, we have obtained a functorial map

$$F_T^*(N^\vee) \xrightarrow[\sim]{\nu_N} F_T^*(N)^\vee \xrightarrow{\gamma_N^\vee} N^\vee$$

and therefore we can define the action of Θ on $\Psi(N) := N^\vee$ by setting, for any $\omega \in N^\vee$,

$$\Theta \cdot \omega := (\gamma_N^\vee \circ \nu_N)(F_* 1 \otimes \omega).$$

Hence, in fact, it turns out that the symbol Ψ defines a contravariant, univariate functor from $\mathrm{Coh}_\gamma(R)$ to ${}_{R[\Theta;F]}\mathcal{A}$.

Next result is the promised equivalence of categories between ${}_{R[\Theta;F]}\mathcal{A}$ and $\mathrm{Coh}_\gamma(R)$ which was established by Katzman in [77, Theorem 3.1]. It is the main result of this subsection. We provide a detailed proof for the convenience of the reader.

Theorem 1.8.12 (Katzman)**.** *Let $d \in \mathbb{N}$, $\mathbb{K}$ will stand for an arbitrary field of characteristic p, $T := \mathbb{K}[\![x_1, \ldots, x_d]\!]$, I is an arbitrary ideal of T and $R := T/I$. Then, the following statements hold.*

- *(i) The functors ${}_{R[\Theta;F]}\mathcal{A} \xrightarrow{\Delta} \mathrm{Coh}_\gamma(R)$ and $\mathrm{Coh}_\gamma(R) \xrightarrow{\Psi} {}_{R[\Theta;F]}\mathcal{A}$ are exact.*
- *(ii) For any cofinite left $R[\Theta; F]$-module M, the cofinite left $R[\Theta; F]$-module $(\Psi \circ \Delta)(M)$ is canonically isomorphic to M.*
- *(iii) For any coherent γ-sheaf (N, γ_N) supported on I, the coherent γ-sheaf $(\Delta \circ \Psi)(N, \gamma_N)$ supported on I is canonically isomorphic to (N, γ_N).*
- *(iv) Both Δ and Ψ are equivalences of categories.*

Proof. First of all, the exactness of the above-mentioned functors stems from the exactness of $(-)^\vee$ and F_T^*, hence part (i) holds.

Now, let M be a cofinite module. We note that $(\Psi \circ \Delta)(M)$ is $M^{\vee\vee}$ which is canonically isomorphic to M as R-module under the evaluation map e_M. In this way, it is enough to check that e_M is actually an isomorphism of left $R[\Theta; F]$-modules. Indeed, let e_m be any element of $M^{\vee\vee}$ and we are to describe $\Theta \cdot e_m$. This is the image of $F_* 1 \otimes e_m$ under the map

$$F_T^*(M^{\vee\vee}) \xrightarrow{e_{F_T^*(M^{\vee\vee})}} F_T^*(M^{\vee\vee})^{\vee\vee} \xrightarrow{(\tau_{M^{\vee\vee}}^{-1})^\vee} F_T^*(M^{\vee\vee\vee})^\vee \xrightarrow{F_T^*(e_{M^\vee}^{-1})^\vee} \tag{1.2}$$

$$\xrightarrow{F_T^*(e_{M^\vee}^{-1})^\vee} F_T^*(M^\vee)^\vee \xrightarrow{\tau_M^\vee} F_T^*(M)^{\vee\vee} \xrightarrow{\alpha_M^{\vee\vee}} M^{\vee\vee}.$$

Moreover, the functoriality of τ implies that the following square commutes:

$$\begin{array}{ccc} F_T^*(M^{\vee\vee\vee})^{\vee} & \xrightarrow{F_T^*(e_{M^\vee}^{-1})^{\vee}} & F_T^*(M^{\vee})^{\vee} \\ \downarrow{\scriptstyle \tau^{\vee}_{M^{\vee\vee}}} & & \downarrow{\scriptstyle \tau^{\vee}_{M}} \\ F_T^*(M^{\vee\vee})^{\vee\vee} & \xrightarrow[e^{-1}_{F_T^*(M^{\vee\vee})}]{} & F_T^*(M^{\vee\vee}). \end{array}$$

With the help of this commutative square, it turns out that we can simplify the chain of maps described in (1.2) obtaining finally the map

$$F_T^*(M^{\vee\vee}) \xrightarrow{\alpha_M^{\vee\vee}} M^{\vee\vee}.$$

In this way, we have checked that part (ii) holds too.

Finally, let (N, γ_N) be a coherent γ-sheaf supported on I. In this case, $(\Delta \circ \Psi)(N, \gamma_N)$ is the γ-sheaf with underlying module $N^{\vee\vee}$ and structural map given by the composition

$$N^{\vee\vee} \xrightarrow{\gamma_N^{\vee\vee}} F_T^*(N)^{\vee\vee} \xrightarrow{F_T^*(e_N)^{\vee\vee}} F_T^*(N^{\vee\vee})^{\vee\vee} \xrightarrow{((\tau^{\vee}_{N^\vee})^{-1})^{\vee}} \qquad (1.3)$$

$$\xrightarrow{((\tau^{\vee}_{N^\vee})^{-1})^{\vee}} F_T^*(N^{\vee})^{\vee\vee\vee} \xrightarrow{e^{-1}_{F_T^*(N^\vee)^\vee}} F_T^*(N^{\vee})^{\vee} \xrightarrow{\tau_{N^\vee}} F_T^*(N^{\vee\vee}).$$

In addition, combining the functoriality of τ joint with the fact that $((\tau^{\vee}_{N^\vee})^{-1})^{\vee} = (\tau^{\vee\vee}_{N^\vee})^{-1}$ one obtains the following commutative square:

$$\begin{array}{ccc} F_T^*(N^{\vee\vee})^{\vee\vee} & \xrightarrow{(\tau^{\vee\vee}_{N^\vee})^{-1}} & F_T^*(N^{\vee})^{\vee\vee\vee} \\ \downarrow{\scriptstyle e^{-1}_{F_T^*(N^{\vee\vee})}} & & \downarrow{\scriptstyle e^{-1}_{F_T^*(N^\vee)^\vee}} \\ F_T^*(N^{\vee\vee}) & \xrightarrow[\tau^{-1}_{N^\vee}]{} & F_T^*(N^{\vee})^{\vee}. \end{array}$$

Therefore, with the aid of this square we rewrite the composition of maps described in (1.3) and we simplify it into

$$N^{\vee\vee} \xrightarrow{\gamma_N^{\vee\vee}} F_T^*(N)^{\vee\vee},$$

just what we finally wanted to show. □

1.8.3 F-finite F-modules

The aim of this subsection is to introduce a functor from the category of coherent γ-sheaves supported on I (we have to note that we preserve the assumptions and notations introduced in the previous subsection) to the category of γ-sheaves. It turns out that the image of such functor is the category of $\mathcal{F}_T$-finite $\mathcal{F}_T$-modules defined by G. Lyubeznik in [94, Section 2]. The below-defined functor was introduced by Blickle in [17, pp. 353].

Construction 1.8.13 (Blickle). Let (N, γ_N) be a γ-sheaf. We have to note that $F_T^* N$ is naturally a γ-sheaf with structural map $F_T^* \gamma_N$. We may iterate this process in order to obtain a direct system

$$N \xrightarrow{\gamma_N} F_T^* N \xrightarrow{F_T^* \gamma_N} F_T^{*2} N \xrightarrow{F_T^{*2} \gamma_N} \cdots$$

In this way, set

$$\operatorname{Gen}(N) := \varinjlim \left(N \xrightarrow{\gamma_N} F_T^* N \xrightarrow{F_T^* \gamma_N} F_T^{*2} N \xrightarrow{F_T^{*2} \gamma_N} \cdots \right).$$

We have to emphasize that the following statements hold.

(a) $\operatorname{Gen}(N)$ is clearly a γ-sheaf with structural map given by

$$\varinjlim_{e \in \mathbb{N}} F_T^{*e} \gamma_N.$$

Moreover, this map is actually an isomorphism. Regarding Lyubeznik's terminology (cf. [94, Section 2]), $\operatorname{Gen}(N)$ is an $\mathcal{F}_T$-finite $\mathcal{F}_T$-module with generating morphism γ_N.

(b) With this notation, Lyubeznik's functor $\mathcal{H}_{R,T}$ (cf. [94, Section 4]) is obtained as the composition $\operatorname{Gen} \circ \Delta$, hence it is clearly a contravariant exact univariate functor.

In this way, one might ask how far the functor Gen (equivalently, the functor $\mathcal{H}_{T,T}$) is of being an equivalence of categories regarding Gen as a functor from $\operatorname{Coh}_\gamma(T)$ to the category of $\mathcal{F}_T$-finite $\mathcal{F}_T$-modules. Albeit we do not want to go into the details of this question, at least we provide the precise statement.

Theorem 1.8.14 (Blickle, Emerton, Kisin, Katzman, Lyubeznik)**.** *The following statements hold.*

(a) An $\mathcal{F}_T$-finite $\mathcal{F}_T$-module $\mathcal{N}$ is precisely a module which is isomorphic to $\operatorname{Gen}(N)$ *for a unique coherent minimal γ-sheaf (N, γ_N). Therefore, the functor* Gen *is dense.*

(b) Let $\mathcal{N}'$ and $\mathcal{N}''$ be $\mathcal{F}_T$-finite $\mathcal{F}_T$-modules. For every $\beta \in \operatorname{Hom}_{\mathcal{F}_T}(\mathcal{N}', \mathcal{N}'')$ there exists a homomorphism

$$\alpha \in \operatorname{Hom}_{\operatorname{Coh}_\gamma(T)}(N', N'')$$

such that $\operatorname{Gen}(N') \cong \mathcal{N}'$, $\operatorname{Gen}(N'') \cong \mathcal{N}''$ *and* $\operatorname{Gen}(\alpha) \cong \beta$. *In this way, the functor* Gen *is full.*

(c) Let $\alpha \in \operatorname{Hom}_{\operatorname{Coh}_\gamma(T)}(N', N'')$*. Then* $\operatorname{Gen}(\operatorname{Im}(\alpha)) = 0$ *if and only if* $\gamma^{*e}_{N''}(\operatorname{Im}(\alpha)) = 0$ *for some* $e \gg 0$*.*

(d) The functor Gen *induces an equivalence of categories between the category of* $\mathcal{F}_T$*-finite* $\mathcal{F}_T$*-modules and the category of minimal* γ*-crystals, where the category of* γ*-crystals (namely,* $\operatorname{Crys}_\gamma(T)$*) is the category obtained from* $\operatorname{Coh}_\gamma(T)$ *by inverting all the morphisms with nilpotent kernel and cokernel.*

Remark 1.8.15. Firstly, part (a) of Theorem 1.8.14 was proved by Lyubeznik in [94, Theorem 3.5] under our assumptions. Later, M. Emerton and M. Kisin in [45, Theorem 6.1.3] proved the same statement (without unicity) dropping the completeness assumption on the base ring under the hypothesis that $\mathbb{K}$ is F-finite.

Secondly, part (b) of Theorem 1.8.14 was proved by Katzman in [80, Theorem 3.3] again under our assumptions building from results obtained by M. Hochster in [67]. Part (c) was also obtained by Katzman (cf. [80, Theorem 3.4]).

Finally, part (d) follows combining parts (a), (b) and (c). Nevertheless, we must observe that part (d) was proved by Blickle in [17, Theorem 2.27] just requiring the F-finiteness assumption of the ground field $\mathbb{K}$.

In the following diagram, we sum up the previously established equivalences of categories. In the below diagram, we recall that $\mathbb{K}$ is an F-finite field, $T := \mathbb{K}[\![x_1, \dots, x_d]\!]$, I is an ideal of T and $R := T/I$.

$$\begin{array}{ccccccc}
{}_{T[\Theta;F]}\mathcal{A} & \xrightarrow[\sim]{\Delta} & \operatorname{Coh}_\gamma(T) & \twoheadrightarrow & \operatorname{Crys}_\gamma(T) & \xrightarrow[\sim]{\operatorname{Gen}} & \\
\uparrow & & \uparrow & & \uparrow & & \mathcal{F}_T - \text{finite} \quad \mathcal{F}_T - \operatorname{Mod} \\
{}_{R[\Theta;F]}\mathcal{A} & \xrightarrow[\Delta]{\sim} & \operatorname{Coh}_\gamma(R) & \twoheadrightarrow & \operatorname{Crys}_\gamma(R) & \xrightarrow[\sim]{\operatorname{Gen}} &
\end{array}$$

Moreover, it is worth noting once more that the composition

$${}_{R[\Theta;F]}\mathcal{A} \xrightarrow[\Delta]{\sim} \operatorname{Coh}_\gamma(R) \twoheadrightarrow \operatorname{Crys}_\gamma(R) \xrightarrow[\sim]{\operatorname{Gen}} \mathcal{F}_T - \text{finite} \quad \mathcal{F}_T - \operatorname{Mod}$$

turns out to be the functor $\mathcal{H}_{R,T}$ introduced by G. Lyubeznik in [94, Section 4].

Some concrete F-finite F-modules

We are to conclude this section describing explicitly the images of some distinguished modules under the previously introduced functors. The first example we treat was provided by M. Katzman in [77, Theorem 4.7].

Example 1.8.16 (Injective hulls). Let $u \in (I^{[p]} :_T I)/I^{[p]}$. So, E_R has an structure as left $R[u\Theta; F]$-module. We have to note that $\Delta(E_R) \cong (R, u\cdot)$; indeed, after identifying $F_T^{*e}(R)$ with $T/I^{[p^e]}$, it is clear that

$$\mathrm{Gen}(R, u\cdot) = \varinjlim \left(T/I \xrightarrow{u\cdot} T/I^{[p]} \xrightarrow{u^p\cdot} T/I^{[p^2]} \xrightarrow{u^{p^2}\cdot} T/I^{[p^3]} \xrightarrow{u^{p^3}\cdot} \ldots \right).$$

In this way, $\mathrm{Gen}(R, u\cdot)$ is the $\mathcal{F}_T$-finite $\mathcal{F}_T$-module minimally generated by the γ-sheaf $(T/I^{F,u}, u\cdot)$, where

$$I^{F,u} := \{t \in T \mid \quad u^{\sigma(p^{e-1})} t^{p^e} \in I^{[p^e]} \quad \text{for some} \quad e \gg 0\}$$

and, for any $e \in \mathbb{N}$,

$$\sigma(p^e) := \sum_{i=0}^{e} p^i.$$

When $u = 1$, $I^{F,u} = I^F$ is the well-known *Frobenius closure* of the ideal I. We have to point out that these facts were established by Katzman in [77, Theorem 4.7]. Moreover, Katzman showed in [77] that if $u \neq 1$ then

$$I^{F,u} = \min\{e \in \mathbb{N} \mid \quad (I^{[p^e]} :_T u^{\sigma(p^{e-1})}) = (I^{[p^a]} :_T u^{\sigma(p^{a-1})}) \quad \text{for all} \quad a \geq e\}.$$

If $I = 0$ then it was already pointed out by Blickle in [17, pp. 365] that

$$\Delta(E) \cong (T, u\cdot) \quad \text{and} \quad \mathrm{Gen}(T, u\cdot) \cong T_u.$$

Example 1.8.17 (Local cohomology modules). Fix $i \in \mathbb{N}$. We have explained in Theorem 1.8.3 that $H_{\mathfrak{m}}^{\dim(T)-i}(R)$ has a natural structure as cofinite $R[\Theta; F]$-module. In this way, local duality implies that

$$\Delta(H_{\mathfrak{m}}^{\dim(T)-i}(R)) \cong (\mathrm{Ext}_T^i(R, T), \mathrm{Ext}_T^i(F, T)),$$

where $\mathrm{Ext}_T^i(R, T) \xrightarrow{\mathrm{Ext}_T^i(F,T)} \mathrm{Ext}_T^i(T/I^{[p]}, T)$ is the natural map induced by

$$\begin{aligned} F_T^* R &\longrightarrow R \\ F_* t \otimes x &\longmapsto t x^p. \end{aligned}$$

Therefore, $\mathrm{Gen}(\mathrm{Ext}_T^i(R, T), \mathrm{Ext}_T^i(F, T)) \cong H_I^i(T)$. This isomorphism was already pointed out by Lyubeznik in [94, Example 4.5].

Although we do not want to go into details, it is worth mentioning that the calculation carried out in Example 1.8.17 was generalized by D. Tobisch. We omit the proof of the following result (which we only state) and refer to [134, Theorem 4.1] for details.

Theorem 1.8.18 (Tobisch)**.** *Let $\mathbb{K}$ be an F-finite field of prime characteristic p, set $T := \mathbb{K}[\![x_1, \dots, x_d]\!]$, I will stand for an arbitrary ideal of T, and let $\mathfrak{a}$ be a cohomologically complete intersection ideal of T with $\operatorname{grade}_T(\mathfrak{a}) := c$. Then, there is an isomorphism*

$$\operatorname{Gen}(H_{\mathfrak{a}}^{c-i}(T/I)) \cong H_I^i(H_{\mathfrak{a}}^c(T)^\vee)$$

for all $i \in \mathbb{N}$.

Our final example deals with the case of the so-called Nagata's ideal transforms.

Example 1.8.19 (Ideal transforms). As T is regular it is, in particular, a domain and therefore $\Gamma_I(T) = 0$; hence the natural exact sequence

$$0 \longrightarrow \Gamma_I(T) \longrightarrow T \longrightarrow D_I(T) \longrightarrow H_I^1(T) \longrightarrow 0$$

becomes into the next short exact sequence:

$$0 \longrightarrow T \longrightarrow D_I(T) \longrightarrow H_I^1(T) \longrightarrow 0.$$

The aim of this example is to endow this sequence with a natural structure as a short exact sequence in the category of $\mathcal{F}_T$-finite $\mathcal{F}_T$-modules. Indeed, we consider the short exact sequence

$$0 \longrightarrow I \longrightarrow T \longrightarrow R \longrightarrow 0.$$

Applying to this sequence the left exact functor $\operatorname{Hom}_T(-, T)$ one obtains the following exact sequence:

$$0 \longrightarrow \operatorname{Hom}_T(R, T) \longrightarrow \operatorname{Hom}_T(T, T) \longrightarrow \operatorname{Hom}_T(I, T) \longrightarrow \operatorname{Ext}_T^1(R, T) \longrightarrow \operatorname{Ext}_T^1(T, T).$$

Nevertheless, as $\operatorname{Hom}_T(R, T) = 0 = \operatorname{Ext}_T^1(T, T)$ and $\operatorname{Hom}_T(T, T) \cong T$ it turns out that this sequence becomes into the following short exact sequence:

$$0 \longrightarrow T \longrightarrow \operatorname{Hom}_T(I, T) \longrightarrow \operatorname{Ext}_T^1(R, T) \longrightarrow 0.$$

In this way, after identifying $F_T^* \operatorname{Hom}_T(I, T)$ and $F_T^* \operatorname{Ext}_T^1(R, T)$ with $\operatorname{Hom}_T(I^{[p]}, T)$ and $\operatorname{Ext}_T^1(T/I^{[p]}, T)$ respectively one has that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & \operatorname{Hom}_T(I, T) & \longrightarrow & \operatorname{Ext}_T^1(R, T) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \operatorname{Hom}_T(F,T)} & & \downarrow{\scriptstyle \operatorname{Ext}_T^1(F,T)} & & \\ 0 & \longrightarrow & T & \longrightarrow & \operatorname{Hom}_T(I^{[p]}, T) & \longrightarrow & \operatorname{Ext}_T^1(T/I^{[p]}, T) & \longrightarrow & 0 \end{array}$$

can be regarded as a short exact sequence in $\operatorname{Coh}_\gamma(T)$. Therefore, applying to this diagram the functor Gen one obtains the following short exact sequence in the category of $\mathcal{F}_T$-finite $\mathcal{F}_T$-modules:

$$\operatorname{Gen}(T, F) \hookrightarrow \operatorname{Gen}(\operatorname{Hom}_T(I, T), \operatorname{Hom}_T(F, T)) \twoheadrightarrow \operatorname{Gen}(\operatorname{Ext}_T^1(R, T), \operatorname{Ext}_T^1(F, T)). \tag{1.4}$$

Morerover, we have checked in Example 1.8.16 and Example 1.8.17 respectively that

$$\mathrm{Gen}(T,F)\cong T \quad \text{and} \quad \mathrm{Gen}(\mathrm{Ext}^1_T(R,T),\mathrm{Ext}^1_T(F,T))\cong H^1_I(T).$$

On the other hand, by the own definition of the Gen functor, it is clear that

$$\mathrm{Gen}(\mathrm{Hom}_T(I,T),\mathrm{Hom}_T(F,T))\cong \varinjlim_{e\in\mathbb{N}} \mathrm{Hom}_T(I^{[p^e]},T).$$

In this way, using once more that the inverse system of ideals $\{I^{[p^e]}\}_{e\in\mathbb{N}}$ is cofinal with respect to the inverse system $\{I^e\}_{e\in\mathbb{N}}$ it turns out that

$$\mathrm{Gen}(\mathrm{Hom}_T(I,T),\mathrm{Hom}_T(F,T))\cong \varinjlim_{e\in\mathbb{N}} \mathrm{Hom}_T(I^e,T)\cong D_I(T).$$

Therefore, combining the foregoing facts it follows that (1.4) becomes into the short exact sequence

$$0\longrightarrow T\longrightarrow D_I(T)\longrightarrow H^1_I(T)\longrightarrow 0$$

in the category of $\mathcal{F}_T$-finite $\mathcal{F}_T$-modules. In particular, Nagata's ideal transform $D_I(T)$ of T with respect to I has a natural structure as $\mathcal{F}_T$-finite $\mathcal{F}_T$-module.

Bibliographical notes

Throughout this mimeograph, we have denoted by F the Frobenius map; that is, the map which raises any element of a ring of prime characteristic p to its pth power, borrowing this notation from [29]; however, we want to prevent the reader that there is no general convention to use this notation. Indeed, in [30, Chapters 5 and 6], the authors use f to refer to the same notion; on the other hand, in [73, Lecture 21] the Frobenius endomorphism is denoted by φ. Finally, in the context of Arithmetic Geometry (e. g. [45]) the Frobenius endomorphism is often denoted by σ.

Cartier algebras were formally introduced by K. Schwede in [117] as an auxiliar tool for solving an open question raised, among others, by R. Lazarsfeld which, roughly speaking, involved the expression of the test ideals as sum of small pieces (cf. [117, Introduction] for more details). Schwede was probably influenced by work due to M. Blickle in [16], S. Takagi (cf. [132, Theorem 3.13]) and by N. Hara and S. Takagi in [58, Lemma 2.1]. Quickly, M. Blickle in [18] presented a systematic study of Cartier algebras from a purely algebraic point of view. His approach builds on Blickle-Böckle's theory of Cartier modules (cf. [19]) and Cartier crystals (cf. [20]). In this case, Blickle mainly uses Cartier algebras for defining with complete generality test modules and generalized test ideals (which were introduced by N. Hara and K.-I. Yoshida in [59]) without any reference to some ideal-adic tight closure theory, although the definition which we have given in this chapter is slightly different (cf. [23, Definition 9.3.7]).

Albeit our approach in this chapter builds from Cartier algebras and only uses Frobenius algebras as an auxiliary tool, it is noteworthy that Frobenius algebras were introduced before Cartier algebras. Indeed, in [96] G. Lyubeznik and K. E. Smith used Frobenius algebras as a tool in order to give a complete characterization of strongly F-regular rings; the interested reader may like to consult [96, Theorem 4.1] for the precise statement.

The connection between Cartier algebras and Frobenius algebras in this chapter based on Matlis duality relies on Theorem 1.5.1, which was proved by F. Enescu and M. Hochster (cf. [46, Discussion 3.4]). As far as we know, this result of Enescu and Hochster generalizes a previous result obtained by Y. Yoshino in [146, Lemma (3.6)].

In this chapter, we have introduced a rough sketch of the theory of algebraic D-modules. We refer to [39] and the references therein for a more detailed treatment.

Theorem 1.6.5 is a result proved by J. Àlvarez Montaner, M. Blickle and G. Lyubeznik in [2, Proposition 2.1]. In fact, their result is deeper. Indeed, they establish an equivalence between the category of R-modules and the category of $D_R^{(e)}$-modules using the Frobenius functor provided R is regular and F-finite. From this point of view, such an equivalence can be regarded as a particular case of the so-called *Morita equivalences* between categories of modules (cf. [114, Theorem 5.55]). On the other hand, it is also worth mentioning that an *arithmetic* version of Theorem 1.6.5 had already previously established by P. Berthelot in [13, 2.3.6 and 2.4.6]. This result stems from ideas worked out, among others, by B. Haastert (cf. [56]).

The terminology *Enochs-Zink theorem* may look a bit strange for some readers; however, it is worth mentioning that Theorem 1.8.10 was originally proved by T. Zink in [147]. Later on, E. Enochs obtained in [48, Proposition 1.3] independently the same result.

Chapter 2

Cartier algebras of Stanley-Reisner rings

In [79], M. Katzman gave an example of an Artinian module M over a complete local ring R of characteristic p for which the algebra of Frobenius operators $\mathcal{F}^M$ is not finitely generated as R-algebra. More precisely, his example R is a non-Cohen-Macaulay quotient of a formal power series ring in three indeterminates by a squarefree monomial ideal and $M = E_R$ is a choice of injective hull of the residue field of R. This example provides a negative answer to a question raised by G. Lyubeznik and K. E. Smith in [96, paragraph preceding Example 3.6].

Bearing in mind the duality established in Chapter 1 (cf. Theorem 1.5.1) between Frobenius and Cartier algebras, one may regard Katzman's example as a case where the Cartier algebra $\mathcal{C}^R$ is infinitely generated as R-algebra.

The main purpose of this chapter will be the study of the generation of the Cartier algebra $\mathcal{C}^R$, where $T := \mathbb{K}[\![x_1, \dots, x_d]\!]$ is a formal power series ring with d indeterminates over a field of characteristic p, I is an ideal minimally generated by squarefree monomials inside $S := \mathbb{K}[x_1, \dots, x_d]$ and $R := T/I$. It turns out that, in this situation, $\mathcal{C}^R$ can only be principally generated or infinitely generated as R-algebra and that such fact just depends on the minimal primary decomposition of I (cf. Theorem 2.3.5).

As a first application of this description of $\mathcal{C}^R$, we prove (cf. Theorem 2.5.3) that the set of F-jumping numbers of pairs $(\operatorname{Spec}(R), \mathbf{V}(\mathfrak{a}))$ forms a discrete subset inside the non-negative real numbers, where $\mathfrak{a}$ is an arbitrary ideal of R.

As a second and final application, we are to provide some description of the image of the pairing $\langle -, - \rangle_e$ introduced in Chapter 1 (cf. Section 1.6); it turns out (not surprisingly, in fact) that one can produce differential operators of level e which does not belong to the image of such pairing (cf. Theorem 2.5.5 and the subsequent discussion).

Now, we move on provide a more detailed overview of contents. First of all, Sections 2.1 and 2.2 are devoted to introduce some preliminary calculations and notations which will be useful later on in this chapter. Second, in Section 2.3 we remind for the reader's benefit the main technical tool used by M. Katzman in [79] (cf. Proposition 2.3.1) because it play a key role in the proof of the main result of this chapter; namely, our main result (cf. Theorem 2.3.5) establishes the fact that the Cartier algebra attached to a complete, Stanley-Reisner ring can only be either principally generated or infinitely generated, and that such fact only depends on the primary decomposition of the corresponding Stanley-Reisner ideal.

On the other hand, in Section 2.4 we discuss several examples of Stanley-Reisner rings at the boundary of the Gorenstein property; this discussion is motivated because it is known that the Cartier algebra attached to a normal, local ring A is principally generated if and only if A is Gorenstein (cf. Proposition 1.4.19). It turns out that we provide examples of even non Cohen-Macaulay rings with principal Cartier algebra; the reader should notice that this does not contradict the previous fact because the case of Stanley-Reisner rings we are dealing with is, in general, non-normal.

Moreover, also in Section 2.4 (cf. 2.4.3) we introduce a generating function and its corresponding generating serie, hoping that both might be interesting in its own right.

Finally, in Section 2.5 we present two applications of Theorem 2.3.5; on one hand, from the explicit description of the Cartier algebra attached to a complete, Stanley-Reisner ring R, we deduce (cf. Theorem 2.5.3) the discreteness of F-jumping numbers of the pair $(\operatorname{Spec}(R), \mathbf{V}(\mathfrak{a}))$, where $\mathfrak{a}$ is any ideal of R. On the other hand, building again from the explicit description of $\mathcal{C}^R$ obtained in Theorem 2.3.5 we exhibit differential operators which does not belong to the natural pairing introduced in Chapter 1 (cf. Section 1.6); in particular, this fact implies that Theorem 1.6.5 is, in general, false when one drops the assumption of regularity.

Special acknowledgement of joint work

The content of this chapter is based in joint work with J. Àlvarez Montaner and S. Zarzuela. Nonetheless, we have rewritten this text in our own words and notation, which is slightly different from [4]. We provide in what follows some details and simplifications which are not found in the paper.

2.1 Some preliminary calculations

The aim of this section is to compute colon ideals of the form $\left(I^{[p^e]} :_T I\right)$, where I stands for an arbitrary ideal of $\mathbb{K}[\![x_1, \ldots, x_d]\!]$ and $\mathbb{K}$ is an arbitrary ground field of prime characteristic p. Recall that Fedder's Theorem (cf. Theorem 1.4.20) establishes the following isomorphism:

$$\mathcal{F}^{E_R} \cong \bigoplus_{e \geq 0} \left\{ (I^{[p^e]} :_T I)/I^{[p^e]} \right\} \Theta^e,$$

where E denotes a choice of injective hull of $\mathbb{K}$ over T, Θ is the standard Frobenius action on E and $E_R := (0 :_E I)$. Thus, our motivation to calculate such quotient ideals stems from this fact.

Next result will be useful in what follows; albeit this result was originally proved by J. Cowden Vassilev in [137, Lemma 2.1], we provide her proof for the sake of completeness.

Lemma 2.1.1. *Let A be a regular (not necessarily local) F-finite ring of prime characteristic and let M be a finitely generated, free A-module. Then, for any family of ideals $\{J_i\}_{i\in I}$ of A, we have that*

$$\bigcap_{i\in I} J_i M = \left(\bigcap_{i\in I} J_i\right) M.$$

In particular, as $F_*^e A$ is a finitely generated, free A-module whenever $\mathbb{K}$ is F-finite it follows that

$$\bigcap_{i\in I} J_i^{[p^e]} = \left(\bigcap_{i\in I} J_i\right)^{[p^e]}.$$

Proof. Let $\{x_j\}_{j\in J}$ be a free basis of M over A. Thus, $M = \oplus_{j\in J} A x_j$. Therefore, one has for any ideal $\mathfrak{a}$ of A that $\mathfrak{a}M = \oplus_{j\in J}\mathfrak{a}x_j$. This fact implies that

$$\bigcap_{i\in I} J_i M = \bigoplus_{j\in J}\left(\bigcap_{i\in I} J_i\right) x_j.$$

In this way, it is clear that $\bigcap_{i\in I} J_i M \supseteq \left(\bigcap_{i\in I} J_i\right) M$. On the other hand, if $x \in \bigcap_{i\in I} J_i M$ then $x \in J_i M$ for all $i \in I$. But again by $J_i M = \oplus_{j\in J} J_i x_j$ it follows that $x = \sum_{j\in J} a_j x_j$, where $a_j \in I$ for all $(i,j) \in I \times J$. Hence $a_j \in \bigcap_{i\in I} J_i$ and therefore $x \in \left(\bigcap_{i\in I} J_i\right) M$. □

In the following result, we collect without proof some well-known general facts which will simplify our later computations. The first three statements are proved in [60, Corollary 1] (see also [44, Exercise 17.13] for more information); the last three statements are proved in [40, Chapter 4, Proposition 10 and Theorem 11].

Proposition 2.1.2. *Let I, J, $I_1, \ldots, I_r$ be ideals of $S = \mathbb{K}[x_1, \ldots, x_d]$, suppose that $J := \langle f_1, \ldots, f_t\rangle$ and let $f \in S$. Then, the following statements hold.*

(i) $(I_1 + \ldots + I_r)T = (I_1 T) + \ldots + (I_r T)$.

(ii) $(I_1 \cap \ldots \cap I_r)T = (I_1 T) \cap \ldots \cap (I_r T)$.

(iii) $(I :_S J)T = ((IT) :_T (JT))$.

(iv) $(\cap_{i=1}^r I_i :_S J) = \cap_{i=1}^r (I_i :_S J)$.

(v) $(I :_S J) = \cap_{i=1}^t (I :_S \langle f_i\rangle)$.

(vi) $(I :_S \langle f\rangle) = \frac{1}{f}(I \cap \langle f\rangle)$.

We conclude this subsection with the following elementary (but very important) fact. We emphasize that in the following result I stands for an arbitrary ideal of T; although we think the below result is well known for experts, we provide a detailed proof because of the lack of a reference.

Proposition 2.1.3. *Let $I = F_1 \cap \ldots \cap F_r$ be a minimal primary decomposition of I. Then*

$$(I^{[p^e]} :_T I) = (F_1^{[p^e]} :_T F_1) \cap \ldots \cap (F_r^{[p^e]} :_T F_r).$$

Proof. Lemma 2.1.1 implies that $I^{[p^e]} = F_1^{[p^e]} \cap \ldots \cap F_r^{[p^e]}$. Moreover, using Proposition 2.1.2 it follows that

$$(I^{[p^e]} :_T I) = (F_1^{[p^e]} :_T I) \cap \ldots \cap (F_r^{[p^e]} :_T I).$$

Now, fix $j \in \{1, \ldots, r\}$ and set $Q_j := \sqrt{F_j}$. We underline that $(F_j^{[p^e]} :_T I)$ is the only Q_j-primary component of $(I^{[p^e]} :_T I)$ and

$$(F_j^{[p^e]} :_T I)_{Q_j} = (F_j^{[p^e]} T_{Q_j} :_{T_{Q_j}} IT_{Q_j}) = (F_j^{[p^e]} T_{Q_j} :_{T_{Q_j}} F_j T_{Q_j}) = (F_j^{[p^e]} :_T F_j) T_{Q_j}.$$

In this way, combining the foregoing equalities we get the desired result. □

2.2 Stanley-Reisner case

From now on, let $\mathbb{K}$ be a perfect field of characteristic p, $q := p^e$ (for some $e \in \mathbb{N}$), $S := \mathbb{K}[x_1, \ldots, x_d]$ is the polynomial ring in d variables over $\mathbb{K}$, $T := \mathbb{K}[\![x_1, \ldots, x_d]\!]$ will be the formal power series ring in d variables over $\mathbb{K}$ and set $I := I'T$, where I' is an ideal of S generated by squarefree monomials $\mathbf{x}^\alpha := x_1^{a_1} \cdots x_d^{a_d}$, $\alpha = (a_1, \ldots, a_d) \in \{0,1\}^d$. Its minimal primary decomposition $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$ is given in terms of face ideals; that is, ideals of the form $I_\beta := \langle x_i \mid \quad b_i \neq 0\rangle$. For simplicity, we shall denote the homogeneous maximal ideal $\mathfrak{m} := I_{\mathbf{1}} = \langle x_1, \ldots, x_d\rangle$, where $\mathbf{1} := (1, \ldots, 1)$. Finally, the support of α is $\operatorname{supp}(\alpha) := \{i \in \{1, \ldots, d\} \mid \quad a_i = 1\}$.

We specialize the facts which have been established in Proposition 2.1.2 in terms of face ideals.

Lemma 2.2.1. *Let I_α be a face ideal. Then $\left(I_\alpha^{[q]} :_T I_\alpha\right) = I_\alpha^{[q]} + \langle \mathbf{x}^{(q-1)\alpha}\rangle$.*

Proof. We only have to note that

$$\left(I_\alpha^{[q]} :_T I_\alpha\right) = \bigcap_{i \in \operatorname{supp}(\alpha)} \left(I_\alpha^{[q]} :_T \langle x_i\rangle\right) = \bigcap_{i \in \operatorname{supp}(\alpha)} \left(I_\alpha^{[q]} + \langle x_i^{q-1}\rangle\right) = I_\alpha^{[q]} + \langle \mathbf{x}^{(q-1)\alpha}\rangle.$$

Hence the result holds. □

Remark 2.2.2. We have to note that, when α has only one non-zero entry,

$$\left(I_{\alpha}^{[q]} :_T I_{\alpha}\right) = I_{\alpha}^{[q]} + \langle \mathbf{x}^{(q-1)\alpha} \rangle = \langle \mathbf{x}^{(q-1)\alpha} \rangle.$$

Therefore, combining Proposition 2.1.3 joint with the previous lemma one obtains the following statement.

Proposition 2.2.3. *Let $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subseteq T$. Then*

$$\left(I^{[q]} :_T I\right) = \left(I_{\alpha_1}^{[q]} + \langle \mathbf{x}^{(q-1)\alpha_1} \rangle\right) \cap \ldots \cap \left(I_{\alpha_s}^{[q]} + \langle \mathbf{x}^{(q-1)\alpha_s} \rangle\right).$$

Remark 2.2.4. From now on, we shall assume that $\mathfrak{m} = I_{\alpha_1} + \ldots + I_{\alpha_s}$ for simplicity; that is, we use all the variables $x_1, \ldots, x_d$.

Discussion 2.2.5. If we take a close look to the equality given in Proposition 2.2.3 then we note that only the following situations are possible.

(i) If $\mathrm{ht}(I_{\alpha_i}) > 1$ for all $i \in \{1, \ldots, s\}$, then $\left(I^{[q]} :_T I\right) = I^{[q]} + J_q + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$, where the generators $\mathbf{x}^{\gamma} := x_1^{c_1} \cdots x_d^{c_d}$ of J_q satisfy $(c_1, \ldots, c_d) \in \{0, q-1, q\}^d$. Moreover, we also notice that $\mathbf{x}^{\gamma} \in \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$ whenever $(c_1, \ldots, c_d) \in \{q-1, q\}^d$ and that we may also have $\mathbf{x}^{\gamma} \in I^{[q]}$ whenever a generator of $I^{[q]}$ divides $\mathbf{x}^{\gamma}$. Summing up, we end up with only two possibilities depending whether J_q is contained in $I^{[q]} + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$ or not.

   (a) $\left(I^{[q]} :_T I\right) = I^{[q]} + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$.

   (b) $\left(I^{[q]} :_T I\right) = I^{[q]} + J_q + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$.

   In the later case, there is a generator $\mathbf{x}^{\gamma}$ of J_q such that $(c_i, c_j, c_k) = (q, q-1, 0)$, where $1 \leq i, j, k \leq d$.

(ii) If $\mathrm{ht}(I) = 1$ and there is $i \in \{1, \ldots, s\}$ such that $\mathrm{ht}(I_{\alpha_i}) > 1$ then $\left(I^{[q]} :_T I\right) = J'_q + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$. In this case, there exists a generator $\mathbf{x}^{\gamma}$ of J'_q such that $(c_i, c_j, c_k) = (q, q-1, 0)$, where $1 \leq i, j, k \leq d$.

(iii) If $\mathrm{ht}(I_{\alpha_i}) = 1$ for all $i \in \{1, \ldots, s\}$ then $\left(I^{[q]} :_T I\right) = \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$. We note that, in this case, R is Gorenstein.

2.3 M. Katzman's criterion

The following result is a key fact in order to give a complete characterization of the generation of $\mathcal{C}^R$. We have to emphasize that, in the following result, I stands for an arbitrary ideal of T. Although the proof of the below result was provided by Katzman in [79, Proposition 2.1], we shall describe it for the sake of completeness.

Proposition 2.3.1 (Katzman)**.** *For any $e \in \mathbb{N}$, set $K_e := \left(I^{[p^e]} :_T I\right)$ and*

$$L_e := \sum_{\substack{1 \le b_1, \ldots, b_s \le e-1 \\ b_1 + \ldots + b_s = e}} K_{b_1} K_{b_2}^{[p^{b_1}]} \cdots K_{b_s}^{[p^{b_1 + \ldots + b_{s-1}}]}.$$

Let $\mathcal{F}_{<e}$ be the R-subalgebra of $\mathcal{F}^{E_R}$ generated by $\mathcal{F}_0^{E_R}, \ldots, \mathcal{F}_{e-1}^{E_R}$. Then $\mathcal{F}_{<e} \cap \mathcal{F}_e^{E_R} = L_e F^e$, where F^e is the e-fold of the natural Frobenius action on the injective hull of the residue field of T.

The proof of this crucial statement requires the following previous calculation.

Lemma 2.3.2. *Let $s \in \mathbb{N}$, let $b_1, \ldots, b_s \in \mathbb{N}$ and set, for each $i \in \{1, \ldots, s\}$,*

$$c_i := \sum_{j=1}^{i-1} b_i.$$

Then

$$J := \prod_{i=1}^{s} \left(I^{[p^{b_i}]} :_T I\right)^{[p^{c_i}]} \subseteq \left(I^{[p^{b_1 + \ldots + b_s}]} :_T I\right).$$

Proof. We proceed by increasing induction on s. First of all, for $s = 1$ there is nothing to prove because of $c_1 = 0$. Secondly, we assume that $s = 2$. Let $\lambda_1 \lambda_2^{p^{b_1}}$ (where $\lambda_i \in \left(I^{[p^{b_i}]} :_T I\right)$ for $i \in \{1, 2\}$) be a generator of J and let $\mu \in I$. We underline that $\lambda_1 \mu \in I^{[p^{b_1}]}$. So, we can write $\lambda_1 \mu = \nu^{p^{b_1}}$ for some $\nu \in I$. Moreover, as $\lambda_2 \nu \in I^{[p^{b_2}]}$ we write $\lambda_2 \nu = \chi^{p^{b_2}}$ for some $\chi \in I$. In this way, it follows that

$$\left(\lambda_1 \lambda_2^{p^{b_1}}\right) \mu = \lambda_2^{p^{b_1}} (\lambda_1 \mu) = \lambda_2^{p^{b_1}} \nu^{p^{b_1}} = (\lambda_2 \nu)^{p^{b_1}} = \chi^{p^{b_1 + b_2}} \in I^{[p^{b_1 + b_2}]},$$

hence the case $s = 2$ holds.

Finally, when $s > 2$ one has, combining the induction hypothesis joint with the case $s = 2$, that

$$J \subseteq \left(I^{[p^{b_1 + \ldots + b_{s-1}}]} :_T I\right) \cdot \left(I^{[p^{b_s}]} :_T I\right)^{[p^{b_1 + \ldots + b_{s-1}}]} \subseteq \left(I^{[p^{b_1 + \ldots + b_s}]} :_T I\right),$$

and therefore we obtain the desired inclusion □

Proof of Proposition 2.3.1. We have that any element of $\mathcal{F}_{<e} \cap \mathcal{F}_e^{E_R}$ can be written as a sum of elements of the form $\phi_1 \cdots \phi_s$ where, for each $j \in \{1, \ldots, s\}$,

$$\phi_j \in \mathcal{F}_{b_j}^{E_R} \quad (1 \le b_j < e) \quad \text{and} \quad b_1 + \ldots + b_s = e.$$

In addition, it follows (for each $1 \leq j \leq s$) from the fact that $\mathcal{F}_{b_j}^{E_R}$ is essentially given by $(I^{[p^{b_j}]} :_T I)\theta^{b_j} = K_{b_j}\theta^{b_j}$ (indeed, bearing in mind Fedder's Theorem) that any such ϕ_j equals $a_j\theta^{b_j}$ (for some $a_j \in K_{b_j}$). Summarizing, we have that

$$\prod_{i=1}^{s} \phi_i = \prod_{i=1}^{s} (a_i\theta^{b_i}). \tag{2.1}$$

Claim 2.3.3. *One has that*

$$\prod_{i=1}^{s}(a_i\theta^{b_i}) = \left(\prod_{i=1}^{s}\left(a_i^{p^{c_i}}\right)\right)\theta^{b_1+\ldots+b_s}, \quad \textit{where} \quad c_i := \sum_{j=1}^{i-1} b_j \quad (1 \leq i \leq s).$$

Proof of Claim 2.3.3. Once more, we proceed by increasing induction on s. First of all, for $s = 1$ one has, as $c_1 = 0$, that

$$\prod_{i=1}^{s}(a_i\theta^{b_i}) = a_1\theta^{b_1} = \left(a_1^{p^{c_1}}\right)\theta^{b_1}.$$

Furthermore, in the case $s = 2$

$$\prod_{i=1}^{s}(a_i\theta^{b_i}) = a_1\theta^{b_1}a_2\theta^{b_2} = a_1\left(a_2^{p^{b_1}}\right)\theta^{b_1+b_2} = \left(a_1^{p^{c_1}}\right)\left(a_2^{p^{c_2}}\right)\theta^{b_1+b_2}.$$

In the general case, combining the induction hypothesis joint with the definition of c_s we only have to note that

$$\begin{aligned}
\prod_{i=1}^{s}(a_i\theta^{b_i}) &= \left(\prod_{i=1}^{s-1}(a_i\theta^{b_i})\right)a_s\theta^{b_s} = \left\{\left(\prod_{i=1}^{s-1}\left(a_i^{p^{c_i}}\right)\right)\theta^{b_1+\ldots+b_{s-1}}\right\}a_s\theta^{b_s} \\
&= \left(\prod_{i=1}^{s-1}\left(a_i^{p^{c_i}}\right)\right)\theta^{b_1+\ldots+b_{s-1}}(a_s\theta^{b_s}) = \left(\prod_{i=1}^{s-1}\left(a_i^{p^{c_i}}\right)\right)\left(a_s^{p^{c_s}}\right)\theta^{b_1+\ldots+b_s} \\
&= \left(\prod_{i=1}^{s}\left(a_i^{p^{c_i}}\right)\right)\theta^{b_1+\ldots+b_s},
\end{aligned}$$

just what we wanted to check. □

In this way, combining the computation given by Claim 2.3.3 joint with (2.1) one has that

$$\prod_{i=1}^{s}\phi_i = a_1\left(a_2^{p^{b_1}}\right)\left(a_3^{p^{b_1+b_2}}\right)\cdots\left(a_s^{p^{b_1+\ldots+b_{s-1}}}\right)\theta^{b_1+\ldots+b_s} \in L_e\theta^e.$$

Thus, we have seen, in fact, that

$$\mathcal{F}_{<e} \cap \mathcal{F}_e^{E_R} \subseteq L_e \theta^e. \tag{2.2}$$

Conversely, we point out that for any $1 \leq b_1, \ldots, b_s < e$ such that $b_1 + \ldots + b_s = e$ we deduce from Lemma 2.3.2 that

$$K_{b_1} K_{b_2}^{[p^{b_1}]} \cdots K_{b_s}^{[p^{b_1+\ldots+b_{s-1}}]} = \prod_{i=1}^{s} \left\{ (I^{[p^{b_i}]} :_T I)^{[p^{c_i}]} \right\} \subseteq (I^{[p^{b_1+\ldots+b_s}]} :_T I) = (I^{[p^e]} :_T I)$$

and therefore

$$L_e = \sum_{\substack{1 \leq b_1,\ldots,b_s < e \\ b_1+\ldots+b_s = e}} K_{b_1} K_{b_2}^{[p^{b_1}]} \cdots K_{b_s}^{[p^{b_1+\ldots+b_{s-1}}]} \subseteq (I^{[p^e]} :_T I),$$

hence

$$L_e \theta^e \subseteq (I^{[p^e]} :_T I) \theta^e = \mathcal{F}_e^{E_R}. \tag{2.3}$$

Besides, doing the computation carried out in Claim 2.3.3 from right to left it follows that

$$L_e \theta^e = \left(\sum_{\substack{1 \leq b_1,\ldots,b_s < e \\ b_1+\ldots+b_s = e}} K_{b_1} K_{b_2}^{[p^{b_1}]} \cdots K_{b_s}^{[p^{b_1+\ldots+b_{s-1}}]} \right) \theta^e \subseteq \mathcal{F}_{<e}. \tag{2.4}$$

In this way, combining (2.3) and (2.4) one has that

$$L_e \theta^e \subseteq \mathcal{F}_{<e} \cap \mathcal{F}_e^{E_R}. \tag{2.5}$$

Joining (2.2) and (2.5) one gets the desired equality. □

Remark 2.3.4. Following Blickle's terminology in [18], Katzman's criterion says essentially that $\{(I^{[p^e]} :_T I)\}_{e \in \mathbb{N}}$ forms an *F-graded system of ideals.* This concept may be regarded as a characterisitic p-analog of the well-known notion of *graded system of ideals.*

Next statement (cf. [4, Theorem 3.5]) is the main result of this chapter.

Theorem 2.3.5 (Àlvarez, Boix, Zarzuela)**.** *Let $\mathbb{K}$ be an F-finite field of prime characteristic p, let T be the power series ring $\mathbb{K}[\![x_1, \ldots, x_d]\!]$ and let I be a squarefree monomial ideal of T such that its minimal primary decomposition*

$$I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$$

verifies that $\mathfrak{m} = I_{\alpha_1} + \ldots + I_{\alpha_s}$. Then, the following statements are equivalent.

(a) $\mathcal{C}^R$ *is a principally generated R-algebra. In this case,* $\mathcal{C}^R \cong R[\varepsilon \mathbf{x}^{(p-1)\mathbf{1}}; F]$*, where* F *denotes the standard Frobenius endomorphism on* T*.*

(b) $\mathcal{F}^{E_R}$ *is a principally generated R-algebra. In this case,* $\mathcal{F}^{E_R} \cong R[\mathbf{x}^{(p-1)\mathbf{1}}\Theta; F]$*, where* F *denotes the standard Frobenius map on* E_T*.*

(c) $(I^{[q]} :_T I) = I^{[q]} + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$ *for all* $q = p^e$*.*

If neither of these conditions hold then $\mathcal{C}^R$ *is an infinitely generated R-algebra.*

Proof. First of all, the isomorphism obtained in Theorem 1.5.1 of Chapter 1

$$\operatorname{Hom}_T(F_*^e R, R)^\vee \cong \operatorname{Hom}_T(E_R, F_*^e E_R)$$

maps $\Phi_e^\vee$ into F^e up to a unit; this fact implies that (a) is equivalent to (b). On the other hand, Fedder's Theorem implies that (b) and (c) are equivalent.

In this way, we only need to show that if neither of these conditions hold then $\mathcal{C}^R$ is infinitely generated.

As $(I^{[q]} :_T I) \neq I^{[q]} + \langle \mathbf{x}^{(q-1)\mathbf{1}} \rangle$ we may assume, without loss of generality, that there exists a generator $\mathbf{x}^\gamma \in (I^{[q]} :_T I)$ with $\mathbf{x}^\gamma = x_1^q x_2^{q-1} x_4^{c_4} \cdots x_d^{c_d}$ such that $(c_4, \ldots, c_d) \in \{0, q-1, q\}^{d-4}$. As L_e is a sum of monomial ideals, $\mathbf{x}^\gamma \in L_e$ if and only if $\mathbf{x}^\gamma$ is in one of the summands. Let $e \in \mathbb{N}$ and let $1 \leq b_1, \ldots, b_s \leq e-1$ be such that $b_1 + \ldots + b_s = e$. If

$$\mathbf{x}^\gamma \in K_{b_1} K_{b_2}^{[p^{b_1}]} \cdots K_{b_s}^{[p^{b_1+\ldots+b_{s-1}}]}$$

then

$$\mathbf{x}^\gamma \in G_{b_1} G_{b_2}^{[p^{b_1}]} \cdots G_{b_s}^{[p^{b_1+\ldots+b_{s-1}}]},$$

where $G_e := \langle x_1^{p^e} x_2^{p^e-1} x_4^{c_4} \cdots x_d^{c_d} \rangle$. But it is impossible because the exponent of x_1 in the generator of last product of ideals is

$$p^{b_1+(b_1+b_2)+\ldots+(b_1+\ldots+b_s)} > p^{b_1+\ldots+b_s} = p^e.$$

In this way, as $\mathbf{x}^\gamma f^e \notin L_e F^e$ and $\mathbf{x}^\gamma F^e \in \mathcal{F}^{E_R}$ we deduce from Proposition 2.3.1 that $\mathbf{x}^\gamma F^e \notin \mathcal{F}_{<e}$ and therefore $\mathcal{C}^R$ is infinitely generated. $\square$

We end up this section with the following important observations.

Remark 2.3.6. Let $\mathbb{K}$ be an F-finite field of prime characteristic p, let T be the power series ring $\mathbb{K}[\![x_1, \ldots, x_d]\!]$, let I be a squarefree monomial ideal of T and let $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$ be its minimal primary decomposition. We have to point out that the formula

$$\left(I^{[q]} :_T I\right) = \left(I_{\alpha_1}^{[q]} + \langle \mathbf{x}^{(q-1)\alpha_1} \rangle\right) \cap \ldots \cap \left(I_{\alpha_s}^{[q]} + \langle \mathbf{x}^{(q-1)\alpha_s} \rangle\right)$$

is exactly the same for any q; in this way, we only have to compute $(I^{[p]} :_T I)$ to describe the whole Cartier algebra.

On the other hand, we have to note as well that such formula does not depend on the characteristic of the field. Therefore, in order to describe the whole Cartier algebra we may reduce to the case $p = 2$ and so we only need to compute $(I^{[2]} :_S I)$.

Remark 2.3.7. Whenever $\mathcal{C}^R$ is an infinitely generated algebra we have some control on the number of generators of each graded piece $\mathcal{C}^R_e$. Indeed, set μ as the minimum number of generators of $\mathcal{C}^R_1$. Our previous calculations imply that $\mu = \nu + 1$, ν generators coming from J_2 or J'_2 and the other one being $\mathbf{x^1}$. So, each graded piece $\mathcal{C}^R_e$ adds up ν new generators, those coming from the corresponding J_2 or J'_2. We shall come back to this observation later on (cf. 2.4.3).

2.4 Examples

We have proved in Chapter 1 (cf. Proposition 1.4.19) that Cartier algebras of Gorenstein rings are principally generated. Moreover, the converse holds for normal rings. However, the case of Stanley-Reisner rings we are dealing with is, in general, non-normal. In this way, we shall discuss examples at the boundary of the Gorenstein property.

2.4.1 Examples with pure height

Cohen-Macaulay rings, in particular Gorenstein rings, are unmixed. In this way, we shall start considering ideals such that all the face ideals in its primary decomposition have the same height. Moreover, we shall only consider ideals involving all the variables; that is, $\mathfrak{m} = I_{\alpha_1} + \ldots + I_{\alpha_s}$.

Table 2.1: Table involving pure squarefree monomial ideals

$d = 3$	p.g.	Gor	i.g.
$\mathrm{ht}(I) = 1$	1	1	-
$\mathrm{ht}(I) = 2$	2	1	-
$\mathrm{ht}(I) = 3$	1	1	-

$d = 4$	p.g.	Gor	i.g.
$\mathrm{ht}(I) = 1$	1	1	-
$\mathrm{ht}(I) = 2$	4	2	3
$\mathrm{ht}(I) = 3$	3	1	-
$\mathrm{ht}(I) = 4$	1	1	-

$d = 5$	p.g.	Gor	i.g.
$\mathrm{ht}(I) = 1$	1	1	-
$\mathrm{ht}(I) = 2$	6	2	13
$\mathrm{ht}(I) = 3$	12	2	10
$\mathrm{ht}(I) = 4$	4	1	-
$\mathrm{ht}(I) = 5$	1	1	-

Building from the computational package described in Appendix A we have constructed Table 2.1 in order to explore the principal generation of all the squarefree monomial ideals such that all its primary components have the same height. In these tables, d is the number of variables of our current formal power series ring. Depending on the height we count the number of ideals (up to relabeling) having principally generated (denoted p.g. in the tables for the sake of brevity) Cartier algebra, how many Gorenstein (Gor) rings we get among them and the number of ideals having infinitely generated (i.g.) Cartier algebra.

In the following, we are to analize the information obtained in these tables.

Distinguished examples in three variables

Before taking a close look of the previous tables, we remind the following concept for the convenience of the reader. We omit its proof and refer to [98, Proposition 2.4] for details.

Theorem/Definition 2.4.1. Let $(A, \mathfrak{m}, \mathbb{K})$ be a local equidimensional unmixed ring which admits a canonical module Ω. Then, the following statements are equivalent.

(i) Ω is abstractly isomorphic to an ideal ω of A.

(ii) A is generically Gorenstein.

Moreover, when these equivalent conditions hold, ω contains at least a non-zerodivisor of A.

Suppose that the previous equivalent conditions hold. We say that A is $\mathbb{Q}$-*Gorenstein* provided $\omega^{(r)}$ is a principal ideal of A for some integer $r \geq 1$, where $(-)^{(r)}$ stands for the rth symbolic power of an ideal. In this case, we refer to the digit

$$\min\{r \in \mathbb{N} \mid \quad \omega^{(r)} \quad \text{is principal}\}$$

as the *index* of A.

It is clear that quasi Gorenstein rings are, in particular, $\mathbb{Q}$-Gorenstein. The converse does not hold, as the following example illustrates; the unjustified calculations were carried out with CoCoA.

Example 2.4.2. Consider the monomial algebra $\mathbb{K}[\![v^3, u^2v, uv^2, u^3]\!]$, where $\mathbb{K}$ is any field of characteristic zero. Moreover, consider the map $\mathbb{Z}^4 \xrightarrow{\psi} \mathbb{Z}^2$ given by matrix $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 3 & 1 & 2 & 3 \end{pmatrix}$; one may check that $\ker(\psi) = \langle e_1 - e_2 - e_3 + e_4, e_1 + e_2 - 2e_3 \rangle_{\mathbb{Z}}$. In this way, we consider the ideal $J := \langle xy - z^2, xw - yz \rangle$. Now, using [103, Lemma 7.6] we produce a lattice ideal as follows:

$$I := (J : \langle xyzw \rangle^{\infty}) = \langle xy - z^2, xw - yz, y^2 - zw, z^3 - x^2w \rangle.$$

So, the algebra $\mathbb{K}[x, y, z, w]/I$ may be viewed as the coordinate ring of an affine toric variety (namely, V). Moreover, we have to point out that $\mathbb{K}[V]$ is not Gorenstein because of $\operatorname{type}(\mathbb{K}[V]) = 2$. However, we claim that $\mathbb{K}[V]$ is $\mathbb{Q}$-Gorenstein.

Firstly, we have to underline that

$$R := \mathbb{K}[\![v^3, u^2v, uv^2, u^3]\!] = \mathbb{K}[\![u, v]\!]^{(3)}$$

is the 3-Veronesean subring of $\mathbb{K}[\![u,v]\!]$ (cf. [30, Section 13.5] and [103, Example 10.6] for more information concerning Veronesean subrings). In this way, if we denote by ω_R the canonical module of R then one may check (using [30, part (v) of 13.5.9 and 14.5.10] for the first isomorphism) that

$$\omega_R \cong \left(\omega_{\mathbb{K}[\![u,v]\!]}\right)^{(3)} = \langle u^3, u^2v\rangle_R.$$

Furthermore, if $\mathfrak{p} := \langle u^3, u^2v, uv^2\rangle_R$, then

$$\mathfrak{p}^{(2)} = \langle u^3, u^2v\rangle_R = \omega_R, \quad \mathfrak{p}^{(3)} = \langle u^3\rangle_R,$$

and therefore

$$\omega_R^{(2)} = \langle u^3\rangle_R \cdot \mathfrak{p} = \mathfrak{p}^{(4)}, \quad \omega_R^{(3)} = \langle u^6\rangle_R = \mathfrak{p}^{(6)}.$$

Whence we have checked that R is a $\mathbb{Q}$-Gorenstein ring of index 3.

Actually, the reader should notice that, in this example, $R = \mathbb{K}[\![u,v]\!]^G$, where G is a group such that $|G| \neq 0$ in $\mathbb{K}$. In such case, the *divisor class group* $\mathrm{Cl}(R)$ (cf. [50] for unexplained terminology) attached to R is isomorphic to $\mathrm{Hom}(G, \mathbb{K}^\times)$, where $\mathbb{K}^\times$ denotes the underlying multiplicative group of $\mathbb{K}$. Therefore, summing up one has that

$$\mathrm{Cl}(R) \cong \mathrm{Hom}(G, \mathbb{K}^\times) \cong \mathbb{Z}/(3).$$

Now, we analize an example of a non-Gorenstein Stanley-Reisner ring with principally generated Cartier algebra.

Example 2.4.3. In three variables we have an example of a Cohen-Macaulay non-Gorenstein ring $R := \mathbb{K}[\![x,y,z]\!]/I$ having principally generated Cartier algebra, where

$$I := \langle x,y\rangle \cap \langle x,z\rangle \cap \langle y,z\rangle = \langle xy, xz, yz\rangle.$$

Indeed, it is not Gorenstein because $\mathrm{type}(R) = 2$. This example is as well interesting because R is not even $\mathbb{Q}$-Gorenstein.

In order to check our last claim, we note that the canonical module ω_R is abstractly isomorphic to the height one ideal $\langle x+y, y+z\rangle/I$ of R. As $\dim(R) = 1$ it follows that $\omega_R^{(n)} = \omega_R^n$ for all $n \geq 1$. We claim that

$$\omega_R^n = \langle x^n, y^n, z^n\rangle/I$$

for all $n \geq 2$ and therefore ω_R can not admit a symbolic power $\omega_R^{(t)}$ such that $\omega_R^{(t)}$ is principal.

We proceed by increasing induction on n. When $n = 2$,

$$\omega_R^2 = \frac{\langle x+y, y+z\rangle}{I} \cdot \frac{\langle x+y, y+z\rangle}{I} = \frac{\langle x^2+y^2, xy+yz+y^2+yz, y^2+z^2\rangle}{I} = \frac{\langle x^2, y^2, z^2\rangle}{I}.$$

When $n > 2$, it follows that

$$\omega_R^n = \omega_R^{n-1} \cdot \omega_R = \frac{\langle x^{n-1}, y^{n-1}, z^{n-1} \rangle}{I} \cdot \frac{\langle x+y, y+z \rangle}{I} = \frac{\langle x^n, y^n, z^n \rangle}{I},$$

just what we finally wanted to show.

We end this subsubsection with the following:

Question 2.4.4. Is it true that, for a Stanley-Reisner ring, being quasi Gorenstein is equivalent of being $\mathbb{Q}$-Gorenstein?

Distinguished examples in four variables

In four variables we have three examples with infinitely generated Cartier algebra; namely,

$$\begin{aligned} &\langle x, y \rangle \cap \langle z, w \rangle = \langle xz, xw, yz, yw \rangle, \\ &\langle x, y \rangle \cap \langle x, w \rangle \cap \langle y, z \rangle = \langle xy, xz, yw \rangle, \\ &\langle x, y \rangle \cap \langle x, w \rangle \cap \langle y, w \rangle \cap \langle z, w \rangle = \langle xyz, xw, yw \rangle. \end{aligned}$$

The first example has disjoint variables and the corresponding colon ideal is

$$\langle x^2z^2, x^2w^2, y^2z^2, y^2w^2, \underline{xyz^2}, \underline{x^2zw}, \underline{y^2zw}, \underline{xyw^2}, xyzw \rangle.$$

In this case, the corresponding Cartier algebra is

$$R[\varepsilon xyzw, z\varepsilon xy, x\varepsilon zw, y\varepsilon zw, w\varepsilon xy, z\varepsilon^2(xy)^3, x\varepsilon^2(zw)^3, y\varepsilon^2(zw)^3, w\varepsilon^2(xy)^3, \ldots; F].$$

The corresponding colon ideal of the second example is

$$\langle x^2y^2, x^2z^2, y^2w^2, \underline{x^2yz}, \underline{xy^2w}, xyzw \rangle.$$

In this case, the corresponding Cartier algebra is

$$R[\varepsilon xyzw, x\varepsilon yz, y\varepsilon xw, x\varepsilon^2(yz)^3, y\varepsilon^2(xw)^3, x\varepsilon^3(yz)^7, y\varepsilon^3(xw)^7, \ldots; F].$$

Finally, the corresponding colon ideal of the third example is

$$\langle x^2y^2z^2, x^2w^2, y^2w^2, \underline{xyw^2}, xyzw \rangle.$$

In this case, the corresponding Cartier algebra is

$$R[\varepsilon xyzw, w\varepsilon xy, w\varepsilon^2(xy)^3, w\varepsilon^3(xy)^7, w\varepsilon^4(xy)^{15}, \ldots; F].$$

Some families of examples having principal Cartier algebra

In this subsubsection, we introduce two general families of pure height ideals with principally generated Cartier algebra.

Examples with pure height $d-1$

The first family of examples are pure squarefree monomial ideals of dimension 1.

Proposition 2.4.5. *Let $I \subseteq T$ be a squarefree monomial ideal of pure height $d-1$. Then $\mathcal{C}^R$ is principally generated.*

Proof. We may suppose, without loss of generality, that there is $k \in \{1, \ldots, d\}$ such that

$$I = \bigcap_{i=1}^{k} \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d \rangle = \langle x_i x_j \mid \quad 1 \leq i < j \leq k \rangle + \langle x_{k+1}, \ldots, x_d \rangle.$$

So, we proceed by increasing induction on k, being the case $k=1$ trivial. For $k=2$, we deduce from Proposition 2.2.3 that

$$(I^{[2]} :_T I) = \langle x_2^2, \ldots, x_d^2, \mathbf{x^1} \rangle \cap \langle x_1^2, x_3^2, \ldots, x_d^2, x_1 x_3 \cdots x_d \rangle = \langle x_1^2 x_2^2, x_3^2, \ldots, x_d^2, \mathbf{x^1} \rangle = I^{[2]} + \langle \mathbf{x^1} \rangle.$$

When $k \geq 3$, combining the induction hypothesis joint with Proposition 2.2.3 and the modular law for monomial ideals (cf. [87, Part (2.3) of Proposition 1]) it follows that

$$\begin{aligned}
&(I^{[2]} :_T I) = \bigcap_{i=1}^{k-1} \langle x_1^2, \ldots, x_{i-1}^2, x_{i+1}^2, \ldots, x_d^2, x_1 \cdots x_{i-1} x_{i+1} \cdots x_d \rangle \\
&\cap \langle x_1^2, \ldots, x_{k-1}^2, x_{k+1}^2, \ldots, x_d^2, x_1 \cdots x_{k-1} x_{k+1} \cdots x_d \rangle \\
&= \big(\langle x_i^2 x_j^2 \mid \quad 1 \leq i < j \leq k-1 \rangle + \langle x_k^2, \ldots, x_d^2, \mathbf{x^1} \rangle \big) \\
&\cap \langle x_1^2, \ldots, x_{k-1}^2, x_{k+1}^2, \ldots, x_d^2, x_1 \cdots x_{k-1} x_{k+1} \cdots x_d \rangle \\
&= \langle x_i^2 x_j^2 \mid \quad 1 \leq i < j \leq k \rangle + \langle x_{k+1}^2, \ldots, x_d^2 \rangle + \langle \mathbf{x^1} \rangle = I^{[2]} + \langle \mathbf{x^1} \rangle,
\end{aligned}$$

just what we wanted to check. □

The following result provides a generalization of Example 2.4.3.

Proposition 2.4.6 (Goto)**.** *Let $d \geq 2$, and set $R := \mathbb{K}[\![x_1, \ldots, x_d]\!]/I$, where*

$$I := \langle x_i x_j \mid \quad 1 \leq i < j \leq d \rangle.$$

Then, the following statements hold.

(i) $\dim(R) = 1$.

(ii) R is Cohen-Macaulay.

(iii) The canonical module ω_R is abstractly isomorphic to the height-one ideal

$$\langle x_1 + x_2, \ldots, x_1 + x_d \rangle / I.$$

In particular, $\operatorname{type}(R) = d-1$ *and therefore R is Gorenstein if and only if $d = 2$.*

From now on, we assume that $d \geq 3$.

(iv) For all $t \geq 2$,

$$\omega_R^{(t)} = \omega_R^t = \frac{\langle x_1^t, \ldots, x_d^t \rangle}{I}.$$

In particular, ω_R can not admit a principal symbolic power; hence R is not $\mathbb{Q}$-Gorenstein.

(v) $\mathcal{C}^R$ is principally generated.

Sketch of proof. Part (i) follows from the fact that

$$I = \bigcap_{i=1}^{d} \langle x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d \rangle.$$

Part (ii) follows combining part (i) joint with the fact that $x_1 + x_2$ is not a zero-divisor on R. Part (v) has been proved in Proposition 2.4.5. Part (iv) is proved performing a similar calculation which has been done in Example 2.4.3. Finally, part (iii) was proved by S. Goto in [53, Example 2.8]. □

Squarefree Veronese ideals

The second family of examples having principal Cartier algebra are the so-called *squarefree Veronese ideals* (cf. [65, Exercise 8.7]).

Proposition 2.4.7. *Let $k \in \{1, \ldots, d\}$ and let $I_{k,d} \subseteq T$ be the squarefree monomial ideal obtained as intersection of all the face ideals of height k; namely,*

$$I_{k,d} := \bigcap_{1 \leq i_1 < \ldots < i_k \leq d} \langle x_{i_1}, \ldots, x_{i_k} \rangle.$$

Then $\mathcal{C}^{T/I_{k,d}}$ is principally generated.

Proof. This proof is again based on the use of Proposition 2.2.3 joint with the modular law for monomial ideals. First of all, we underline that we have a decomposition of the form $I_{k,d} = I_{k,d-1} \cap J_{k,d}$, where

$$J_{k,d} := \bigcap_{1 \leq i_1 < \ldots < i_{k-1} \leq d-1} \langle x_{i_1}, \ldots, x_{i_{k-1}}, x_d \rangle = \langle x_d \rangle + I_{k-1,d-1}.$$

In particular, $\mathcal{C}^{T/J_{k,d}}$ is principally generated if and only if $\mathcal{C}^{T/I_{k-1,d-1}}$ is principally generated. In this way, the result now follows by increasing induction on d. Indeed, the case $d = 1$ is obvious. On the other hand, when $d > 1$ we may write

$$(I_{k,d}^{[2]} :_T I_{k,d}) = (I_{k,d-1}^{[2]} :_T I_{k,d-1}) \cap (J_{k,d}^{[2]} :_T J_{k,d}).$$

As $\mathcal{C}^{T/I_{k,d-1}}$ and $\mathcal{C}^{T/J_{k,d}}$ are both principally generated by induction hypothesis it follows that

$$(I_{k,d}^{[2]} :_T I_{k,d}) = \left(I_{k,d-1}^{[2]} + \langle x_1 \cdots x_{d-1}\rangle\right) \cap \left(J_{k,d}^{[2]} + \langle \mathbf{x}^{\mathbf{1}}\rangle\right) = I_{k,d}^{[2]} + \left(\langle x_1 \cdots x_{d-1}\rangle \cap J_{k,d}^{[2]}\right) + \langle \mathbf{x}^{\mathbf{1}}\rangle.$$

Nevertheless, since $\langle x_1 \cdots x_{d-1}\rangle \cap J_{k,d}^{[2]} \subseteq I_{k,d}^{[2]} + \langle \mathbf{x}^{\mathbf{1}}\rangle$ one finally gets $(I_{k,d}^{[2]} :_T I_{k,d}) = I_{k,d}^{[2]} + \langle \mathbf{x}^{\mathbf{1}}\rangle$. Hence $\mathcal{C}^{T/I_{k,d}}$ is principal. □

2.4.2 Examples with no pure height

We have seen that the Gorensteinness of the Stanley-Reisner ring R is not the property that tackles when $\mathcal{C}^R$ is principally generated. It turns out that one may even find examples of non Cohen-Macaulay rings with $\mathcal{C}^R$ principally generated. In 4 and 5 variables we have the following examples.

$$\begin{aligned}
I &= \langle x, w\rangle \cap \langle x, z\rangle \cap \langle x, y\rangle \cap \langle y, z, w\rangle,\\
I &= \langle x, u\rangle \cap \langle x, w\rangle \cap \langle x, z\rangle \cap \langle x, y\rangle \cap \langle y, z, w, u\rangle,\\
I &= \langle x, z\rangle \cap \langle x, w\rangle \cap \langle x, u\rangle \cap \langle y, z\rangle \cap \langle y, w\rangle \cap \langle y, u\rangle \cap \langle z, w, u\rangle,\\
I &= \langle x, y, u\rangle \cap \langle x, y, w\rangle \cap \langle x, y, z\rangle \cap \langle y, z, w, u\rangle,\\
I &= \langle x, y, w\rangle \cap \langle x, y, u\rangle \cap \langle x, z, w\rangle \cap \langle x, z, u\rangle \cap \langle y, z, w, u\rangle.
\end{aligned}$$

In general, we may find examples in any dimension. Indeed, we can take the following families of ideals in d variables as such as a point.

$$\begin{aligned}
I &= \langle x_1, \ldots, x_r, x_{r+1} \cdots x_d\rangle \cap \langle x_{r+1}, \ldots, x_d\rangle,\\
I &= \langle x_1, \ldots, x_r, x_{r+1} \cdots x_{r_1}, x_{r_1+1} \cdots x_{r_2}, \ldots, x_{r_t+1} \cdots x_d\rangle \cap \langle x_{r+1}, \ldots, x_d\rangle,
\end{aligned}$$

where $1 \leq r < r_1 < \ldots < r_t \leq d$.

Examples with no pure height: some families

In this subsection, we shall exhibit a family of squarefree monomial ideals with the property that almost all its members have infinitely generated Cartier algebra.

We start introducing the concept of ideal with disjoint variables.

Definition 2.4.8. Let $d \in \mathbb{N}$, let $\mathbb{K}$ be any field, let $S := \mathbb{K}[x_1, \ldots, x_d]$ and let I be a squarefree monomial ideal of S. We say that I has *disjoint variables* if there is $t \in \mathbb{N}$ and integers $d_1 \geq d_2 \geq \ldots \geq d_t > 0$ with $d_1 + \ldots + d_t = d$ such that

$$I = \bigcap_{j=1}^{t} \left\langle x_{b_{j-1}+1}, \ldots, x_{b_j} \right\rangle,$$

where, for each $i \in \{1, \dots, t\}$,

$$b_0 := 0 \quad \text{and} \quad b_i := \sum_{j=1}^{i} d_j.$$

As we have announced, the main result of this subsection is the following:

Proposition 2.4.9. *Let $d \in \mathbb{N}$, let $\mathbb{K}$ be any F-finite field, let $S := \mathbb{K}[x_1, \dots, x_d]$, let J be a squarefree monomial ideal with disjoint variables, set $T := \mathbb{K}[\![x_1, \dots, x_d]\!]$ and set $I := JT$. Then, the following statements are equivalent.*

(a) $\mathcal{C}^R$ is principally generated.

(b) $J = \langle x_1, \dots, x_d \rangle$ or $J = \langle x_1 \cdots x_d \rangle$.

Proof. It is clear that (b)$\Longrightarrow$ (a) because, in any case, R is Gorenstein. In this way, we have only to check that the converse implication holds.

Let $(d_1, \dots, d_t)$ be as in Definition 2.4.8 and suppose that $J \neq \langle x_1, \dots, x_d \rangle, \langle x_1 \cdots x_d \rangle$. Moreover, as we know that non-pure ideals of height one have infinitely generated Cartier algebra we are also allowed to assume that $d_t > 1$. In this case, the monomial $x_1^2 (x_{b_1+1} \cdots x_d)$ is a minimal monomial generator of $(J^{[2]} :_S J)$ which does not belong to $J^{[2]} + \langle \mathbf{x}^{\mathbf{1}} \rangle$ because of $d_t > 1$. □

Finally, we end this subsection proving the following result which also involves ideals with disjoint variables.

Proposition 2.4.10. *Let I be a squarefree monomial ideal with disjoint variables of S such that its minimal primary decomposition*

$$I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$$

verifies that $\operatorname{ht}(I_{\alpha_i}) > 1$ for all $i \in \{1, \dots, s\}$. Then, the following statements hold.

(a) One has that

$$\mu\left(\left(I^{[2]} :_S I\right)\right) = \prod_{j=1}^{s} (\operatorname{ht}(I_{\alpha_j}) + 1).$$

(b) It follows that

$$\mu(J_2) = \left(\prod_{j=1}^{s} (\operatorname{ht}(I_{\alpha_j}) + 1) \right) - \left(\prod_{j=1}^{s} \operatorname{ht}(I_{\alpha_j}) \right) - 1.$$

Proof. Let $(b_0, \dots, b_t)$ be as in Definition 2.4.8. As I has disjoint variables,

$$\left(I^{[2]} :_S I\right) = \bigcap_{j=1}^{t} \left\langle x_{b_{j-1}+1}^2, \dots, x_{b_j}^2, x_{b_{j-1}+1} \cdots x_{b_j} \right\rangle = \prod_{j=1}^{t} \left\langle x_{b_{j-1}+1}^2, \dots, x_{b_j}^2, x_{b_{j-1}+1} \cdots x_{b_j} \right\rangle.$$

In this way, as the exponent set of the product of monomial ideals equals the Minkowski sum of the exponent set of each factor one obtains part (a). Part (b) follows from part (a) directly. □

2.4.3 A numerical function attached to Cartier algebras

The aim of this subsection is to introduce a generating function which is directly related with the number of generators of each graded piece $\mathcal{C}_e^R$. We hope that such generating function (and the corresponding generating serie) may be interesting in its own right.

Throughout this section, $\mathbb{K}$ will denote an F-finite field of characteristic p and T will be $\mathbb{K}[\![x_1, \dots, x_d]\!]$.

Definition 2.4.11. Let $\mathcal{L}_T$ be the lattice of ideals of T. Set, for each $e \in \mathbb{N}$,

$$\begin{aligned} c_{e,p} : \quad & \mathcal{L}_T \longrightarrow \mathbb{N} \\ & I \longmapsto \dim_{\mathbb{K}} \left(\frac{\operatorname{Hom}_T(F_*^e(T/I), T/I)}{\mathfrak{m} \operatorname{Hom}_T(F_*^e(T/I), T/I)} \right). \end{aligned}$$

We shall refer to $c_{e,p}$ as the (e,p)-*Cartier function* of T.

Now, we collect the previously defined digits in a power serie as follows.

Definition 2.4.12. Let I be an ideal of T. The *Cartier-Frobenius generating function with respect to* I is defined in the following way:

$$\operatorname{CFG}(I; X) := \sum_{e \geq 0} c_{e,p}(I) X^e \in \mathbb{N}[\![X]\!].$$

Next result summarizes some elementary properties of the previous formal power series. All of such properties follow immediately from its definition.

Proposition 2.4.13. *Let I be an ideal of T. Then, the following statements hold.*

(i) $c_{0,p}(I) = 1$.

(ii) The following statements are equivalent.

(a) $\mathcal{C}^{T/I}$ *is principally generated.*

(b) $c_{e,p}(I) = 1$ *for all* $e \in \mathbb{N}$.

(c) *One has that*

$$\mathrm{CFG}(I;X) = \frac{1}{1-X}.$$

In case of I is a squarefree monomial ideal we can provide a more precise description of such generating function.

Theorem 2.4.14. *Let I be a squarefree monomial ideal of T and let*

$$I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$$

be its minimal primary decomposition in terms of face ideals. Then, the following statements hold.

(i) *For all $e \geq 1$, $c_{e,p}(I) = c_{e,2}(I)$.*

(ii) *One has that*

$$c_{e,2}(I) = \begin{cases} \mu(J_2)+1, & \text{if} \quad e=1, \\ \mu(J_2), & \text{if} \quad e>1. \end{cases}$$

(iii) *One has that*

$$\mathrm{CFG}(I;X) = \frac{1+\mu(J_2)X - X^2}{1-X}.$$

(iv) *If, in addition, I has disjoint variables and $\mathrm{ht}(I_{\alpha_i}) > 1$ for any $i \in \{1,\ldots,s\}$, then*

$$c_{1,2}(I) = \left(\prod_{j=1}^{s}(\mathrm{ht}(I_{\alpha_j})+1)\right) - \left(\prod_{j=1}^{s}\mathrm{ht}(I_{\alpha_j})\right).$$

Bearing in mind the previous result, one might ask whether $c_{e,p}(I) = c_{1,p}(I)$ for all $e \in \mathbb{N}$. The answer is in general negative, as the following result illustrates.

Proposition 2.4.15. *Let $T := \mathbb{F}_p[\![x_1,x_2,x_3,x_4,x_5,x_6]\!]$ and let I be the ideal generated by the 2×2 minors of matrix*

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

Then, the following statements hold.

(i) $c_{e,p}(I) = \binom{p^e+1}{2}$.

(ii) *One has that*

$$\mathrm{CFG}(I;X) = \frac{2-p(p+1)X}{2(1-pX)(1-p^2X)}.$$

Proof. First of all, we deduce part (ii) from part (i) in the following manner:

$$\begin{aligned}\operatorname{CFG}(I;X) &= \sum_{e\geq 0}\binom{p^e+1}{2}X^e = \sum_{e\geq 0}\frac{(p^e+1)p^e}{2}X^e = \frac{1}{2}\sum_{e\geq 0}(1+p^e)(pX)^e\\ &= \frac{1}{2}\sum_{e\geq 0}(pX)^e + \frac{1}{2}\sum_{e\geq 0}(p^2X)^e = \frac{1}{2}\left(\frac{1}{1-pX}+\frac{1}{1-p^2X}\right)\\ &= \frac{2-p(p+1)X}{2(1-pX)(1-p^2X)};\end{aligned}$$

whence part (ii) holds. Therefore, it only remains to check part (i). The reader should remind that

$$I = \langle \Delta_1, \Delta_2, \Delta_3\rangle = \langle x_1x_5 - x_2x_4, x_1x_6 - x_3x_4, x_2x_6 - x_3x_5\rangle.$$

Now, using [84, part (2) of Proposition 5.1] it follows that

$$(I^{[p^e]} :_T I) = I^{[p^e]} + \langle f_{a,b} \mid \ a+b \leq p^e - 1\rangle,$$

where each $f_{a,b}$ is such that

$$x_5^a x_6^b(\Delta_2\Delta_3)^{p^e-1} \equiv x_4^{a+b} f_{a,b} \pmod{I^{[p^e]}}.$$

From this fact, we deduce that $c_{e,p}(I)$ turns out to be the cardinality of the set of the $f_{a,b}$'s, which is exactly

$$\sum_{i=0}^{p^e-1}(i+1) = \sum_{i=1}^{p^e} i = \frac{p^e(p^e+1)}{2} = \binom{p^e+1}{2},$$

just what we finally wanted to prove. □

These previous computations lead us to raise the following:

Question 2.4.16. Is it true that $\operatorname{CFG}(I;X)$ is always a rational function?

We finish this part with the following:

Remark 2.4.17. It is worth mentioning that Question 2.4.16 is closely related with the notion of the so-called *Frobenius complexity*, which has been introduced by F. Enescu and Y. Yao (cf. [47, Definition 2.13]).

Counting new generators of infinitely generated Cartier algebras

Now, we introduce another numerical function.

Definition 2.4.18. Let $\mathcal{L}_{\Delta,T,\infty}$ be the lattice of squarefree monomial ideals of T such that $\mathcal{C}^{T/I}$ is infinitely generated for any $I \in \mathcal{L}_{\Delta,T,\infty}$. In this way, set

$$M_\Delta(e,p) := \max_{I\in\mathcal{L}_{\Delta,T,\infty}} \{c_{e,p}(I)-1\}.$$

Roughly speaking, once it is fixed the number of formal indeterminates in our formal power series ring T, M_Δ may be regarded as the function which counts the maximum number of generators which are added in an infinitely generated Cartier algebra attached to a complete Stanley-Reisner ring.

Remark 2.4.19. We have to point out that, in fact, M_Δ is a constant function. Indeed, we have seen that if I is a squarefree monomial ideal then $c_{e,p}(I) = c_{1,2}(I)$ for all $e \in \mathbb{N}$. In this way, it follows that

$$M_\Delta = \max_{I\in\mathcal{L}_{\Delta,T,\infty}} \{c_{1,2}(I)-1\}.$$

In this way, one might ask the following question regarding M_Δ.

Question 2.4.20. Is it true that there exists a unique (up to relabeling) squarefree monomial ideal I such that $M_\Delta = c_{1,2}(I) - 1$?

Example 2.4.21. When $d = 3$, $M_\Delta = 2 = c_{1,2}(\langle x_1x_2, x_1x_3\rangle) - 1$. In this case, M_Δ is reached in a squarefree monomial ideal with disjoint variables.

In fact, this example suggests the following:

Question 2.4.22. Is it true that $M_\Delta = c_{1,2}(I) - 1$, where I is a squarefree monomial ideal with disjoint variables?

In Appendix A, we provide some numerical evidence (cf. Table A.1) that Question 2.4.22 might be true.

2.4.4 Behaviour of Cartier algebras under Alexander duality

In this section, we briefly explore the behaviour of the generation of $\mathcal{C}^R$ with respect to the so-called *Alexander duality*.

We start recalling a well-known notion.

Definition 2.4.23. Let I be a squarefree monomial ideal of S and let $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$ be its minimal primary decomposition in terms of face ideals such that $\mathfrak{m} = I_{\alpha_1} + \ldots + I_{\alpha_s}$. Set

$$I^{\mathbf{1}} := \langle \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_s}\rangle$$

and we refer to the ideal $I^{\mathbf{1}}$ to be the the *Alexander dual* of I.

Remark 2.4.24. In general, having principally generated Cartier algebra does not behave well with respect to Alexander duality. Indeed, take for instance $I = \langle xy, xz\rangle$ and $I^{\mathbf{1}} = \langle x, yz\rangle$. While $\mathcal{C}^{T/I}$ is infinitely generated, $\mathcal{C}^{T/I^{\mathbf{1}}}$ is principally generated.

On the other hand, it is straightforward to check that $I^{\mathbf{1}}_{k,d} = I_{d-k+1,d}$. This fact leads us to raise the following:

Question 2.4.25. Let I be a squarefree monomial ideal such that $\mathcal{C}^{T/I}$ and $\mathcal{C}^{T/I^{\mathbf{1}}}$ are both principally generated. Is it true that $I = I_{k,d}$ for some $k \in \{1, \dots, d\}$?

We have checked using CoCoA (cf. [38]) that this question has an affirmative answer for $d \leq 6$.

We conclude this part with the following:

Remark 2.4.26. It is worth mentioning that, in [7, Theorem 4], the authors obtained an elementary combinatorial characterization of complete Stanley-Reisner rings having principally generated Cartier algebra; in particular, they show that being principally generated is not a topological property of the simplicial complex attached to any Stanley-Reisner ring.

2.5 Applications

In this section we shall use our description of the Cartier algebra of a complete Stanley-Reisner rings in the ways described as follows.

2.5.1 Discreteness of F-jumping numbers

In Chapter 1, we have recalled the notion of *gauge boundedness* of a Cartier algebra introduced by M. Blickle in [18, Definition 4.8] (cf. Definition 1.7.14). This notion is mainly introduced because Blickle proved that the set of F-jumping numbers attached to the pair $(\operatorname{Spec}(R), \mathbf{V}(\mathfrak{a}))$ forms a discrete subset inside the non-negative real numbers provided $\mathcal{C}^{\mathfrak{a}}$ is gauge bounded (cf. Proposition 1.7.15). He also proved that finitely generated Cartier algebras are gauge bounded.

So far in this chapter, we have seen that Cartier algebras of complete Stanley-Reisner rings can be either principally generated or infinitely generated. It is clear that principally generated Cartier algebras are gauge bounded.

The main result of this section is the following:

Theorem 2.5.1 (Àlvarez, Boix, Zarzuela)**.** *Let $\mathbb{K}$ be a perfect field of characteristic p, let T be $\mathbb{K}[\![x_1, \dots, x_d]\!]$, let I be a squarefree monomial ideal of T and set $R := T/I$. Then, the R-Cartier algebra $\mathcal{C}^R$ is gauge bounded.*

Proof. Let $s \in S$ and write

$$s = \sum_{0 \leq ||\varepsilon||_\infty \leq ||s||_\infty} s_\varepsilon \mathbf{x}^\varepsilon, \quad s_\varepsilon \in \mathbb{K}, \quad \varepsilon \in \mathbb{N}^d.$$

On the other hand, let $\mathbf{x}^\gamma := x_1^{c_1} \cdots x_d^{c_d}$ be a minimal monomial generator of $J_{p^e} + \langle \mathbf{x}^{(p^e-1)\mathbf{1}} \rangle$ and write $\gamma = p^e\alpha + (p^e-1)\beta$, where $\alpha := (a_1, \dots, a_d)$ and

$$a_i := \begin{cases} 1, & \text{if } c_i = p^e, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $\beta = (b_1, \dots, b_d)$, where

$$b_i := \begin{cases} 1, & \text{if } c_i = p^e - 1, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\Phi_{e,\gamma}$ the composition $\Phi_e \circ {}_{\mathbf{x}^\gamma}\mu$, where ${}_{\mathbf{x}^\gamma}\mu$ denotes left multiplication by $\mathbf{x}^\gamma$ and Φ_e denotes the unique p^{-e}-linear map which is the projection onto the direct summand $S\mathbf{x}^{(p^e-1)\mathbf{1}}$. Our foregoing results imply that $\{\Phi_{e,\gamma}\}_{e,\gamma}$ (where e runs over $\mathbb{N}$ and γ runs over the exponent set of $J_{p^e} + \langle \mathbf{x}^{(p^e-1)\mathbf{1}} \rangle$) generates $\mathcal{C}^R$ as right T-module. In this way, we underline that

$$\Phi_{e,\gamma}(s) = \Phi_e(\mathbf{x}^\gamma s) = \sum_{0 \leq ||\varepsilon||_\infty \leq ||s||_\infty} s_\varepsilon \mathbf{x}^{\frac{\varepsilon + p^e\alpha + (p^e-1)(\beta - \mathbf{1})}{p^e\mathbf{1}}}$$

So, we need to distinguish two cases. If $\alpha = \mathbf{0}$ then $\beta = \mathbf{1}$ and therefore

$$||\Phi_{e,\gamma}(s)||_\infty \quad \leq \frac{||s||_\infty}{p^e}.$$

If $\alpha \neq \mathbf{0}$ then $\beta - \mathbf{1}$ has, at least, a strictly negative entry, hence

$$||\Phi_{e,\gamma}(s)||_\infty \quad \leq \frac{||s||_\infty}{p^e} + 1.$$

In any case, one obtains that

$$||\Phi_{e,\gamma}(s)||_\infty \quad \leq \frac{||s||_\infty}{p^e} + 1$$

and therefore $\mathcal{C}^R$ is gauge bounded. □

We also want to provide here a different way of proving Theorem 2.5.1. This alternative proof involves the following statement, which was obtained by M. Katzman and W. Zhang; the reader is encouraged to consult [86, Lemma 2.2] for further details.

Proposition 2.5.2 (Katzman, Zhang). *Let $\mathbb{K}$ be any field of prime characteristic p, let S be the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$, and let I be an ideal of S such that $(I^{[p^e]} :_S I)$ is generated by elements of infinity norm at most Cp^e for any $e \geq 1$ (and some constant C which does not depend neither on e, nor on the number of generators of $(I^{[p^e]} :_S I)$). Then, the Cartier algebra $\mathcal{C}^{S/I}$ is gauge bounded.*

Now, using Proposition 2.5.2 we give an alternative proof of Theorem 2.5.1.

Alternative proof of Theorem 2.5.1. Suppose that I is a Stanley-Reisner ideal and let $e \geq 1$. Keeping in mind Discussion 2.2.5, one has that

$$(I^{[p^e]} :_S I) = I^{[p^e]} + J_{p^e} + \langle (x_1 \cdots x_d)^{p^e-1} \rangle,$$

where $||(x_1 \cdots x_d)^{p^e-1}||_\infty = p^e - 1$, $I^{[p^e]}$ is generated by monomials of infinity norm equal to p^e, and J_{p^e} is also generated by monomials of infinity norm less or equal than p^e; summing up, $(I^{[p^e]} :_S I)$ is generated by elements of infinity norm less or equal that p^e for any $e \geq 1$. In this way, Proposition 2.5.2 implies that $\mathcal{C}^{T/I}$ is gauge bounded; the proof is therefore completed. □

As a direct consequence of Theorem 2.5.1 we obtain the following:

Theorem 2.5.3 (Àlvarez, Boix, Zarzuela)**.** *Let $\mathbb{K}$ be a perfect field of characteristic p, let T be $\mathbb{K}[\![x_1, \ldots, x_d]\!]$, let I be a squarefree monomial ideal of T and set $R := T/I$. Let $\mathfrak{a}$ be any ideal of R. Then, the F-jumping numbers of the pair* $(\operatorname{Spec}(R), \mathbf{V}(\mathfrak{a}))$ *are a discrete set inside the non-negative real numbers.*

We end this section with the following observations.

Remark 2.5.4. Theorem 2.5.1 was observed by M. Blickle in a particular case. Indeed, in [18, Remark 4.20] he pointed out that the example studied by M. Katzman in [79] is gauge bounded.

On the other hand, we want to emphasize that Theorem 2.5.3 is not covered by the results obtained in [18] since we have seen that the Cartier algebra of a complete Stanley-Reisner ring might be infinitely generated.

It is natural to ask for the rationality of these F-jumping numbers. As far as we know, the best argument to prove discreteness and rationality of these digits was obtained by M. Katzman, G. Lyubeznik and W. Zhang in [81]. Unfortunately, their argument is only valid for regular rings; as the reader can see, our framework in this chapter is far from being regular.

2.5.2 Cartier algebras, Frobenius algebras and differential operators revisited

In Chapter 1 (cf. Section 1.6), we have introduced a pairing $\langle -, - \rangle_e$ which is just the composition of an element of $\mathcal{C}_e^R$ followed by an element of $\mathcal{F}_e^R$; such composition yields a differential operator of level e. More precisely, we have a natural map

$$\operatorname{Hom}_R(R, F_*^e R) \otimes_R \operatorname{Hom}_R(F_*^e R, R) \xrightarrow{\langle -,- \rangle_e} \operatorname{Hom}_R(F_*^e R, F_*^e R)$$

given by the assignment $\phi_e \otimes \psi_e \longmapsto \phi_e \circ \psi_e$. In addition, we have also seen (cf. Theorem 1.6.5) that this pairing is an isomorphism for regular rings. However, in general, this pairing is far from being an isomorphism; therefore, one may naturally ask for its image inside the differential operators of level e.

The aim of this section is to show that, for a complete Stanley-Reisner ring, we are able to check out how far this pairing might be from being surjective. More precisely, the following statement is the main result of this section.

Theorem 2.5.5 (Àlvarez, Boix, Zarzuela)**.** *Let $R := \mathbb{K}[\![x_1, \ldots, x_d]\!]/I$ be a complete, Stanley-Reisner ring. Then, one has that the image of $\langle -, - \rangle_e$ is generated, as abelian group, by the differential operators*

$$\{\mathbf{x}^{p^e\alpha} \circ \partial^{(p^e-1)\mathbf{1}} \circ \mathbf{x}^{(p^e-1)\beta}\}_{\gamma\in\Gamma},$$

where $\gamma := p^e\alpha + (p^e - 1)\beta$ runs over a minimal monomial generating set of the ideal $J_e + \langle \mathbf{x}^{(p^e-1)\mathbf{1}} \rangle$; that is, $\gamma \in \Gamma$ if and only if $\mathbf{x}^\gamma$ is a minimal monomial generator of $J_e + \langle \mathbf{x}^{(p^e-1)\mathbf{1}} \rangle$.

Proof. Let $\gamma \in \Gamma$. We have seen that $\psi_{e,\gamma}$ is a right R-generator of $\mathcal{C}_e^R$. In this way, as a left R-generator of $\mathcal{F}_e^R$ is the eth Frobenius map F^e it follows that

$$\begin{aligned}&\langle F^e, \psi_{e,\gamma} \rangle_e = F^e \circ \psi_{e,\gamma} = F^e \circ (\psi_e \circ \mathbf{x}^\gamma) = F^e \circ (\psi_e \circ \mathbf{x}^{p^e\alpha+(p^e-1)\beta})\\ &= F^e \circ (\mathbf{x}^\alpha \circ \psi_e \circ \mathbf{x}^{(p^e-1)\beta}) = \mathbf{x}^{p^e\alpha} \circ (F^e \circ \psi_e) \circ \mathbf{x}^{(p^e-1)\beta} = \mathbf{x}^{p^e\alpha} \circ \partial^{(p^e-1)\mathbf{1}} \circ \mathbf{x}^{(p^e-1)\beta},\end{aligned}$$

just what we wanted to show. □

In order to exhibit differential operators in $D_R^{(e)}$ that do not belong to the image of $\langle -, - \rangle_e$ we need to recall the following explicit presentation of D_R whenever R is a Stanley-Reisner ring. The below result is due to W. N. Traves; we omit the proof and refer to [136, Theorem 3.5] for additional details.

Proposition 2.5.6 (Traves)**.** *Let $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s} \subseteq S := \mathbb{K}[x_1, \ldots, x_d]$ be a squarefree monomial ideal and let $R' := S/I$ be the corresponding Stanley-Reisner ring. A monomial $\mathbf{x}^\beta\partial^\alpha \in D_S$ is in $D_{R'}$ if and only if, for each face ideal I_{α_i} in the minimal primary decomposition of I, we have either $\mathbf{x}^\beta \in I_{\alpha_i}$ or $\mathbf{x}^\alpha \notin I_{\alpha_i}$. In particular, $D_{R'}$ is generated as a $\mathbb{K}$-algebra by*

$$\{\mathbf{x}^\beta\partial^\alpha \mid \quad \mathbf{x}^\beta \in I_{\alpha_i} \text{ or } \mathbf{x}^\alpha \notin I_{\alpha_i} \quad \forall i \in \{1, \ldots, s\}\}.$$

Remark 2.5.7. We underline that the expression for the elements in the image of $\langle -, - \rangle_e$ given in Theorem 2.5.5 is not the same as the one given in Traves' result. Nevertheless, applying the relations defining D_R it is not hard to get it.

In this way, combining Theorem 2.5.5 joint with Traves' result we can find differential operators in $D_R^{(e)}$ that do not belong to the image of $\langle -, - \rangle_e$.

Example 2.5.8. Set $R := \mathbb{K}[\![x,y,z]\!]/I$, where $I := \langle y \rangle \cap \langle x,z \rangle$ and $\mathbb{K}$ is any field of characteristic p. Traves' result says that the ring of differential operators D_R is the R-algebra generated by

$$\{x\partial_1^n\partial_3^m, z\partial_1^n\partial_3^m, y\partial_2^m\}_{(n,m)\in\mathbb{N}}.$$

Moreover, $\mathcal{C}^R$ is generated by $\{x^{p^e}\Phi_e y^{p^e-1}, z^{p^e}\Phi_e y^{p^e-1}, \Phi_e x^{p^e-1}y^{p^e-1}z^{p^e-1}\}_{e\in\mathbb{N}}$ that correspond via $\langle -,- \rangle_e$ to the differential operators

$$\left\{x^{p^e}\partial_1^{p^e-1}\partial_2^{p^e-1}\partial_3^{p^e-1}y^{p^e-1}, z^{p^e}\partial_1^{p^e-1}\partial_2^{p^e-1}\partial_3^{p^e-1}y^{p^e-1}, \partial_1^{p^e-1}\partial_2^{p^e-1}\partial_3^{p^e-1}x^{p^e-1}y^{p^e-1}z^{p^e-1}\right\}_{e\in\mathbb{N}}$$

We note that $x\partial_1^{p^e-1}$ does not even belong to the R-algebra generated by this set.

This example illustrates a general way to build differential operators in $D_R^{(e)}$ that do not belong to the image of $\langle -,- \rangle_e$. Indeed, just take $x_i\partial_i^{p^e-1}$, where $x_i \in I_\alpha$ for some face ideal I_α in the minimal primary decomposition of I.

Bibliographical notes

Albeit the proof of Proposition 2.1.3 presented in this chapter is so straightforward, we point out that this result was obtained in the unmixed case by R. Fedder in [49]. The proof given in [4] is quite different; in fact, the proof presented here follows closely the argument used by R. Y. Sharp in order to prove [123, Proposition 2.8]. On the other hand, another, a priori, innocent tool used is Lemma 2.1.1. The proof presented in this chapter is due to J. Cowden Vassilev (cf. [137, Lemma 2.1]). We underline that this result has been generalized by M. Blickle, M. Mustaţă and K. E. Smith in [21, Lemma 2.3] and by M. Katzman in [77, Proposition 5.3].

Perhaps, the most distinguished family of squarefree monomial ideals with principal Cartier algebra are the one given in Proposition 2.4.7. These ideals are called *squarefree Veronese ideals* (we use the terminology from [65, Exercise 8.7 and page 200]). The importance of these ideals (from a combinatorial point of view) stems from the fact that these ideals are the only squarefree monomial ideals which are simultaneously Cohen-Macaulay and polymatroidal (cf. [65, Theorem 12.6.7] for more details).

The formal introduction of F-jumping numbers was made by M. Mustaţă, S. Takagi and K. -I. Watanabe in [107] under the name F-pure thresholds. Moreover, in [107, Remark 2.12] the authors proved that the test ideal is constant in intervals of the form $[a,b)$ whenever the ambient ring is regular. On the other hand, the first results on discreteness and rationality of F-jumping numbers were obtained by N. Hara in [57]. Generalizations of results of Hara has been recently obtained in [21] and [22]. The study of these digits is definitely a hot topic of research nowadays (cf. [121] and [120]). It is worth mentioning here that, when the ambient ring is not regular, there is no clear connection between the F-threshold previously mentioned and the first non-zero F-jumping number; the interested reader may like to consult [99] for concrete examples.

Chapter 3

An algorithm for producing F-pure ideals

In Chapter 1, the notion of Cartier algebra introduced by K. Schwede and M. Blickle was established (cf. Definition 1.4.6). Moreover, in Chapter 2 we have given a complete characterization of $\mathcal{C}^R$ whenever R is a complete Stanley-Reisner ring. We have seen, among other things, that $\mathcal{C}^R$ can only be either principally generated or infinitely generated as R-algebra (cf. Theorem 2.3.5).

From now on in this chapter, unless otherwise is specified, we shall denote by A a fixed regular ring which is either the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$, the localization $\mathbb{K}[x_1, \dots, x_d]_{\mathfrak{m}}$ (where $\mathfrak{m}$ is the ideal of $\mathbb{K}[x_1, \dots, x_d]$ generated by all the variables), or the formal power series ring $\mathbb{K}[\![x_1, \dots, x_d]\!]$, where $\mathbb{K}$ is any F-finite field of prime characteristic p. Moreover, all the ideals of A which we shall consider will be of the form JA, where J is an ideal of $\mathbb{K}[x_1, \dots, x_d]$.

Before going on, we introduce the objects of our study in this chapter.

Definition. Let $\mathcal{C}$ be a graded A-subalgebra of $\mathcal{C}^A$. An ideal I of A is called *F-pure* (with respect to $\mathcal{C}$) provided $\mathcal{C}_+ I = I$.

Our purpose is to introduce an effective method (cf. Theorem 3.2.5) in order to calculate all the F-pure ideals of A contained in $\mathfrak{m}$ with respect to the subalgebra of $\mathcal{C}^A$ generated by a single p^{-e}-linear map ϕ under the additional assumption that its ground field $\mathbb{K}$ is *finite*. This procedure has been implemented in Macaulay2 (cf. [25]) in case $\mathbb{K} = \mathbb{F}_p$ and $A = \mathbb{K}[x_1, \dots, x_d]$; the interested reader may like to consult Appendix B for further details about our implementation of this method.

This work is motivated by the study of F-pure ideals and their properties carried out by M. Blickle in [18]; for instance, under our assumptions one has that A itself is an F-pure ideal (with respect to $\mathcal{C}^A$) if and only if $\mathcal{C}^A_+$ contains a splitting of a certain power of the Frobenius map on A (cf. [18, Proposition 3.5]); therefore, these F-pure ideals turn out to be a generalization of the F-purity property.

Furthermore, among the main results obtained in [18] (see also [19, Corollary 4.20 and Proposition 5.4]) is the fact that the set of F-pure ideals in A is finite, and that the big test ideal is the minimal element of such set. Apart from their usefulness in describing big test ideals, we believe that the set of F-pure ideals provides an interesting set of invariants of A yielding information about the ring which is not yet fully understood.

The reader should contrast this with the situation one encounters when studying the set of $\mathcal{C}_+^A$-*compatible ideals*; that is, ideals $I \subseteq A$ for which $\mathcal{C}_+^A I \subseteq I$. One might hope to list all F-pure ideals by listing all compatible ideals and checking which ones are F-pure. However, the set of compatible ideals need not be finite, and one can only describe algorithmically the radical ones among these; such task was carried out in [82].

Our contribution to the understanding of F-pure ideals is to provide an effective procedure to calculate all the F-pure ideals contained in $\mathfrak{m}$ of the subalgebra $\mathcal{C} = \mathcal{C}^{\phi}$ of $\mathcal{C}^A$ generated by a single p^{-e}-linear map ϕ under the additional assumption that the coefficient field $\mathbb{K}$ is finite.

Now, we provide an overview of the contents of this chapter. Firstly, in Section 3.1 we introduce compatible and fixed ideals, stressing its connection with Cartier algebras given by a result originally due to O. Gabber and later on extended by M. Blickle (cf. Theorem 3.1.1); moreover, we show that the so-called *eth root ideal* (cf. Definition 3.1.6) plays a key role in their calculation. Second, Section 3.2 contains the main result of this chapter; namely, the algorithm previously referred (cf. Theorem 3.2.5); by the way, we introduce a new operation on ideals (cf. Definition 3.2.2), hoping that it may be interesting in its own right. On the other hand, in Section 3.3 we provide examples in order to illustrate how our method works; most of these specific computations were carried out with an implementation of this procedure in Macaulay2.

Finally, it is worth mentioning that ,when $\mathbb{K}$ is not F-finite, we have no explicit methods for solving such task; nevertheless, in the final part of this chapter (cf. Section 3.4), we shall give theoretical evidence that the algorithm presented here might be of some help in order to tackle this problem. For that purpose, we use the so-called Γ-*construction* introduced by M. Hochster and C. Huneke in [68].

Special acknowledgement of joint work

The content of this chapter is a submitted joint work with Mordechai Katzman (cf. [26]). However, it is worth mentioning that in what follows we are to give some details and results which are not found in the article.

3.1 Basic notions

Our motivation for studying F-pure ideals stems from the following result. We omit the proof and refer to [18, Proposition 2.13] and [51, Lemma 13.1] for details.

Theorem 3.1.1 (Gabber, Blickle). *Let $\mathcal{C} \subseteq \mathcal{C}^A$ be a Cartier subalgebra of $\mathcal{C}^A$ and let M be a finitely generated A-module that is also a left $\mathcal{C}$-module. Then, the chain of submodules*

$$M \supseteq \mathcal{C}_+ M \supseteq \mathcal{C}_+^2 M \supseteq \ldots \supseteq \mathcal{C}_+^i M \supseteq \mathcal{C}_+^{i+1} M \supseteq \ldots$$

eventually stabilizes; that is, there is $e_0 \gg 0$ such that $\mathcal{C}_+^{e_0} M = \mathcal{C}_+^e M$ for all $e \geq e_0$.

This non-trivial result motivates the introduction of the objects of study in this chapter.

Definition 3.1.2. Let M be a left module over $\mathcal{C}$. We say that M is *F-pure* (with respect to $\mathcal{C}$) if $\mathcal{C}_+ M = M$. In particular, we say that an ideal I of A is *F-pure* (with respect to $\mathcal{C}$) provided $\mathcal{C}_+ I = I$.

In fact, we are interested in case $M = A$ and $\mathcal{C}$ a principal Cartier subalgebra of $\mathcal{C}^A$.

Before going on, we shall recall the explicit description of $\mathcal{C}^A$ in this setup. Unless otherwise is specified, $\mathbb{K}$ will denote an F-finite field of characteristic p.

Under our assumptions, $F_*^e A$ is a free A-module of finite rank. Indeed, if $\mathcal{B}_e$ is a $\mathbb{K}^{p^e}$-basis for $\mathbb{K}$ then a free basis is given by

$$\{b\mathbf{x}^\alpha \mid \quad b \in \mathcal{B}_e, \quad 0 \leq ||\alpha||_\infty \leq p^e - 1\}.$$

Moreover, we recall that the trace map $\Phi_e \in \operatorname{Hom}_A(F_*^e A, A)$, which is just the unique p^{-e}-linear map which is the projection onto the direct summand $A\mathbf{x}^{(p^e-1)\mathbf{1}}$, generates the $F_*^e A$-module $\operatorname{Hom}_A(F_*^e A, A)$ (cf. Theorem/Definition 1.4.15). In this way, any homogeneous element $\phi \in \operatorname{Hom}_A(F_*^e A, A)$ can be written as $u\Phi_e$ (interpreted as the composition of multiplication by u and Φ_e) for some $u \in F_*^e A$. All these facts imply that if $\mathcal{C}$ is the Cartier subalgebra of $\mathcal{C}^A$ generated by such ϕ then the problem of finding the F-pure ideals of A is equivalent of computing ideals $I \subseteq A$ such that $\phi(F_*^e I) = I$.

The previous discussion motivates the introduction of the following notions.

Definition 3.1.3. Let I be an ideal of A and let $\phi \in \operatorname{Hom}_A(F_*^e A, A)$.

(i) We say that I is *ϕ-compatible* if $\phi(F_*^e I) \subseteq I$.

(ii) We say that I is *ϕ-fixed* if $\phi(F_*^e I) = I$.

It is clear that all ϕ-fixed ideals are, in particular, ϕ-compatible. The converse holds, for instance, when ϕ is a Frobenius splitting. Before proving so, we recall that:

Definition 3.1.4. It is said that $\phi \in \operatorname{Hom}_A(F_*^e A, A)$ is a *Frobenius splitting* if $\phi(F_*^e 1) = 1$.

Lemma 3.1.5. *If ϕ is a Frobenius splitting then an ideal $I \subseteq A$ is ϕ-fixed if and only if I is ϕ-compatible.*

Proof. Suppose that I is ϕ-compatible and let $r \in I$. We have also to note that

$$r = r \cdot 1 = r\phi(F_*^e 1) = \phi(r \cdot F_*^e 1) = \phi(F_*^e r^{p^e}) \in I.$$

Indeed, in the second equality we have used that ϕ is a Frobenius splitting and in the last inclusion we have used that I is ϕ-compatible. □

From now on, we shall suppose that $\phi = u\Phi_e$, where $u \in F_*^e A$.

3.1.1 The root ideal

Our next goal is to express in an equivalent way the condition of being ϕ-fixed in order to carry out explicit calculations. Such equivalent expression requires us to introduce the following concept (cf. [21, Definition 2.2] and [77, Section 5]).

Definition 3.1.6. Let J be an ideal of A. We set $I_e(J)$ as the smallest ideal I such that $I^{[p^e]} \supseteq J$. We shall refer to $I_e(J)$ as the *e-th root ideal* of J. It is often denoted $J^{[1/p^e]}$.

In the forthcoming result, we are to focus on collecting some elementary (but important) properties which the e-th root ideal verifies. We shall omit its proof and we refer to [77, Section 5] for details.

Proposition 3.1.7 (Katzman)**.** *Let $J, J_1, \dots, J_r$ be ideals of A. Then, the following statements hold.*

(a) If $J_1 \subseteq J_2$ then $I_e(J_1) \subseteq I_e(J_2)$.

(b) One has that

$$I_e\left(\sum_{i=1}^{r} J_i\right) = \sum_{i=1}^{r} I_e(J_i).$$

Note that this fact implies that it is enough to know how to calculate $I_e(J)$ when J is a principal ideal.

(c) Let $g \in A$. If

$$g = \sum_{\substack{b \in \mathcal{B}_e \\ 0 \leq ||\alpha||_\infty \leq p^e-1}} g_{\alpha b}^{p^e} b \mathbf{x}^\alpha$$

then $I_e(g)$ is the ideal of A generated by all the $g_{\alpha b}$'s.

(d) If A' is any faithfully flat A-algebra then $I_e(JA') = I_e(J)A'$.

We refer to Appendix B in order to give a concrete algorithm for computing $I_e(g)$.

Remark 3.1.8. Albeit we shall not exploit it further in this chapter, we have to underline that in [22] was proved that $I_e(J)^{[p^e]} = D_A^{(e)} \cdot J$, where $D_A^{(e)} := \operatorname{Hom}_A(F_*^e A, F_*^e A)$ is the ring of differential operators of level e. This fact had been already pointed out in [2] in case J is a principal ideal.

Now, we are ready for expressing the condition of being ϕ-fixed in computational terms. This is the main result of this section.

Theorem 3.1.9. *Let $J \subseteq A$ be any ideal and let $\phi = u\Phi_e \in \operatorname{Hom}_A(F_*^e A, A)$. Then, the following statements hold.*

(a) The image of $F_*^e J$ under ϕ is $I_e(uJ)$.

(b) J is ϕ-compatible if and only if $I_e(uJ) \subseteq J$.

(c) J is ϕ-fixed if and only if $I_e(uJ) = J$.

Proof. Parts (b) and (c) follow directly form part (a). So, it is enough to prove part (a).

Proposition 3.1.7 implies that, in order to compute $I_e(uJ)$, one may choose a set of generators $g_1, \dots, g_t$ of $F_*^e J$ and then compute $I_e(ug_1)+\dots+I_e(ug_t)$. Now, fix $i \in \{1, \dots, t\}$ and write

$$ug_i = \sum_{\substack{b \in \mathcal{B}_e \\ 0 \leq ||\alpha||_\infty \leq p^e-1}} r_{i\alpha b}^{p^e} b \mathbf{x}^\alpha.$$

Applying once more Proposition 3.1.7, it follows that $I_e(ug_i)$ is the ideal generated by all the previous $r_{i\alpha b}$'s. Moreover, taking into account this fact, we have to notice that

$$r_{i\alpha b} = \Phi_e \left(F_*^e \left(b^{-1} \mathbf{x}^{(p^e-1)\mathbf{1}-\alpha} \right) ug_i \right) \in \phi(F_*^e J).$$

This argument shows that $I_e(ug_i) \subseteq \phi(F_*^e J)$ for any $i \in \{1, \dots, t\}$; in this way, it follows that

$$I_e(uJ) \subseteq \phi(F_*^e J).$$

Conversely, we have to note that $\phi(y) = \Phi_e(uy) \in I_e(uJ)$ for any $y \in F_*^e J$, hence $\phi(F_*^e J) \subseteq I_e(uJ)$ and therefore we obtain the desired conclusion. $\square$

Before going on, we are to fix some additional notation.

Notation 3.1.10. Hereafter, set $S := \mathbb{K}[x_1, \dots, x_d]$ and set S_l as the $\mathbb{K}$-vector space generated by monomials $\mathbf{x}^\alpha$ with $||\alpha||_\infty \leq l$.

The following result will guarantee that the algorithm we shall introduce later on terminates after a finite number of steps.

Proposition 3.1.11. *The following statements hold.*

(i) For any $y \in S$, the ideal $I_e(y)$ can be generated by elements $g \in S$ such that

$$||g||_\infty \leq \frac{||y||_\infty}{p^e}.$$

(ii) If J is $u\Phi_e$-fixed then there exists a set of generators of J such that if g belongs to such set then

$$||g||_\infty \leq \frac{||u||_\infty}{p^e-1}.$$

Proof. We begin proving part (i). Indeed, we write

$$y = \sum_{\substack{b \in \mathcal{B}_e \\ 0 \leq ||\alpha||_\infty \leq p^e - 1}} y_{\alpha b}^{p^e} b \mathbf{x}^\alpha.$$

In this way, for any α and b as above it follows that

$$p^e ||y_{\alpha b}||_\infty \leq ||y_{\alpha b}^{p^e}||_\infty \leq ||y_{\alpha b}^{p^e} \mathbf{x}^\alpha||_\infty \leq ||y||_\infty,$$

hence part (i) holds.

Now, we prove part (b). Let $M \geq 0$ be the minimal integer for which a set of generators of J have infinity norm at most M. Part (i) shows that $I_e(uJ)$ can be generated by polynomials with infinity norm at most $(||u||_\infty + M)/p^e$. In addition, as $I_e(uJ) = J$ we deduce, by the minimality of M, that $M \leq (||u||_\infty + M)/p^e$ and therefore we conclude that $M \leq ||u||_\infty/(p^e - 1)$, just what we finally wanted to check. □

3.2 The algorithm through the hash operation

The aim of this section is to describe a computational method to produce all the $u\Phi_e$-fixed ideals of S. As the reader will appreciate, our procedure is based on a new operation on ideals (cf. Definition 3.2.2), which we hope to be of some interest in its own right.

We start with the following elementary statement, which we provide a proof for the sake of completeness. It may be regarded as an elementary consequence of Nakayama's Lemma.

Lemma 3.2.1. *Let $I \subseteq \mathfrak{m}$ be an ideal minimally generated by s elements. Then, any ideal $J \subsetneq I$ is contained in some ideal V, where $\mathfrak{m}I \subseteq V \subseteq I$ and $\dim_{\mathbb{K}} I/V = 1$.*

Proof. Nakayama's Lemma implies that there are $g_1, \ldots, g_s \in S$ with $I = Sg_1 + \ldots + Sg_s$ such that $g_1, \ldots, g_s \pmod{\mathfrak{m}K}$ is a basis of the s-dimensional $\mathbb{K}$-vector space $I/\mathfrak{m}I$. In this way, it follows that any ideal $J \subsetneq I$ is contained in some $V := SW + \mathfrak{m}I$, where W is a $(s-1)$-dimensional $\mathbb{K}$-vector subspace of $I/\mathfrak{m}I$. Moreover, we have to note as well that $\dim_{\mathbb{K}} I/V = 1$. □

From now on, we shall assume that $u \in F_*^e S$ is fixed and set

$$D_e := \left\lceil \frac{||u||_\infty}{p^e - 1} \right\rceil.$$

The following construction will become in the crucial building block of our method.

Definition 3.2.2. Given any ideal $J \subseteq S$, we define the sequence of ideals

$$J_0 := J, \quad J_{i+1} := \left(J_i \cap \left(J_i^{[p^e]} :_S u \right) \cap I_e(uJ_i) \cap S_{D_e} \right) S,$$

and set

$$J^{\#_e} := \bigcap_{i \geq 0} J_i.$$

When $e = 1$, we shall write $J^{\#}$ instead of $J^{\#_1}$ for the sake of brevity. Hereafter, we are to refer to this construction as the *hash operation*.

The introduction of the hash operation is motivated by the following result.

Lemma 3.2.3. *For any ideal $J \subseteq S$, $J^{\#_e}$ contains all the $u\Phi_e$-fixed ideals which are contained in J.*

Proof. Let $I \subseteq J$ be any $u\Phi_e$-fixed ideal. We shall show by increasing induction on $i \geq 0$ that $I \subseteq J_i$, where J_i is as in Definition 3.2.2. This is clearly true for $i = 0$.

Now, we assume that $i \geq 0$ and that $I \subseteq J_i$. First of all, as I is $u\Phi_e$-fixed it is, in particular, $u\Phi_e$- compatible and therefore

$$uI \subseteq I^{[p^e]} \subseteq J_i^{[p^e]},$$

hence $I \subseteq \left(J_i^{[p^e]} :_S u\right)$. Secondly, using once more that I is $u\Phi_e$-fixed it follows that $I = I_e(uI) \subseteq I_e(uJ_i)$ (indeed, we are simultaneously using that $I \subseteq J_i$ and that the eth root operation on ideals preserves inclusions). In this way, the previous two facts allow us to say that

$$I \subseteq J_i \cap \left(J_i^{[p^e]} :_S u\right) \cap I_e(uJ_i).$$

Finally, bearing in mind that $I = (I \cap S_{D_e})S$ (indeed, here we are applying part (ii) of Proposition 3.1.11) and the foregoing it follows that

$$I = (I \cap S_{D_e})S \subseteq J_{i+1},$$

just what we finally wanted to check. □

Remark 3.2.4. As we have seen in Lemma 3.2.3, the hash operation produces the smallest compatible ideal $J^{\#}$ contained in a given one (namely, J) such that all the ϕ-fixed ideals contained in J are also contained in $J^{\#}$. This fact allows us to regard this construction as a kind of round down operation on ideals.

3.2.1 The statement of the algorithm

Now, we introduce our promised algorithm. More precisely, next result is a recursive procedure for producing all the $u\Phi_e$-fixed ideals of S.

This is the main result of this chapter.

Theorem 3.2.5 (Boix, Katzman)**.** *Let $I \subseteq \mathfrak{m}$. The set $\mathrm{FP}_e(I)$ of all $u\Phi_e$-fixed ideals contained in I is given recursively as $\mathrm{FP}_e(\langle 0\rangle) = \{\langle 0\rangle\}$ and, for $I \neq \langle 0\rangle$, defined as the union of $\{I^{\#_e}\}$ (whenever $I^{\#_e}$ is $u\Phi_e$-fixed) and*

$$\bigcup \left\{ \mathrm{FP}_e(V) \mid \quad \mathfrak{m} I^{\#_e} \subseteq V \subseteq I^{\#_e}, \quad \dim_{\mathbb{K}} I^{\#_e}/V = 1 \right\}.$$

Moreover, if $\mathbb{K}$ is finite then this recursion is finite in the sense that the resulting execution tree is finite.

Proof. First of all, we shall show that if $J \subseteq I$ is $u\Phi_e$-fixed then $J \in \mathrm{FP}_e(I)$. We shall proceed by increasing induction on $t := \dim_{\mathbb{K}}(I \cap S_{D_e})$.

Indeed, if $t = 0$ then $J \subseteq I^{\#_e} = \langle 0\rangle$ and therefore $J \in \{\langle 0\rangle\} = \mathrm{FP}_e(I)$.

Now, let $J \subseteq I$ be such that $t \geq 1$. If $J = I^{\#_e}$ then we are done by Lemma 3.2.3. Thus, we assume that $J \subsetneq I^{\#_e}$. Since $I \subseteq \mathfrak{m}$, Lemma 3.2.1 says us that we can find an ideal $\mathfrak{m} I^{\#_e} \subseteq V \subsetneq I^{\#_e}$ such that $\dim_{\mathbb{K}} I^{\#_e}/V = 1$ and $J \subseteq V$. Furthermore, by construction, $I^{\#_e}$ can be generated by elements in S_{D_e}, hence $V \cap S_{D_e} \subsetneq I^{\#_e} \cap S_{D_e}$ and therefore the induction hypothesis implies that $J \in \mathrm{FP}_e(V) \subseteq \mathrm{FP}_e(I)$.

Finally, we have to point out that our foregoing inductive argument shows that the chains of $V^{\#_e}$'s produced in this recursion have length at most $\dim_{\mathbb{K}} S_{D_e}$, hence the second statement follows too. □

In this way, we can turn Theorem 3.2.5 into an effective method to calculate all the $u\Phi_e$-fixed ideals of any polynomial ring having a finite field as ring of coefficients as follows.

Algorithm 3.2.6. Let $\mathbb{K}$ be a finite field of prime characteristic p, $S := \mathbb{K}[x_1, \ldots, x_d]$ and let $u \in S$. These data act as the input of the procedure. Moreover, we initialize I as the whole ring S and L as the empty list $\{\}$.

(i) Compute $I^{\#_e}$. Assign to I the value of $I^{\#_e}$.

(ii) If $I_e(uI) = I$, then add I to the list L.

(iii) If $I = 0$, then stop and output the list L.

(iv) If $I \neq 0$ but principal, assign to I the value of $\mathfrak{m} I$ and loop over the previous steps.

(v) If $I \neq 0$ and not principal, then compute

$$\{V \quad \text{ideal} \mid \quad \mathfrak{m} I \subseteq V \subseteq I, \quad \dim_{\mathbb{K}} I/V = 1\}.$$

For each element V of such set, loop over the previous steps.

At the end of this method, the list L contain all the $u\Phi_e$-fixed ideals of S.

Remark 3.2.7. The reader should notice that step (v) of the previous method is the only reason for which we have to assume that our coefficient field $\mathbb{K}$ is finite; otherwise, the set $\{V \quad \text{ideal} \mid \quad \mathfrak{m}I \subseteq V \subseteq I, \quad \dim_{\mathbb{K}} I/V = 1\}$ is not finite.

Remark 3.2.8. We underline that Theorem 3.2.5 also follows if we consider in Definition 3.2.2 J_{i+1} to be $\left(J_i \cap \left(J_i^{[p^e]} :_S u\right) \cap S_{D_e}\right) S$ rather than

$$\left(J_i \cap \left(J_i^{[p^e]} :_S u\right) \cap I_e(uJ_i) \cap S_{D_e}\right) S.$$

Nevertheless, such smaller definition has the potential to decrease the size of the recursion tree drastically.

We illustrate this remark with the following:

Example 3.2.9. We consider the ring $S := \mathbb{F}_2[x, y, z]$ and we set $u := x^3yz + x^2yz = x^2yz(x + 1)$. If we apply the algorithm described in Theorem 3.2.5 defining J_{i+1} as

$$\left(J_i \cap \left(J_i^{[p^e]} :_S u\right) \cap S_{D_e}\right) S$$

in the hash operation, then the program does not terminate. Regardless, applying the algorithm described in Theorem 3.2.5 defining J_{i+1} as

$$\left(J_i \cap \left(J_i^{[p^e]} :_S u\right) \cap I_e(uJ_i) \cap S_{D_e}\right) S$$

in the hash operation we obtain quite quickly all the $u\Phi_1$-fixed ideals of S. Namely, we obtain the following thirty non-zero $u\Phi_1$-fixed ideals:

(i) Nineteen ideals of the form $\langle x\rangle \cdot I$, where I runs over all the possible non-zero squarefree monomial ideals of S.

(ii) The ideals $\langle u\rangle$ and $\langle x^2y, xy(x + z^2), xyz(x + 1)\rangle$.

(iii) Five ideals of the form $\langle x^2y(x + 1)\rangle + J$, where J can be either

$$\langle xz\rangle, \langle xyz, x^2z\rangle, \langle xyz\rangle, \langle x^2yz\rangle \text{ or } \langle x^2z\rangle.$$

(iv) Four ideals of the form $\langle x^2z(x + 1)\rangle + K$, where K can be either

$$\langle xyz\rangle, \langle x^2yz\rangle, \langle x^2y\rangle \text{ or } \langle xy\rangle.$$

We end this section with the following result, which may be regarded as an elementary consequence of the very definition of the hash operation.

Corollary 3.2.10. *Let $\mathbb{K}$ be any F-finite field of prime characteristic p, set S as the polynomial ring $\mathbb{K}[x_1, \ldots, x_d]$, and let $u \in S$. Then, the ideal $\langle u\rangle$ is a minimal ϕ-fixed ideal, where $\phi := u^{p-1}\Phi_1$.*

Proof. We have to check that $\langle u \rangle$ is a minimal non-zero ϕ-fixed ideal of S. Firstly, we are to show that $\langle u \rangle$ is ϕ-fixed; indeed,

$$I_1(u^{p-1} \cdot \langle u \rangle) = I_1(u^p) = \langle u \rangle,$$

whence $\langle u \rangle$ is ϕ-fixed. So, it only remains to prove that $\langle u \rangle$ is a minimal ϕ-fixed ideal of S. First of all, notice that

$$D_1 = \left\lceil \frac{||u^{p-1}||_\infty}{p-1} \right\rceil = ||u||_\infty.$$

On the other hand, as $\langle u \rangle$ is principal it follows, according to Lemma 3.2.3, that if $I \subseteq \langle u \rangle$ is ϕ-fixed, then $I \subseteq (\langle u \rangle \cdot \mathfrak{m})^{\#}$. Nevertheless, it implies that any element $g \in \langle u \rangle \cdot \mathfrak{m}$ which forms part of a system of generators for $\langle u \rangle \cdot \mathfrak{m}$ is such that

$$||g||_\infty \geq ||u||_\infty + 1 > ||u||_\infty.$$

But this strict lower inequality implies, taking into account the very definition of the hash operation, that $(\langle u \rangle \cdot \mathfrak{m})^{\#} = 0$ and therefore $I = 0$, just what we finally wanted to check. □

3.3 Examples

In this section, we present some interesting calculations carried out with an implementation of the algorithm presented in this chapter. We refer to Appendix B for more algorithmic details. Macaulay2 (cf. [25]) have been used extensively in the writing of such procedure, both in constructing and exploring examples, as well as implementing the method described herein. It is worth mentioning here that we are to divide the examples considered in this section into two parts; namely, the first one is concerned with examples in which the map $u\Phi_1$ defines a Frobenius splitting. On the other hand, in the second part the homomorphism $u\Phi_1$ won't define a Frobenius splitting.

3.3.1 F-split examples

First of all, we include an example where we develop the algorithm step by step for the convenience of the reader. This example is also interesting because it illustrates a particular case of a general fact which we shall show later (cf. Proposition 3.3.2).

Example 3.3.1. We consider the ring $S := \mathbb{F}_2[x, y]$ and set $u := xy$. We compute $\mathrm{FP}_1(S)$.

(a) Start with $I = S = I^{\#}$. As $I_1(uI) = I$ add S to the list $\mathrm{FP}_1(S)$.

(b) As I is principal, go on with $I = \mathfrak{m} = I^{\#}$. Since $I_1(uI) = I$ add $\mathfrak{m}$ to the list $\mathrm{FP}_1(S)$. Moreover, we have to note that

$$\left\{\mathfrak{m}^2 \subseteq V \subseteq \mathfrak{m} \mid \quad \dim_{\mathbb{F}_2} \mathfrak{m}/V = 1\right\} = \left\{\langle x, y^2\rangle, \langle y, x^2\rangle, \langle x^2, xy, x + y\rangle\right\}.$$

We have to emphasize that in the calculation of this set is when we are using that we are working with characteristic two.

Thus, we need to compute $\mathrm{FP}_1(\langle x^2, xy, x+y\rangle)$, $\mathrm{FP}_1(\langle x, y^2\rangle)$ and $\mathrm{FP}_1(\langle y, x^2\rangle)$.

(i) Since $\langle x^2, xy, x+y\rangle^{\#} = \langle xy\rangle$ and $I_1(u\langle xy\rangle) = \langle xy\rangle$, we add $\langle xy\rangle$ to the list $\mathrm{FP}_1(\langle x^2, xy, x+y\rangle)$. Moreover, as $\langle xy\rangle$ is principal go on with $\langle x^2y, xy^2\rangle$. Nevertheless, since $\langle x^2y, xy^2\rangle^{\#} = \langle 0\rangle$ we deduce that $\mathrm{FP}_1(\langle x^2, xy, x+y\rangle) = \{\langle xy\rangle, \langle 0\rangle\}$.

(ii) As $\langle x, y^2\rangle^{\#} = \langle x\rangle$ and $I_1(u\langle x\rangle) = \langle x\rangle$ add $\langle x\rangle$ to the list $\mathrm{FP}_1(\langle x, y^2\rangle)$. In addition, as $\langle x\rangle$ is principal go on with $\langle x^2, xy\rangle$. However, since $\langle x^2, xy\rangle^{\#} = \langle xy\rangle$ we can use the foregoing calculations and therefore we conclude that $\mathrm{FP}_1(\langle x, y^2\rangle) = \{\langle x\rangle, \langle xy\rangle, \langle 0\rangle\}$.

A similar calculation shows that $\mathrm{FP}_1(\langle y, x^2\rangle) = \{\langle y\rangle, \langle xy\rangle, \langle 0\rangle\}$.

In this way, it follows that $\mathrm{FP}_1(S) = \{\mathbb{F}_2[x,y], \langle x, y\rangle, \langle x\rangle, \langle y\rangle, \langle xy\rangle, \langle 0\rangle\}$.

As we have previously explained, this specific computation is just a particular case of the following general fact.

Proposition 3.3.2 (Schwede, Tucker)**.** *When $u := (x_1 \cdots x_d)^{p^e-1}$, the only $u\Phi_e$-fixed ideals of $S := \mathbb{F}_p[x_1, \ldots, x_d]$ are the squarefree monomial ideals of S.*

Proof. First of all, we shall check that if I is a squarefree monomial ideal of S then $I_e(uI) = I$. Indeed, as

$$I_e(uI) = I_e(um_1) + \ldots + I_e(um_t)$$

(where $m_1, \ldots, m_t$ is a finite collection of squarefree monomial ideals which generate I) it is enough to check that $I_e(um) = \langle m\rangle$, where m is a single squarefree monomial ideal. However, in this case there is $1 \leq i_1 < \ldots < i_t \leq d$ such that $m = x_{i_1} \cdots x_{i_t}$, hence

$$um = x_{i_1}^{p^e} \cdots x_{i_t}^{p^e} x_{j_1}^{p^e-1} \cdots x_{j_{d-t}}^{p^e-1}$$

(where $\{i_1, \ldots, i_t, j_1, \ldots, j_{d-t}\} = \{1, \ldots, d\}$) and therefore $I_e(um) = \langle m\rangle$. The converse inclusion follows from [118, Proposition 5.3]. $\square$

In the below example, we focus on an slightly more difficult computation which looks in greater detail at [78, Example 5.6].

Example 3.3.3. Consider the ring $S := \mathbb{F}_2[x, y, z]$ and set $u := y(y+z)(x+z)$. It is straightforward to check that $u\Phi_1$ is a Frobenius splitting. Our algorithm produces a complete list of nineteen non-zero $u\Phi_1$-fixed ideals as follows.

(i) The whole ring S, the homogeneous maximal ideal $\mathfrak{m}$ and the principal ideal $\langle u\rangle$.

(ii) Three prime ideals generated by two elements; namely, $\langle x, y\rangle$, $\langle y, z\rangle$ and $\langle x+z, y+z\rangle$.

(iii) Three prime ideals generated by a single element; namely, $\langle y\rangle$, $\langle x+y\rangle$ and $\langle y+z\rangle$

(iv) The remainder ten ideals are suitable intersections of the previous ones.

We have to underline that in our previous examples all the $u\Phi_e$-fixed ideals are radical. Indeed, these particular examples illustrate the following general fact. Although it is an adaptation of [29, Proposition 1.2.1] we shall give a proof for the sake of completeness.

Proposition 3.3.4. *Let $u \in F_*^e S$ be such that $u\Phi_e$ is a Frobenius splitting. Then any $u\Phi_e$-fixed ideal is radical.*

Proof. Denote by F^e the e-fold of the Frobenius map, set $q := p^e$ and let $y \in \sqrt{J}$, where $J \supseteq I_e(uJ)$. Thus, $\{a \in \mathbb{N} \mid \quad y^{q^a} \in J\} \neq \emptyset$. In this way, set

$$b := \min \left\{a \in \mathbb{N} \mid \quad y^{q^a} \in J\right\}.$$

If $b = 0$ then we are done. Assume, to get a contradiction, that $b > 0$. Thus, using that $u\Phi_e$ is a Frobenius splitting it follows that

$$y^{q^{b-1}} = (u\Phi_e \circ F^e)\left(y^{q^{b-1}}\right) = (u\Phi_e)\left(y^{q^b}\right) \in I_e(uJ) \subseteq J,$$

hence $y^{q^{b-1}} \in J$, which contradicts our choice of b. □

Finally, we end this subsection carrying out a more involved calculation which looks in greater detail at [77, Section 9].

Example 3.3.5. Consider the matrix of variables

$$A := \begin{pmatrix} x_1 & x_2 & x_2 & x_5 \\ x_4 & x_4 & x_3 & x_1 \end{pmatrix}$$

and set $S := \mathbb{F}_2[x_1, x_2, x_3, x_4, x_5]$. Furthermore, for any $1 \leq i < j \leq 4$ we denote by M_{ij} the minor of A of size 2 obtained from columns i and j. In addition, set

$$u := x_1^3x_2x_3 + x_1^3x_2x_4 + x_1^2x_3x_4x_5 + x_1x_2x_3x_4x_5 + x_1x_2x_4^2x_5 + x_2^2x_4^2x_5 + x_3x_4^2x_5^2 + x_4^3x_5^2.$$

Our procedure produces the following 84 proper ϕ-fixed ideals of S, where $\phi := u\Phi_1$.

(i) One prime ideal generated by five elements; namely, the ideal $\mathfrak{m}$ generated by all the variables of S.

(ii) Four prime ideals generated by four elements; namely,

$$\langle x_1, x_2, x_3, x_4\rangle, \langle x_1, x_2, x_4, x_5\rangle, \langle x_1, x_3, x_4, x_5\rangle, \langle x_1, x_2, x_3 + x_4, x_5\rangle.$$

(iii) Five prime ideals generated by three elements; namely,

$$\langle x_1, x_2, x_5\rangle, \langle x_1, x_3, x_4\rangle, \langle x_1, x_2, x_4\rangle, \langle x_1, x_4, x_5\rangle, \langle x_1 + x_2, x_3 + x_4, x_2^2 + x_4x_5\rangle.$$

(iv) Two prime ideals generated by two elements; namely, $\langle x_1, x_4\rangle$ and $\langle x_1+x_2, x_2^2+x_4x_5\rangle$. The reader should notice that

$$\langle x_1 + x_2, x_2^2 + x_4x_5\rangle = \langle x_1 + x_2, x_1^2 + x_4x_5\rangle$$

because of we are working on characteristic two.

(v) One prime ideal generated by just one element; namely, the ideal $\langle u\rangle$.

(vi) Twenty-nine ideals which contains in their set of minimal generators some M_{ij} for some $1 \leq i < j \leq 4$.

(vii) The remainder fourty-two ideals define arrangements of linear varieties. Among these 42 ideals, there is one distinguished element; namely, the ideal $\langle x_1, x_2, x_3 + x_4, x_4x_5\rangle$. In [77, Section 9] it was shown that this ideal is the parameter test ideal of the quotient ring S/I, where I is the ideal of S generated by the 2×2 minors of A.

The reader should notice that, in this case, the set of ϕ-fixed ideals equals the set of ϕ-compatible ideals; indeed, this is due to the fact that, in this case, the map $\phi = u\Phi_1$ is a Frobenius splitting. In particular, we recover the thirteen non-zero ϕ-compatible primes obtained by M. Katzman and K. Schwede in [82, Example 7.2].

We conclude this part with the following:

Remark 3.3.6 (Algorithmic). In case ϕ defines a Frobenius splitting, we have seen that the set of ϕ-fixed ideals agrees with the set of ϕ-compatible ideals; therefore, in this setting, the method introduced in Theorem 3.2.5 performs the same task than the procedures described in [82]. It is also worth mentioning, albeit we do not provide a formal proof, that we have checked out in examples that the methods described in [82] are faster than our algorithm. This fact is not surprising, taking into account that, whereas our procedure has as crucial step the calculation of all the vector subspaces of a given one over a finite field, the methods described in [82] only rely (essentially) on primary decomposition and elimination theory.

3.3.2 Non F-split examples

As we have previously explained, the aim of this section is to analize some examples in which $u\Phi_1$ does not define a Frobenius splitting.

Example 3.3.7. Consider the ring $S := \mathbb{F}_2[x, y, z]$ and set $u := x^3yz + x^2yz$. Our algorithm produces a complete list of $u\Phi_1$-fixed ideals; namely, the ones given in example 3.2.9. We have to emphasize that, in this example, the ideal $\langle x, y, z\rangle$ is not $u\Phi_1$-fixed.

In this last example, $u\Phi_1$ does not define a Frobenius splitting but u is a reducible polynomial. But, of course, one may produce examples with u irreducible. The next one is such as a point.

Example 3.3.8. Consider the ring $S := \mathbb{F}_2[x, y, z, w]$ and set $u := x^3 + y^3 + z^3 + w^3$. Our algorithm produces a complete list of $u\Phi_1$-fixed ideals as follows.

$$\mathbb{F}_2[x, y, z, w], \quad \langle x^3 + y^3 + z^3 + w^3\rangle, \quad \langle 0\rangle.$$

Remark 3.3.9. It is worth mentioning that example 3.3.8 was pointed out by K. Schwede on

`http://mathoverflow.net/questions/107062/frobenius-splitting-of-fano-varieties`

The interest for such example, from a geometric point of view, stems from the fact that

$$X := \operatorname{Proj}\left(\frac{\mathbb{F}_2[x, y, z, w]}{\langle x^3 + y^3 + z^3 + w^3\rangle}\right)$$

is an example of a Fano variety reduced to characteristic 2 which does not admit a Frobenius splitting. It constrasts with the fact that Fano varieties, when reduced to characteristic $p \gg 0$, possess a Frobenius splitting (cf. [29, 1.6.E]).

Thirdly, we include an example where the characteristic of our ground field is greater than two.

Example 3.3.10. Let $S := \mathbb{F}_5[x, y, z]$, and $u = x^4 + y^4 + z^4$. The aim of this example is to compute, using our algorithm, all the ϕ-fixed ideals of S, where $\phi := u^4\Phi_1$. Our method produces the following sixty-five non-zero ϕ-fixed ideals.

(i) The ideal $\mathfrak{m} := \langle x, y, z\rangle \subseteq S$, its square $\mathfrak{m}^2$ and the principal ideal $\langle u\rangle$.

(ii) Thirty-one ideals of the form $\mathfrak{m}^2 + H$, where H is an ideal of S generated by a single linear form.

(iii) Thirty-one ideals of the form $\mathfrak{m}^2 + G$, where G is an ideal of S generated by two linear forms.

It is worth noting that this specific calculation is also interesting because it provides an example where our method provides more information than the procedures worked out in [82]; indeed, if one uses [83] here, then one only gets the ideals $\langle u\rangle$ and $\mathfrak{m}$. As we have explained in the Introduction, the reader should remember that, whereas our algorithm produces all the ϕ-fixed ideals, the procedures described in [82] describes algorithmically the radical ϕ-compatible ideals.

Finally, we end this subsection with a determinantal example. The main interest for considering such example stems from the fact that, in [84, Section 5] the authors have shown that the Cartier algebra attached to such ring is infinitely generated, raising positively a question posed by M. Katzman in [79, Section 2].

Example 3.3.11. We fix the 2×3 matrix of indeterminates

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

and we let S to be the polynomial ring $\mathbb{F}_2[x_1, x_2, x_3, y_1, y_2, y_3]$. Moreover, for any $1 \leq i < j \leq 3$ Δ_{ij} will stand for the 2×2 minor obtained from columns i and j. In this way, taking into account this notation, we set

$$u := \Delta_{12}\Delta_{13} = (x_1 y_2 - x_2 y_1)(x_1 y_3 - x_3 y_1).$$

Our procedure produces the following seven proper ϕ-fixed ideals, where $\phi := u\Phi_1$; namely,

$$\langle x_1, y_1, \Delta_{23}\rangle, \langle x_1, y_1\rangle, \langle \Delta_{12}, \Delta_{13}, \Delta_{23}\rangle, \langle \Delta_{12}, \Delta_{13}\rangle, \langle \Delta_{12}\rangle, \langle \Delta_{13}\rangle, \langle \Delta_{12}\Delta_{13}\rangle.$$

In particular, we obtain the following five proper ϕ-fixed prime ideals:

$$\langle \Delta_{12}\rangle, \langle \Delta_{13}\rangle, \langle x_1, y_1\rangle, \langle x_1, y_1, \Delta_{23}\rangle, \langle \Delta_{12}, \Delta_{13}, \Delta_{23}\rangle.$$

Such list of ϕ-fixed prime ideals turns out to be the complete list of proper ϕ-compatible prime ideals, as the reader can check using [83]. As we have roughly explained in the paragraph which preceds this calculation, the interest of this example comes from the fact that the Cartier algebra attached to the quotient ring S/I is infinitely generated, where I is the ideal generated by all the Δ's. For additional details about such result, the reader is encouraged to consult [84, Section 5].

3.4 The non F-finite case

So far, we have produced an algorithm for computing all the $u\Phi_e$-fixed ideals of S, where $u \in F_*^e S$, $S := \mathbb{K}[x_1, \ldots, x_d]$ and $\mathbb{K}$ is an F-finite field.

The aim of this section is to give theoretical evidence that this case might be of some help in order to calculate the $u\Phi_e$-fixed ideals of S, where $u \in F_*^e S$, $S := \mathbb{K}[x_1, \ldots x_d]$ and now $\mathbb{K}$ is a (not necessarily F-finite) field of characteristic p.

We begin with the following result. It takes care about the behaviour of $u\Phi_e$-fixed ideals under a special faithfully flat extension.

Proposition 3.4.1. *Let S' be the faithfully flat S-algebra $\mathbb{L}[x_1, \ldots, x_d]$, where $\mathbb{L}$ is any overfield of $\mathbb{K}$. Then, the following statements hold.*

(a) If I is a $u\Phi_e$-fixed ideal of S then IS' is a $u\Phi_e$-fixed ideal of S'.

(b) If J is a $u\Phi_e$-fixed ideal of S' such that $J = IS'$ for some ideal I of S, then I is a $u\Phi_e$-fixed ideal of S.

(c) There is a bijective correspondence between the $u\Phi_e$-fixed ideals of S and the $u\Phi_e$-fixed ideals of S' (namely, J) such that $J = IS'$ for some $u\Phi_e$-fixed ideal I of S. This correspondence is given by extension and contraction of ideals with respect to the inclusion $S \hookrightarrow S'$.

Proof. Part (c) follows directly combining parts (a) and (b). Thus, it is enough to check that (a) and (b) holds.

First of all, let I be a $u\Phi_e$-fixed ideal of S. So, bearing in mind part (f) of Proposition 3.1.7 one has that

$$I_e(u(IS')) = I_e((uI)S') = I_e(uI)S' = IS',$$

hence part (a) holds.

Finally, let J be a $u\Phi_e$-fixed ideal of S' such that $J = IS'$ for some (non-necessarily $u\Phi_e$-fixed) ideal I of S. In this way, as $S \hookrightarrow S'$ is faithfully flat it follows that

$$I_e(uI) = I_e(uI)S' \cap S = I_e(u(IS')) \cap S = I_e(uJ) \cap S = J \cap S = IS' \cap S = I,$$

and therefore part (b) holds too. □

Remark 3.4.2. Actually, much more is true. T. J. Stadnik Jr. has proved in [130, Proposition 2.16] that $I_e(JS') = I_e(J)S'$ provided $S \longrightarrow S'$ is étale. Moreover, it is also known (cf. [85, Lemma 3.5]) that the operation $I_e(-)$ commutes with localization. We refer to [105, Proposition 9] for more information concerning the behaviour of the eth root ideal.

The main tool for showing, at least theoretically, that the algorithm constructed in the F-finite case might be of some help in order to compute all the fixed ideals in general polynomial rings is the so-called Γ-*construction*.

In what follows, we are to recall such construction with some detail. Keeping in mind the notion of p-basis (cf. Definition 1.4.9), we introduce the following preliminary:

Definition 3.4.3. Let $\mathbb{L}$ be a field of prime characteristic p and let Λ be a p-base for $\mathbb{L}$. It is said that a subset Γ of Λ is *cofinite* provided $\Lambda - \Gamma$ is a finite set.

Now, we are ready for introducing the gamma construction inside the following result, albeit the proof of such result will be omitted (cf. [66, pp. 136–137] for details).

Theorem/Definition 3.4.4 (Hochster, Huneke)**.** Let $\mathbb{K}$ be a fixed field of characteristic p, let Λ be a fixed p-base for $\mathbb{K}$ and let Γ be a cofinite subset of Λ. Now, set

$$\mathbb{K}_e^{\Gamma} := \mathbb{K}[\lambda^{1/q} \mid \quad \lambda \in \Gamma].$$

Moreover, set $T := \mathbb{K}[\![x_1, \ldots, x_d]\!]$ and set

$$T^{\Gamma} := \bigcup_{e \geq 0} \mathbb{K}_e^{\Gamma}[\![x_1, \ldots, x_d]\!].$$

We shall refer to T^{Γ} as being obtained from T by the *gamma construction*. Moreover, the following statements hold.

(i) $T \hookrightarrow T^{\Gamma}$ is a flat local homomorphism of rings and therefore T^{Γ} is a faithfully flat T-algebra.

(ii) T^{Γ} is a local regular ring of dimension d with maximal ideal $\mathfrak{m}T^{\Gamma}$ and residue field

$$\mathbb{K}^{\Gamma} := \bigcup_{e \geq 0} \mathbb{K}_e^{\Gamma}.$$

(iii) T^{Γ} is purely inseparable over T.

(iv) T^{Γ} is F-finite, which is equivalent to say that $\mathbb{K}^{\Gamma}$ is F-finite.

In this way, we can now turn all these constructions into a theoretical algorithm to calculate all the $u\Phi_e$-fixed ideals as follows.

(a) Choose Γ a cofinite subset of Λ such that the coefficients of u can be expressed as polynomials in $\Lambda - \Gamma$.

(b) Compute the $u\Phi_e$-fixed ideals of $\mathbb{K}^{\Gamma}[x_1, \ldots, x_d]$ using the algorithm presented in the F-finite case.

(c) Use part (c) of Proposition 3.4.1 in order to recover the $u\Phi_e$-fixed ideals of S.

We end this section with the following:

Remark 3.4.5. As it has already pointed out in the introduction of this chapter, in this section we have just given some theoretical evidence that the algorithm constructed in the F-finite case might be useful when one drops the F-finiteness assumption. We emphasize that, at least for us, it is not clear how we can turn the theoretical constructions presented in this section into an effective procedure for computing all the $u\Phi_e$-fixed ideals of a polynomial ring over any non F-finite ground field of prime characteristic.

Bibliographical notes

The Γ-construction was introduced by M. Hochster and C. Huneke in [68, 6.11] as a main ingredient to prove the existence of test elements in almost all the rings which mainly arise in practice on Algebraic Geometry. At that time, it was unclear for the authors what was the necessary and sufficient assumption for a ring in order to guarantee the existence of such elements. As far as we know, the best result in that direction has been recently obtained by R. Y. Sharp in [124, Theorem 8.4] using his theory of graded annihilator submodules over the Frobenius Ore extension ring.

Going back to the Γ-construction, the presentation of such construction given in this chapter is almost verbatim the one given by M. Hochster in [66, pp. 135–137].

Finally, we have to point out that T. J. Stadnik Jr. in [130, Definition 2.14] has proposed a sheafified version of the eth root ideal. In a more algebraic framework, M. Katzman and W. Zhang (cf. [85, Theorem 3.2]) have generalized the eth root operation for submodules of free modules over $\mathbb{K}[\![x_1, \dots, x_d]\!]$, where $\mathbb{K}$ is any field of prime characteristic.

Chapter 4

Extension problems attached to some spectral sequences

In [6], the authors established the existence of a *Mayer-Vietoris spectral sequence* of local cohomology modules in case the base ring is a polynomial ring over a ground field $\mathbb{K}$ and such local cohomology modules are supported on ideals defining an arrangement of linear varieties. It was also given in [6, Theorem 1.2] a sufficient condition in order to guarantee that such spectral sequence degenerates at the E_2-page. Finally, they determined precisely the extension problems attached to the filtration produced by such degeneration in the category of straight modules (cf. [6, Proposition of page 50]).

In Chapter 1, we reviewed the functor $\mathcal{H}_{R,A}$ (where $A = \mathbb{K}[\![x_1, \ldots, x_d]\!]$, $\mathbb{K}$ is of prime characteristic p, and $R = A/I$ for some ideal I of A) introduced by G. Lyubeznik in [94, Section 4]; this functor has the following behaviour (cf. Example 1.8.17) with respect to local cohomology modules:

$$\mathcal{H}_{R,A}\left(H_{\mathfrak{m}}^{d-i}(R)\right) \cong H_I^i(A).$$

In this way, regarding the existence of the Mayer-Vietoris spectral sequence of local cohomology modules

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} H_{I_p}^j(A) \underset{i}{\Longrightarrow} H_I^{j-i}(A) \qquad (4.1)$$

(where P is the poset of all the possible sums of the prime components of I ordered by reverse inclusion) firstly established in the setting of linear arrangements by J. Àlvarez Montaner, R. García López and S. Zarzuela in [6, Section 2] and later on generalized by G. Lyubeznik in [95, Theorem 2.1], it might seem natural to ask about the existence of a certain spectral sequence

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p\in P} H_{\mathfrak{m}}^j(A/I_p) \underset{i}{\Longrightarrow} H_{\mathfrak{m}}^{i+j}(A/I)$$

which should correspond to the previous Mayer-Vietoris one under $\mathcal{H}_{R,A}$. The problem turns out to be that such dual spectral sequence does not exist in this *nice* formulation; indeed, if such spectral sequence existed, then it would have to collapse according to results obtained by C. U. Jensen (cf. [74, Proposition 7.1]). Regardless, this collapse would lead to the canonical isomorphism

$$H_I^r(A) \cong \bigoplus_{p \in P} H_{I_p}^{\operatorname{ht}(I_p)}(A),$$

which would contradict Hochster's decomposition of local cohomology modules of Stanley-Reisner rings (cf. [103, Theorem 13.13]).

We shall see (cf. Theorem 4.3.15) that there is a spectral sequence which is the most similar to the previous false one, albeit unfortunately this true spectral sequence does not correspond to (4.1) under $\mathcal{H}_{R,A}$.

Now, we provide a brief overview of the contents of this chapter for the reader's benefit; our first aim will be to establish several results (cf. Theorem 4.2.17, Theorem 4.2.25 and Theorem 4.3.15) in order to build several spectral sequences through a common formalism. It is worth mentioning that this fact was partially pointed out in [6, Remark 1.4 (iii)]; in this way, we are able to develop such observation. Since all the spectral sequences which appear in this chapter involve the left (respectively, the right) derived functors of the direct (respectively, the inverse) limit functors, we review in Section 4.1 the facts we need later on about the categories of direct and inverse systems; albeit all the material presented in Section 4.1 is known, we present, as far as possible, a self contained treatment for the convenience of the reader.

Once we establish our spectral sequences, our second goal will be to provide sufficient conditions in order to determine when these spectral sequences degenerate at the E_2-page (cf. Theorem 4.2.30 and Theorem 4.3.15); it is noteworthy that Theorem 4.2.30 recovers and generalizes [6, Theorem 1.2].

Our final objective will be the study of the extension problems attached to the filtration produced by the degeneration of the local cohomology spectral sequences given in Example 4.2.18 and Theorem 4.3.18; on one hand, the spectral sequence obtained in Example 4.2.18 is exactly the Mayer-Vietoris one produced in full generality by Lyubeznik in [95, Theorem 2.1]. In this way, we show that the extension problems associated to the filtration produced by the degeneration of this Mayer-Vietoris spectral sequence are, in general, non-trivial (cf. Counterargument 2 and Counterargument 3); it is worth mentioning that our way of arguing is different with respect to the one adopted in [6].

On the other hand, we see that the extension problems attached to the filtration produced by the degeneration of the local cohomology spectral sequence given in Theorem 4.3.15 are non-trivial; as a final application of this study, we recover and generalize (cf. Theorem 4.3.30) the so-called *Gräbe's formula*, which was obtained by H.-G. Gräbe in [54, Theorem 2].

Special acknowledgement of joint work

The content of this chapter turns out to be an ongoing joint work with J. Àlvarez Montaner and S. Zarzuela (cf. [3]).

4.1 The categories of inverse and direct systems

As pointed out in the introduction of this chapter, the purpose of this section is to review the facts we shall need later on about the categories of inverse and direct systems; we try to present, as far as possible, a self contained exposition of this topic for the reader's profit.

Let $(P, \leq)$ be a partially ordered set (from now on, *poset* for the sake of brevity) and let $\mathcal{A}$ be the category of A-modules, where A denotes a commutative Noetherian ring. We shall regard P as a small category which has as objects the elements of P and, given $p, q \in P$, there is one morphism $p \to q$ if $p \leq q$. If P contains a unique minimal (respectively, maximal) element then this is called the *initial* (respectively, *terminal*) element of P and it will be denoted by 0_P (respectively, 1_P). Adding an initial and a terminal element to P (even in case P have them) we may consider the poset $(\widehat{P}, \leq)$, where $\widehat{P} := P \cup \{0_{\widehat{P}}, 1_{\widehat{P}}\}$.

(a) A *direct system over P valued on $\mathcal{A}$* is a covariant functor $P \longrightarrow \mathcal{A}$.

(b) An *inverse system over P valued on $\mathcal{A}$* is a contravariant functor $P \longrightarrow \mathcal{A}$.

Before going on, we fix some additional notation.

Notation 4.1.1. Hereafter, $\operatorname{Dir}(P, \mathcal{A})$ (respectively, $\operatorname{Inv}(P, \mathcal{A})$) will denote the category of direct systems (respectively, inverse systems) valued on $\mathcal{A}$; the reader should remind that both are abelian categories (cf. [114, Corollary 5.94]), albeit it is worth noting here that we recover indirectly this fact later on in this chapter (cf. Proposition 4.1.19 and Proposition 4.1.20).

Remark 4.1.2. A brief remark about terminology; in [6], the authors used the phrase *diagram* to refer to what here is called direct system. We prefer the terminology of direct and inverse systems because, on one hand, is more classical; on the other hand, according to [114, page 18] a *diagram* is a covariant functor such that its source category is small. Since any poset can be regarded as a small category, it is clear that direct systems are a particular example of diagrams; in fact, it is also known that any small category is of the form $P(Q)/\sim$, where Q is a certain quiver, $P(Q)$ is its categorification, and $\sim$ denotes a certain equivalence relation.

We conclude this introductory part stressing the following:

Assumption 1. Hereafter, we shall always assume that P is a finite poset.

4.1.1 The Roos complexes

Given an inverse system of modules, J. E. Roos and G. Nöbeling independently introduced in [112] and [108] a cochain complex which has as ith cohomology the ith right derived functor of the inverse limit functor. In this subsection, we shall review their definition as well as their dual notion for direct systems.

The homological Roos complex

We consider a direct system over P valued on $\mathcal{A}$ given by a covariant functor $P \xrightarrow{F} \mathcal{A}$. Then, we construct a chain complex (cf. [74, page 33])

$$\dots \longrightarrow \mathrm{Roos}_k(F) \xrightarrow{d_k} \mathrm{Roos}_{k-1}(F) \longrightarrow \dots$$

in the following way:

(a) The spots of the complex are

$$\mathrm{Roos}_k(F) := \bigoplus_{p_0 < \ldots < p_k} F_{p_0 \ldots p_k},$$

where $F_{p_0 \ldots p_k} := F(p_0) \in \mathcal{A}$.

(b) The boundary map $\mathrm{Roos}_k(F) \xrightarrow{d_k} \mathrm{Roos}_{k-1}(F)$ is defined on each direct summand $F_{p_0 \ldots p_k}$ as

$$j_{p_1 \ldots p_k} \circ F(p_0 \to p_1) + \sum_{l=1}^{k} (-1)^l j_{p_0 \ldots \widehat{p_l} \ldots p_k},$$

where $j_{p_0 \ldots p_k}$ denotes the natural inclusion map $F_{p_0 \ldots p_k} \hookrightarrow \mathrm{Roos}_k(F)$.

From now on, we denote by $\mathrm{Roos}_*(F)$ this chain complex. We collect in the following result the main feature of this construction, which we reprove later on in this Chapter (cf. Proposition 4.1.42) for the sake of completeness.

Lemma 4.1.3. *The following statements hold.*

(i) There is an augmented chain complex

$$\mathrm{Roos}_*(F) \longrightarrow \varinjlim_{p \in P} F(p) \longrightarrow 0$$

in the category $\mathcal{A}$.

(ii) The homology of this chain complex gives the left derived functors of the direct limit; that is,

$$H_i(\mathrm{Roos}_*(F)) = \mathbb{L}_i \varinjlim_{p\in P} F(p).$$

In particular, when $i=0$ one has that

$$H_0(\mathrm{Roos}_*(F)) = \varinjlim_{p\in P} F(p).$$

Remark 4.1.4. It is worth mentioning here that the left derived functors of the direct limit $\mathbb{L}_i \varinjlim_{p\in P} F(p)$ are actually objects of $\mathcal{A}$. Indeed, as $\mathcal{A}$ is the category of modules over a commutative ring A, the classical construction of these groups through the Roos chain complex can be carried out inside $\mathcal{A}$; recall that

$$\varinjlim_{p\in P} F(p) = \left(\bigoplus_{p\in P} F(p)\right)/N,$$

where $N := \langle (j_q \circ F(p\to q))(m_p) - j_p(m_p) \mid \quad p\leq q, \quad m_p \in F_p\rangle$ and, for each $p\in P$, j_p denotes the natural inclusion map $F(p) \overset{j_p}{\hookrightarrow} \mathrm{Roos}_0(F)$.

Although the following fact is so elementary, we want to state it because it will play a key role later on (e. g. Construction 4.2.10).

Lemma 4.1.5. *Assume, in addition, that $F(1_{\widehat{P}})$ is defined; that is, that F is not only defined on P but also on $P\cup\{1_{\widehat{P}}\}$. Then, there is a unique functorial map*

$$\varinjlim_{p\in P} F(p) \xrightarrow{\psi} F\left(1_{\widehat{P}}\right)$$

such that the diagram of chain complexes

$$\begin{array}{ccccc}
\mathrm{Roos}_*(F) & \xrightarrow{d_0} & \varinjlim_{p\in P} F(p) & \longrightarrow & 0 \\
\| & & \downarrow \psi & & \\
\mathrm{Roos}_*(F) & \xrightarrow{\psi\circ d_0} & F\left(1_{\widehat{P}}\right) & \longrightarrow & 0
\end{array}$$

is commutative.

Proof. For any $p\in P$, $p<1_{\widehat{P}}$ and therefore there is an arrow $F(p)\longrightarrow F\left(1_{\widehat{P}}\right)$. In this way, the universal property of the direct limit produces such ψ with the desired properties. □

Remark 4.1.6. Preserving the assumptions of Lemma 4.1.5, it is unclear for us whether ψ defines an isomorphism or not. Equivalently, it is unclear for us whether the natural map

$$\varinjlim_{p\in P} F(p) \xrightarrow{\psi} \varinjlim_{p\in \widehat{P}} F(p) = F\left(1_{\widehat{P}}\right)$$

is bijective or not. It is noteworthy that the equality

$$\varinjlim_{p\in \widehat{P}} F(p) = F\left(1_{\widehat{P}}\right)$$

is well known (cf. [114, Part (iii) of Exercise 5.22]).

We end this subsubsection with the following:

Remark 4.1.7. We denote by $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes of objects of $\mathcal{A}$. In this way, we have built a functor $\mathrm{Dir}(P,\mathcal{A}) \xrightarrow{\mathrm{Roos}_*(-)} \mathrm{Ch}(\mathcal{A})$ which is exact and commutes with arbitrary direct sums; that is, it is straightforward to check that, on one hand, given any short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

of direct systems, the induced sequence of chain complexes

$$0 \longrightarrow \mathrm{Roos}_*(F') \longrightarrow \mathrm{Roos}_*(F) \longrightarrow \mathrm{Roos}_*(F'') \longrightarrow 0$$

is also exact; on the other hand, given an arbitrary set of indexes (namely, Q) and given any family $\{F_q\}_{q\in Q}$ of direct systems indexed by Q, one has that there is a canonical isomorphism

$$\mathrm{Roos}_*\left(\bigoplus_{q\in Q} F_q\right) \cong \bigoplus_{q\in Q} \mathrm{Roos}_*(F_q)$$

of direct systems.

The cohomological Roos complex

Now, we consider an inverse system over P valued on $\mathcal{A}$ given by a contravariant functor $P \xrightarrow{G} \mathcal{A}$. Thus, we build a cochain complex of inverse systems (cf. [74, pp. 31-32] or [140, Vista 3.5.12])

$$\ldots \longrightarrow \mathrm{Roos}^k(G) \xrightarrow{d^k} \mathrm{Roos}^{k+1}(G) \longrightarrow \ldots$$

as follows.

(a) The pieces of the complex are

$$\mathrm{Roos}^k(G) := \prod_{p_0<\ldots<p_k} G_{p_0\ldots p_k},$$

where $G_{p_0\ldots p_k} := G(p_0)$.

(b) The coboundary map $\mathrm{Roos}^k(G) \xrightarrow{d^k} \mathrm{Roos}^{k+1}(G)$ is defined on each factor $G_{p_0\ldots p_k}$ as

$$G(p_0 \to p_1) \circ \pi_{p_1\ldots p_{k+1}} + \sum_{l=1}^{k+1} (-1)^l \pi_{p_0\ldots\widehat{p_l}\ldots p_k},$$

where $\pi_{p_0\ldots p_k}$ denotes the natural projection $\mathrm{Roos}^k(G) \twoheadrightarrow G_{p_0\ldots p_k}$.

Hereafter, we shall denote by $\mathrm{Roos}^*(G)$ this cochain complex. As in the homological case, the main feature of this cohomological construction is recalled in the following result; later on in this Chapter (cf. Proposition 4.1.32), we provide a detailed proof of this fact.

Lemma 4.1.8. *The following statements hold.*

(i) There is a coaugmented cochain complex

$$0 \longrightarrow \varprojlim_{p\in P} G(p) \longrightarrow \mathrm{Roos}^*(G)$$

in the category $\mathcal{A}$.

(ii) The cohomology of this cochain complex yields the right derived functors of the inverse limit; that is,

$$H^i(\mathrm{Roos}^*(G)) = \mathbb{R}^i \varprojlim_{p\in P} G(p).$$

In particular, when $i = 0$ one has that

$$H^0(\mathrm{Roos}^*(G)) = \varprojlim_{p\in P} G(p).$$

Remark 4.1.9. As in the homological framework, it is worth mentioning the fact that $\mathbb{R}^i \varprojlim_{p\in P} G(p)$ are objects of $\mathcal{A}$; in this case, the details are left to the interested reader.

We state the analogous cohomological of Lemma 4.1.5. We skip the details.

Lemma 4.1.10. *Suppose, in addition, that $G(0_{\widehat{P}})$ is defined; that is, that G is not only defined on P but also in $P \cup \{0_{\widehat{P}}\}$. Then, there is a unique functorial map*

$$G(0_{\widehat{P}}) \xrightarrow{\alpha} \varprojlim_{p\in P} G(p)$$

such that the diagram of cochain complexes

$$\begin{array}{ccccc} 0 & \longrightarrow & G(0_{\widehat{P}}) & \xrightarrow{d^0\circ\alpha} & \mathrm{Roos}^*(G) \\ & & \downarrow{\scriptstyle\alpha} & & \| \\ 0 & \longrightarrow & \varprojlim_{p\in P} G(p) & \xrightarrow{d^0} & \mathrm{Roos}^*(G) \end{array}$$

is commutative.

Remark 4.1.11. Preserving the assumptions of Lemma 4.1.10, it is unclear for us whether α defines an isomorphism or not. In other words, we do not know whether the natural restriction map

$$G(0_{\widehat{P}}) = \varprojlim_{p\in\widehat{P}} G(p) \xrightarrow{\alpha} \varprojlim_{p\in P} G(p)$$

is an isomorphism. It is known (cf. [31, Example 3.3]) to be an isomorphism in some particular situations. We shall come back later on to this point (cf. Subsection 4.2.1).

As in the homological case, we end this subsubsection with the following:

Remark 4.1.12. If we denote by $\mathrm{CoCh}(\mathcal{A})$ the category of cochain complexes of objects of $\mathcal{A}$, then we have produced a functor $\mathrm{Inv}(P,\mathcal{A})\xrightarrow{\mathrm{Roos}^*(-)}\mathrm{CoCh}(\mathcal{A})$ which is exact and commutes with arbitrary direct products; that is, it is straightforward to check that, on one hand, given any short exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

of inverse systems, the induced sequence of cochain complexes

$$0 \longrightarrow \mathrm{Roos}^*(G') \longrightarrow \mathrm{Roos}^*(G) \longrightarrow \mathrm{Roos}^*(G'') \longrightarrow 0$$

is also exact; on the other hand, given an arbitrary set of indexes (namely, Q) and given any family $\{F_q\}_{q\in Q}$ of inverse systems indexed by Q, one has that there is a canonical isomorphism

$$\mathrm{Roos}^*\left(\prod_{q\in Q} G_q\right) \cong \prod_{q\in Q} \mathrm{Roos}^*(G_q)$$

of inverse systems.

4.1.2 Equivalent approaches

Throughout this chapter, we decided to choose the previous approach to study derived functors of direct and inverse limits. Nevertheless, there are other equivalent approaches to this subject. Some of them are reviewed in what follows. The reader is encouraged to follow his/her own preferences.

The category of sheafs on posets

We start recalling once again our setup for the reader's benefit. Let $(P, \leq)$ be a poset such that, given $p, q \in P$, there is one morphism $p \to q$ if $p \leq q$. If P contains a unique minimal (respectively, maximal) element then this is called the *initial* (respectively, *terminal*) element of P and it will be denoted by 0_P (respectively, 1_P). Adding an initial and a terminal element to P (even in case P have them) we may consider the poset $(\widehat{P}, \leq)$, where $\widehat{P} := P \cup \{0_{\widehat{P}}, 1_{\widehat{P}}\}$. The closed interval of elements between p and q will be denoted

$$[p, q] := \{z \in P \mid \quad p \leq z \leq q\}$$

and form a sub-poset of P. In a similar way, we can also construct the intervals $(p, q]$, $[p, q)$ and (p, q).

The *Alexandrov topology* on P is the topology where the open sets are the subsets U of P such that $p \in U$ and $p \leq q$ implies $q \in U$. In fact, this is the unique topology which one can attach to P verifying this property. Moreover, the subsets of the form $[p, 1_{\widehat{P}})$ form a open basis for this topology.

On the other hand, the *dual Alexandrov topology* on P is the topology where the open sets are the subsets U of P such that $p \in U$ and $q \leq p$ implies $q \in U$. Once more, this is the unique topology which one can attach to P verifying this property and the subsets of the form $(0_{\widehat{P}}, p]$ form a open basis for this topology. We underline that this can be viewed as the Alexandrov topology on the opposite poset $P^{op} = (P, \preceq)$, where $p \preceq q$ if and only if $q \leq p$.

We shall fix some additional notation before going on.

Notation 4.1.13. In the sequel, we shall denote by $\mathrm{Sh}(P, \mathcal{A})$ (respectively, $\mathrm{Sh}(P^{op}, \mathcal{A})$) the category of sheaves on P (respectively, P^{op}) valued on $\mathcal{A}$.

We conclude this subsubsection with the following elementary statement, which will be useful in what follows; although we think the below result is well known, we provide a proof because of the lack of a reference.

Lemma 4.1.14. *Let $(P, \leq)$ be a poset regarded as a topological space with the Alexandrov topology. Then, for any $p \in P$, the basic open set $[p, 1_{\widehat{P}})$ is contractible.*

Proof. Let $[p, 1_{\widehat{P}}) \xrightarrow{r} \{p\}$ be the constant map, let $\{p\} \xhookrightarrow{j} [p, 1_{\widehat{P}})$ be the inclusion map and set

$$\begin{aligned} H: \quad & [p, 1_{\widehat{P}}) \times [0, 1] \longrightarrow [p, 1_{\widehat{P}}) \\ & (q, t) \longmapsto \begin{cases} p, & \text{if} \quad t = 0, \\ q, & \text{if} \quad t \neq 0. \end{cases} \end{aligned}$$

We have to point out that the following statements hold.

(i) $r \circ j = \mathbb{1}_{\{p\}}$, $H(p,t) = p$ for any $t \in [0,1]$, $H(q,0) = p = (r \circ j)(p)$ for any $q \in [p, 1_{\widehat{P}})$ and $H(q,1) = q = \mathbb{1}_{[p,1_{\widehat{P}})}(q)$.

(ii) H is a continuous map. Indeed, let $q \in P$ and let $[q, 1_{\widehat{P}})$ be a basic open subset of $[p, 1_{\widehat{P}})$. We have to underline that

$$H^{-1}([q, 1_{\widehat{P}})) = \begin{cases} [p, 1_{\widehat{P}}) \times [0,1], & \text{if} \quad q = p, \\ [q, 1_{\widehat{P}}) \times (0,1], & \text{if} \quad q > p. \end{cases}$$

In any case, $H^{-1}([q, 1_{\widehat{P}}))$ is an open subset of $[p, 1_{\widehat{P}}) \times [0,1]$.

The foregoing shows that $\{p\}$ is a strong deformation retract of $[p, 1_{\widehat{P}})$. $\square$

We also want to establish now the analogous of Lemma 4.1.14 regarding P as a topological space with the dual Alexandrov topology; since the proof of such result is almost verbatim the one of Lemma 4.1.14 (with few minor changes), we omit it.

Lemma 4.1.15. *Let $(P, \leq)$ be a poset regarded as a topological space with the dual Alexandrov topology. Then, for any $p \in P$, the basic open set $(0_{\widehat{P}}, p]$ is contractible.*

The interested reader on Alexandrov spaces and its topological properties may like to consult [9] (and the references therein) for additional information.

The category of AP-modules

The purpose of this part is to review the category of left AP-modules, as presented in [31, Section 6]; in fact, whereas in op. cit. the authors established this notion in the context of inverse systems, here our definition deals with direct systems. However, it is clear that both notions can be mutually recovered just by taking the opposite order on the poset P.

Definition 4.1.16. A left *AP-module* M is a system $(M_p)_{p \in P}$ of left A-modules and, for $p \leq q$, homomorphisms $M_p \xrightarrow{M_{pq}} M_q$ with the property that, for all $p \leq q \leq z$,

$$M_{pp} = \mathbb{1}_{M_p} \text{ and } M_{pq} \circ M_{qz} = M_{pz}.$$

A *homomorphism* $M \xrightarrow{f} N$ of left AP-modules consists of, for $p \in P$, homomorphisms $M_p \xrightarrow{f_p} N_p$ of left A-modules such that, for any $p \leq q$ in P, the following square commutes.

$$\begin{array}{ccc} M_p & \xrightarrow{f_p} & N_p \\ {\scriptstyle M_{pq}}\downarrow & & \downarrow{\scriptstyle N_{pq}} \\ M_q & \xrightarrow[f_q]{} & N_q \end{array}$$

We shall denote the group of homomorphisms from M to N by $\mathrm{Hom}_{AP}(M,N)$. Moreover, we shall denote by $AP-\mathrm{Mod}$ the category of left AP-modules.

In a similar way, just reversing the convenient morphisms, we can also construct the category $AP^{op}-\mathrm{Mod}$ of left AP^{op}-modules.

Remark 4.1.17. Albeit we do not exploit it in what follows, it is worth mentioning that in [31, Section 3] it is also defined the notion of *AP-algebra*; the interested reader on this notion may like to consult op. cit. and [32] for additional details.

Modules over the incidence algebra

The reader should remind here that P is a finite poset (cf. Assumption 1) and that A is a commutative ring. We review the following construction (cf. [129, Definition 1.2.1]).

Definition 4.1.18 (Incidence algebra)**.** We define the *incidence algebra* $I(P,A)$ of P over A as follows; $I(P,A)$ is the A-algebra with underlying A-module

$$I(P,A) := \bigoplus_{p\leq q} A\cdot \mathbf{e}_{p\leq q}$$

endowed with the following multiplication rule:

$$(\mathbf{e}_{p\leq q})\cdot(\mathbf{e}_{p'\leq q'}) := \begin{cases} \mathbf{e}_{p\leq q'}, \text{ if } q=p', \\ 0, \text{ otherwise.} \end{cases}$$

It is worth mentioning that the elements $\mathbf{e}_{p\leq p}$ are idempotent in $I(P,A)$ and that, since P is finite, the element

$$\sum_{p\in P} \mathbf{e}_{p\leq p}$$

is the multiplicative unit in $I(P,A)$.

The four previous categories are equivalent

In this part, we justify that all the foregoing approaches are equivalent and we make the equivalence explicit. It is worth mentioning that such statement may be regarded a a dual version of [31, Remark 6.5 and Proposition 6.6]. We are to provide a proof for the reader's benefit.

Proposition 4.1.19. *The following categories are equivalent:*

(a) $\mathrm{Dir}(P,\mathcal{A})$*, the category of direct systems on* P *valued in* $\mathcal{A}$*.*

(b) $\mathrm{Sh}(P,\mathcal{A})$*, the category of sheaves on* P *valued in* $\mathcal{A}$ *regarding* P *as a topological space with the Alexandrov topology.*

(c) $AP-\mathrm{Mod}$, the category of left AP-modules.

(d) The category of left modules over the incidence algebra $I(P^{op}, A)$.

Proof. The equivalence between (a) and (c) is tautological. Now, we are to describe the equivalence between (a) and (d).

Firstly, given a left $I(P^{op}, A)$-module $\mathcal{M}$ we produce a direct system as follows; indeed, set $M := (M_p)_{p\in P}$, where

$$M_p := (\mathbf{e}_{p\leq p})\mathcal{M}.$$

On the other hand, given $p \leq q$ we also set

$$\begin{aligned} &M_p \xrightarrow{M_{pq}} M_q \\ &(\mathbf{e}_{p\preceq p})m \longmapsto (\mathbf{e}_{q\preceq p})(\mathbf{e}_{p\preceq p})m = (\mathbf{e}_{q\preceq p})m = (\mathbf{e}_{q\preceq q})(\mathbf{e}_{q\preceq p})m. \end{aligned}$$

Conversely, given a direct system M we can define a left $I(P^{op}, A)$-module $\mathcal{M}$ as follows: set

$$\mathcal{M} := \bigoplus_{p\in P} M_p$$

with multiplication defined in the following way:

$$(\mathbf{e}_{q\preceq p})m_p := \begin{cases} M_{pq}(m_p), \text{ if } m_p \in M_p, \\ 0, \text{ otherwise.} \end{cases}$$

It turns out that the foregoing two correspondences defines an equivalence of categories between left $I(P^{op}, A)$-modules and direct systems valued on $\mathcal{A}$.

In this way, it only remains to establish an equivalence between (b) and (c).

Let $\mathcal{F}$ be a sheaf on P valued in $\mathcal{A}$. We set $\Gamma(P, \mathcal{F})$ as the left AP-module with

$$\Gamma(P, \mathcal{F})_p := \mathcal{F}([p, 1_{\widehat{P}}))$$

(where $p \in P$) and with $\Gamma(P, \mathcal{F})_{pq}$ (where $p \leq q$) equal to the restriction homomorphism attached to the inclusion $[q, 1_{\widehat{P}}) \subseteq [p, 1_{\widehat{P}})$. This defines a functor $\Gamma(P, -)$ from the category of sheaves of left A-modules on P to the category of left AP-modules.

Conversely, let M be a left AP-module and let $U \subseteq P$ be an open subset of the topological space P. Set

$$\widetilde{M}(U) := \varinjlim_{p\in U} M_p.$$

Moreover, given an inclusion $V \subseteq U$ of open subsets of P, the natural restriction map $\widetilde{M}(U) \longrightarrow \widetilde{M}(V)$ is just the natural restriction map of direct limits. In addition, since $[p, 1_{\widehat{P}})$ is contained in any neighbourhood of p (indeed, it follows from the definition of the Alexandrov topology on P) one has that the presheaf $\widetilde{M}$ is isomorphic to the associated sheaf of left A-modules on P. Therefore, the symbol $\widetilde{(-)}$ defines a functor from $AP-\mathrm{Mod}$ to $\mathrm{Sh}(P,\mathcal{A})$.

Now, we are going to check that $\Gamma(P,\widetilde{(-)}) \cong \mathbb{1}_{AP-\mathrm{Mod}}$ and $\widetilde{\Gamma(P,-)} \cong \mathbb{1}_{\mathrm{Sh}(P,\mathcal{A})}$.

Let M be a left AP-module. The homomorphism

$$M_p \longrightarrow \Gamma(P,\widetilde{M})_p = \varinjlim_{q\in[p,1_{\widehat{P}})} M_q$$

induced by the structural maps $M_p \xrightarrow{M_{pq}} M_q$ is a natural isomorphism. It shows that

$$\Gamma(P,\widetilde{(-)}) \cong \mathbb{1}_{AP-\mathrm{Mod}}.$$

On the other hand, let $\mathcal{F}$ be a sheaf of left A-modules on P and let U be an open subset of P. Fix $q \in P$. Since the homomorphism

$$\mathcal{F}([q,1_{\widehat{P}})) \longrightarrow \widetilde{\Gamma(P,\mathcal{F})}([q,1_{\widehat{P}})) = \varinjlim_{p\in[q,1_{\widehat{P}})} \mathcal{F}([p,1_{\widehat{P}}))$$

is an isomorphism and q is any element of U, it follows that there is an isomorphism

$$\mathcal{F}(U) \longrightarrow \widetilde{\Gamma(P,\mathcal{F})}(U) = \varinjlim_{p\in U} \mathcal{F}([p,1_{\widehat{P}})) \tag{4.2}$$

gluing the previous ones (indeed, we are using here the second sheaf axiom on $\mathcal{F}$ with respect to the open covering $\{[q,1_{\widehat{P}})\}_{q\in U}$ of U). However, U is an arbitrary open subset of P and therefore (4.2) defines, in fact, a natural isomorphism of sheaves $\mathcal{F} \cong \widetilde{\Gamma(P,\mathcal{F})}$, whence one obtains the equivalence of functors $\widetilde{\Gamma(P,-)} \cong \mathbb{1}_{\mathrm{Sh}(P,\mathcal{A})}$, just what we finally wanted to check. □

The following result also establishes an equivalence among several categories. We omit its proof and refer to [31, Remark 6.5 and Proposition 6.6] for details.

Proposition 4.1.20. *The following categories are equivalent:*

(a) $\mathrm{Inv}(P,\mathcal{A})$*, the category of inverse systems on P valued in $\mathcal{A}$.*

(b) $\mathrm{Sh}(P^{op},\mathcal{A})$*, the category of sheaves on P^{op} valued in $\mathcal{A}$ regarding P as a topological space with the Alexandrov topology.*

(c) $AP^{op}-\mathrm{Mod}$*, the category of left AP^{op}-modules.*

(d) The category of left modules over the incidence algebra $I(P,A)$.

We conclude this part with the following:

Remark 4.1.21. Although we do not use it in what follows, it is noteworthy that, under the equivalence between inverse systems and sheaves of A-modules over P^{op}, the resolution of (weakly) flasque inverse systems (cf. Definition 4.1.27) used in the proof of Proposition 4.1.32 (namely, Π^*) corresponds to the so-called *Godement resolution* (cf. [114, Proposition 6.73 and Definition of page 381]) for computing sheaf cohomology on the topological space P^{op}; on the other hand, it was already pointed out in [31, Remark 7.2] that the equivalence between sheaves and left AP^{op}-modules transforms sheaf cohomology on the topological space P^{op} into the right derived functors of $\mathrm{Hom}_{AP^{op}}(A,-)$ (namely, $\mathrm{Ext}^*_{AP^{op}}(A,-)$). Summing up, all the equivalences in Proposition 4.1.20 are preserved in the derived category $D(\mathcal{A})$ of $\mathcal{A}$.

4.1.3 Injective and projective objects in the category of inverse systems

Proposition 4.1.20 implies that $\mathrm{Inv}(P,\mathcal{A})$ is equivalent to the category of sheaves on P^{op} valued on $\mathcal{A}$, where P^{op} is regarded as a topological space with the Alexandrov topology. From this result, we might deduce that the existence of enough injective objects in $\mathrm{Inv}(P,\mathcal{A})$ is equivalent to the existence of enough injective objects in $\mathrm{Sh}(P^{op},\mathcal{A})$; albeit this last fact is well known (see, for instance, [61, Chapter 1]), we want to provide here a proof of this result, which we think is interesting in its own right, not only for the sake of completeness, but also because it provides a explicit description of the injectives of $\mathrm{Inv}(P,\mathcal{A})$ which will be useful later on for our purposes (cf. Construction 4.3.1).

Before doing so, we review the following technical statement about adjoint functors between abelian categories. Because the only available reference we know of the below result are a set of unpublished lecture notes by B. Keller (see also [113, Theorem 11.8] for a partial statement), we provide a proof for the convenience of the reader.

Proposition 4.1.22 (Keller)**.** *Let $\mathcal{C},\mathcal{C}'$ be abelian categories and let $\mathcal{C}\xrightarrow{R}\mathcal{C}'$, $\mathcal{C}'\xrightarrow{L}\mathcal{C}$ be a pair of adjoint functors. Then, the following assertions hold.*

(i) If L is exact, then R preserves injectives.

(ii) If L is exact, faithful, and $\mathcal{C}$ has enough injective objects, then $\mathcal{C}'$ has also enough injective objects; in such case, any injective object of $\mathcal{C}'$ is a direct summand of an injective of the form $R(I)$, where I is an injective object of $\mathcal{C}$.

Proof. Let I be an injective object of $\mathcal{C}$. By the adjointness of the pair (L,R), one gets the following natural equivalence of functors:

$$\mathrm{Hom}_{\mathcal{C}}(L(-),I)\cong \mathrm{Hom}_{\mathcal{C}'}(-,R(I)).$$

Moreover, since L is exact by assumption, $\mathrm{Hom}_{\mathcal{C}}(L(-),I)$ is an exact functor because it is a composition of two exact ones; namely, firstly apply L and then apply $\mathrm{Hom}_{\mathcal{C}}(-,I)$.

Whence $\operatorname{Hom}_{\mathcal{C}'}(-, R(I))$ is exact and therefore we can conclude that $R(I)$ is an injective object of $\mathcal{C}'$; in this way, part (i) holds.

Now, we want to prove part (ii) under the additional assumption that L is faithful, which we suppose henceforth. Let A' be an object of $\mathcal{C}'$; since $\mathcal{C}$ has enough injectives, there is an injective object I of $\mathcal{C}$ and a monomorphism $L(A') \hookrightarrow I$. In addition, since (L, R) is an adjoint pair, R is automatically left exact; therefore, applying R to the previous monomorphism we get a new one in $\mathcal{C}'$; namely,

$$0 \longrightarrow RL(A') \xrightarrow{\ j\ } R(I).$$

We have to point out that, by part (i), $R(I)$ is an injective object of $\mathcal{C}'$; on the other hand, the natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}}\big(L(A'), L(A')\big) \cong \operatorname{Hom}_{\mathcal{C}'}\big(A', RL(A')\big)$$

maps $\mathbb{1}_{L(A')}$ isomorphically into a certain $\psi \in \operatorname{Hom}_{\mathcal{C}'}(A', RL(A'))$. In this way, we consider the composition

$$A' \xrightarrow{\ \psi\ } RL(A') \xrightarrow{\ j\ } R(I);$$

thus, as j is a monomorphism we only need to check out that ψ is a monomorphism. Indeed, let B' be an object of $\mathcal{C}'$ and let $B' \xrightarrow{\ h\ } A'$ be an element of $\operatorname{Hom}_{\mathcal{C}'}(B', A')$ such that $\psi h = 0$. Now, using once more the naturality of the adjoint pair (L, R) we obtain the following commutative square:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{C}}(L(A'), L(A')) & \xrightarrow[\sim]{\tau_{A', L(A')}} & \operatorname{Hom}_{\mathcal{C}'}(A', RL(A')) \\
{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(L(h), L(A'))}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}'}(h, RL(A'))} \\
\operatorname{Hom}_{\mathcal{C}}(L(B'), L(A')) & \xrightarrow[\sim]{\tau_{B', L(A')}} & \operatorname{Hom}_{\mathcal{C}'}(B', RL(A')).
\end{array}$$

Now, we want to single out how the previous commutative square acts on $\mathbb{1}_{L(A')}$; namely,

$$\begin{array}{ccc}
\mathbb{1}_{L(A')} & \longmapsto & \psi \\
\downarrow & & \downarrow \\
h & \longmapsto & L(h) = \psi h = 0.
\end{array}$$

Summing up, $L(h) = \psi h = 0$, whence $h = 0$ because of L is faithful by assumption; therefore, it implies that ψ is a monomorphism. Thus, we have produced a monomorphism $A' \hookrightarrow R(I)$, where A' is an arbitrary object of $\mathcal{C}'$ and $R(I)$ is an injective object of $\mathcal{C}'$; this shows that $\mathcal{C}'$ has enough injectives, as claimed.

In this way, it only remains to check out that, under the assumptions of part (ii), any injective object of $\mathcal{C}'$ is a direct summand of an injective of the form $R(I)$, where I is an injective object of $\mathcal{C}$; indeed, let I' be an injective object of $\mathcal{C}'$. Since $\mathcal{C}$ has enough injectives, there exists an injective object I of $\mathcal{C}$ and a monomorphism $L(I') \hookrightarrow I$. Applying to this monomorphism the left exact functor R, we obtain a monomorphism $RL(I') \hookrightarrow R(I)$; moreover, as in the previous part we can show that the natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\left(L(I'), L(I')\right) \cong \mathrm{Hom}_{\mathcal{C}'}\left(I', RL(I')\right)$$

maps $\mathbb{1}_{L(I')}$ isomorphically into a certain monomorphism $I' \hookrightarrow RL(I')$. In this way, composing we get the following monomorphism $I' \hookrightarrow RL(I') \hookrightarrow R(I)$; however, this monomorphism splits because I' is injective and therefore we can ensure that I' is a direct summand of $R(I)$. This is exactly what we finally wanted to show. □

Our next aim is to show, with the help of Proposition 4.1.22, that the category of sheaves valued on A-modules has enough injective objects; as the reader can easily notice, the proof of the below result turns out to be nothing but a refinement of the one presented in [114, Proposition 5.97].

Theorem 4.1.23. *Given any topological space X, the category $\mathrm{Sh}(X, \mathcal{A})$ has enough injective objects.*

Proof. Firstly, we review some classical constructions from sheaf theory; indeed, given an A-module M and fixed $x \in X$ we consider the so-called *skyscraper sheaf* $x_*(M)$, which can be defined in the following way: given U an open subset of X, one sets

$$x_*(M)(U) := \begin{cases} M, & \text{if } x \in U, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand, we can also consider the so-called *stalk functor* with respect to x; namely, the functor whichs maps a sheaf F to

$$F_x := \varinjlim_{U \in U_x} F(U),$$

where U_x denotes all the open subsets of X which contain x.

Now, we have two exact functors $\mathcal{A} \xrightarrow{x_*} \mathrm{Sh}(X, \mathcal{A})$ and $\mathrm{Sh}(X, \mathcal{A}) \xrightarrow{(-)_x} \mathcal{A}$; the problem turns out to be that the stalk functor is not faithful (indeed, because a non-zero sheaf might have some stalks zero), so we need to work a bit more in order to apply Proposition 4.1.22. But we can arrange this situation with the so-called *product category* $\mathcal{A}^X$; the reader should remind that $\mathcal{A}^X$ has enough injectives because $\mathcal{A}$ has so (indeed, slightly loosely speaking all in $\mathcal{A}^X$ is defined componentwise). In this way, we have two exact functors $\mathcal{A}^X \xrightarrow{x_*} \mathrm{Sh}(X, \mathcal{A})$ and $\mathrm{Sh}(X, \mathcal{A}) \xrightarrow{\prod_{x \in X} (-)_x} \mathcal{A}^X$; however, we have to point out that $\prod_{x \in X} (-)_x$ is also faithful becase a sheaf is zero if and only if all of its stalks vanish.

Therefore, we only have to check out that $\left(\prod_{x\in X}(-)_x, x_*\right)$ is an adjoint pair; in such case, Proposition 4.1.22 will imply that $\mathrm{Sh}(X,\mathcal{A})$ has enough injective objects; regardless, the adjointness of the previous functors boils down to the following chain of natural isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{A}^X}\left(\prod_{x\in X} F_x, \prod_{x\in X} M_x\right) \cong \prod_{x\in X} \mathrm{Hom}_{\mathcal{A}}\left(F_x, M_x\right) \cong \prod_{x\in X} \mathrm{Hom}_{\mathrm{Sh}(X,\mathcal{A})}\left(F, x_*(M_x)\right)\\ &\cong \mathrm{Hom}_{\mathrm{Sh}(X,\mathcal{A})}\left(F, x_*\left(\prod_{x\in X} M_x\right)\right).\end{aligned}$$

We have to stress here that the only non-trivial part is the known fact that, for any $x\in X$, $((-)_x, x_*)$ is an adjoint pair; the proof is therefore completed. $\square$

Before going on, we want to single out the following result, which has been obtained during the proof of Theorem 4.1.23, because it will play a crucial role later on in this chapter.

Theorem 4.1.24. *Any injective object of* $\mathrm{Inv}(P,\mathcal{A})$ *is a direct summand of a direct sum of injectives of the form* $E_{\geq q}$ *for some* $q\in P$*, where* $E = E_A(A/\mathfrak{p})$ *is an indecomposable injective* A*-module, and*

$$\left(E_{\geq q}\right)_p := \begin{cases} E, & \text{if } p\in [q, 1_{\widehat{P}}),\\ 0, & \text{otherwise.}\end{cases}$$

We conclude this part with the following:

Remark 4.1.25. The statement of Theorem 4.1.24 is not fully satisfactory in the sense that one would wish to have an structure theorem of injectives in $\mathrm{Inv}(P,\mathcal{A})$; in other words, we would want to know what are the indecomposable injectives of $\mathrm{Inv}(P,\mathcal{A})$. We do not know whether such description exists.

Existence of enough flasque inverse systems

The goal of this part is to provide a detailed proof of Lemma 4.1.8 (being precise, of part (ii)); that is, we want to show that the cohomology of the cochain Roos complex agree with the right derived functors of the inverse limit. First of all, we have to point out that, since the category of inverse systems has enough injectives, one has that the right derived functors of the inverse limit can be defined in the usual way through injective resolutions; regardless, the calculation of these right derived functors through injective resolutions is not suitable in this framework. That is, injective resolutions are necessary in order to define in the usual way the right derived functors of the inverse limit; however, in this case, it turns out that one needs to compute explicitly such derived functors and, for this purpose, injective inverse systems are not useful.

In this way, we have to restrict our attention to the following subclass of objects of $\mathrm{Inv}(P, \mathcal{A})$.

Definition 4.1.26. We say that an object G of $\mathrm{Inv}(P, \mathcal{A})$ is *flasque* if, for any pair $U \subseteq V \subseteq P$ of open subsets of P (regarding P as a topological space with the dual Alexandrov topology), the natural restriction map

$$\varprojlim_{v \in V} G(v) \longrightarrow \varprojlim_{u \in U} G(u)$$

is surjective.

In fact, we also have to consider the following bigger subclass of objects in $\mathrm{Inv}(P, \mathcal{A})$ (cf. [74, Lemme 1.3]):

Definition 4.1.27. It is said that an object G of $\mathrm{Inv}(P, \mathcal{A})$ is *weakly flasque* if, for any $p_0 \leq q_0$, the natural restriction map

$$\varprojlim_{q \in (0_{\widehat{P}}, q_0]} G(q) \longrightarrow \varprojlim_{p \in (0_{\widehat{P}}, p_0]} G(p)$$

is surjective.

Remark 4.1.28. Regarding its very definition, it is clear that flasque inverse systems are, in particular, weakly flasque; it is worth noting that the converse is, in general, not true. Essentially, the difference stems from the fact that, whereas weakly flasque objects forms a class which is closed under short exact sequences, flasque objects are not. In particular, as pointed out by C. U. Jensen in [74, Chapter 1], the quotient I/G of an injective inverse system I by a flasque one G is, in general, not flasque; however, I/G is always weakly flasque.

As we have roughly explained in the previous remark, the main reason for considering weakly flasque inverse systems is given formally in the next result, whose proof is omitted; the interested reader may like to consult [74, Corollaire 1.7] for details.

Proposition 4.1.29. *Let* $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ *be a short exact sequence in* $\mathrm{Inv}(P, \mathcal{A})$*. If* G' *and* G *are weakly flasque, then so is* G''*.*

Now, we want to single out the following result because it will play some role soon (cf. Proposition 4.1.32).

Lemma 4.1.30. *Any direct summand of a flasque (respectively, weakly flasque) inverse system is also flasque (respectively, weakly flasque).*

Proof. We only prove the flasque piece of the statement (because the remainder weakly flasque part can be proved exactly in the same way). Indeed, let G'' be an inverse system such that $G = G'\Pi G''$ for some flasque inverse system G and for some (a priori non necessarily flasque) inverse system G'; so, the natural splitted short exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$$

induces, for any pair $U \subseteq V$ of open subsets of P, the following commutative diagram with exact rows, where the vertical columns are the corresponding natural restriction maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim_{v\in V} G'(v) & \longrightarrow & \varprojlim_{v\in V} G(v) & \longrightarrow & \varprojlim_{v\in V} G''(v) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varprojlim_{u\in U} G'(u) & \longrightarrow & \varprojlim_{u\in U} G(u) & \longrightarrow & \varprojlim_{u\in U} G''(u) & \longrightarrow & 0. \end{array}$$

Since the central vertical arrow is surjective (because of G is flasque) the Snake's Lemma ensures that the righmost vertical one is also surjective. In particular, this shows that G'' is also flasque; the proof is therefore completed. □

The last technical fact we need is the following result, which was proved by G. Nöbeling in [108, Satz 6]; we provide a detailed proof for the convenience of the reader.

Proposition 4.1.31. *Let I be an injective inverse system. Then, for any $j \geq 1$ one has that $H^j(\operatorname{Roos}^*(I)) = 0$.*

Proof. Fix $p \in P$. Since I_p is an A-module and the category of A-modules has enough injectives, there exists an injective A-module J_p and a monomorphism $I_p \hookrightarrow J_p$ (say, h_p). In this way, we can define an inverse system (namely, $J_{\geq p}$) in the following manner; on one hand, for any $q \in P$ set

$$(J_{\geq p})_q := \begin{cases} J_p, \text{ if } q \in [p, 1_{\widehat{P}}), \\ 0, \text{ otherwise.} \end{cases}$$

On the other hand, if $q' \leq q$ then set

$$(h_{\geq p})_{qq'} := \begin{cases} \mathbb{1}_{J_p}, \text{ if } q \in [p, 1_{\widehat{P}}), \\ 0, \text{ otherwise.} \end{cases}$$

In this way, $J_{\geq p} = \left\{(J_{\geq p})_q, (h_{\geq p})_{qq'}\right\}$ defines an inverse system; moreover, we can produce a map $I \xrightarrow{\beta_{\geq p}} J_{\geq p}$ in the next manner; indeed, given $q \in P$ one sets

$$(\beta_{\geq p})_q := \begin{cases} h_p \circ i_{qp}, \text{ if } q \in [p, 1_{\widehat{P}}), \\ 0, \text{ otherwise.} \end{cases}$$

Here, $I_q \xrightarrow{i_{qp}} I_p$ denotes the corresponding structural moprphism of the inverse system I. Thus, it is straightforward to check that, for any $p \in P$, $\beta_{\geq p}$ defines a morphism of inverse systems; therefore, setting

$$J := \prod_{p \in P} J_{\geq p}$$

one has that the universal property of the direct product ensures the existence of a unique map $I \xrightarrow{\beta} J$ such that $\pi_p \circ \beta = \beta_{\geq p}$, where $J \xrightarrow{\pi_p} J_{\geq p}$ is the canonical projection map. Moreover, β is injective because h_p is so; however, since I is injective it follows that I is a direct summand of J. In this way, combining this fact with the known result that Roos^* commutes with direct products, one obtains, for any $k \geq 0$, that $H^k(\text{Roos}^*(I))$ is a direct summand of

$$H^k(\text{Roos}^*(J)) = \prod_{p \in P} H^k(\text{Roos}^*(J_{\geq p})).$$

So, given $k \geq 1$ it is enough to show that $H^k(\text{Roos}^*(J_{\geq p})) = 0$; regardless, $\text{Roos}^*(J_{\geq p})$ turns out to be the cochain complex for computing the simplicial cohomology of the topological space $[p, 1_{\widehat{P}})$ with coefficients in a certain abelian group. But, once more, Lemma 4.1.14 ensures that $[p, 1_{\widehat{P}})$ is contractible; this concludes the proof. □

Flasque and weakly flasque inverse systems are useful for our purposes because of the following:

Proposition 4.1.32. *The following statements hold.*

(i) Any inverse system can be embedded into a flasque one; whence $\text{Inv}(P, \mathcal{A})$ *has enough flasque and weakly flasque objects.*

(ii) Any injective inverse system is flasque.

(iii) Any weakly flasque inverse system is acyclic with respect to the inverse limit functor.

(iv) The right derived functors of the inverse limit can be computed through either a flasque or weakly flasque resolution.

(v) Given an object G of $\text{Inv}(P, \mathcal{A})$*, there exists a flasque resolution*

$$0 \longrightarrow G \longrightarrow \prod{}^*(G).$$

(vi) For any $i \geq 0$, one has that

$$H^i\left(\text{Roos}^*(G)\right) = \mathbb{R}^i \varprojlim_{p \in P} G(p);$$

the reader should notice that this part is exactly the content of part (ii) of Lemma 4.1.8.

Proof. Let G be an object of $\operatorname{Inv}(P, \mathcal{A})$ and fix $p \in P$. We consider the following inverse system:

$$\Pi^0(p) := \Pi^0(G)(p) := \prod_{p_0 \leq p} G(p_0).$$

Moreover, given $p \leq q$ we set $\Pi^0(q) \longrightarrow \Pi^0(p)$ as the natural projection. In this way, Π^0 defines an inverse system which we claim is flasque.

Before showing so, we check that, for any open subset W of P, one has that

$$\varprojlim_{w \in W} \Pi^0(w) \cong \prod_{w \in W} G(w).$$

Indeed, given $w_0 \in W$ consider the natural projection $\prod_{w \in W} G(w) \xrightarrow{\pi_{w_0}} \prod_{w \leq w_0} G(w)$; on the other hand, we also consider the natural restriction map

$$\varprojlim_{w \in W} \Pi^0(w) \xrightarrow{p_{w_0}} \Pi^0(w_0).$$

In this way, the universal property of the inverse limit guarantees the existence of a unique homomorphism of A-modules

$$\prod_{w \in W} G(w) \xrightarrow{\varphi} \varprojlim_{w \in W} \Pi^0(w)$$

such that $p_{w_0} \varphi = \pi_{w_0}$; on the other hand, the universal property of the direct product ensures the existence of a unique homomorphism of A-modules

$$\varprojlim_{w \in W} \Pi^0(w) \xrightarrow{\psi} \prod_{w \in W} G(w)$$

such that $\pi_{w_0} \psi = p_{w_0}$. We claim that ψ and φ are mutually inverses; indeed, we only have to point out that

$$\pi_{w_0} (\psi \varphi) = (\pi_{w_0} \psi) \varphi = p_{w_0} \varphi = \pi_{w_0}$$

and

$$p_{w_0} (\varphi \psi) = (p_{w_0} \varphi) \psi = \pi_{w_0} \psi = p_{w_0},$$

whence $\psi\varphi$ and $\varphi\psi$ satisfy respectively the same universal problem as the identity on

$$\prod_{w \in W} G(w) \text{ and } \varprojlim_{w \in W} \Pi^0(w),$$

whence they are mutually inverses; in particular, one has a canonical isomorphism

$$\varprojlim_{w \in W} \Pi^0(w) \cong \prod_{w \in W} G(w).$$

Now, we want to show that Π^0 is flasque; let $U \subseteq V$ be open subsets of P. Under the previous isomorphism, the natural restriction map

$$\varprojlim_{v \in V} \Pi^0(v) \longrightarrow \varprojlim_{u \in U} \Pi^0(u)$$

boils down to the natural projection

$$\prod_{v \in V} G(v) \longrightarrow \prod_{u \in U} G(u),$$

which is clearly surjective since $U \subseteq V$. Summing up, we have checked that Π^0 is a flasque inverse system; finally, the natural map $G \longrightarrow \Pi^0$ clearly defines a monomorphism of A-modules; in this way, part (i) holds.

Now, we prove part (ii). Let I be an injective inverse system; by part (i), I can be embedded into a flasque inverse system $\Pi^0(I)$. Regardless, since I is injective one has that I is a direct factor of $\Pi^0(I)$, whence it is also flasque according to Lemma 4.1.30. In this way, part (ii) also holds.

Now, we provide a sketch of proof of part (iii), referring to [74, Proposition 1.6 and Thèoréme 1.8] for full details. Let G be a weakly flasque inverse system and let

$$0 \longrightarrow G \longrightarrow I \longrightarrow I/G \longrightarrow 0$$

be a short exact sequence in $\operatorname{Inv}(P, \mathcal{A})$, where I is an injective inverse system; since G and I are weakly flasque, Proposition 4.1.29 implies that I/G is also weakly flasque. Therefore, the long exact sequence of right derived functors of the inverse limit yields, for any $j \in \mathbb{N}$, the following exact sequence:

$$\mathbb{R}^j \varprojlim_{p \in P} I \longrightarrow \mathbb{R}^j \varprojlim_{p \in P} I/G \longrightarrow \mathbb{R}^{j+1} \varprojlim_{p \in P} G \longrightarrow \mathbb{R}^{j+1} \varprojlim_{p \in P} I.$$

If $j \geq 1$, then

$$\mathbb{R}^j \varprojlim_{p \in P} I = 0 = \mathbb{R}^{j+1} \varprojlim_{p \in P} I$$

(indeed, injective inverse systems are clearly acyclic with respect to the inverse limit functor) and therefore one has, for any $j \geq 1$, a canonical isomorphism

$$\mathbb{R}^j \varprojlim_{p \in P} I/G \cong \mathbb{R}^{j+1} \varprojlim_{p \in P} G.$$

In this way, by applying increasing induction on j it is enough to check out that the exact sequence

$$\varprojlim_{p \in P} I \longrightarrow \varprojlim_{p \in P} I/G \longrightarrow \mathbb{R}^1 \varprojlim_{p \in P} G \longrightarrow \mathbb{R}^1 \varprojlim_{p \in P} I = 0$$

implies that

$$\mathbb{R}^1 \varprojlim_{p\in P} G = 0;$$

in other words, one should show that the natural map

$$\varprojlim_{p\in P} I \longrightarrow \varprojlim_{p\in P} I/G$$

is surjective; however, this surjectivity follows very easily from the fact that P is finite and both I and I/G are weakly flasque. The interested reader may like to consult [74, Proposition 1.6] for details.

On the other hand, since part (iv) follows directly combining parts (i) and (iii), we go on proving part (v). Given G an object of $\mathrm{Inv}(P, \mathcal{A})$, we have constructed in part (i) a flasque inverse system $\Pi^0(G)$. In this way, setting $Q^0 := \mathrm{Coker}\left(G \longrightarrow \Pi^0(G) \right)$ we have the following short exact sequence of inverse systems:

$$0 \longrightarrow G \longrightarrow \Pi^0(G) \longrightarrow Q^0 \longrightarrow 0.$$

Now, replacing G by Q^0 and setting $\Pi^1(G) := \Pi^0(Q^0)$ one obtains another short exact sequence:

$$0 \longrightarrow Q^0 \longrightarrow \Pi^1(G) \longrightarrow Q^1 \longrightarrow 0.$$

Iterating this process, we build a (possibly infinite) flasque resolution $0 \longrightarrow G \longrightarrow \Pi^*(G)$; whence part (v) also holds.

In this way, it only remains to prove part (vi). Actually, what we show is that the right derived functors of the inverse limit $(U^j)_{j\geq 0} := \left(\mathbb{R}^j \varprojlim_{p\in P}\right)_{j\geq 0}$ and the cohomology of the Roos cochain complex $(V^j)_{j\geq 0} := \left(H^j(\mathrm{Roos}^*(-))\right)_{j\geq 0}$ are isomorphic universal δ-functors.

Indeed, on one hand it is clear that both $(U^j)_{j\geq 0}$ and $(V^j)_{j\geq 0}$ are universal δ-functors with

$$U^0 = \mathbb{R}^0 \varprojlim_{p\in P} = \varprojlim_{p\in P} = H^0(\mathrm{Roos}^*(-)) = V^0.$$

On the other hand, given any $j \geq 1$ and any injective inverse system I it is also clear that

$$U^j(I) = \mathbb{R}^j \varprojlim_{p\in P} I_p = 0 = H^j(\mathrm{Roos}^*(I)) = V^j(I);$$

the reader should notice that the equality $0 = H^j(\mathrm{Roos}^*(I))$ is a direct application of Proposition 4.1.31. Summing up, combining the previous two facts one obtains a canonical isomorphism

$$\left(\mathbb{R}^j \varprojlim_{p\in P}\right)_{j\geq 0} \cong \left(H^j(\mathrm{Roos}^*(-))\right)_{j\geq 0}$$

of universal δ-functors, just what we finally wanted to check. □

Existence of enough projective inverse systems

Our next goal is to review, for the convenience of the reader, the fact that the category of inverse systems has enough projective objects and describe explicitly a certain class of projectives which will play certain role later on in this chapter.

The below result is just [108, Satz 2].

Theorem 4.1.33 (Nöbeling)**.** *For any inverse system G, there exists a projective one F and an epimorphism $F \twoheadrightarrow G$.*

Proof. Fix $p \in P$. Since G_p is an A-module and the category of A-modules has enough projectives, there exists a projective A-module F_p and an epimorphism $F_p \xrightarrow{\pi_p} G_p$. In this way, we can define an inverse system (namely, $F_{\leq p}$) in the following manner; on one hand, for any $q \in P$ set

$$(F_{\leq p})_q := \begin{cases} F_p, \text{ if } q \in (0_{\widehat{P}}, p], \\ 0, \text{otherwise.} \end{cases}$$

On the other hand, if $q' \leq q$ then set

$$(\pi_{\leq p})_{qq'} := \begin{cases} \mathbb{1}_{F_p}, \text{ if } q \in (0_{\widehat{P}}, p], \\ 0, \text{ otherwise.} \end{cases}$$

In this way, $F_{\leq p} = \left\{ (F_{\leq p})_q , (\pi_{\leq p})_{qq'} \right\}$ defines an inverse system; moreover, we can produce a map $F_{\leq p} \xrightarrow{\mu_{\leq p}} G$ in the next way; indeed, given $q \in P$ one sets

$$(\mu_{\leq p})_q := \begin{cases} g_{pq} \circ \pi_p, \text{ if } q \in (0_{\widehat{P}}, p], \\ 0, \text{ otherwise.} \end{cases}$$

Here, $G_p \xrightarrow{g_{pq}} G_q$ is one of the structural morphisms of the inverse system G. Moreover, it is straightforward to check that, for any $p \in P$, $\mu_{\leq p}$ defines a morphism of inverse systems. Therefore, setting

$$F := \bigoplus_{p \in P} F_{\leq p},$$

one has that the universal property of the direct sum ensures the existence of a unique map $F \xrightarrow{\mu} G$ such that $\mu \circ i_p = \mu_{\leq p}$, where $F_{\leq p} \xrightarrow{i_p} F$ are the canonical insertion maps. In this way, it only remains to check that μ is an epimorphism; however, the surjectivity of μ is equivalent to the surjectivity of μ_p (for any $p \in P$). But, fixed $p \in P$, the surjectivity of μ_p boils down to the surjectivity of π_p, which is clear; the proof is therefore completed. □

Before going on, we want to single out the following fact, which has been obtained during the proof of Theorem 4.1.33, because it will play some role in this chapter.

Theorem 4.1.34. *Given a projective A-module F and given $p \in P$, the inverse system $F_{\leq p}$ defined by*

$$(F_{\leq p})_q := \begin{cases} F_p, & \text{if } q \in (0_{\widehat{P}}, p], \\ 0, & \text{otherwise} \end{cases}$$

is a projective object of $\mathrm{Inv}(P, \mathcal{A})$*; in particular, $A_{\leq p}$ is so. Furthermore, any inverse system G admits a projective (not necessarily minimal) resolution*

$$\ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow G \longrightarrow 0$$

such that, for any $i \in \mathbb{N}$, F_i can be expressed as a direct sum of projectives of the form $A_{\leq p}$.

4.1.4 Existence of enough injective and projective direct systems

On one hand, it is known that any category of modules over an arbitrary ring has enough injective and projective objects; on the other hand, we have shown in Proposition 4.1.19 that, whenever P is a finite poset, the category $\mathrm{Dir}(P, \mathcal{A})$ is equivalent to the category of left modules over the incidence algebra $I(P^{op}, A)$. Therefore, combining these facts one obtains the following:

Theorem 4.1.35. *If P is a finite poset, then the category* $\mathrm{Dir}(P, \mathcal{A})$ *has enough injective and projective objects.*

Remark 4.1.36. The reader should notice that Theorem 4.1.35 guarantees the existence of enough projective objects, but it does not provide a priori any information about how such projectives look like.

Existence of coflasque direct systems

Now, we encounter a similar problem with respect to the previous case; indeed, projective direct systems are not suitable for our homological purposes. For this reason, we have to introduce the following subclass of objects of $\mathrm{Dir}(P, \mathcal{A})$; it is worth noting that the following terminology is not completely standard.

Definition 4.1.37. Let $P \xrightarrow{F} \mathcal{A}$ be a direct system; moreover, we regard P as a topological space with the Alexandrov topology. It is said that F is *coflasque* if, for any open subsets $U \subseteq V \subseteq P$, the natural insertion map

$$\varinjlim_{u \in U} F(u) \longrightarrow \varinjlim_{v \in V} F(v)$$

is injective.

Again, we have to introduce a bigger subclass in $\mathrm{Dir}(P, \mathcal{A})$.

Definition 4.1.38. It is said that a direct system F is *weakly coflasque* if, for any $p_0 \leq q_0$, the natural insertion map

$$\varinjlim_{q \in [q_0, 1_{\widehat{P}})} F(q) \longrightarrow \varinjlim_{p \in [p_0, 1_{\widehat{P}})} F(p)$$

is injective.

The following elementary property which weakly coflasque direct systems verify will play some role later on (cf. Proposition 4.1.42); it may be regarded as the analogue in this setting of Lemma 4.1.30.

Lemma 4.1.39. *Any direct summand of a coflasque (respectively, weakly coflasque) direct system is also coflasque (respectively, weakly coflasque).*

Proof. We only deal with the coflasque piece of the statement, because the reaminder weakly coflasque part can be proved in exactly the same way. Indeed, let F' be a direct system such that $F = F' \oplus F''$ for some coflasque direct system F and for some (a priori non necessarily coflasque) direct system F''; so, the natural splitted short exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

induces, for any pair $U \subseteq V$ of open subsets of P, the following commutative diagram with exact rows, where the vertical columns are the corresponding natural insertion maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varinjlim_{u \in U} F'(u) & \longrightarrow & \varinjlim_{u \in U} F(u) & \longrightarrow & \varinjlim_{u \in U} F''(u) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \varinjlim_{v \in V} F'(v) & \longrightarrow & \varinjlim_{v \in V} F(v) & \longrightarrow & \varinjlim_{v \in V} F''(v) & \longrightarrow & 0. \end{array}$$

Since the central vertical arrow is injective (because of F is coflasque) the Snake's Lemma ensures that the leftmost vertical one is also injective. In particular, this shows that F' is also coflasque; the proof is therefore completed. □

We also want to state now the *homological* analogues of Proposition 4.1.29 and Proposition 4.1.31; in both cases, the proof is left to the interested reader.

Proposition 4.1.40. *Let* $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ *be a short exact sequence in* $\mathrm{Dir}(P, \mathcal{A})$. *If* F *and* F'' *are weakly coflasque, then so is* F'.

Proposition 4.1.41. *Let* F *be a projective direct system. Then, for any* $j \geq 1$ *one has that* $H_j(\mathrm{Roos}_*(F)) = 0$.

Next result shows, in particular, that $\mathrm{Dir}(P, \mathcal{A})$ has enough coflasque objects and that such class of direct systems can be used in order to calculate the left derived functors of the direct limit.

Proposition 4.1.42. *The following statements hold.*

(i) Any direct system can be expressed as a homomorphic image of a coflasque direct system.

(ii) Any projective direct system is coflasque.

(iii) Any weakly coflasque direct system is acyclic with respect to the direct limit functor.

(iv) The left derived functors of the direct limit can be computed through either coflasque or weakly coflasque resolutions.

(v) Given a direct system F valued on $\mathcal{A}$, there exists a coflasque resolution

$$\Pi_*(F) \longrightarrow F \longrightarrow 0.$$

(vi) For any $i \geq 0$, one has that

$$H_i(\mathrm{Roos}_*(F)) = \mathbb{L}_i \varinjlim_{p \in P} F(p).$$

The reader should notice that this part is exactly part (ii) of Lemma 4.1.3.

Proof. Let F be an object of $\mathrm{Dir}(P, \mathcal{A})$ and fix $p \in P$. We consider the following direct system:

$$\Pi_0(p) := \Pi_0(F)(p) := \bigoplus_{p \leq p_0} F(p_0).$$

Moreover, given $p \leq q$ we set $\Pi_0(p) \longrightarrow \Pi_0(q)$ as the natural insertion. In this way, Π_0 defines a direct system which we claim is coflasque.

Before showing so, we check that, for any open subset W of P, one has that

$$\varinjlim_{w \in W} \Pi_0(w) \cong \bigoplus_{w \in W} F(w).$$

Indeed, given $w_0 \in W$ consider the natural inclusion $\oplus_{w \leq w_0} F(w) \xrightarrow{j_{w_0}} \oplus_{w \in W} F(w)$; on the other hand, we also consider the natural insertion map

$$\Pi_0(w_0) \xrightarrow{i_{w_0}} \varinjlim_{w \in W} \Pi_0(w).$$

In this way, the universal property of the direct limit guarantees the existence of a unique homomorphism of A-modules

$$\varinjlim_{w \in W} \Pi_0(w) \xrightarrow{\varphi} \bigoplus_{w \in W} F(w)$$

such that $\varphi i_{w_0} = j_{w_0}$; on the other hand, the universal property of the direct sum ensures the existence of a unique homomorphism of A-modules

$$\bigoplus_{w \in W} F(w) \xrightarrow{\psi} \varinjlim_{w \in W} \Pi_0(w)$$

such that $\psi j_{w_0} = i_{w_0}$. We claim that ψ and φ are mutually inverses; indeed, we only have to point out that

$$(\psi\varphi)\, i_{w_0} = \psi\,(\varphi i_{w_0}) = \psi j_{w_0} = i_{w_0}$$

and

$$(\varphi\psi)\, j_{w_0} = \varphi\,(\psi j_{w_0}) = \varphi i_{w_0} = j_{w_0},$$

whence $\psi\varphi$ and $\varphi\psi$ satisfy respectively the same universal problem as the identity on

$$\bigoplus_{w \in W} F(w) \text{ and } \varinjlim_{w \in W} \Pi_0(w),$$

whence they are mutually inverses; in particular, one has a canonical isomorphism

$$\varinjlim_{w \in W} \Pi_0(w) \cong \bigoplus_{w \in W} F(w).$$

Now, we want to show that Π_0 is coflasque; let $U \subseteq V$ be open subsets of P. Under the previous isomorphism, the natural insertion map

$$\varinjlim_{u \in U} \Pi_0(u) \longrightarrow \varinjlim_{v \in V} \Pi_0(v)$$

boils down to the natural inclusion

$$\bigoplus_{u \in U} F(u) \longrightarrow \bigoplus_{v \in V} F(v),$$

which is clearly injective since $U \subseteq V$. Summing up, we have checked that Π_0 is a coflasque direct system; finally, the natural map $\Pi_0 \longrightarrow F$ clearly defines an epimorphism of A-modules; in this way, part (i) holds.

Now, we prove part (ii). Let R be a projective direct system; by part (i), R can be expressed as the homomorphic image of a coflasque direct system $\Pi_0(R)$. Regardless, since R is projective one has that R is a direct summand of $\Pi_0(R)$, whence it is also coflasque by a direct application of Lemma 4.1.39. In this way, part (ii) also holds.

Now, we provide a sketch of proof of part (iii). Let F be a weakly coflasque direct system and let

$$0 \longrightarrow K \longrightarrow R \longrightarrow F \longrightarrow 0$$

be a short exact sequence in $\mathrm{Dir}(P, \mathcal{A})$, where R is a projective direct system; since F and R are weakly coflasque, the analogous coflasque of Proposition 4.1.29 (aka Proposition 4.1.40) implies that K is also weakly coflasque. Therefore, the long exact sequence of left derived functors of the direct limit yields, for any $j \in \mathbb{N}$, the following exact sequence:

$$\mathbb{L}_{j+1} \varinjlim_{p\in P} R \longrightarrow \mathbb{L}_{j+1} \varinjlim_{p\in P} F \longrightarrow \mathbb{L}_j \varinjlim_{p\in P} K \longrightarrow \mathbb{L}_j \varinjlim_{p\in P} R.$$

If $j \geq 1$, then

$$\mathbb{L}_{j+1} \varinjlim_{p\in P} R = 0 = \mathbb{L}_j \varinjlim_{p\in P} R$$

(indeed, projective direct systems are clearly acyclic with respect to the direct limit functor) and therefore one has, for any $j \geq 1$, a canonical isomorphism

$$\mathbb{L}_{j+1} \varinjlim_{p\in P} F \cong \mathbb{L}_j \varinjlim_{p\in P} K.$$

In this way, by applying increasing induction on j it is enough to check out that the exact sequence

$$0 = \mathbb{L}_1 \varinjlim_{p\in P} R \longrightarrow \mathbb{L}_1 \varinjlim_{p\in P} F \longrightarrow \varinjlim_{p\in P} K \longrightarrow \varinjlim_{p\in P} R$$

implies that

$$\mathbb{L}_1 \varinjlim_{p\in P} F = 0;$$

in other words, one should show that the natural map

$$\varinjlim_{p\in P} K \longrightarrow \varinjlim_{p\in P} R$$

is injective; however, this surjectivity follows from the fact that P is finite and both K and R are weakly coflasque.

On the other hand, since part (iv) follows directly combining parts (i) and (iii), we go on proving part (v). Given F an object of $\mathrm{Dir}(P, \mathcal{A})$, we have constructed in part (i) a coflasque direct system $\Pi_0(F)$. In this way, setting $K_0 := \ker \Big(\Pi_0(F) \longrightarrow F \Big)$ we have the following short exact sequence of direct systems:

$$0 \longrightarrow K_0 \longrightarrow \Pi_0(F) \longrightarrow F \longrightarrow 0.$$

Now, replacing F by K_0 and setting $\Pi_1(F) := \Pi_0(K_0)$ one obtains another short exact sequence:

$$0 \longrightarrow K_1 \longrightarrow \Pi_1(F) \longrightarrow K_0 \longrightarrow 0.$$

Iterating this process, we build a (possibly infinite) coflasque resolution $0 \longrightarrow \Pi_*(F) \longrightarrow F$; whence part (v) also holds.

In this way, the only thing it remains to prove is part (vi); in fact, what we show is that the left derived functors of the direct limit $(U_j)_{j\geq 0} := \left(\mathbb{L}_j \varinjlim_{p\in P}\right)_{j\geq 0}$ and the homology of the Roos chain complex $(V_j)_{j\geq 0} := (H_j(\mathrm{Roos}_*(-)))_{j\geq 0}$ are isomorphic universal δ-functors.

Indeed, on one hand it is clear that both $(U_j)_{j\geq 0}$ and $(V_j)_{j\geq 0}$ are universal δ-functors with

$$U_0 = \mathbb{L}_0 \varprojlim_{p\in P} = \varprojlim_{p\in P} = H_0(\mathrm{Roos}_*(-)) = V_0.$$

Moreover, given any $j \geq 1$ and any projective direct system F it is also clear that

$$U_j(F) = \mathbb{L}_j \varprojlim_{p\in P} F_p = 0 = H_j(\mathrm{Roos}_*(F)) = V_j(F).$$

Here, the reader should point out that the equality $0 = H_j(\mathrm{Roos}_*(F))$ is a direct consequence of Proposition 4.1.41. In this way, combining the previous two facts one obtains a canonical isomorphism

$$\left(\mathbb{L}_j \varprojlim_{p\in P}\right)_{j\geq 0} \cong (H_j(\mathrm{Roos}_*(-)))_{j\geq 0}$$

of universal δ-functors, just what we finally wanted to check. □

4.2 Homological spectral sequences

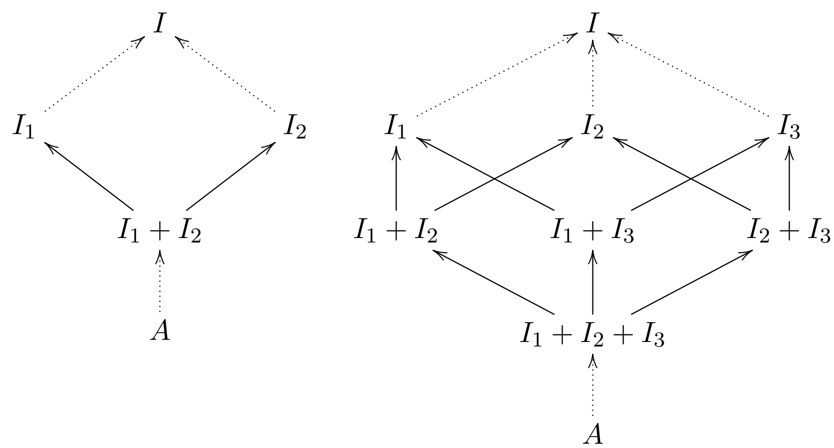
Hereafter, A will denote a commutative Noetherian ring. To any ideal $I \subseteq A$ one may associate a poset P as follows: let $I = I_1 \cap \ldots \cap I_n$ be its minimal primary decomposition, then P is the poset given by all the possible sums of the ideals $I_1, \ldots, I_n$ ordered by reverse inclusion but we underline that we have to identify these sums when they describe the same ideal. More generally, building from this poset P we can construct, for each A-module M, the inverse system $M/[*]M$ given by $(M/I_pM)_{p\in P}$.

The aim of this section is to construct several spectral sequences even though we will be mostly interested on those involving local cohomology modules. The method will be as follows. Firstly, given any object M of $\mathcal{A}$, we shall construct a direct system over P of objects of $\mathcal{A}$. Then, using the homological Roos complex we shall build a double complex with a finite number of non-zero columns (or rows) that will rise to a spectral sequence that, with the help of a technical lemma, will converge to a certain object of $\mathcal{A}$. We refer to [114, Chapter 10] for any unexplained fact and terminology about spectral sequences. Finally, we are to study the degeneration of such spectral sequences and their attached extension problems in the spirit of [6].

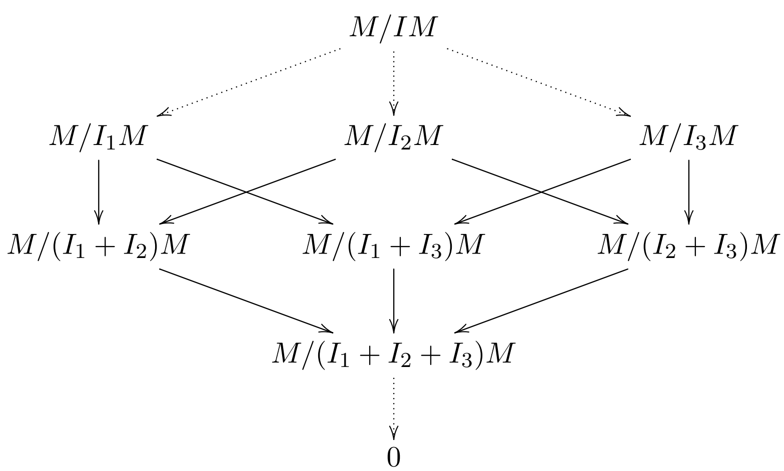
The following concept may be attached to any finite poset. It will be useful later on in order to illustrate some of our ideas and results.

Definition 4.2.1 (Hasse, Voght)**.** Let P be a finite poset. The *Hasse-Voght diagram* of P is obtained by drawing the elements of P as dots, with x drawn lower than y if $x < y$, and with an edge between x and y whenever y *covers* x; that is, if $x < y$ and no $z \in P$ satisfies $x < z < y$.

Example 4.2.2. Perhaps, some Hasse-Voght diagrams illustrate our ideas. In case $n = 2$, the picture is evident. Regardless, it turns out that the situation gets more subtle in case $n = 3$; the reader should notice that the below pictures correspond to the direct system F given by the assignment $F(p) = I_p$ (remind that we are ordering these ideals by reverse inclusion, for this reason we obtain a direct system).

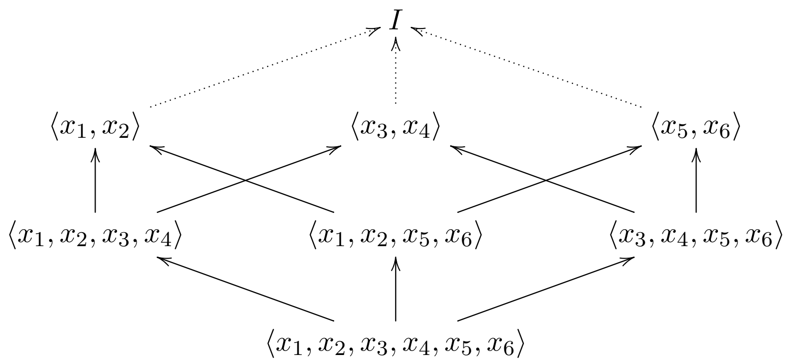


A similar picture is obtained in case we draw the Hasse-Voght diagram attached to the inverse system $M/[*]M$; by simplicity, we only illustrate the picture in case $n = 3$:

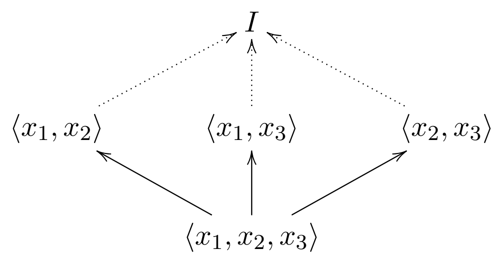


From now onward, we omit the initial element of the poset $\widehat{P}$ in our Hasse-Voght diagrams. For a concrete realization of the previous general pictures, just take the ring

$A = \mathbb{K}[x_1, \ldots, x_6]$, where $\mathbb{K}$ is any field. Firstly, if $I = \langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle$ then our poset can be drawn in the following way:



Regardless, if $I = \langle x_1, x_2\rangle \cap \langle x_1, x_3\rangle \cap \langle x_2, x_3\rangle$ then it turns out that our picture becomes more simple:



Here, we are using our convention that we identify these sums when they describe the same ideal.

4.2.1 Some obstructions

Before constructing our homological spectral sequences, we want to come back to Remark 4.1.11; as we explained there, given an inverse system G it is unclear for us whether the natural restriction map

$$G(0_{\widehat{P}}) = \varprojlim_{p \in \widehat{P}} G(p) \longrightarrow \varprojlim_{p \in P} G(p)$$

is an isomorphism. In fact, for the purposes of this chapter, we are mostly interested in the following particular case of such question.

Question 4.2.3. Let M be an object of $\mathcal{A}$. We shall denote by $M/[*]M$ the inverse system defined by the family $(M/I_pM)_{p\in P}$ and homomorphisms $M/I_qM \twoheadrightarrow M/I_pM$ for $p \leq q$. Is it true that

$$\varprojlim_{p\in P} M/I_pM \cong M/IM,$$

where $I := I_1 \cap \ldots \cap I_n$?

Remark 4.2.4. We want to stress again that Question 4.2.3 is nothing but a particular case of the one raised implicitly in Remark 4.1.11.

It is worth mentioning here that Question 4.2.3 was partially answered in [31, Example 3.3] in case $M = A$. We state the result but we refer to op. cit. for details.

Theorem 4.2.5 (Brun, Bruns, Römer)**.** *Assume that P is a subset of a distributive lattice of ideals of A (with respect to sum and intersection). Then, there is an isomorphism*

$$\varprojlim_{p\in P} A/I_p \cong A/I.$$

Actually, a slightly more general result holds again in case $M = A$. Once more, we shall refer to [31, page 216] for details.

Theorem 4.2.6 (Brun, Bruns, Römer)**.** *We suppose that the following assumptions hold.*

(a) $A/[*]$ is flasque.

(b) The natural homomorphism

$$A \longrightarrow \varprojlim_{p\in P} A/I_p$$

is surjective.

Then, there is a canonical isomorphism

$$A/I \cong \varprojlim_{p\in P} A/I_p.$$

Our next goal is to show that it might have some obstructions to a positive answer in full generality of Question 4.2.3; the first step in that direction is the next elementary statement, whose proof is left to the interested reader.

Lemma 4.2.7. *Let M be an object of $\mathcal{A}$. Then, the following statements hold.*

(i) The map

$$\begin{aligned} r_{n,M} : M &\longrightarrow \prod_{i=1}^{n} M/I_iM \\ m &\longmapsto (m + I_1M, \ldots, m + I_nM) \end{aligned}$$

has kernel IM and image isomorphic to M/IM.

(ii) $M/IM = \mathrm{Im}(r_{n,M}) \subseteq \ker(\Delta_{n,M})$, *where*

$$\begin{aligned} \Delta_{n,M} : \prod_{i=1}^{n} M/I_iM &\longrightarrow \prod_{1\le i<j\le n} M/(I_i+I_j)M \\ (m_i + (I_iM) \mid \quad 1 \le i \le n) &\longmapsto (m_i - m_j + (I_i+I_j)M \mid \quad 1 \le i < j \le n). \end{aligned}$$

In other words, Lemma 4.2.7 implies that, if $\mathrm{Im}(r_{n,M}) = \ker(\Delta_{n,M})$, then Question 4.2.3 has a positive answer. Now, we want to show that this equality works when $n = 2$, but for $n \geq 3$ it might be not true.

Remark 4.2.8. When $n = 2$, one has the short exact sequence of A-modules

$$0 \longrightarrow M/IM \xrightarrow{r_{2,M}} M/I_1M \times M/I_2M \xrightarrow{\Delta_{2,M}} M/(I_1+I_2)M \longrightarrow 0$$

and therefore the equality $\mathrm{Im}(r_{2,M}) = \ker(\Delta_{2,M})$ holds. However, in case $n \geq 3$ it is unclear for us whether the equality $\mathrm{Im}(r_{n,M}) = \ker(\Delta_{n,M})$ holds or not.

In case $n = 3$, we can bound more precisely our doubts.

Example 4.2.9 (Case $n = 3$). In this case, the diagonal map adopts the following concrete aspect:

$$\begin{aligned} &M/I_1M \oplus M/I_2M \oplus M/I_3M \xrightarrow{\Delta_{3,M}} M/(I_1+I_2)M \oplus M/(I_1+I_3)M \oplus M/(I_2+I_3)M \\ &(m_1 + I_1M, m_2 + I_2M, m_3 + I_3M) \longmapsto (m_2 - m_1 + (I_1+I_2)M, m_1 - m_3 + (I_1+I_3)M, \\ &m_3 - m_2 + (I_2+I_3)M). \end{aligned}$$

Thus, $(m_1 + I_1M, m_2 + I_2M, m_3 + I_3M)$ belongs to $\ker(\Delta_{3,M})$ if and only if there are $(x_j, y_j) \in I_jM \times I_jM$ (where $j \in \{1,2,3\}$) such that

$$\begin{cases} m_2 - m_1 = x_1 + x_2 \\ m_1 - m_3 = y_1 + x_3 \\ m_3 - m_2 = y_2 + y_3. \end{cases}$$

On the other hand, $(m_1 + I_1M, m_2 + I_2M, m_3 + I_3M)$ lies on $\mathrm{Im}(r_{3,M})$ if and only if there are $z_j \in I_jM$ (where $j \in \{1,2,3\}$) such that $m_1 + z_1 = m_2 + z_2 = m_3 + z_3$. In this way, we may consider the linear system with variables z_1, z_2, z_3; namely,

$$\begin{cases} z_1 - z_2 = x_1 + x_2 \\ z_3 - z_1 = y_1 + x_3 \\ z_2 - z_3 = y_2 + y_3. \end{cases}$$

This linear system has as matrix

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & x_1+x_2 \\ -1 & 0 & 1 & y_1+x_3 \\ 0 & 1 & -1 & y_2+y_3 \end{array}\right).$$

After performing some elementary transformations, we obtain the following matrix in reduced row echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -x_3-y_1 \\ 0 & 1 & -1 & -x_1-x_2-x_3-y_1 \\ 0 & 0 & 0 & x_1+x_2+x_3+y_1+y_2+y_3 \end{array}\right).$$

4.2.2 Construction of direct limit spectral sequences

As we have previously announced, we proceed to construct several spectral sequences using the homological Roos functor as a crucial building block. Recall that $\mathcal{A}$ stands for the category of A-modules, where A is a commutative Noetherian ring.

Construction 4.2.10. Let $\widehat{P} \times \mathcal{A} \xrightarrow{T_{[*]}} \mathcal{A}$ be an additive bivariate functor which verifies the following requirements:

(i) For any $p \in \widehat{P}$, T_p is a covariant, left exact, univariate functor which commutes with arbitrary direct sums.

(ii) If $p \leq q$ then there exists a natural transformation of derived functors $\mathbb{R}^i T_p \longrightarrow \mathbb{R}^i T_q$.

In addition, we also have to suppose that $T_{[*]}$ verifies one (and only one) of the following two assumptions.

(a) For any $\mathfrak{p} \in \operatorname{Spec}(A)$ and for any maximal ideal $\mathfrak{m}$ of A, there exists an A-module X such that, for any $p \in \widehat{P}$,

$$T_p(E(A/\mathfrak{p}))_{\mathfrak{m}} = \begin{cases} X, & \text{if } \mathfrak{p} \in \mathbf{W}(I_p, J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

It must be mentioned that X may depend on $\mathfrak{p}$ and $\mathfrak{m}$, but not on p. Moreover, here

$$\mathbf{W}(I_p, J) := \left\{\mathfrak{q} \in \operatorname{Spec}(A) \mid \ I_p^n \subseteq \mathfrak{q} + J \text{ for some integer } n \geq 1\right\},$$

and J is an ideal of A which does not depend on any of the previous choices.

(b) For any $\mathfrak{p} \in \operatorname{Spec}(A)$ and for any maximal ideal $\mathfrak{m}$ of A, there exists an A-module Y such that, for any $p \in \widehat{P}$,

$$T_p(E(A/\mathfrak{p}))_{\mathfrak{m}} = \begin{cases} Y, & \text{if } \mathfrak{p} \notin \mathbf{W}(I_p, J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

It must be mentioned that Y may depend on $\mathfrak{p}$ and $\mathfrak{m}$, but not on p. Once again, J is an ideal of A which does not depend on any of the previous choices.

Hereafter in this subsection, we set $T := T_{1_{\widehat{P}}}$; moreover, we also define the *cohomological dimension* of T, denoted $\mathrm{cd}(T)$, in the following way:

$$\mathrm{cd}(T) := \max\{i \in \mathbb{N} \mid \mathbb{R}^i T(X) \neq 0 \text{ for some } A-\text{module } X\}.$$

We want to single out in the next result some elementary properties which the subsets of the form $\mathbf{W}(I_p, J)$ verify because they will play a crucial role later on (cf. Lemma 4.2.15).

Lemma 4.2.11. *Let $p, p' \in P$, let $\mathfrak{p} \in \mathrm{Spec}(A)$, and let J be an ideal of A. Then, the following statements hold.*

(i) If $\mathfrak{p} \in \mathbf{W}(I_p, J) \cap \mathbf{W}(I_{p'}, J)$, then $\mathfrak{p} \in \mathbf{W}(I_p + I_{p'}, J)$.

(ii) If $p \leq p'$ and $\mathfrak{p} \in \mathbf{W}(I_p, J)$, then $\mathfrak{p} \in \mathbf{W}(I_{p'}, J)$.

Proof. First of all, suppose that $\mathfrak{p} \in \mathbf{W}(I_p, J) \cap \mathbf{W}(I_{p'}, J)$; by the very definition of the subsets $\mathbf{W}$'s, there are integers $n \geq 1$ and $m \geq 1$ such that $I_p^n \subseteq \mathfrak{p} + J$ and $I_{p'}^m \subseteq \mathfrak{p} + J$. Now, setting $k := n + m$ it follows that

$$(I_p + I_{p'})^k = \sum_{j=1}^{k} I_p^{n-j} I_{p'}^j = \sum_{j=1}^{m} I_p^{n-j} I_{p'}^j + \sum_{j=m+1}^{k} I_p^{n-j} I_{p'}^j \subseteq I_p^n + I_{p'}^m \subseteq \mathfrak{p} + J;$$

whence $\mathfrak{p} \in \mathbf{W}(I_p + I_{p'}, J)$ and therefore part (i) holds.

On the other hand, assume that $p \leq p'$ and $\mathfrak{p} \in \mathbf{W}(I_p, J)$. Thus, there is an integer $l \geq 1$ such that $I_p^l \subseteq \mathfrak{p} + J$; however, $I_{p'} \subseteq I_p$ and therefore it follows that $I_{p'}^l \subseteq I_p^l \subseteq \mathfrak{p} + J$, whence $\mathfrak{p} \in \mathbf{W}(I_{p'}, J)$, just what we finally wanted to prove. □

Let M be any A-module. We shall denote by $\mathbb{R}^i T_{[*]}(M)$ the direct system defined by the system $(\mathbb{R}^i T_p(M))_{p \in P}$ and homomorphisms $\mathbb{R}^i T_p(M) \longrightarrow \mathbb{R}^i T_q(M)$ for $p \leq q$ given by the previously mentioned natural transformations.

Basic examples

Before going on, we provide examples that there are functors verifying the established conditions in Construction 4.2.10. In all of these examples, let N be a finitely generated A-module with finite projective dimension.

Our first example is concerned with the so-called *generalized local cohomology modules.*

Example 4.2.12 (Generalized local cohomology). The direct system of functors $T_{[*]} = \Gamma_{[*]}(N, -)$ of *generalized torsion functors* given by $T_p(M) := \Gamma_{I_p}(N, M)$ verifies all the foregoing requirements established in Construction 4.2.10; moreover, in this case $T = \Gamma_I(N, -)$. Indeed, first of all we remind that, for any $p \in \widehat{P}$ and for any A-module M,

$$H^j_{I_p}(N, M) := \varinjlim_{k \in \mathbb{N}} \operatorname{Ext}^j_A \left(N/I_p^k N, M\right);$$

in particular,

$$\Gamma_{I_p}(N, M) = \varinjlim_{k \in \mathbb{N}} \operatorname{Hom}_A \left(N/I_p^k N, M\right).$$

Now, fix $k \in \mathbb{N}$. Using the adjoint associativity between Hom and tensor product (cf. [114, Theorem 2.76]) one has that

$$\operatorname{Hom}_A \left(N/I_p^k N, E(A/\mathfrak{p})\right) \cong \operatorname{Hom}_A \left(N, \operatorname{Hom}_A \left(A/I_p^k, E(A/\mathfrak{p})\right)\right)$$

On the other hand, since N is finitely related it follows that

$$\varinjlim_{k \in \mathbb{N}} \operatorname{Hom}_A \left(N, \operatorname{Hom}_A \left(A/I_p^k, E(A/\mathfrak{p})\right)\right) \cong \operatorname{Hom}_A \left(N, \varinjlim_{k \in \mathbb{N}} \operatorname{Hom}_A \left(A/I_p^k, E(A/\mathfrak{p})\right)\right).$$

Summing up, one has that

$$\varinjlim_{k \in \mathbb{N}} \operatorname{Hom}_A \left(N/I_p^k N, E(A/\mathfrak{p})\right) \cong \operatorname{Hom}_A \left(N, \Gamma_{I_p} \left(E(A/\mathfrak{p})\right)\right).$$

Moreover, it is well known (cf. [30, 10.1.11]) that

$$\Gamma_{I_p}(E(A/\mathfrak{p})) = \begin{cases} E(A/\mathfrak{p}), & \text{if } \mathfrak{p} \in \mathbf{V}(I_p), \\ 0, & \text{if } \mathfrak{p} \notin \mathbf{V}(I_p). \end{cases}$$

In this way, combining these two facts it follows that

$$\Gamma_{I_p}(N, E(A/\mathfrak{p})) \cong \operatorname{Hom}_A(N, \Gamma_{I_p}(E(A/\mathfrak{p}))) = \begin{cases} \operatorname{Hom}_A(N, E(A/\mathfrak{p})), & \text{if } \mathfrak{p} \in \mathbf{V}(I_p), \\ 0, & \text{if } \mathfrak{p} \notin \mathbf{V}(I_p). \end{cases}$$

Finally, given any $\mathfrak{m} \in \operatorname{Max}(A)$ one has, as a direct application of [30, 4.1.7] (indeed, N is finitely generated and the localization map $A \longrightarrow A_{\mathfrak{m}}$ is flat), that

$$\Gamma_{I_p}(N, E(A/\mathfrak{p}))_{\mathfrak{m}} = \begin{cases} \operatorname{Hom}_{A_{\mathfrak{m}}} \left(N_{\mathfrak{m}}, E(A/\mathfrak{p})_{\mathfrak{m}}\right), & \text{if } \mathfrak{p} \in \mathbf{V}(I_p) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

Our next example revolves around the so-called *generalized Nagata's ideal transforms*.

Example 4.2.13 (Generalized ideal transform). The direct system of functors $T_{[*]} = D_{[*]}(N, -)$ of *generalized Nagata's ideal transforms* verifies all the previous requirements (cf. [43]). In this case, $T = D_I(N, -)$. Indeed, fix $p \in \widehat{P}$; firstly, the reader should remind that, by definition, for any A-module M,

$$D_{I_p}(N, M) := \varinjlim_{k \in \mathbb{N}} \operatorname{Hom}_A \left(I_p^k N, M \right).$$

By [43, Lemma 2.1], there is an exact sequence

$$0 \longrightarrow \Gamma_{I_p}(N, E(A/\mathfrak{p})) \longrightarrow \operatorname{Hom}_A(N, E(A/\mathfrak{p})) \longrightarrow D_{I_p}(N, E(A/\mathfrak{p})) \longrightarrow H^1_{I_p}(N, E(A/\mathfrak{p})).$$

So, since $E(A/\mathfrak{p})$ is injective, it follows that

$$H^1_{I_p}(N, E(A/\mathfrak{p})) = \varinjlim_{k \in \mathbb{N}} \operatorname{Ext}^1_A \left(N/I_p^k N, E(A/\mathfrak{p}) \right) = 0,$$

whence one can arrange the previous exact sequence in the following way:

$$0 \longrightarrow \Gamma_{I_p}(N, E(A/\mathfrak{p})) \longrightarrow \operatorname{Hom}_A(N, E(A/\mathfrak{p})) \longrightarrow D_{I_p}(N, E(A/\mathfrak{p})) \longrightarrow 0.$$

Therefore, combining this short exact sequence with the calculation carried out in the previous part it follows that

$$D_{I_p}(N, E(A/\mathfrak{p})) = \begin{cases} \operatorname{Hom}_A(N, E(A/\mathfrak{p})), \text{ if } \mathfrak{p} \notin \mathbf{V}(I_p), \\ 0, \text{ if } \mathfrak{p} \in \mathbf{V}(I_p). \end{cases}$$

Finally, given any $\mathfrak{m} \in \operatorname{Max}(A)$ one has, as a direct application of [30, 4.1.7] (indeed, N is finitely generated and the localization map $A \longrightarrow A_{\mathfrak{m}}$ is flat), that

$$D_{I_p}(N, E(A/\mathfrak{p}))_{\mathfrak{m}} = \begin{cases} \operatorname{Hom}_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}, E(A/\mathfrak{p})_{\mathfrak{m}}), \text{ if } \mathfrak{p} \notin \mathbf{V}(I_p) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, \text{ otherwise.} \end{cases}$$

Our final example takes care about the so-called *local cohomology with respect to pairs of ideals.*

Example 4.2.14 (Local cohomology with respect to pairs of ideals). Let J be an arbitrary ideal of A. The direct system of *torsion functors with respect to pairs of ideals* $T_{[*]} = \Gamma_{[*],J}$ is given by $T_p(M) := \Gamma_{I_p,J}(M)$; that is, the (I_p, J)-torsion module with respect to M. The reader should remind that, for any ideal K of A, the torsion functor $\Gamma_{K,J}$ is defined in the following manner:

$$\Gamma_{K,J}(M) := \left\{ m \in M \mid \quad K^l m \subseteq Jm \text{ for some } l \in \mathbb{N} \right\}.$$

Furthermore, it is known (cf. [133, Proposition 1.11]) that $T_{[*]}$ verifies the previous requirements. In particular, $T = \Gamma_{I,J}$. Moreover, once more from [133, Proposition 1.11] one deduces that

$$\Gamma_{I_p,J}(E(A/\mathfrak{p})) = \begin{cases} E(A/\mathfrak{p}), \text{ if } \mathfrak{p} \in \mathbf{W}(I_p, J), \\ 0, \text{ if } \mathfrak{p} \notin \mathbf{W}(I_p, J). \end{cases}$$

Here, $\mathbf{W}(I_p, J) := \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \quad I_p^n \subseteq \mathfrak{p} + J \text{ for some integer } n \geq 1\}$.

Main result

Next fact computes the abutment of our homological spectral sequences. It may be regarded as a generalization of [6, Lemma of page 38].

Lemma 4.2.15. *If E is any injective object of $\mathcal{A}$ then the augmented homological Roos complex* $\operatorname{Roos}_*\left(T_{[*]}(E)\right) \longrightarrow T(E) \longrightarrow 0$ *is exact.*

Proof. First of all, suppose that $T_{[*]}$ verifies requirement (a) of Construction 4.2.10; so, as $\operatorname{Roos}_*(-)$ and $T_{[*]}(-)$ commutes with arbitrary direct sums we deduce from the Matlis-Gabriel Theorem (cf. [131, 3.2.3 and 3.2.5]) that we may assume, without loss of generality, that there is $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $E = E_A(A/\mathfrak{p})$, a choice of injective hull of $A/\mathfrak{p}$ over A. Moreover, as being exact is a local property it is enough to check that, for any maximal ideal $\mathfrak{m}$ of A, the chain complex

$$\operatorname{Roos}_*\left(T_{[*]}(E)\right)_{\mathfrak{m}} \longrightarrow T(E)_{\mathfrak{m}} \longrightarrow 0$$

is exact. Indeed, we can express P as the disjoint union of Q and Q', where

$$Q := \{q \in P \mid \quad I_q \subseteq \mathfrak{m}\},$$
$$Q' := \{q \in P \mid \quad I_q \not\subseteq \mathfrak{m}\}.$$

Notice that Q is clearly a subposet of P. In this way, we have to distinguish two cases. Firstly, if $\mathfrak{p} \not\subseteq \mathfrak{m}$, then the previous chain complex is identically zero, whence we are done. Otherwise, suppose that $\mathfrak{p} \subseteq \mathfrak{m}$; in this case, we split Q as the disjoint union $Q = Q_1 \cup Q_2$, where

$$Q_1 := \{p \in Q \mid \ \mathfrak{p} \in \mathbf{W}(I_p, J)\},$$
$$Q_2 := \{p \in Q \mid \ \mathfrak{p} \notin \mathbf{W}(I_p, J)\}.$$

Now, we have to distinguish two cases. Indeed, if $\mathfrak{p} \notin \mathbf{W}(I_p, J)$ then the previous chain complex is identically zero and therefore we are done. Otherwise, suppose that $\mathfrak{p} \in \mathbf{W}(I_p, J)$ for at least one p; in this case, this assumption combined with Lemma 4.2.11 ensure that Q_1 is a non-empty subposet of P of the form $[r, 1_{\widehat{P}})$, where $r \in P$ such that

$$I_r = \sum_{q \in Q_1} I_q := J.$$

Indeed, since the ideal J is clearly the greatest ideal among the ideals of Q_1 (we want to stress that here is where we are using the finiteness of Q_1 combined with Lemma 4.2.11), it turns out that there is an element $r \in P$ such that $I_r = J$ and therefore $Q_1 = [r, 1_{\widehat{P}})$. Summing up, our chain complex

$$\mathrm{Roos}_*\left(T_{[*]}(E)\right)_{\mathfrak{m}} \longrightarrow T(E)_{\mathfrak{m}} \longrightarrow 0$$

agrees with the one obtained considering the Roos chain complex on Q_1 instead of P; finally, this augmented chain complex where we only consider the subposet Q_1 in the construction of the Roos chain complex equals the augmented one for computing the simplicial homology of the topological space $[r, 1_{\widehat{P}})$ with coefficients in X. But we have checked in Lemma 4.1.14 that this topological space is contractible; this concludes the proof provided $T_{[*]}$ verifies assumption (a) of Construction 4.2.10.

Hereafter in this proof, we assume that $T_{[*]}$ verifies assumption (b) of Construction 4.2.10 and that $\mathfrak{p} \subseteq \mathfrak{m}$; now, consider the short exact sequence of direct systems

$$0 \longrightarrow |Y|_{|P_1} \longrightarrow |Y| \longrightarrow |Y|_{|P-P_1} \longrightarrow 0. \tag{4.3}$$

Here, $P_1 := \{p \in P \mid \ \mathfrak{p} \in \mathbf{W}(I_p, J),\ I_p \subseteq \mathfrak{m}\}$ and $|Y|_{|P_1}$ (respectively, $|Y|_{|P-P_1}$) denotes the direct system with constant value Y in all the points of P_1 (respectively, $P - P_1$) and zero elsewhere; moreover, its non-zero structural homomorphisms are all identities on Y. As he have seen in the first part of this proof, we can write $P_1 = [r, 1_{\widehat{P}})$ for some $r \in P$. On the other hand, (4.3) induces the following short exact sequence of chain complexes:

$$0 \longrightarrow \mathrm{Roos}_*\left(|Y|_{|P_1}\right) \longrightarrow \mathrm{Roos}_*\left(|Y|\right) \longrightarrow \mathrm{Roos}_*\left(|Y|_{|P-P_1}\right) \longrightarrow 0. \tag{4.4}$$

Furthermore, regarding that $T_{[*]}$ verifies assumption (b) of Construction 4.2.10 it follows that $\mathrm{Roos}_*\left(|Y|_{|P-P_1}\right) = \mathrm{Roos}_*\left(T_{[*]}(E)\right)_{\mathfrak{m}}$, whence we can rewrite (4.4) in the following manner:

$$0 \longrightarrow \mathrm{Roos}_*\left(|Y|_{|P_1}\right) \longrightarrow \mathrm{Roos}_*\left(|Y|\right) \longrightarrow \mathrm{Roos}_*\left(T_{[*]}(E)\right)_{\mathfrak{m}} \longrightarrow 0. \tag{4.5}$$

However, $\mathrm{Roos}_*\left(|Y|_{|P_1}\right)$ (respectively, $\mathrm{Roos}_*\left(|Y|\right)$) turns out to be the chain complex for computing the reduced simplicial homology of the topological space $P_1 = [r, 1_{\widehat{P}})$ (respectively, P); since both P_1 and P are contractible (cf. Lemma 4.1.14) one has that both $\mathrm{Roos}_*\left(|Y|_{|P_1}\right)$ and $\mathrm{Roos}_*\left(|Y|\right)$ are exact.

Summing up, all the foregoing implies that (4.5) is a short exact sequence of chain complexes with two of them exact; whence the remainder one (namely, $\mathrm{Roos}_*\left(T_{[*]}(E)\right)_{\mathfrak{m}}$) is so, just what we finally wanted to prove. □

Remark 4.2.16. The reader is encouraged to compare the first part of the proof of Lemma 4.2.15 with the argument used by S. Yassemi in order to obtain a Mayer-Vietoris long exact sequence for his so-called *generalized section functors* (cf. [143, Theorem 2.11]). On the other hand, it is worth noting that the argument employed in the second part of the proof of Lemma 4.2.15 turns out to be a straightforward generalization of the one used by A. Castaño Domínguez in his sheafified setting (cf. [36, Proposition 3.4]). As the reader can easily point out, both are particular cases of Lemma 4.2.15.

Next statement is the main result of this subsection.

Theorem 4.2.17. *Let M be any A-module. Then, we have the following spectral sequence*

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_p(M) \underset{i}{\Longrightarrow} \mathbb{R}^{j-i} T(M)$$

in the category of A-modules.

Proof. Let $0 \longrightarrow M \longrightarrow E^*$ be an injective resolution of M in the category $\mathcal{A}$. Applying to this resolution the functor $T_{[*]}$ one gets the following cochain complex of direct systems:

$$0 \longrightarrow T_{[*]}(M) \longrightarrow T_{[*]}(E^0) \longrightarrow T_{[*]}(E^1) \longrightarrow \dots$$

Now, we use the Roos chain complex in order to produce the bicomplex $\mathrm{Roos}^{-i}\left(T_{[*]}\left(E^j\right)\right) = \mathrm{Roos}_i\left(T_{[*]}\left(E^j\right)\right)$; the reader should point out that we put a minus in the i index because we want to work with a bicomplex where the two variables are cohomological. We hope the following picture illustrates all this paragraph:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Roos}_1\left(T_{[*]}\left(M\right)\right) & \longrightarrow & \mathrm{Roos}_1\left(T_{[*]}\left(E^0\right)\right) & \longrightarrow & \mathrm{Roos}_1\left(T_{[*]}\left(E^1\right)\right) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Roos}_0\left(T_{[*]}\left(M\right)\right) & \longrightarrow & \mathrm{Roos}_0\left(T_{[*]}\left(E^0\right)\right) & \longrightarrow & \mathrm{Roos}_0\left(T_{[*]}\left(E^1\right)\right) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T\left(M\right) & \longrightarrow & T\left(E^0\right) & \longrightarrow & T\left(E^1\right) & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Moreover, we have to stress that the vertical differentials are the ones of the Roos chain complex and the horizontal ones are the induced by the injective resolution of M; so, the bicomplex $\mathrm{Roos}_i\left(T_{[*]}\left(E^j\right)\right)$ produces two spectral sequences; namely, the ones provided respectively by the first and the second filtration of the previous bicomplex. In this way, the first thing one should ensure is that both spectral sequences converge and calculate their common abutment.

On one hand, the reader should notice that the E_2-page of one of such spectral sequences is obtained by firstly computing the homology of the columns and then computing the cohomology of the rows; regardless, Lemma 4.2.15 guarantees that all the columns of the bicomplex $\operatorname{Roos}_i\left(T_{[*]}\left(E^j\right)\right)$ are exact up to the column $\operatorname{Roos}_*\left(T_{[*]}\left(M\right)\right)$. Therefore, this fact implies that this spectral sequence collapses, whence its abutment turns out to be $\mathbb{R}^*T(M)$.

On the other hand, the other spectral sequence that we can produce is the one obtained by firstly taking cohomology on the rows and then calculating the homology of the columns; in such case, one obtains as E_1-page

$$E_1^{-i,j} = \operatorname{Roos}_i\left(\mathbb{R}^j T_{[*]}\left(M\right)\right).$$

In addition, since the boundary map of the E_1-page is the one of the Roos chain complex, and such chain complex computes the ith left derived functor of the direct limit, its E_2-page turns out to be

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_p(M).$$

Summing up, combining all the foregoing facts one obtains the spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_p(M) \underset{i}{\Longrightarrow} \mathbb{R}^{j-i}T(M)$$

in the category of A-modules; the proof is therefore completed. □

Examples revisited

Now, we specialize our general construction to the specific functors which have been previously introduced; as usual, in all these examples N will always stand for a finitely generated A-module with finite projective dimension.

Example 4.2.18 (Generalized local cohomology). When $T_{[*]} = \Gamma_{[*]}(N, -)$ is the system of generalized torsion functors, we get the following spectral sequence:

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} H^j_{I_p}(N,M) \underset{i}{\Longrightarrow} H_I^{j-i}(N,M).$$

This spectral sequence is a generalization of the Mayer-Vietoris spectral sequence obtained in [6]. In case the ideal has two components $I = I_1 \cap I_2$, we recover partially the Mayer-Vietoris long exact sequence obtained by Yassemi in [143, Corollary 2.14]:

$$\ldots \longrightarrow H^j_{I_1}(N,M) \oplus H^j_{I_2}(N,M) \longrightarrow H^j_I(N,M) \longrightarrow H^{j+1}_{I_1+I_2}(N,M) \longrightarrow \ldots$$

For the sake of simplicity, we single out the version of it that we shall consider in this chapter, especially for the case when $N = A$ is the ring itself.

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} H^j_{I_p}(M) \underset{i}{\Longrightarrow} H^{j-i}_I(M).$$

We have to recall that for the particular case when the ideal has two components $I = I_1 \cap I_2$, i. e., the associated poset has three vertices corresponding to I_1, I_2 and $I_1 + I_2$, we recover the usual Mayer-Vietoris long exact sequence of local cohomology modules:

$$\cdots \longrightarrow H^j_{I_1}(M) \oplus H^j_{I_2}(M) \longrightarrow H^j_I(M) \longrightarrow H^{j+1}_{I_1+I_2}(M) \longrightarrow \cdots$$

It is also worth mentioning that the Mayer-Vietoris spectral sequence considered in [95, Theorem 2.1] by G. Lyubeznik is slightly different at the E_1-page (because he used a different poset associated to the ideal I) but they coincide at the E_2-page; indeed, it follows from the fact that our poset is cofinal with respect to Lyubeznik's one. It has been previously illustrated with the Hasse-Voght diagrams of Example 4.2.2.

Example 4.2.19 (Generalized ideal transforms)*.* When $T_{[*]} = D_{[*]}(N, -)$ is the direct system of generalized Nagata's ideal transform functors, we obtain the following spectral sequence:

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} \mathbb{R}^j D_{I_p}(N, M) \underset{i}{\Longrightarrow} \mathbb{R}^{j-i} D_I(N, M).$$

We have to point out that, when $N = A$,

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} \mathbb{R}^j D_{I_p}(M) \underset{i}{\Longrightarrow} \mathbb{R}^{j-i} D_I(M).$$

In fact, such spectral sequence for ordinary Nagata's ideal transforms may be regarded as the module version obtained by A. Castaño Domínguez in [36, Theorem 3.5]. On the other hand, this spectral sequence is very closely related with the previous Mayer-Vietoris spectral sequence for local cohomology modules because of the well-known isomorphism (cf. [30, Theorem 2.2.6])

$$\mathbb{R}^j D_J(M) \cong H^{j+1}_J(M)$$

for any $j \geq 1$ and for any ideal J of A.

Finally, we have to underline that, when $n = 2$, we recover the long exact sequence

$$\begin{aligned} &0 \longrightarrow D_{I_1+I_2}(M) \longrightarrow D_{I_1}(M) \oplus D_{I_2}(M) \longrightarrow D_I(M) \longrightarrow H^2_{I_1+I_2}(M) \\ &\cdots \longrightarrow H^j_{I_1}(M) \oplus H^j_{I_2}(M) \longrightarrow H^j_I(M) \longrightarrow H^{j+1}_{I_1+I_2}(M) \longrightarrow \cdots \end{aligned}$$

obtained in [30, Exercise 3.2.5].

Example 4.2.20 (Local cohomology with respect to pairs of ideals). Let J be an arbitrary ideal of A. If $T_{[*]} = \Gamma_{[*],J}$ is the direct system of torsion functors with respect to pairs of ideals, then we obtain the following spectral sequence:

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} H^j_{I_p,J}(M) \underset{i}{\Longrightarrow} H^{j-i}_{I,J}(M).$$

It turns out that, when $n = 2$, this spectral sequence degenerates into the following long exact sequence:

$$\ldots \longrightarrow H^j_{I_1,J}(M) \oplus H^j_{I_2,J}(M) \longrightarrow H^j_{I,J}(M) \longrightarrow H^{j+1}_{I_1+I_2,J}(M) \longrightarrow \ldots$$

This long exact sequence may be regarded as a Mayer-Vietoris long exact sequence for local cohomology modules with respect to pairs of ideals; it is worth mentioning that, at the best of our knowledge, this is the first time that such long exact sequence appears in the literature.

An amended counterexample

We conclude this subsection by showing that the previous formalism can NOT be applied in case $T_{[*]} = \operatorname{Hom}_A(A/[*], -)$. In particular, we shall check in a particularly simple example that the spectral sequence established by the authors in [6, Remark 1.4 (iii)] can not be recovered using our foregoing formalism. Later on, we shall see (cf. Theorem 4.2.25) that there is a spectral sequence which is the most similar (but not exactly the same) to the one obtained in [6, Remark 1.4 (iii)].

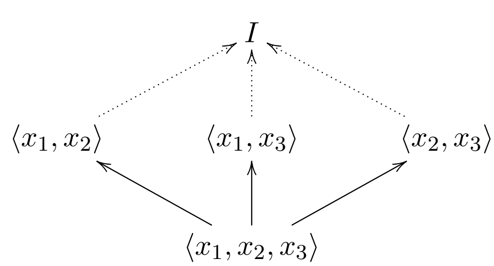
Counterargument 1. Let $\mathbb{K}$ be any field, set $A := \mathbb{K}[\![x, y, z]\!]$ and

$$I := \langle xy, xz, yz\rangle = \langle x, y\rangle \cap \langle x, z\rangle \cap \langle y, z\rangle.$$

Moreover, set $E := E_A(\mathbb{K})$ as a choice of injective hull of $\mathbb{K}$ over A. In this case, the reader should notice that

$$\begin{aligned} &I_1 = \langle x, y\rangle,\ I_2 = \langle x, z\rangle,\ I_3 = \langle y, z\rangle,\\ &I_1 + I_2 = I_1 + I_3 = I_2 + I_3 = \langle x, y, z\rangle,\\ &I_1 + I_2 + I_3 = \langle x, y, z\rangle. \end{aligned}$$

So, in such case, the poset P attached to I has the following Hasse-Voght diagram:



Our goal in this example is to compute explicitly the following augmented chain complex:

$$\mathrm{Roos}_*(\mathrm{Hom}_A(A/[*], E)) \longrightarrow \mathrm{Hom}_A(A/I, E) \longrightarrow 0. \tag{4.6}$$

In addition, since it is noteworthy that, for any ideal J of A, there is a canonical isomorphism of A-modules

$$\begin{aligned} \mathrm{Hom}_A(A/J, E) &\longrightarrow (0 :_E J) \\ f &\longmapsto f(\mathrm{cls}(1)) \end{aligned}$$

it turns out that (4.6) is canonically isomorphic to the next augmented chain complex:

$$\mathrm{Roos}_*((0 :_E [*])) \longrightarrow (0 :_E I) \longrightarrow 0,$$

which, in this case, is nothing but

$$0 \longrightarrow \mathrm{Roos}_1((0 :_E [*])) \xrightarrow{d_1} \mathrm{Roos}_0((0 :_E [*])) \xrightarrow{d_0} (0 :_E I) \longrightarrow 0. \tag{4.7}$$

So, our aim is to calculate explicitly (4.7). Firsly, we determine its spots:

(a) Its 0th spot is

$$\mathrm{Roos}_0((0 :_E [*])) = (0 :_E \mathfrak{m}) \oplus (0 :_E I_3) \oplus (0 :_E I_2) \oplus (0 :_E I_1).$$

Geometrically, this term corresponds to the vertices of the Hasse-Voght diagram attached to P.

(b) By the very definition of the homological Roos complex, it follows that its 1th piece is

$$\mathrm{Roos}_1((0 :_E [*])) = (0 :_E \mathfrak{m}) \oplus (0 :_E \mathfrak{m}) \oplus (0 :_E \mathfrak{m}).$$

Geometrically, this term corresponds to the edges of the previous picture of P.

Secondly, we have to compute its differentials; namely, d_0 and d_1.

(i) The 0th differential turns out to be

$$\begin{aligned} (0 :_E \mathfrak{m}) \oplus (0 :_E I_3) \oplus (0 :_E I_2) \oplus (0 :_E I_1) &\xrightarrow{d_0} (0 :_E I) \\ (a, a_1, a_2, a_3) &\longmapsto -a + a_3 - a_2 + a_1. \end{aligned}$$

(ii) The first differential d_1 is given by

$$\begin{aligned} (0 :_E \mathfrak{m}) \oplus (0 :_E \mathfrak{m}) \oplus (0 :_E \mathfrak{m}) &\xrightarrow{d_1} (0 :_E \mathfrak{m}) \oplus (0 :_E I_3) \oplus (0 :_E I_2) \oplus (0 :_E I_1) \\ (b_3, b_2, b_1) &\longmapsto (0, b_2 - b_3, b_1 - b_3, b_1 - b_2). \end{aligned}$$

Summing up, the augmented chain complex (4.7) is the one induced by the augmented chain complex

$$0 \longrightarrow E^{\oplus 3} \xrightarrow{A_1} E^{\oplus 4} \xrightarrow{A_0} E \longrightarrow 0,$$

where $A_0 := \begin{pmatrix} -1 & 1 & -1 & 1 \end{pmatrix}$ and

$$A_1 := \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Applying Matlis duality $(-)^{\vee}$, one obtains the following coaugmented cochain complex:

$$0 \longrightarrow A/I \longrightarrow A/\mathfrak{m} \oplus A/I_3 \oplus A/I_2 \oplus A/I_1 \longrightarrow A/\mathfrak{m} \oplus A/\mathfrak{m} \oplus A/\mathfrak{m} \longrightarrow 0.$$

As the reader can notice, such coaugmented cochain complex is the induced one given by the next complex:

$$0 \longrightarrow A \xrightarrow{A_0^t} A^{\oplus 4} \xrightarrow{A_1^t} A^{\oplus 3} \longrightarrow 0,$$

where

$$A_0^t = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } A_1^t = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Regardless, neither the previous lifted complex nor the induced one are exact. Indeed, we have checked that the lifted complex is not exact using Macaulay2 (cf. [55]). Of course, the reader might think that perhaps the lifted complex is not exact, but the induced complex after taking equivalence classes is so. Unfortunately, this is not the case, because of the element $(\mathrm{cls}(1), \mathrm{cls}(x), \mathrm{cls}(y), \mathrm{cls}(z))$ is a member of the kernel of the map given by A_1^t which does not belong to the image of the map given by A_0^t.

Our next goal is to show that there is a spectral sequence which is quite similar with the one obtained in [6, Remark 1.4 (iii)].

Construction 4.2.21. Let $\mathcal{A} \xrightarrow{T} \mathcal{A}$ be a contravariant, left exact, univariate functor. Building over T, we produce a new functor (namely, $\mathcal{T}$) in the following manner:

$$\begin{aligned} \operatorname{Inv}(P, \mathcal{A}) &\xrightarrow{\mathcal{T}} \operatorname{Dir}(P, \mathcal{A}) \\ G = (G_p)_{p \in P} &\longmapsto (T(G_p))_{p \in P}. \end{aligned}$$

Moreover, we also assume that T commutes with finite direct sums and that $T(A) = Z$ for some A-module Z; in particular, one has that

$$\varinjlim_{p \in P} \mathcal{T}(A_{\leq q}) = Z,$$

where $q \in P$ (notice that Z only depends on T, but not on q).

Of course, the example on which are mostly interested is the next one:

Example 4.2.22. Let N be an arbitrary A-module. Then, the functor $\mathrm{Hom}_A(-, N)$ is clearly left exact, contravariant, and commutes with finite direct sums; whence $\mathrm{Hom}_A(-, N)$ can be regarded as a particular case of Construction 4.2.21.

The reader should remind that our aim is to build an spectral sequence; next lemma provides its abutment.

Lemma 4.2.23. *Let G be a projective object of $\mathrm{Inv}(P, \mathcal{A})$ of the form*

$$G = \bigoplus_{j \in J} A_{\leq q_j},$$

where J is a finite index set and $q_j \in P$ (cf. Theorem 4.1.34). Then, the augmented chain complex

$$\mathrm{Roos}_*(\mathcal{T}(G)) \longrightarrow \left(\varinjlim_{p \in P} \circ \mathcal{T}\right)(G) \longrightarrow 0$$

is exact.

Proof. Since Roos_*, $\mathcal{T}$ and the direct limit functor commutes with finite direct sums we may suppose, without loss of generality, that $G = A_{\leq q}$ for some fixed $q \in P$; regardless, in this case, our augmented chain complex is exactly the one for computing the simplicial homology of the interval $[q, 1_{\widehat{P}})$ (indeed, notice that we have to take this interval because $\mathcal{T}$ is contravariant) with coefficients in Z. But $[q, 1_{\widehat{P}})$ is contractible by Lemma 4.1.14; the proof is therefore completed. □

The following result provides the announced spectral sequence.

Theorem 4.2.24. *There is a first quadrant spectral sequence*

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} \mathbb{R}^j \mathcal{T}(A/[*]) \underset{i}{\Longrightarrow} H^{j-i}\left(\varinjlim_{p \in P} \circ \mathcal{T}\right)(A/[*]),$$

where the abutment denotes the cohomology of the cochain complex

$$0 \longrightarrow \varinjlim_{p \in P} \mathcal{T}(A/[*]) \longrightarrow \varinjlim_{p \in P} \mathcal{T}(F_0) \longrightarrow \varinjlim_{p \in P} \mathcal{T}(F_1) \longrightarrow \dots$$

and

$$\dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A/[*] \longrightarrow 0$$

denotes a projective resolution of $A/[*]$ in $\mathrm{Inv}(P, \mathcal{A})$, where any F_i is made up by direct summands of the form $A_{\leq p}$ ($p \in P$).

Proof. Let $F_\bullet \longrightarrow A/[*] \longrightarrow 0$ be a projective resolution of $A/[*]$ as described in the statement of the result; applying to this resolution the functor $\mathcal{T}$, we obtain the following cochain complex of direct systems:

$$0 \longrightarrow \mathcal{T}\left(A/[*]\right) \longrightarrow \mathcal{T}\left(F_\bullet\right).$$

Now, we produce the bicomplex $\mathrm{Roos}^{-i}\left(\mathcal{T}\left(F_j\right)\right) = \mathrm{Roos}_i\left(\mathcal{T}\left(F_j\right)\right)$; the reader should point out that we put a minus in the i index because we want to work with a bicomplex where the two variables are cohomological. We hope the following picture illustrates all this paragraph:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Roos}_1\left(\mathcal{T}\left(A/[*]\right)\right) & \longrightarrow & \mathrm{Roos}_1\left(\mathcal{T}\left(F_0\right)\right) & \longrightarrow & \mathrm{Roos}_1\left(\mathcal{T}\left(F_1\right)\right) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Roos}_0\left(\mathcal{T}\left(A/[*]\right)\right) & \longrightarrow & \mathrm{Roos}_0\left(\mathcal{T}\left(F_0\right)\right) & \longrightarrow & \mathrm{Roos}_0\left(\mathcal{T}\left(F_1\right)\right) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varinjlim_{p\in P}\mathcal{T}\left(A/[*]\right) & \longrightarrow & \varinjlim_{p\in P}\mathcal{T}\left(F_0\right) & \longrightarrow & \varinjlim_{p\in P}\mathcal{T}\left(F_1\right) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Moreover, we have to stress that the vertical differentials are the ones of the Roos chain complex and the horizontal ones are the induced by the projective resolution of $A/[*]$; so, the bicomplex $\mathrm{Roos}_i\left(\mathcal{T}\left(F_j\right)\right)$ produces two spectral sequences; namely, the ones provided respectively by the first and the second filtration of the previous bicomplex. In this way, the first thing one should ensure is that both spectral sequences converge and calculate their common abutment.

On one hand, the reader should notice that the E_2-page of one of such spectral sequences is obtained by firstly computing the homology of the columns and then computing the cohomology of the rows; regardless, Lemma 4.2.23 guarantees that all the columns of the bicomplex $\mathrm{Roos}_i\left(\mathcal{T}\left(F_j\right)\right)$ are exact up to the column $\mathrm{Roos}_*\left(\mathcal{T}\left(A/[*]\right)\right)$. Therefore, this fact implies that this spectral sequence collapses, whence its abutment turns out to be the announced one.

On the other hand, the other spectral sequence that we can produce is the one obtained by firstly taking cohomology on the rows and then calculating the homology of the columns; in such case, one obtains as E_1-page

$$E_1^{-i,j} = \mathrm{Roos}_i\left(\mathbb{R}^j\mathcal{T}\left(A/[*]\right)\right).$$

In addition, since the boundary map of the E_1-page is the one of the Roos chain complex, and such chain complex computes the ith left derived functor of the direct limit, its E_2-page turns out to be

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j \mathcal{T}\left(A/[*]\right).$$

Summing up, combining all the foregoing facts one obtains the spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j \mathcal{T}\left(A/[*]\right) \underset{i}{\Longrightarrow} H^{j-i}\left(\varinjlim_{p\in P} \circ \mathcal{T}\right)\left(A/[*]\right)$$

in the category of A-modules; the proof is therefore completed. □

In case $T = \operatorname{Hom}_A(-, N)$, where N is an arbitrary A-module, we obtain a more transparent result; namely:

Theorem 4.2.25. *The following assertions hold.*

(i) There is a canonical isomorphism

$$\operatorname{Hom}_A\left(\varprojlim_{p\in P} A/I_p, N\right) \cong \varinjlim_{p\in P} \mathcal{H}\mathrm{om}_A\left(A/[*], |N|\right),$$

where $|N|$ is the constant inverse system given by N with identities as structural maps.

(ii) There is a first quadrant spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathcal{E}\mathrm{xt}_A^j\left(A/[*], |N|\right) \underset{i}{\Longrightarrow} \operatorname{Ext}_A^{j-i}\left(\varprojlim_{p\in P} A/I_p, N\right).$$

(iii) If, in addition, $A/[*]$ is flasque, then the previous spectral sequence can be written in the following way:

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathcal{E}\mathrm{xt}_A^j\left(A/[*], |N|\right) \underset{i}{\Longrightarrow} \operatorname{Ext}_A^{j-i}\left(A/I, N\right).$$

Proof. On one hand, part (ii) follows directly combining part (i) jointly with Theorem 4.2.24; on the other hand, part (i) is well known. The proof is therefore completed. □

We conclude this part with the following:

Example 4.2.26. When $I = I_1 \cap I_2$, the spectral sequence obtained in Theorem 4.2.25 boils down to the long exact sequence

$$\ldots \longrightarrow \operatorname{Ext}_A^i\left(A/\left(I_1+I_2\right), N\right) \longrightarrow \operatorname{Ext}_A^i\left(A/I_1, N\right) \oplus \operatorname{Ext}_A^i\left(A/I_2, N\right) \longrightarrow \operatorname{Ext}_A^i\left(A/I, N\right)$$
$$\longrightarrow \operatorname{Ext}_A^{i+1}\left(A/\left(I_1+I_2\right), N\right) \longrightarrow \ldots$$

obtained after applying the functor $\operatorname{Hom}_A(-, N)$ to the natural short exact sequence:

$$0 \longrightarrow A/I \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/\left(I_1+I_2\right) \longrightarrow 0.$$

4.2.3 Degeneration of homological spectral sequences

So far in this chapter, we have constructed several spectral sequences which involve the left derived functors of the direct limit functor. The goal of this section is to provide sufficient conditions in order to guarantee that the previously mentioned homological spectral sequences degenerate at the E_2-page.

Preliminary calculations

We shall collect in this subsubsection some preliminary facts which will simplify the proofs of the main result of this subsection.

Definition 4.2.27. Let $q \in P$ and let M be an object of $\mathcal{A}$. The direct system *represented by M on q* (namely, M_q) is defined as follows: for any $p \in P$,

$$(M_q)_p := \begin{cases} M, & \text{if} \quad p = q, \\ 0, & \text{otherwise.} \end{cases}$$

Next result computes the direct limit of this special construction. The reader is encouraged to compare such statement with [31, Lemma 8.7].

Lemma 4.2.28. *Let $q \in P$ and let M be an object of $\mathcal{A}$. For any $i \in \mathbb{N}$,*

$$\mathbb{L}_i \varinjlim_{p \in P} (M_q)_p \cong \widetilde{H}_{i-1}((q, 1_{\widehat{P}}); M),$$

where the tilde denotes reduced simplicial homology. We agree that the reduced homology of the empty simplicial complex is M in degree -1 and zero otherwise.

Proof. First of all, set the following direct systems:

$$(M_{>q})_p := \begin{cases} M, \text{ if } p \in (q, 1_{\widehat{P}}), \\ 0, \text{ otherwise,} \end{cases} \quad \text{and} \quad (M_{\geq q})_p := \begin{cases} M, \text{ if } p \in [q, 1_{\widehat{P}}), \\ 0, \text{ otherwise.} \end{cases}$$

In this way, one obtains the following short exact sequence in $\operatorname{Dir}(P, \mathcal{A})$:

$$0 \longrightarrow M_{>q} \longrightarrow M_{\geq q} \longrightarrow M_q \longrightarrow 0.$$

Therefore, we can consider the long exact sequence of reduced homology attached to the previous one; regardless, since $[q, 1_{\widehat{P}})$ is contractible (cf. Lemma 4.1.14) it follows that

$$\widetilde{H}_i \left(P; M_{\geq q}\right) \cong \widetilde{H}_i \left([q, 1_{\widehat{P}}); M\right) = 0$$

for all $i \geq 0$. In this way, such long exact sequence of reduced homology boils down to the following isomorphisms:

$$\widetilde{H}_0 \left(P; M_q\right) = 0 \text{ and } \widetilde{H}_{i-1} \left(P; M_{>q}\right) \cong \widetilde{H}_i \left(P; M_q\right) \text{ for any } i \geq 1,$$

where the non-zero isomorphisms are given by the connecting homomorphism. Moreover, we also have, for any $i \geq 1$, canonical isomorphisms

$$\widetilde{H}_{i-1}(P; M_{>q}) \cong \widetilde{H}_{i-1}\left((q, 1_{\widehat{P}}); M\right) \text{ and } \widetilde{H}_i(P; M_q) \cong H_i(P; M_q) \cong \mathbb{L}_i \varinjlim_{p \in P} (M_q)_p.$$

Summing up, for any $i \geq 1$ one obtains a natural isomorphism

$$\mathbb{L}_i \varinjlim_{p \in P} (M_q)_p \cong \widetilde{H}_{i-1}((q, 1_{\widehat{P}}); M),$$

just what we finally wanted to show. □

Another important ingredient in our later proofs will be the so-called *universal coefficients theorem.* We omit the proof and refer to [114, Theorem 7.55] for additional details.

Theorem 4.2.29 (Universal coefficients theorem for homology)**.** *Let B a (not necessarily commutative) ring, let M be a left B-module, and let $\mathbf{K}_\bullet$ be a chain complex of flat right B-modules whose chain subcomplex of boundaries has all terms flat. Then, for all $n \in \mathbb{N}$ there is a short exact sequence*

$$0 \longrightarrow H_n(\mathbf{K}_\bullet) \otimes_B M \xrightarrow{\lambda_n} H_n(\mathbf{K}_\bullet \otimes_B M) \xrightarrow{\mu_n} \operatorname{Tor}_1^B(H_{n-1}(\mathbf{K}_\bullet), M) \longrightarrow 0,$$

where

$$\begin{aligned} H_n(\mathbf{K}_\bullet) \otimes_B M &\xrightarrow{\lambda_n} H_n(\mathbf{K}_\bullet \otimes_B M) \\ \operatorname{cls}(z) \otimes m &\longmapsto \operatorname{cls}(z \otimes m) \end{aligned}$$

and both λ_n and μ_n are canonical.

Main result of degeneration involving homological spectral sequences

Now, we introduce the main result of this subsection. It is worth mentioning here that the assumptions imposed in the below result are a slight generalization of the ones imposed in [6, Theorem 1.2] and [31, Theorem 1.1].

Theorem 4.2.30. *Let A be a commutative Noetherian ring containing a field $\mathbb{K}$, let $T_{[*]}$ be the functor introduced in Section 4.2 and let M be an object of $\mathcal{A}$ verifying the following requirements.*

(a) For any $p \in P$, $\mathbb{R}^j T_p(M) = 0$ up to a unique value of j (namely, h_p).

(b) For any pair of distinct elements p and q of P, $\operatorname{Hom}_A(\mathbb{R}^{h_p} T_p(M), \mathbb{R}^{h_q} T_q(M)) = 0$.

Then, the spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_p(M) \underset{i}{\Longrightarrow} \mathbb{R}^{j-i}T(M)$$

degenerates at the E_2-page.

Proof. Part (a) of our assumptions implies that there is a canonical isomorphism of direct systems

$$\mathbb{R}^j T_{[*]}(M) \cong \bigoplus_{j=h_q} \Big(\mathbb{R}^{h_q} T_q(M)\Big)_q .$$

Fix $i \in \mathbb{N}$. Applying to this previous isomorphism the ith left derived functor of the direct limit over P, we get the following canonical isomorphism:

$$\mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_{[*]}(M) \cong \bigoplus_{j=h_q} \mathbb{L}_i \varinjlim_{p\in P} \Big(\mathbb{R}^{h_q} T_q(M)\Big)_q .$$

Moreover, Lemma 4.2.28 implies that there is a canonical isomorphism:

$$\mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_{[*]}(M) \cong \bigoplus_{j=h_q} \widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{R}^{h_q} T_q(M)).$$

In this way, the universal coefficients theorem for homology implies that there is a canonical isomorphism

$$\mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_{[*]}(M) \cong \bigoplus_{j=h_q} \Big(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M)\Big) .$$

Now, set $t_i := \dim_{\mathbb{K}}(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}))$. As the natural map

$$\mathbb{K} \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M) \longrightarrow \mathbb{R}^{h_q} T_q(M)$$

given by the assignment $r \otimes x \longmapsto rx$ is a canonical isomorphism of A-modules and $\mathbb{R}^{h_q} T_q(M)$ is an object of $\mathcal{A}$, it follows that $\mathbb{K} \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M)$ can be regarded as an object of $\mathcal{A}$ and therefore the abstract isomorphism

$$\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M) \cong \mathbb{R}^{h_q} T_q(M)^{\oplus t_i}$$

implies that we can regard $\mathbb{R}^{h_q} T_q(M)^{\oplus t_i}$ also as an object of $\mathcal{A}$, whence

$$\mathbb{L}_i \varinjlim_{p\in P} \mathbb{R}^j T_{[*]}(M) \cong \bigoplus_{j=h_q} \mathbb{R}^{h_q} T_q(M)^{\oplus t_i}$$

can be considered as an abstract isomorphism in the category $\mathcal{A}$. In this way, combining the previous isomorphism joint with part (b) of our assumptions one obtains the announced degeneration. □

Remark 4.2.31. It is noteworthy that, during the proof of Theorem 4.2.30, we do not need to consider the Tor term appearing in the general statement of the universal coefficients theorem for homology because such term vanishes; indeed, this is due to the fact that our coefficient ring is just the field $\mathbb{K}$.

When a spectral sequence degenerates at the E_2-page, it is natural to ask for the corresponding filtration which such degeneration provides. This is the content of the following direct consequence of Theorem 4.2.30.

Corollary 4.2.32. *Let A be a commutative Noetherian ring containing a field $\mathbb{K}$, let $T_{[*]}$ be the functor introduced in Section 4.2 and let M be an object of $\mathcal{A}$ verifying the following requirements.*

(a) For any $p \in P$, $\mathbb{R}^j T_p(M) = 0$ up to a unique value of j (namely, h_p).

(b) For any pair of distinct elements p and q of P, $\operatorname{Hom}_A(\mathbb{R}^{h_p} T_p(M), \mathbb{R}^{h_q} T_q(M)) = 0$.

Then, for each $0 \leq r \leq \operatorname{cd}(T)$ there is an increasing filtration $\{G_k^r\}_{r \leq k \leq \dim(A)}$ of $\mathbb{R}^r T(M)$ by A-modules such that

$$G_k^r / G_{k-1}^r \cong \bigoplus_{\{q \in P \ \mid \ k+r=h_q\}} \left(\widetilde{H}_{k-1}((q, 1_{\widehat{P}}); \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M) \right)$$

for all $r \leq k \leq \dim(A)$.

Proof. Under these assumptions, Theorem 4.2.30 ensures that the spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} \mathbb{R}^j T_p(M) \underset{i}{\Longrightarrow} \mathbb{R}^{j-i} T(M)$$

degenerates at the E_2-page; moreover, if one inspects carefully the proof of Theorem 4.2.30 then one notices that we have shown, in fact, that

$$E_2^{-i,j} = \bigoplus_{\{q \in P \ \mid \ j=h_q\}} \left(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M) \right).$$

Now, fix $0 \leq r \leq \operatorname{cd}(T)$. In this way, combining the degeneration at the E_2-page obtained in Theorem 4.2.30 joint with the very definition of convergence of a spectral sequence (as formulated, for instance, in [114, pp. 626–627]), one gets a bounded filtration $\{G_k^r\}_{r \leq k \leq \dim(A)}$ of $\mathbb{R}^r T(M)$ by A-modules such that

$$G_k^r / G_{k-1}^r \cong E_2^{-k,k+r}.$$

Therefore, combining such isomorphism with the description of the E_2-page which we have provided during this proof, one finally obtains that

$$G_k^r / G_{k-1}^r \cong \bigoplus_{\{q \in P \ \mid \ k+r=h_q\}} \left(\widetilde{H}_{k-1}((q, 1_{\widehat{P}}); \mathbb{K}) \otimes_{\mathbb{K}} \mathbb{R}^{h_q} T_q(M) \right)$$

for all $r \leq k \leq \dim(A)$, just what we finally wanted to show. $\square$

Specific examples

Our next aim is to exhibit particular settings on which the results obtained in Theorem 4.2.30 and Corollary 4.2.32 can be applied. Our first example is concerned with ordinary local cohomology modules.

Example 4.2.33 (Ordinary local cohomology modules). When $T_{[*]} = \Gamma_{[*]}$ is the ordinary torsion functor, the conclussions obtained in Theorem 4.2.30 and Corollary 4.2.32 can be applied in case, for any $p \in P$, I_p is a cohomologically complete intersection ideal (cf. Definition 4.2.37); for a more down-to-earth situation, just take I either as a Stanley-Reisner ideal or, more generally, suppose that I defines an arrangement of linear varieties. From this point of view, our results in this specific setting recover and extend the ones obtained by Àlvarez Montaner, García López and Zarzuela in [6, Section 2].

Our next example revolves around Nagata's ideal transforms.

Example 4.2.34 (Ideal transforms). When $T_{[*]} = D_{[*]}$ is the ordinary ideal transform functor, the conclussions obtained in Theorem 4.2.30 and Corollary 4.2.32 can be applied in case, for any $p \in P$, I_p is a principal ideal; regardless, in such case all this business boils down to the following well known short exact sequence:

$$0 \longrightarrow \Gamma_{aA}(A) \longrightarrow A \longrightarrow A_a \longrightarrow H^1_{aA}(A) \longrightarrow 0.$$

Here, $I = aA$ for some $a \in A$.

4.2.4 Extension problems in the homological framework

In the spirit of [6, Section 3], the aim of this section is to focus on the study of the extension problems attached to the corresponding filtrations produced by the degeneration of our previously introduced homological spectral sequences.

General starting setup

Our starting point is the following collection of short exact sequences:

$$\begin{array}{ccccccccc}
0 \longrightarrow G^r_{k-1} \longrightarrow G^r_k \longrightarrow G^r_k/G^r_{k-1} \longrightarrow 0 \\
0 \longrightarrow G^r_k \longrightarrow G^r_{k+1} \longrightarrow G^r_{k+1}/G^r_k \longrightarrow 0 \\
\vdots \quad \vdots \quad \ddots \quad \ddots \quad \ddots \\
0 \longrightarrow G^r_{\dim(A)-1} \longrightarrow G^r_{\dim(A)} \longrightarrow G^r_{\dim(A)}/G^r_{\dim(A)-1} \longrightarrow 0.
\end{array}$$

Hereafter, we omit the superscript r. Moreover, we have to point out that, for any k,

$$(s_k): \quad 0 \longrightarrow G_{k-1} \longrightarrow G_k \longrightarrow G_k/G_{k-1} \longrightarrow 0$$

may be regarded as an element of $\mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_{k-1})$.

From now onward in this subsection, $T_{[*]} = (T_p)_{p\in P}$ is the direct system of functors which has been previously introduced and $T := T_{1_{\widehat{P}}}$. In this way, the following preliminary calculation is just a mild generalization of [6, Lemma of page 47].

Lemma 4.2.35. *We assume, in addition, that* $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{R}^{h_p}T_p(M), \mathbb{R}^{h_q}T_q(M)) = 0$ *provided* $h_p \geq h_q + 2$. *Then, the natural maps*

$$\mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_{k-1}) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_{k-1}/G_{k-2})$$

are injective for all $k \geq 2$.

Proof. Consider the short exact sequence

$$(s_{k-1}): \quad 0 \longrightarrow G_{k-2} \longrightarrow G_{k-1} \longrightarrow G_{k-1}/G_{k-2} \longrightarrow 0.$$

In this way, applying to (s_{k-1}) the functor $\mathrm{Hom}_{\mathcal{A}}(G_k/G_{k-1}, -)$ one obtains the following exact sequence:

$$\mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_{k-2}) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_{k-1}) \longrightarrow \mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_{k-1}/G_{k-2}).$$

Therefore, applying once more $\mathrm{Hom}_{\mathcal{A}}(G_k/G_{k-1}, -)$ to the short exact sequence (s_l) for $l \leq k-2$ and descending induction, it turns out that we are only required to check that $\mathrm{Ext}^1_{\mathcal{A}}(G_k/G_{k-1}, G_l/G_{l-1}) = 0$ for any $l \leq k-2$. However, applying Corollary 4.2.32 it is enough to show that $\mathrm{Ext}^1_{\mathcal{A}}(\mathbb{R}^{h_p}T_p(M), \mathbb{R}^{h_q}T_q(M)) = 0$, where $h_q \leq j-2$ and $h_p = j$. But such vanishing holds by assumption. $\square$

Mayer-Vietoris spectral sequence of local cohomology modules

In the sequel, we restrict our attention to study the extension problems attached to the Mayer-Vietoris spectral sequence of local cohomology modules in case it degenerates; namely,

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} H^j_{I_p}(A) \underset{i}{\Longrightarrow} H^{j-i}_I(A).$$

Here, A is any regular ring containing a field $\mathbb{K}$.

Extension problems inside the ring of differential operators

Our next aim is to show that such extension problems are non-trivial with a slightly different argument from the one used in [6]. Before doing so, we review the following notion; the interested reader is referred to [93] for further details.

Theorem/Definition 4.2.36 (Lyubeznik)**.** Let $(R, \mathfrak{m}, \mathbb{K})$ be an equicharacteristic local ring, so that its completion $\widehat{R}$ admits a surjective ring homomorphism $A \overset{\pi}{\twoheadrightarrow} \widehat{R}$, where $A := \mathbb{K}[\![x_1, \ldots, x_d]\!]$ for some $d \in \mathbb{N}$, and set $I := \ker(\pi)$. Then, the following statements hold.

(a) The digit $\lambda_{i,j}(R) := \dim_{\mathbb{K}} \operatorname{Hom}_A(\mathbb{K}, H^i_{\mathfrak{m}}(H^{d-j}_I(A)))$ is finite and just depends on R and (i, j), but neither on A nor on π.

(b) One has that $\lambda_{i,j}(R) = \operatorname{length}_{D_{A|\mathbb{K}}}(H^i_{\mathfrak{m}}(H^{d-j}_I(A)))$.

We refer to $\lambda_{i,j}(R)$ as the (i, j)-*Lyubeznik number* of R.

We still have to introduce another preliminary concept (cf. [63]).

Definition 4.2.37 (Hellus, Schenzel)**.** Let R be a commutative Noetherian ring and let I be an ideal of R. It is said that I is *cohomologically complete intersection* provided $H^j_I(R) = 0$ for all $j \neq \operatorname{ht}(I)$.

Remark 4.2.38. Just a brief comment about terminology. In several papers (see, for instance, [111]), the phrase *A ring R is relative Cohen-Macaulay with respect to an ideal I of R* is the same as saying that I is cohomologically complete intersection.

Now, we are ready to illustrate that, in general, the extension problems attached to the Mayer-Vietoris spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p\in P} H^j_{I_p}(A) \underset{i}{\Longrightarrow} H^{j-i}_I(A)$$

are non-trivial.

Counterargument 2. In case $n = 2$, suppose that I_1 and I_2 are both cohomologically complete intersection ideals, with $\operatorname{grade}_A(I_i) := c$ for all $i \in \{1, 2\}$, such that $I_1 + I_2$ is a cohomologically complete intersection ideal with $\operatorname{grade}_A(I_1 + I_2) = c + 1$. In this case, it is straightforward to check that the attached filtration (because, in this case, the spectral sequence degenerates without further assumptions) boils down to the following short exact sequence:

$$0 \longrightarrow H^c_{I_1}(A) \oplus H^c_{I_2}(A) \longrightarrow H^c_I(A) \longrightarrow H^{c+1}_{I_1+I_2}(A) \longrightarrow 0.$$

If such exact sequence could be splitted in the category of $D_{A|\mathbb{K}}$-modules, then there would be an isomorphism

$$H_I^c(A) \cong H_{I_1}^c(A) \oplus H_{I_2}^c(A) \oplus H_{I_1+I_2}^{c+1}(A)$$

in the category of $D_{A|\mathbb{K}}$-modules and therefore, for any i,

$$\lambda_{i,d-c}(A/I) = \lambda_{i,d-c}(A/I_1) + \lambda_{i,d-c}(A/I_2) + \lambda_{i,d-c-1}(A/I_1 + I_2).$$

But this equality is, in general, not true. For a concrete example, just take the square-free monomial ideal $I := \langle x, yz \rangle$ inside $\mathbb{K}[\![x, y, z]\!]$, where $\mathbb{K}$ is any field. The unjustified calculations were carried out with the Macaulay2 package [5]. In this case, $d = 3$, $c = 2$, $I_1 = \langle x, y \rangle$, $I_2 = \langle x, z \rangle$, and $I_1 + I_2 = \langle x, y, z \rangle$. The ring $R := \mathbb{K}[\![x, y, z]\!]/I$ is a complete 1-dimensional Gorenstein Stanley-Reisner ring; whence

$$\lambda_{1,1}(R) = 1 \neq 2 = \lambda_{1,1}\left(\frac{\mathbb{K}[\![x,y,z]\!]}{I_1}\right) + \lambda_{1,1}\left(\frac{\mathbb{K}[\![x,y,z]\!]}{I_2}\right) + \lambda_{1,0}\left(\frac{\mathbb{K}[\![x,y,z]\!]}{\mathfrak{m}}\right)$$

and therefore the extension problems attached to the Mayer-Vietoris spectral sequence

$$E_2^{-i,j} = \mathbb{L}_i \varinjlim_{p \in P} H_{I_p}^j(A) \underset{i}{\Longrightarrow} H_I^{j-i}(A)$$

are, in general, non-trivial in the category of $D_{A|\mathbb{K}}$-modules.

We end this part with the following:

Remark 4.2.39. It turns out that, when the ground field $\mathbb{K}$ has characteristic zero, the Ext groups have only an structure as finite dimensional $\mathbb{K}$-vector spaces. Indeed, it comes from the fact that the Ext groups between two (either algebraic or analytic) holonomic D-modules have just an structure as finite dimensional vector spaces (cf. [75, Theorem 3.1] and [76, Theorem 4.8]). Actually, these Ext groups can be effectively computed using Macaulay2 (cf. [109]).

Extension problems in the category of F-modules

The goal of this subsection is to use again Counterargument 2 in order to show that, in general, the extension problems in the category of F_A-modules are non-trivial. Firstly, we need to review the following:

Construction 4.2.40 (Lyubeznik). Set

$$U_{A|\mathbb{K}} := \bigcup_{e \geq 0} \operatorname{Hom}_A(F_*^e A, F_*^e A).$$

It is well known that $D_{A|\mathbb{K}} \subseteq U_{A|\mathbb{K}}$. Now, consider an F_A-module $M \xrightarrow[\sim]{\psi} F^* M$ and set

$$\psi_e := F^{*e-1}(\psi) \circ F^{*e-2}(\psi) \circ \ldots \circ \psi : \quad M \xrightarrow[\sim]{} F^{*e} M.$$

Now, for any $u \in \mathrm{Hom}_A(F^e_* A, F^e_* A)$ and any $m \in M$, set

$$u \cdot m := \left(\psi_e^{-1} \circ (u \otimes \mathbb{1}_M) \circ \psi_e\right)(m).$$

It is straightforward to check (cf. [94, page 116]) that this action is well defined and compatible with the addition and multiplication operations on $U_{A|\mathbb{K}}$ and makes M a $U_{A|\mathbb{K}}$-module. Moreover, an F_A-module homomorphism $M \longrightarrow M'$ is automatically an $U_{A|\mathbb{K}}$-module homomorphism; in this way, we have an additive, covariant, faithful (but not necessarily full), univariate functor

$$\mathcal{F}_A - \mathrm{Mod} \longrightarrow U_{A|\mathbb{K}} - \mathrm{Mod}\,.$$

In addition, as $D_{A|\mathbb{K}} \subseteq U_{A|\mathbb{K}}$ composing the previously constructed functor with the forgetful one given by such inclusion of rings one finally gets a functor

$$\mathcal{F}_A - \mathrm{Mod} \xrightarrow{\xi_{A,\mathbb{K}}} D_{A|\mathbb{K}} - \mathrm{Mod}\,.$$

Roughly speaking, $\xi_{A,\mathbb{K}}$ may be regarded as a sort of analitification functor in prime characteristic. We are to use this construction to check that the extension problems in the category of $\mathcal{F}_A$-modules are non-trivial.

Counterargument 3. In case $n = 2$, suppose that I_1 and I_2 are both cohomologically complete intersection ideals, with $\mathrm{grade}_A(I_i) := c$ for all $i \in \{1, 2\}$, such that $I_1 + I_2$ is a cohomologically complete intersection ideal with $\mathrm{grade}_A(I_1 + I_2) = c + 1$. In this case, it is straightforward to check that the attached filtration (because, in this case, the spectral sequence degenerates without further assumptions) boils down to the following short exact sequence:

$$0 \longrightarrow H^c_{I_1}(A) \oplus H^c_{I_2}(A) \longrightarrow H^c_I(A) \longrightarrow H^{c+1}_{I_1+I_2}(A) \longrightarrow 0.$$

If such exact sequence could be splitted in the category of $\mathcal{F}_A$-modules, then there would be an isomorphism

$$H^c_I(A) \cong H^c_{I_1}(A) \oplus H^c_{I_2}(A) \oplus H^{c+1}_{I_1+I_2}(A)$$

in the category of $\mathcal{F}_A$-modules. Applying the functor $\xi_{A,\mathbb{K}}$ we would obtain an isomorphism in the category of $D_{A,\mathbb{K}}$-modules. But this is impossible taking into account Counterargument 2.

4.3 Cohomological spectral sequences

As in the homological setup, the goal of this section is to build some spectral sequences, in spite of the fact that we are mostly interested on those involving local cohomology modules.

4.3.1 A formalism for producing cohomological spectral sequences

The reader is encouraged to compare the following construction with the ones carried out in 4.2.10 and 4.2.21.

Construction 4.3.1. Let $\mathcal{A}\xrightarrow{T}\mathcal{A}$ be a covariant, left exact, univariate functor. Building from T, we produce the following endofunctor on $\operatorname{Inv}(P,\mathcal{A})$; namely,

$$\begin{aligned}\operatorname{Inv}(P,\mathcal{A}) &\xrightarrow{\mathcal{T}} \operatorname{Inv}(P,\mathcal{A})\\ M=(M_p)_{p\in P} &\longmapsto \mathcal{T}(M):=(T(M_p))_{p\in P}.\end{aligned}$$

In addition, we suppose that T commutes with arbitrary direct sums and that T verifies one (and only one) of the following two assumptions.

(a) For any $\mathfrak{p}\in\operatorname{Spec}(A)$ and for any maximal ideal $\mathfrak{m}$ of A, there exists an A-module X such that

$$T\left(E\left(A/\mathfrak{p}\right)\right)_{\mathfrak{m}}=\begin{cases}X, & \text{if } \mathfrak{p}\in\mathbf{W}(J,K) \text{ and } \mathfrak{p}\subseteq\mathfrak{m},\\ 0, & \text{otherwise.}\end{cases}$$

It is worth noting that X only depends on $\mathfrak{p}$ and $\mathfrak{m}$. Here, J and K are ideals of A which do not depend on any of the previous choices.

(b) For any $\mathfrak{p}\in\operatorname{Spec}(A)$ and for any maximal ideal $\mathfrak{m}$ of A, there exists an A-module Y such that

$$T\left(E\left(A/\mathfrak{p}\right)\right)_{\mathfrak{m}}=\begin{cases}Y, & \text{if } \mathfrak{p}\notin\mathbf{W}(J,K) \text{ and } \mathfrak{p}\subseteq\mathfrak{m},\\ 0, & \text{otherwise.}\end{cases}$$

It is worth noting that Y only depends on $\mathfrak{p}$ and $\mathfrak{m}$. Here, J and K are ideals of A which do not depend on any of the previous choices.

Specific examples

Before introducing examples where the previous assumptions are fulfilled, we have to review the following notion.

Definition 4.3.2 (Bijan-Zadeh)**.** Let Φ be a non-empty set of ideals of A. It is said that Φ is a *system of ideals* of A if, whenever $\mathfrak{a},\mathfrak{b}\in\Phi$, there is an ideal $\mathfrak{c}$ in Φ such that $\mathfrak{c}\subseteq\mathfrak{a}\mathfrak{b}$. The reader should notice that, regarding a system of ideals Φ as poset ordered by reverse inclusion, Φ turns out to be a filtered poset. Keeping this in mind, one can define the bivariate functor $H^i_\Phi(-,-)$ by

$$H^i_\Phi(N,M):=\varinjlim_{\mathfrak{a}\in\Phi}\operatorname{Ext}^i_A(N/\mathfrak{a}N,M).$$

As it was already pointed out by Bijan-Zadeh in [14, page 174], when $N = A$ and, for some $j \in \mathbb{Z}$,

$$\Phi = \{\mathfrak{a} \in \mathcal{L}_A \mid \quad \dim(A/\mathfrak{a}) \leq j\},$$

$H^i_\Phi(A, -)$ is naturally equivalent with the functor $H^i_j(-)$ (respectively, $H^i_{[j]}(-)$) studied in [10] (respectively, used in [37, Definition 5.1]). Here, $\mathcal{L}_A$ denotes the lattice of all the ideals of A.

Example 4.3.3. Before going on, we present several instances where the previous assumptions are fullfilled. In what follows, J, K will denote arbitrary ideals of A and N will stand for a finitely generated object of $\mathcal{A}$ with finite projective dimension. In this case, we are interested in the inverse system $M = A/[*]$. In what follows, we shall use the fact that filtered direct limits commute with finite inverse limits.

(i) $\operatorname{Hom}_A(N, -)$ verifies such assumptions. Indeed, it is enough to point out that

$$\operatorname{Hom}_A\left(N, \varprojlim_{p\in P} A/I_p\right) \cong \varprojlim_{p\in P} \mathcal{H}\mathrm{om}_A\left(N, A/[*]\right).$$

Moreover, given $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{m} \in \operatorname{Max}(A)$ it follows, again as a direct consequence of [30, 4.1.7], that

$$\operatorname{Hom}_A(N, E(A/\mathfrak{p}))_\mathfrak{m} = \begin{cases} \operatorname{Hom}_{A_\mathfrak{m}}\left(N_\mathfrak{m}, E(A/\mathfrak{p})_\mathfrak{m}\right), & \text{if } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

(ii) The ordinary torsion functor $\Gamma_J(-)$ also verifies the previous assumptions. This fact follows from the next chain of isomorphisms:

$$\Gamma_J\left(\varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{t\in\mathbb{N}} \operatorname{Hom}_A\left(A/J^t, \varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{t\in\mathbb{N}} \varprojlim_{p\in P} \mathcal{H}\mathrm{om}_A\left(A/J^t, A/[*]\right)$$
$$\cong \varprojlim_{p\in P} \varinjlim_{t\in\mathbb{N}} \mathcal{H}\mathrm{om}_A\left(A/J^t, A/[*]\right) \cong \varprojlim_{p\in P} \mathcal{H}^0_J\left(A/[*]\right).$$

Furthermore, the reader should also remind, given $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{m} \in \operatorname{Max}(A)$, that

$$\Gamma_J\left(E(A/\mathfrak{p})\right)_\mathfrak{m} = \begin{cases} E(A/\mathfrak{p})_\mathfrak{m}, & \text{if } \mathfrak{p} \in \mathbf{V}(J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

(iii) The generalized torsion functor $\Gamma_J(N, -)$ verifies these requirements too. It may be verified in the following way:

$$\Gamma_J\left(N, \varprojlim_{p\in P} A/I_p\right) \cong \operatorname{Hom}_A\left(N, \Gamma_J\left(\varprojlim_{p\in P} A/I_p\right)\right) \cong \operatorname{Hom}_A\left(N, \varprojlim_{p\in P} \mathcal{H}^0_J\left(A/[*]\right)\right)$$
$$\cong \varprojlim_{p\in P} \mathcal{H}\mathrm{om}_A\left(N, \mathcal{H}^0_J\left(A/[*]\right)\right) \cong \varprojlim_{p\in P} \mathcal{H}^0_J\left(N, A/[*]\right).$$

In addition, we also have to point out, for any $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{m} \in \operatorname{Max}(A)$, that

$$\Gamma_J\left(N, E(A/\mathfrak{p})\right)_{\mathfrak{m}} = \begin{cases} \operatorname{Hom}_{A_{\mathfrak{m}}}\left(N_{\mathfrak{m}}, E(A/\mathfrak{p})_{\mathfrak{m}}\right), & \text{if } \mathfrak{p} \in \mathbf{V}(J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

(iv) The generalized Nagata's ideal transform functor $D_J(N, -)$ also verifies the previous assumptions. Indeed, we only have to notice that

$$D_J\left(N, \varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{t\in\mathbb{N}} \operatorname{Hom}_A\left(J^t N, \varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{t\in\mathbb{N}} \varprojlim_{p\in P} \mathcal{H}\mathrm{om}_A\left(J^t N, A/[*]\right)$$
$$\cong \varprojlim_{p\in P} \mathcal{D}_J(N, A/[*]).$$

In addition, we also have to point out, for any $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{m} \in \operatorname{Max}(A)$, that

$$D_J\left(N, E(A/\mathfrak{p})\right)_{\mathfrak{m}} = \begin{cases} \operatorname{Hom}_{A_{\mathfrak{m}}}\left(N_{\mathfrak{m}}, E(A/\mathfrak{p})_{\mathfrak{m}}\right), & \text{if } \mathfrak{p} \notin \mathbf{V}(J) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

(v) The torsion functor $\Gamma_{J,K}$ with respect to (J, K) verifies the previous requirements. Indeed, set $\widetilde{W}(J,K)$ as the set of ideals $\mathfrak{a}$ of A such that $J^t \subseteq \mathfrak{a} + K$ for some $t \in \mathbb{N}$. We regard $\widetilde{W}(J,K)$ as poset with order given by reverse inclusion of ideals. In this way, applying [133, Theorem 3.2] it follows that

$$\Gamma_{J,K}\left(\varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(J,K)} \Gamma_{\mathfrak{a}}\left(\varprojlim_{p\in P} A/I_p\right).$$

In this way, combining this previous isomorphism joint with the fact that $\widetilde{W}(J,K)$ is filtered it follows that

$$\Gamma_{J,K}\left(\varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(J,K)} \Gamma_{\mathfrak{a}}\left(\varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{\mathfrak{a}\in\widetilde{W}(J,K)} \varprojlim_{p\in P} \mathcal{H}^0_{\mathfrak{a}}\left(A/[*]\right)$$
$$\cong \varprojlim_{p\in P} \mathcal{H}^0_{J,K}\left(A/[*]\right).$$

Moreover, we also notice, for any $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{m} \in \operatorname{Max}(A)$, that

$$\Gamma_{J,K}\left(E(A/\mathfrak{p})\right)_{\mathfrak{m}} = \begin{cases} E(A/\mathfrak{p})_{\mathfrak{m}}, & \text{if } \mathfrak{p} \in \mathbf{W}(J,K) \text{ and } \mathfrak{p} \subseteq \mathfrak{m}, \\ 0, & \text{otherwise.} \end{cases}$$

(vi) Let Φ be a system of ideals. We claim that $\Gamma_\Phi(N,-)$ also verifies these requirements; indeed, it is enough to point out that

$$\Gamma_\Phi\left(N, \varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Hom}_A\left(N/\mathfrak{a}N, \varprojlim_{p\in P} A/I_p\right) \cong \varinjlim_{\mathfrak{a}\in\Phi}\varprojlim_{p\in P} \mathcal{H}\mathrm{om}_A\left(N/\mathfrak{a}N, A/[*]\right)$$
$$\cong \varprojlim_{p\in P}\varinjlim_{\mathfrak{a}\in\Phi} \mathcal{H}\mathrm{om}_A\left(N/\mathfrak{a}N, A/[*]\right) \cong \varprojlim_{p\in P} \mathcal{H}^0_\Phi\left(N, A/[*]\right).$$

Furthermore, we have to point out, for any $\mathfrak{p}\in\operatorname{Spec}(A)$ and $\mathfrak{m}\in\operatorname{Max}(A)$, that

$$\Gamma_\Phi\left(N, E(A/\mathfrak{p})\right)_\mathfrak{m} = \begin{cases} \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Hom}_{A_\mathfrak{m}}\left(N_\mathfrak{m}/\mathfrak{a}_\mathfrak{m}N_\mathfrak{m}, E(A/\mathfrak{p})_\mathfrak{m}\right), & \text{if } \mathfrak{p}\subseteq\mathfrak{m},\\ 0, & \text{otherwise.}\end{cases}$$

4.3.2 Construction of cohomological spectral sequences

As we have previously explained, our goal is to construct an spectral sequence which involves the right derived functors of the inverse limit. The following lemma turns out to be the first step in such construction.

Lemma 4.3.4. *Let Υ be an injective inverse system of the form*

$$\bigoplus_{j\in J}(E_jP)_{\geq q_j},$$

where J is a (not necessarily finite) index set, $q_j\in P$, E_j is an indecomposable injective A-module, and

$$\left[(E_jP)_{\geq q_j}\right]_p := \begin{cases} E_j, & \text{if } p\in[q_j, 1_{\widehat{P}}),\\ 0, & \text{otherwise.}\end{cases}$$

Then, the coaugmented cochain complex

$$0 \longrightarrow \left(\varprojlim_{p\in P}\circ\mathcal{T}\right)(\Upsilon) \longrightarrow \operatorname{Roos}^*(\mathcal{T}(\Upsilon))$$

is exact.

Proof. As $\mathcal{T}$ and $\operatorname{Roos}^*(-)$ commutes with arbitrary direct sums we may assume, without loss of generality, that $\Upsilon = (E(A/\mathfrak{p})P)_{\geq q}$ for some $(\mathfrak{p}, q)\in\operatorname{Spec}(A)\times P$.

Now, we carry out a similar strategy as the one employed in the proof of Lemma 4.2.15; indeed, fix a maximal ideal $\mathfrak{m}$ of A. By the usual generalities, it is enough to show that the cochain complex

$$0 \longrightarrow \left(\varprojlim_{p\in P}\circ\mathcal{T}\right)(\Upsilon)_\mathfrak{m} \longrightarrow \operatorname{Roos}^*(\mathcal{T}(\Upsilon))_\mathfrak{m}$$

is exact. If $\mathfrak{p} \nsubseteq \mathfrak{m}$ then the previous coaugmented cochain complex is zero and we are done; therefore, from now on we suppose that $\mathfrak{p} \subseteq \mathfrak{m}$.

First of all, suppose that T verifies requirement (a) of Construction 4.3.1; on one hand, if $\mathfrak{p} \notin \mathbf{W}(J,K)$, then our coaugmented cochain complex is identically zero, whence we are done. On the other hand, if $\mathfrak{p} \in \mathbf{W}(J,K)$, then such coaugmented cochain complex turns out to be equal to the one for computing the simplicial cohomology of the topological space $[q, 1_{\widehat{P}})$ with coefficients in X. But such topological space is contractible by Lemma 4.1.14; this fact concludes the proof just in case T verifies assumption (a) of Construction 4.3.1.

In this way, from now on in this proof we suppose that T verifies hypothesis (b) of Construction 4.3.1 (and also that $\mathfrak{p} \subseteq \mathfrak{m}$). On one hand, if $\mathfrak{p} \in \mathbf{W}(J,K)$, then our coaugmented cochain complex is identically zero, whence we are done. On the other hand, if $\mathfrak{p} \notin \mathbf{W}(J,K)$, then such coaugmented cochain complex turns out to be equal to the one for computing the simplicial cohomology of the topological space $[q, 1_{\widehat{P}})$ with coefficients in Y. But such topological space is contractible by Lemma 4.1.14; therefore, the proof is completed. □

Next spectral sequence will play a key role in what follows.

Theorem 4.3.5. *The following statements hold.*

(i) There is a first quadrant spectral sequence

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p \in P} \mathbb{R}^j \mathcal{T}(A/[*]) \underset{i}{\Longrightarrow} \mathbb{R}^{i+j} \left(\varprojlim_{p \in P} \circ \mathcal{T} \right) (A/[*]) .$$

(ii) If, in addition, there is a natural equivalence of functors

$$\varprojlim_{p \in P} \circ \mathcal{T} \cong T \circ \varprojlim_{p \in P},$$

then the previous spectral sequence can be arranged in the following manner:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p \in P} \mathbb{R}^j \mathcal{T}(A/[*]) \underset{i}{\Longrightarrow} \mathbb{R}^{i+j} T \left(\varprojlim_{p \in P} A/I_p \right) .$$

(iii) If, furthermore, $A/[*]$ is flasque, then the previous spectral sequence becomes into the next one:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p \in P} \mathbb{R}^j \mathcal{T}(A/[*]) \underset{i}{\Longrightarrow} \mathbb{R}^{i+j} T (A/I) .$$

Proof. Let $0 \longrightarrow A/[*] \longrightarrow \mathcal{I}^\bullet$ be an injective resolution of $A/[*]$ in the category of inverse systems; regarding Theorem 4.1.24, we can choose any spot of $\mathcal{I}^\bullet$ (say, $\mathcal{I}^i$) such that

$$\mathcal{I}^i \cong \bigoplus_{j \in J_i} (E_{j_i} P)_{\geq q_{j_i}},$$

as in Lemma 4.3.4. Moreover, applying to such resolution the functor $\mathcal{T}$ one gets the following cochain complex of inverse systems:

$$0 \longrightarrow \mathcal{T}\left(A/[*]\right) \longrightarrow \mathcal{T}\left(\mathcal{I}^{\bullet}\right).$$

Thus, solving each spot of this cochain complex through the Roos cochain complex, we obtain the bicomplex $\mathrm{Roos}^i\left(\mathcal{T}\left(\mathcal{I}^j\right)\right)$; we hope the following picture explains our last sentence:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Roos}^1\left(\mathcal{T}\left(A/[*]\right)\right) & \longrightarrow & \mathrm{Roos}^1\left(\mathcal{T}\left(\mathcal{I}^0\right)\right) & \longrightarrow & \mathrm{Roos}^1\left(\mathcal{T}\left(\mathcal{I}^1\right)\right) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Roos}^0\left(\mathcal{T}\left(A/[*]\right)\right) & \longrightarrow & \mathrm{Roos}^0\left(\mathcal{T}\left(\mathcal{I}^0\right)\right) & \longrightarrow & \mathrm{Roos}^0\left(\mathcal{T}\left(\mathcal{I}^1\right)\right) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \varprojlim_{p\in P}\left(\mathcal{T}\left(A/[*]\right)\right) & \longrightarrow & \varprojlim_{p\in P}\left(\mathcal{T}\left(\mathcal{I}^0\right)\right) & \longrightarrow & \varprojlim_{p\in P}\left(\mathcal{T}\left(\mathcal{I}^1\right)\right) & \longrightarrow & \dots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In this way, we produce two spectral sequences; namely, the ones obtained respectively through the horizontal and vertical filtrations associated to the bicomplex $\mathrm{Roos}^i\left(\mathcal{T}\left(\mathcal{I}^j\right)\right)$. Furthermore, the reader should notice that both spectral sequences converge because both stem from a bicomplex concentrated in the first quadrant; therefore, our next aim is to calculate their common abutment.

On one hand, the E_2-page of one of our spectral sequences is obtained by firstly computing cohomology on columns and then calculating cohomology of the resulting rows; regardless, Lemma 4.3.4 ensures that all the columns of $\mathrm{Roos}^i\left(\mathcal{T}\left(\mathcal{I}^j\right)\right)$ are acyclic up to $\mathrm{Roos}^*\left(\mathcal{T}\left(A/[*]\right)\right)$, whence such spectral sequence collapses, providing a natural isomorphism between its E_2-sheet and

$$\mathbb{R}^*\left(\varprojlim_{p\in P}\circ\mathcal{T}\right)\left(A/[*]\right).$$

This is exactly the abutment we are looking for.

On the other hand, the E_2-sheet of our remainder spectral sequence is the one obtained by firstly computing cohomology on rows and then computing cohomology on the resulting columns; it is left to the reader to check out that such E_2-page turns out to be

$$E_2^{i,j} = \mathbb{R}^i\varprojlim_{p\in P}\mathbb{R}^j\mathcal{T}\left(A/[*]\right).$$

Summing up, one obtains the following spectral sequence

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p\in P} \mathbb{R}^j \mathcal{T}\left(A/[*]\right) \underset{i}{\Longrightarrow} \mathbb{R}^{i+j} T \left(\varprojlim_{p\in P} A/I_p \right),$$

just what we finally wanted to show in part (i). Finally, part (ii) follows immediately from the well known theory of Grothendieck spectral sequences (cf. [114, Theorem 10.47] for further details). □

Remark 4.3.6. It is worth mentioning here that part (ii) of Proposition 4.3.5 can be regarded as an extension of the argument pointed out by M. Brun, W. Bruns and T. Römer in [31, Remark 8.8].

Examples revisited

The goal of this part is to specialize Theorem 4.3.5 on the functors which have been previously considered at the beginning of Subsection 4.3.1; as usual, throughout these examples N will stand for a finitely generated A-module with finite projective dimension.

The first example is concerned with the Hom functor.

Example 4.3.7 (Covariant Hom). When $T = \operatorname{Hom}_A(N, -)$, we obtain the following spectral sequence:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p\in P} \mathcal{E}\mathrm{xt}_A^j\left(|N|, A/[*]\right) \underset{i}{\Longrightarrow} \operatorname{Ext}_A^{i+j}\left(N, A/I\right),$$

where, as usual, $|N|$ denotes the constant inverse system given by N with identities on N as structural morphisms. On the other hand, when $I = I_1 \cap I_2$ this spectral sequence degenerates without assumptions to the long exact sequence

$$\begin{aligned} \ldots &\longrightarrow \operatorname{Ext}_A^j\left(N, A/I\right) \longrightarrow \operatorname{Ext}_A^j\left(N, A/I_1\right) \oplus \operatorname{Ext}_A^j\left(N, A/I_2\right) \longrightarrow \operatorname{Ext}_A^j\left(N, A/(I_1+I_2)\right) \\ &\longrightarrow \operatorname{Ext}_A^{j+1}\left(N, A/I\right) \longrightarrow \ldots \end{aligned}$$

obtained after applying the functor $\operatorname{Hom}_A(N, -)$ to the natural short exact sequence

$$0 \longrightarrow A/I \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/\left(I_1+I_2\right) \longrightarrow 0.$$

The second one revolves around generalized local cohomology.

Example 4.3.8 (Generalized local cohomology). When $T = \Gamma_J(N, -)$ for some ideal J of A, we obtain the following spectral sequence:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p\in P} \mathcal{H}_J^j\left(|N|, A/[*]\right) \underset{i}{\Longrightarrow} H_J^{i+j}\left(N, A/I\right).$$

Moreover, if $I = I_1 \cap I_2$ then such spectral sequence boils down to the long exact sequence

$$\begin{aligned} \ldots &\longrightarrow H^j_J(N, A/I) \longrightarrow H^j_J(N, A/I_1) \oplus H^j_J(N, A/I_2) \longrightarrow H^j_J(N, A/(I_1 + I_2)) \\ &\longrightarrow H^{j+1}_J(N, A/I) \longrightarrow \ldots \end{aligned}$$

obtained after applying the functor $\Gamma_J(N, -)$ to the natural short exact sequence

$$0 \longrightarrow A/I \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/(I_1 + I_2) \longrightarrow 0.$$

The third one treats the case of generalized ideal transforms.

Example 4.3.9 (Generalized ideal transforms). When $T = D_J(N, -)$ for some ideal J of A, we obtain the following spectral sequence:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p \in P} \mathcal{D}^j_J(|N|, A/[*]) \underset{i}{\Longrightarrow} D^{i+j}_J(N, A/I).$$

In addition, if $I = I_1 \cap I_2$ then the previous spectral sequence becomes into the long exact sequence

$$\begin{aligned} \ldots &\longrightarrow D^j_J(N, A/I) \longrightarrow D^j_J(N, A/I_1) \oplus D^j_J(N, A/I_2) \longrightarrow D^j_J(N, A/(I_1 + I_2)) \\ &\longrightarrow D^{j+1}_J(N, A/I) \longrightarrow \ldots \end{aligned}$$

obtained after applying the functor $D_J(N, -)$ to the natural short exact sequence

$$0 \longrightarrow A/I \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/(I_1 + I_2) \longrightarrow 0.$$

We go on our particular collection of examples with local cohomology with respect to pairs of ideals.

Example 4.3.10 (Local cohomology with respect to pairs of ideals). When $T = \Gamma_{J,K}$ for some ideals J and K of A, we obtain the following spectral sequence:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p \in P} \mathcal{H}^j_{J,K}(A/[*]) \underset{i}{\Longrightarrow} H^{i+j}_{J,K}(A/I).$$

Moreover, if $I = I_1 \cap I_2$ then such spectral sequence boils down to the long exact sequence

$$\begin{aligned} \ldots &\longrightarrow H^j_{J,K}(A/I) \longrightarrow H^j_{J,K}(A/I_1) \oplus H^j_{J,K}(A/I_2) \longrightarrow H^j_{J,K}(A/(I_1 + I_2)) \\ &\longrightarrow H^{j+1}_{J,K}(A/I) \longrightarrow \ldots \end{aligned}$$

obtained after applying the functor $\Gamma_{J,K}$ to the natural short exact sequence

$$0 \longrightarrow A/I \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/(I_1 + I_2) \longrightarrow 0.$$

Our final example involves local cohomology with respect to inverse systems of ideals.

Example 4.3.11 (Local cohomology with respect to inverse systems of ideals). When $T = \Gamma_\Phi$ for some inverse system of ideals Φ of A, we obtain the next spectral sequence:

$$E_2^{i,j} = \mathbb{R}^i \varprojlim_{p \in P} \mathcal{H}^j_\Phi\left(|N|, A/[*]\right) \underset{i}{\Longrightarrow} H^{i+j}_\Phi\left(N, A/I\right).$$

Moreover, if $I = I_1 \cap I_2$ then such spectral sequence boils down to the long exact sequence

$$\begin{aligned} \ldots \longrightarrow H^j_\Phi\left(N, A/I\right) \longrightarrow H^j_\Phi\left(N, A/I_1\right) \oplus H^j_\Phi\left(N, A/I_2\right) \longrightarrow H^j_\Phi\left(N, A/(I_1+I_2)\right) \\ \longrightarrow H^{j+1}_\Phi\left(N, A/I\right) \longrightarrow \ldots \end{aligned}$$

obtained after applying the functor $\Gamma_\Phi(N, -)$ to the natural short exact sequence

$$0 \longrightarrow A/I \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/\left(I_1 + I_2\right) \longrightarrow 0.$$

Preliminary results

Our next aim is to give an alternative description of the E_2-page of the spectral sequence provided by Proposition 4.3.5. Such description involves some previous technical results.

The first one is the so-called *universal coefficients theorem for cohomology.* As in the homological framework, we omit its proof and refer to [114, Theorem 7.59] for further explanations.

Theorem 4.3.12 (Universal coefficients theorem for cohomology)**.** *Let B a (not necessarily commutative) ring, let M be a left B-module, and let $\mathbf{K}_\bullet$ be a chain complex of projective left B-modules whose chain subcomplex of boundaries has all terms projective. Then, for all $n \in \mathbb{N}$ there is a short exact sequence*

$$0 \longrightarrow \operatorname{Ext}^1_B(H_{n-1}(\mathbf{K}_\bullet), M) \xrightarrow{\lambda_n} H^n(\operatorname{Hom}_B(\mathbf{K}_\bullet, M)) \xrightarrow{\mu_n} \operatorname{Hom}_B(H_n(\mathbf{K}_\bullet), M) \longrightarrow 0,$$

and both λ_n and μ_n are canonical.

The second technical tool which we shall need later on is the following result, whose proof we omit because of it is the dual version of Lemma 4.2.28.

Lemma 4.3.13. *Let M be an object of $\mathcal{A}$. Then, for any $i \in \mathbb{N}$,*

$$\left(\mathbb{R}^i \varprojlim_{p \in P}\right)((M_q)_p) \cong \widetilde{H}^{i-1}((q, 1_{\widehat{P}}); M),$$

where the tilde denotes reduced simplicial cohomology. As usual, we agree that the reduced cohomology of the empty simplicial cochain complex is M in degree -1 and zero otherwise.

Now, we are ready for the promised calculation of the E_2-sheet of the spectral sequence given by Theorem 4.3.5.

Proposition 4.3.14. *There is a natural isomorphism*

$$\left(\mathbb{R}^i \varprojlim_{p\in P}\right)(\mathbb{R}^j\mathcal{T}(A/[*])) \cong \bigoplus_{j=d_q} \mathbb{R}^{d_q}T(A/I_q)^{\oplus t_i},$$

where

$$t_i := \dim_{\mathbb{K}}\left(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K})\right).$$

Proof. Since $\mathbb{R}^jT(A/I_p) = 0$ up to a single value of j it follows that there is a canonical isomorphism of inverse systems

$$\mathbb{R}^j\mathcal{T}(A/[*]) \cong \bigoplus_{j=d_q}(\mathbb{R}^{d_q}T(A/I_q))_q.$$

Fix $i \in \mathbb{N}$. Applying to this previous isomorphism the ith right derived functor of the inverse limit over P, we get the following canonical isomorphism:

$$\mathbb{R}^i \varprojlim_{p\in P} \mathbb{R}^j\mathcal{T}(A/[*]) \cong \bigoplus_{j=d_q} \mathbb{R}^i \varprojlim_{p\in P}\left(\mathbb{R}^{d_q}T(A/I_q)\right)_q.$$

Moreover, Lemma 4.3.13 implies that there is a canonical isomorphism:

$$\mathbb{R}^i \varprojlim_{p\in P} \mathbb{R}^j\mathcal{T}(A/[*]) \cong \bigoplus_{j=d_q} \widetilde{H}^{i-1}((q, 1_{\widehat{P}}); \mathbb{R}^{d_q}T(A/I_q)).$$

In this way, the universal coefficients theorem for cohomology implies that there is a canonical isomorphism

$$\mathbb{R}^i \varprojlim_{p\in P} \mathbb{R}^j\mathcal{T}(A/[*]) \cong \bigoplus_{j=d_q} \operatorname{Hom}_{\mathbb{K}}\left(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}), \mathbb{R}^{d_q}T(A/I_q)\right).$$

Now, set $t_i := \dim_{\mathbb{K}}(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}))$. As the evaluation at 1 map

$$\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{R}^{d_q}T(A/I_q)) \longrightarrow \mathbb{R}^{d_q}T(A/I_q)$$

is a canonical isomorphism of A-modules and $\mathbb{R}^{d_q}T(A/I_q)$ is an object of $\mathcal{A}$, it follows that $\operatorname{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{R}^{d_q}T(A/I_q))$ can be regarded as an object of $\mathcal{A}$ and therefore the abstract isomorphism

$$\operatorname{Hom}_{\mathbb{K}}\left(\widetilde{H}_{i-1}((q, 1_{\widehat{P}}); \mathbb{K}), \mathbb{R}^{d_q}T(A/I_q)\right) \cong \mathbb{R}^{d_q}T(A/I_q)^{\oplus t_i}$$

implies that we can regard $\mathbb{R}^{d_q}T(A/I_q)^{\oplus t_i}$ also as an object of $\mathcal{A}$, whence

$$\mathbb{R}^i \varprojlim_{p\in P} \mathbb{R}^j\mathcal{T}(A/[*]) \cong \bigoplus_{j=d_q} \mathbb{R}^{d_q}T(A/I_q)^{\oplus t_i}$$

can be considered as an abstract isomorphism in the category $\mathcal{A}$. □

Main result in the cohomological framework

In this way, we can finally establish the spectral sequence on which we are mainly interested in this cohomological framework; more precisely, next result can be regarded as a refinement of Proposition 4.3.5.

Theorem 4.3.15. *Let $\mathbb{K}$ be any field, let A be a commutative Noetherian ring containing $\mathbb{K}$, let I be an ideal of A with minimal primary decomposition given by*

$$I = I_1 \cap \ldots \cap I_n,$$

and let P be the poset given by all the possible sums of the ideals $I_1, \ldots, I_n$ ordered by reverse inclusion. We further assume (cf. Theorem 4.2.6) that the inverse system $A/[*]$ is flasque and that the natural map

$$A \longrightarrow \varprojlim_{p \in P} A/I_p$$

induces a natural isomorphism

$$A/I \cong \varprojlim_{p \in P} A/I_p.$$

Moreover, we also suppose that, for any $p \neq q$,

$$\operatorname{Hom}_A \Big(\mathbb{R}^{d_p} T\left(A/I_p\right), \mathbb{R}^{d_q} T\left(A/I_q\right) \Big) = 0$$

and that, for any $p \in P$, $\mathbb{R}^j T\left(A/I_p\right) = 0$ up to a single value of j (namely, d_p). Then, there exists a third quadrant spectral sequence of the form:

$$E_2^{i,j} = \bigoplus_{j=d_q} \mathbb{R}^{d_q} T\left(A/I_q\right)^{\oplus t_i} \underset{i}{\Longrightarrow} \mathbb{R}^{i+j} T\left(A/I\right),$$

where

$$t_i := \dim_{\mathbb{K}} \Big(\widetilde{H}_{i-1}\left(\left(q, 1_{\widehat{P}}\right); \mathbb{K}\right) \Big).$$

Moreover, such spectral sequence degenerates at the E_2-sheet.

Moreover, we also want to state the cohomological analogue of Corollary 4.2.32; in this case, the details are left to the interested reader.

Corollary 4.3.16. *Under the assumptions of Theorem 4.3.15, for each $0 \leq r \leq \operatorname{cd}(T)$ there is an increasing filtration $\{H_k^r\}_{r \leq k \leq \dim(A)}$ of $\mathbb{R}^r T(A/I)$ by A-modules such that*

$$H_k^r / H_{k-1}^r \cong \bigoplus_{\{q \in P \ \mid \ r-k=d_q\}} \Big(T^{d_q}\left(A/I_q\right) \otimes_{\mathbb{K}} \widetilde{H}_{k-1}\left(\left(q, 1_{\widehat{P}}\right); \mathbb{K}\right) \Big).$$

4.3.3 Extension problems in the cohomological framework

In Subsection 4.2.4, we studied the extension problems attached to the filtration produced in Theorem 4.2.30 and Corollary 4.2.32. Our next goal is to carry out a similar business with the filtration produced in Theorem 4.3.15; indeed, under its assumptions Theorem 4.3.15 implies that we have a collection of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^r_{k-1} & \longrightarrow & H^r_k & \longrightarrow & H^r_k/H^r_{k-1} & \longrightarrow & 0 \\ 0 & \longrightarrow & H^r_k & \longrightarrow & H^r_{k+1} & \longrightarrow & H^r_{k+1}/H^r_k & \longrightarrow & 0 \\ \vdots & & \vdots & & \ddots & & \ddots & & \ddots \\ 0 & \longrightarrow & H^r_{\dim(A)-1} & \longrightarrow & H^r_{\dim(A)} & \longrightarrow & H^r_{\dim(A)}/H^r_{\dim(A)-1} & \longrightarrow & 0. \end{array}$$

Maybe, it is convenient to remind here that, according to Corollary 4.3.16, for each r the quotients H^r_k/H^r_{k-1} can be decomposed in the following manner:

$$H^r_k/H^r_{k-1} \cong \bigoplus_{\{q\in P \ \mid \ r-k=d_q\}} \Big(T^{d_q}\left(A/I_q\right) \otimes_{\mathbb{K}} \widetilde{H}_{k-1}\left(\left(q, 1_{\widehat{P}}\right); \mathbb{K}\right)\Big).$$

From now on, we omit the superscript r; in addition, the reader should point out that, for each k,

$$(h_k): \quad 0 \longrightarrow H_{k-1} \longrightarrow H_k \longrightarrow H_k/H_{k-1} \longrightarrow 0$$

can be considered as a member of $\operatorname{Ext}^1_{\mathcal{A}}\left(H_k/H_{k-1}, H_{k-1}\right)$.

Next result is just a reformulation of Lemma 4.2.35 in this setup; since its proof is exactly the same as the one of Lemma 4.2.35, we omit the details.

Lemma 4.3.17. *We assume, in addition, that* $\operatorname{Ext}^1_{\mathcal{A}}\left(\mathbb{R}^{d_p}T\left(A/I_p\right), \mathbb{R}^{d_q}T\left(A/I_q\right)\right) = 0$ *provided* $d_p \geq d_q + 2$. *Then, the natural maps*

$$\operatorname{Ext}^1_{\mathcal{A}}(H_k/H_{k-1}, H_{k-1}) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(H_k/H_{k-1}, H_{k-1}/H_{k-2})$$

are injective for all $k \geq 2$.

In other words, Lemma 4.3.17 tells us that in order to determine the extension problems associated to the filtration given in Theorem 4.3.15, it is enough to look at the extension groups $\operatorname{Ext}^1_{\mathcal{A}}\left(\mathbb{R}^{d_p}T\left(A/I_p\right), \mathbb{R}^{d_q}T\left(A/I_q\right)\right)$, where $d_p = d_q + 1$.

4.3.4 Extension problems attached to a local cohomology spectral sequence

The final aim of this chapter is to study the extension problems associated to the spectral sequence produced in Theorem 4.3.15 in case $T = \Gamma_{\mathfrak{m}}$; we think it will be illustrative to single out such result in this very particular case for the convenience of the reader.

Theorem 4.3.18. *Let $\mathbb{K}$ be any field, let A be any commutative Noetherian ring containing $\mathbb{K}$, let I be an ideal of A with minimal primary decomposition given by*

$$I = I_1 \cap \ldots \cap I_n,$$

and let P be the poset given by all the possible sums of the ideals $I_1, \ldots, I_n$ ordered by reverse inclusion; suppose that all of them are contained in a certain maximal ideal of A (namely, $\mathfrak{m}$). We further assume (cf. Theorem 4.2.6) that the inverse system $A/[*]$ is flasque and that the natural map

$$A \longrightarrow \varprojlim_{p \in P} A/I_p$$

induces a natural isomorphism

$$A/I \cong \varprojlim_{p \in P} A/I_p.$$

Moreover, we also suppose that, for any $p \in P$, A/I_p is a Cohen-Macaulay domain. Then, there exists a third quadrant spectral sequence of the form:

$$E_2^{i,j} = \bigoplus_{j=d_q} H_{\mathfrak{m}}^{d_q}\left(A/I_q\right)^{\oplus t_i} \underset{i}{\Longrightarrow} H_{\mathfrak{m}}^{i+j}\left(A/I\right),$$

where

$$t_i := \dim_{\mathbb{K}}\left(\widetilde{H}_{i-1}\left(\left(q, 1_{\widehat{P}}\right); \mathbb{K}\right)\right).$$

Moreover, such spectral sequence degenerates at the E_2-sheet.

Discussion 4.3.19. At this point, maybe it is convenient to provide specific situations where the assumptions of Theorem 4.3.18 are fulfilled.

(i) When I is a squarefree monomial ideal inside a polynomial ring over a field; indeed, in this case I admits a minimal primary decomposition in terms of face ideals and these specific prime ideals are closed with respect to sum; whence the poset P is, under these assumptions, made up of face ideals, which clearly satisfy all the requirements of Theorem 4.3.18.

(ii) When I defines an arrangement of linear varieties; in this case, I admits a minimal primary decomposition in terms of prime ideals generated by linear forms, which is a family of prime ideals closed under sum. Therefore, under these assumptions P is made up by prime ideals generated by linear forms, which also verify all the assumptions of Theorem 4.3.18.

(iii) Let Q be an affine semigroup (cf. [103, Theorem 7.4]), let $\mathbb{K}$ be any field, and suppose that the semigroup ring $A = \mathbb{K}[Q]$ is either:

   (a) normal; or

(b) simplicial and Cohen-Macaulay.

Moreover, let $I \subseteq \mathbb{K}[Q]$ be a squarefree monomial ideal (cf. [103, Definition 7.9]). We claim that Theorem 4.3.18 can also be applied in this case; indeed, I admits a minimal primary decomposition in terms of monomial prime ideals (cf. [103, Example 7.13]). In addition, for any monomial prime ideal $\mathfrak{p}$ it is known (see [33, Theorem 6.3.5], [141, Paragraph below Remark 3.4] in the normal case, and [142, Lemma 2.4] in the remainder case) that $A/\mathfrak{p}$ is a Cohen-Macaulay ring. Therefore, the poset P is made up by Cohen-Macaulay monomial prime ideals, which is just what we need to check.

(iv) Let A be a commutative Noetherian ring containing a field $\mathbb{K}$, and let $y_1, \dots, y_n$ be an A-regular sequence contained in the Jacobson radical of A. Moreover, let $I := J\mathbb{K}[Y_1, \dots, Y_n]$, where $\mathbb{K}[Y_1, \dots, Y_n] \xrightarrow{\psi} A$ is the map of $\mathbb{K}$-algebras which sends each indeterminate Y_i to y_i, and J is a squarefree monomial ideal in the usual sense. Thus, since ψ is flat (cf. [116, Theorem 2.1]), it follows that Theorem 4.3.18 can also be applied in this setting.

The reader should appreciate some difference between the statement of Theorem 4.3.18 and the one of Theorem 4.3.15; indeed, we do not need to require any vanishing of Hom's because of the following:

Lemma 4.3.20. *Let $\mathbb{K}$ be any field, let A be any commutative Noetherian ring containing $\mathbb{K}$, and let $\mathfrak{p}, \mathfrak{q}$ be two prime ideals of A contained in a fixed maximal one (say, $\mathfrak{m}$) such that $\mathfrak{q} \not\subseteq \mathfrak{p}$. Then, one has that*

$$\operatorname{Hom}_A \left(H_{\mathfrak{m}}^{d_p} (A/\mathfrak{p}), H_{\mathfrak{m}}^{d_q} (A/\mathfrak{q}) \right) = 0.$$

Proof. First of all, in order to simplify notation, set $H_p := H_{\mathfrak{m}}^{d_p} (A/\mathfrak{p})$ and $H_q := H_{\mathfrak{m}}^{d_q} (A/\mathfrak{q})$; we assume, to get a contradiction, that there is

$$0 \neq \psi \in \operatorname{Hom}_A (H_p, H_q).$$

By [30, 7.3.2], $\operatorname{Att}_A (H_p) = \{\mathfrak{p}\}$ and $\operatorname{Att}_A (H_q) = \{\mathfrak{q}\}$, where Att denotes the set of attached primes as defined, for instance, in [30, 7.2]; in this way, we get the following commutative square:

$$\begin{array}{ccc} H_p & \xrightarrow{\psi} & H_q \\ \downarrow & & \uparrow \\ Q & \xrightarrow[\sim]{} & \operatorname{Im}(\psi). \end{array}$$

Here, $Q := H_p / \ker(\psi)$ and the bottom isomorphism is the one provided by the First Isomorphism Theorem. So, since $\operatorname{Att}_A(Q) \subseteq \operatorname{Att}_A(H_p)$ (indeed, this fact follows from [30, 7.2.6]) and $\operatorname{Att}_A(H_p) = \{\mathfrak{p}\}$ one has that $\operatorname{Att}_A(Q)$ is either empty or the singleton set $\{\mathfrak{p}\}$; however, $Q \neq 0$ because we are assuming that $\psi \neq 0$, whence $\operatorname{Att}_A(Q) = \{\mathfrak{p}\}$. Moreover, as $Q \cong \operatorname{Im}(\psi)$ this implies that $\operatorname{Att}_A(\operatorname{Im}(\psi)) = \{\mathfrak{p}\}$. On the other hand, since $\operatorname{Im}(\psi) \subseteq H_q$ it is clear that

$$\sqrt{(0 :_A H_q)} \subseteq \sqrt{(0 :_A \operatorname{Im}(\psi))};$$

regardless, combining [30, 7.2.11] and the foregoing facts it follows that

$$\mathfrak{q} = \sqrt{(0 :_A H_q)} \subseteq \sqrt{(0 :_A \operatorname{Im}(\psi))} = \mathfrak{p}.$$

But this contradicts our assumption that $\mathfrak{q} \not\subseteq \mathfrak{p}$, whence ψ must be zero; the proof is therefore completed. □

Before going on, we want to raise the following:

Question 4.3.21. Let $\mathbb{K}$ be any field, let A be $\mathbb{K}[\![x_1, \ldots, x_d]\!]$, and let $\mathfrak{p}, \mathfrak{q}$ be two prime ideals of A such that both $A/\mathfrak{p}$ and $A/\mathfrak{q}$ are Cohen-Macaulay and that $d_p = d_q + 1$. Is it true that

$$\operatorname{Ext}^1_A\left(H^{d_p}_{\mathfrak{m}}(A/\mathfrak{p}), H^{d_q}_{\mathfrak{m}}(A/\mathfrak{q})\right) = 0?$$

The reader should point out that an affirmative answer to the previous question, combined with Lemma 4.3.17, would imply that the calculations carried out in the next subsubsection determines all the extension problems attached to the filtration produced by the degeneration of the spectral sequence provided by Theorem 4.3.18.

In this way, the final aim of this chapter is to show that the extension problems associated to the filtration produced by the degeneration of the spectral sequence provided by Theorem 4.3.18 are non-trivial; regarding our previous comments, it is clear that we can not ensure that we are determining all the extension problems attached to such filtration.

Extension problems in the ungraded setting

Our goal in this part is to show that the extension problems attached to filtration produced by the degeneration of the spectral sequence provided by Theorem 4.3.18 are, in general, non-trivial.

Firstly, we review the well known duality between Ext and Tor groups obtained through Matlis duality; we omit the proof and refer to [131, 3.4.14] for details.

Proposition 4.3.22. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local ring, let $j \in \mathbb{Z}$ and let X_0, X_1 be R-modules. As usual, $(-)^\vee$ denotes the Matlis duality functor $\operatorname{Hom}_R(-, E_R)$, where E_R denotes a choice of injective hull of $\mathbb{K}$ over R. Then, the following statements hold.*

(a) $\operatorname{Tor}^R_j(X_0, X_1)^\vee \cong \operatorname{Ext}^j_R(X_0, X_1^\vee)$.

(b) If, in addition X_0 is finitely generated, then one has that

$$\operatorname{Tor}_j^R(X_0, X_1^\vee) \cong \operatorname{Ext}_R^j(X_0, X_1)^\vee.$$

In any case, both isomorphisms are canonical.

Now, we are ready to show that the extension problems extension problems associated to the filtration produced by the degeneration of the spectral sequence provided by Theorem 4.3.18 in this ungraded setting are non-trivial. This fact follows directly from the next technical result, which is interesting in its own right.

Proposition 4.3.23. *Let $\mathbb{K}$ be any field, set $A := \mathbb{K}[\![x_1, \ldots, x_d]\!]$ and let $y_1, \ldots, y_n$ be an A-regular sequence. Moreover, for each $1 \leq t \leq n$, set $I_t := \langle y_1, \ldots, y_t\rangle$ and $H_t := H_{\mathfrak{m}}^{d-t}(A/I_t)$. Then, the following statements hold.*

(i) $\operatorname{Hom}_A(A/I_{t+1}, A/I_t) = 0$.

(ii) $\operatorname{Hom}_A(A/I_t, A/I_t) = A/I_t$.

(iii) $\operatorname{Ext}_A^1(A/I_{t+1}, A/I_t) = A/I_{t+1}$.

(iv) $\operatorname{Ext}_A^1((0 :_E I_t), (0 :_E I_{t+1})) = A/I_{t+1}$.

(v) $\operatorname{Ext}_A^1(H_t, H_{t+1}) = A/I_{t+1}$.

Proof. Let $Y_1, \ldots, Y_n$ be indeterminates and, moreover, consider the natural map of $\mathbb{K}$-algebras $S := \mathbb{K}[Y_1, \ldots, Y_n] \longrightarrow A$ which sends Y_j into y_j for each $1 \leq j \leq n$. Thus, using Hartshorne's results (cf. [60, Proposition 1 and Corollary 1]) it follows that

$$(I_t :_A I_{t+1}) = (\langle Y_1, \ldots, Y_t\rangle :_S \langle Y_1, \ldots, Y_{t+1}\rangle)\, A = \langle Y_1, \ldots, Y_t\rangle A = I_t,$$

whence

$$\operatorname{Hom}_A(A/I_{t+1}, A/I_t) \cong \frac{(I_t :_A I_{t+1})}{I_t} = 0,$$

just what we firstly wanted to check.

On the other hand, consider the short exact sequence

$$0 \longrightarrow A/I_t \xrightarrow{\cdot y_{t+1}} A/I_t \longrightarrow A/I_{t+1} \longrightarrow 0.$$

In this way, applying the functor $\operatorname{Hom}_A(-, A/I_t)$ to this short exact sequence the functor $\operatorname{Hom}_A(-, A/I_t)$ one obtains the following exact one:

$$0 \longrightarrow \operatorname{Hom}_A(A/I_{t+1}, A/I_t) \longrightarrow \operatorname{Hom}_A(A/I_t, A/I_t) \xrightarrow{\cdot y_{t+1}} \operatorname{Hom}_A(A/I_t, A/I_t)$$

$$\longrightarrow \operatorname{Ext}_A^1(A/I_{t+1}, A/I_t) \longrightarrow \operatorname{Ext}_A^1(A/I_t, A/I_t) \xrightarrow{\cdot y_{t+1}} \operatorname{Ext}_A^1(A/I_t, A/I_t).$$

Thus, since

$$\mathrm{Hom}_A(A/I_t, A/I_t) \cong \frac{(I_t :_A I_t)}{I_t} = A/I_t$$

and $\mathrm{Hom}_A(A/I_{t+1}, A/I_t) = 0$ one can rewrite the previous exact sequence in the following way:

$$0 \longrightarrow A/I_t \xrightarrow{\cdot y_{t+1}} A/I_t \longrightarrow \mathrm{Ext}^1_A(A/I_{t+1}, A/I_t) \longrightarrow \mathrm{Ext}^1_A(A/I_t, A/I_t) \xrightarrow{\cdot y_{t+1}} \mathrm{Ext}^1_A(A/I_t, A/I_t).$$

Therefore, since the cokernel of $A/I_t \xrightarrow{\cdot y_{t+1}} A/I_t$ is A/I_{t+1} one obtains the following exact sequence:

$$0 \longrightarrow A/I_{t+1} \longrightarrow \mathrm{Ext}^1_A(A/I_{t+1}, A/I_t) \longrightarrow \mathrm{Ext}^1_A(A/I_t, A/I_t) \xrightarrow{\cdot y_{t+1}} \mathrm{Ext}^1_A(A/I_t, A/I_t). \quad (4.8)$$

Now, we claim that $\mathrm{Ext}^1_A(A/I_t, A/I_t) = (A/I_t)^{\oplus t}$; indeed, $\mathrm{Ext}^1_A(A/I_t, A/I_t)$ can be computed as $H^1(\mathrm{Hom}_A(K_\bullet(y_1, \ldots, y_t), A/I_t))$; that is, the first cohomology group of the cochain complex obtained by applying the functor $\mathrm{Hom}_A(-, A/I_t)$ to the Koszul resolution of A/I_t (namely, $K_\bullet(y_1, \ldots, y_t)$). Regardless, taking into account the very definition of the Koszul complex, we know that all the matrices which represent the differentials in $K_\bullet(y_1, \ldots, y_t)$ have all their entries in I_t; thus, this single fact implies that all the differentials of the cochain complex $\mathrm{Hom}_A(K_\bullet(y_1, \ldots, y_t), A/I_t)$ vanish and therefore one obtains that

$$\mathrm{Ext}^1_A(A/I_t, A/I_t) = H^1(\mathrm{Hom}_A(K_\bullet(y_1, \ldots, y_t), A/I_t)) = (A/I_t)^{\oplus t}.$$

In this way, bearing in mind this fact we can arrange the exact sequence (4.8) in the following way:

$$0 \longrightarrow A/I_{t+1} \longrightarrow \mathrm{Ext}^1_A(A/I_{t+1}, A/I_t) \longrightarrow (A/I_t)^{\oplus t} \xrightarrow{\cdot y_{t+1}} (A/I_t)^{\oplus t}.$$

But the endomorphism on $(A/I_t)^{\oplus t}$ given by multiplication by y_{t+1} is injective; whence one finally obtains that

$$A/I_{t+1} \cong \mathrm{Ext}^1_A(A/I_{t+1}, A/I_t).$$

In particular, part (iii) holds.

In addition, we have to point out that part (iv) follows combining part (iii) joint with Proposition 4.3.22 and Matlis duality in the following way: indeed, we have to notice that

$$(A/I_{t+1})^\vee = \mathrm{Ext}^1_A(A/I_{t+1}, A/I_t)^\vee \cong \mathrm{Tor}_1^A\left(A/I_{t+1}, (A/I_t)^\vee\right) \cong \mathrm{Tor}_1^A\left((A/I_t)^\vee, A/I_{t+1}\right)$$

and therefore

$$A/I_{t+1} \cong (A/I_{t+1})^{\vee\vee} \cong \mathrm{Tor}_1^A\left((A/I_t)^\vee, A/I_{t+1}\right)^\vee \cong \mathrm{Ext}^1_A\left((0 :_E I_t), (0 :_E I_{t+1})\right),$$

whence part (iv) also holds.

Finally, since A/I_t is a complete intersection ring for any t it is, in particular, quasi Gorenstein. In this way, combining this fact joint with part (iv) one has that

$$\mathrm{Ext}^1_A(H_t, H_{t+1}) \cong \mathrm{Ext}^1_A\left((0 :_E I_t), (0 :_E I_{t+1})\right) = A/I_{t+1},$$

just what we finally wanted to show. □

Extension problems in the graded setting

The main purpose of this part is to determine explicitly the extension problems attached to the spectral sequence provided by Theorem 4.3.18 in case our ambient is a polynomial ring graded in a suitable way. As the reader will easily see, our way of proceeding will follow the same steps used in the ungraded setting, making the appropriate changes.

Firstly, we are to review the following auxiliary notions (cf. [88, Definition 4.1.6 and Definition 4.1.17]).

Definition 4.3.24 (Kreuzer, Robbiano)**.** Let $\mathbb{K}$ be a field, let S be the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$ and let $m \geq 1$ be an integer.

(i) Given a matrix $W \in \mathcal{M}_{m\times d}(\mathbb{Z})$, we can consider the $\mathbb{Z}^m$-grading on S for which $\mathbb{K} \subseteq S_0$ and the indeterminates are homogeneous elements whose degrees are given by the columns of W. In this case, it is said that S *is graded by* W. Moreover, we refer to the rows of W as the *weight vectors* of the indeterminates $x_1, \dots, x_d$.

(ii) Now, suppose that S is graded by a matrix $W \in \mathcal{M}_{m\times d}(\mathbb{Z})$ of rank m and let $w_1, \dots, w_m$ be the weight vectors. It is said that the grading on S given by W is *of positive type* provided there exist $a_1, \dots, a_m \in \mathbb{Z}$ such that all the entries of $a_1 w_1 + \dots + a_m w_m$ are positive. In this case, it is also said that W is a matrix *of positive type.*

Example 4.3.25. We exhibit some examples of positive type matrices.

(a) The standard grading on $\mathbb{Z}$ (that is, $\deg(x_i) = 1$ for all i) is given by matrix $\begin{pmatrix}1 & \dots & 1\end{pmatrix}$, which is clearly of positive type.

(b) The standard $\mathbb{Z}^d$-grading on S (that is, $\deg(x_i) = \mathbf{e}_i$, where $\mathbf{e}_i$ denotes the element of $\mathbb{Z}^d$ which has all its components 0 up to a 1 in the ith position) is given by $W =$ the identity matrix of size d. It is also clear in this case that W is of positive type; indeed, just take $a_i = 1$ for all i in the definition.

The reason for which we consider matrices of positive type is the following result, which says that polynomial rings with gradings of positive type and finitely generated graded modules over them have finite dimensional homogeneous components. We omit its proof and refer to [88, Proposition 4.1.19] for details.

Proposition 4.3.26 (Kreuzer, Robbiano)**.** *Let $\mathbb{K}$ be a field, let S be the polynomial ring $\mathbb{K}[x_1, \dots, x_d]$ graded by a matrix $W \in \mathcal{M}_{m\times d}(\mathbb{Z})$ of positive type, and let M be a finitely generated W-graded S-module. Then, the following statements hold.*

(a) We have $S_0 = \mathbb{K}$.

(b) For all $a \in \mathbb{Z}^m$, we have $\dim_{\mathbb{K}}(M_a) < +\infty$.

Remark 4.3.27. It is worth mentioning that the conclussion of the previous proposition also works in greater generality; the interested reader may like to consult [103, Theorem 8.6] for additional details.

In this way, hereafter $\mathbb{K}$ will denote a field and S will stand for the polynomial ring $\mathbb{K}[x_1, \ldots, x_d]$ graded by a positive type matrix $W \in \mathcal{M}_{m\times d}(\mathbb{Z})$. Moreover, let $y_1, \ldots, y_n$ be homogeneous elements of S (with $\deg(y_j) = D_j \in \mathbb{Z}^m$) which forms an S-regular sequence and, for any $1 \leq t \leq n$, set $I_t := \langle y_1, \ldots, y_t\rangle$. On the other hand, borrowing notation from [33, 1.5] (see also [30, 13.1.8]) ${}^*\operatorname{Hom}_S(-,-)$ will stand for the internal Hom in the category of W-graded S-modules, and set $(-)^\vee := {}^*\operatorname{Hom}_{\mathbb{K}}(-, \mathbb{K})$.

Next result gives the W-graded analogue of Proposition 4.3.22. Albeit its proof is the adaptation in this graded context of [131, Proof of 3.4.14], we provide it for the convenience of the reader.

Proposition 4.3.28. *Let $j \in \mathbb{Z}$ and let X_0, X_1 be W-graded S-modules. Then, the following statements hold.*

(a) $\operatorname{Tor}_j^S(X_0, X_1)^\vee \cong {}^*\operatorname{Ext}_S^j(X_0, X_1^\vee)$.

(b) If, in addition, X_0 is finitely generated, then one has that

$$\operatorname{Tor}_j^S(X_0, X_1^\vee) \cong {}^*\operatorname{Ext}_S^j(X_0, X_1)^\vee.$$

In any case, both isomorphisms are canonical.

Proof. Firstly, we prove part (a). Indeed, set T^j and U^j to be the functors $\operatorname{Tor}_j^S(-, X_1)^\vee$ and ${}^*\operatorname{Ext}_S^j(-, X_1^\vee)$. Since $(-)^\vee$ is exact and contravariant, it follows that both $(T^j)_{j\in\mathbb{N}}$ and $(U^j)_{j\in\mathbb{N}}$ form a positive strongly connected sequences of contravariant functors. Moreover, it is well known that

$${}^*\operatorname{Hom}_{\mathbb{K}}(X_0 \otimes_S X_1, \mathbb{K}) \cong {}^*\operatorname{Hom}_S(X_0, {}^*\operatorname{Hom}_{\mathbb{K}}(X_1, \mathbb{K}))$$

for any W-graded S-module X_0; on the other hand, it is also clear that $T^j P = 0 = U^j P$ for $j \geq 1$ and any *projective module P. Therefore, applying the appropriate dual of [30, 13.3.5] one has that there exist uniquely determined natural equivalences of functors $T^j \Longrightarrow U^j$; whence part (a) follows directly from this fact.

Finally, we prove part (b). In this case, we set T^j and U^j as the functors $\operatorname{Tor}_j^S(-, X_1^\vee)$ and ${}^*\operatorname{Ext}_S(-, X_1)^\vee$. In this case, $(T^j)_{j\in\mathbb{N}}$ and $(U^j)_{j\in\mathbb{N}}$ both form a positive strongly connected sequence of covariant functors. In addition, for a finitely generated W-graded S-module X_0 one has a canonical isomorphism

$$X_0 \otimes_S {}^*\operatorname{Hom}_{\mathbb{K}}(X_1, \mathbb{K}) \cong {}^*\operatorname{Hom}_{\mathbb{K}}({}^*\operatorname{Hom}_S(X_0, X_1), \mathbb{K}).$$

Again, $T^jP = 0 = U^jP$ for $j \geq 1$ and any *projective module P. Therefore, applying the appropriate dual of [30, 13.3.5] one has that there exist uniquely determined natural equivalences of functors $T^j \Longrightarrow U^j$; whence one has that part (b) also holds. $\square$

The following statement can be regarded as the W-graded analogue of Proposition 4.3.23.

Proposition 4.3.29. *Preserving the foregoing assumptions and notations, the following statements hold.*

(i) $^*\operatorname{Hom}_S(S/I_{t+1}, S/I_t) = 0$.

(ii) $^*\operatorname{Hom}_S(S, S) \cong S$.

(iii) $^*\operatorname{Hom}_S(S/I_t, S/I_t) \cong S/I_t$.

(iv) $^*\operatorname{Hom}_S(S/I_t, (S/I_t)(-D_{t+1})) \cong (S/I_t)(-D_{t+1})$.

(v) $^*\operatorname{Ext}^1_S(S/I_{t+1}, S/I_t) \cong (S/I_{t+1})(D_{t+1})$.

(vi) $^*\operatorname{Ext}^1_S((S/I_{t+1})^\vee, (S/I_t)^\vee) \cong (S/I_{t+1})(D_{t+1})$.

(vii) $^*\operatorname{Ext}^1_S(H_t, H_{t+1}) \cong (S/I_{t+1})(D_{t+1})$, *where H_t (respectively, H_{t+1}) stands for the local cohomology module $H^{d-t}_{\mathfrak{m}}(S/I_t)$ (respectively, $H^{d-t-1}_{\mathfrak{m}}(S/I_{t+1})$).*

Proof. First of all, we have to point out that, since S/I_t is finitely generated, one has that $^*\operatorname{Hom}_S(S/I_{t+1}, S/I_t)$ is nothing but $\operatorname{Hom}_S(S/I_{t+1}, S/I_t)$ in case the grading is forgotten. Regardless, we have checked in part (i) of Proposition 4.3.23 that $\operatorname{Hom}_S(S/I_{t+1}, S/I_t) = 0$; whence part (i) follows directly from this fact.

Second, as $^*\operatorname{Hom}_S(S, S)$ (respectively, $^*\operatorname{Hom}_S(S/I_t, S/I_t)$) are nothing but $\operatorname{Hom}_S(S, S)$ (respectively, $\operatorname{Hom}_S(S/I_t, S/I_t)$) when the grading is forgotten, we obtain that both part (ii) and (iii) hold. Moreover, we can also get part (iv) in the below way:

$$^*\operatorname{Hom}_S(S/I_t, (S/I_t)(-D_{t+1})) = {}^*\operatorname{Hom}_S(S/I_t, S/I_t)(-D_{t+1}) = (S/I_t)(-D_{t+1}).$$

Now, consider the next short exact sequence of W-graded S-modules and homogeneous homomorphisms:

$$0 \longrightarrow (S/I_t)(-D_{t+1}) \xrightarrow{\cdot y_{t+1}} S/I_t \longrightarrow S/I_{t+1} \longrightarrow 0.$$

Applying to such short exact sequence the functor $^*\operatorname{Hom}_S(-, (S/I_t)(-D_{t+1}))$ one obtains the following exact sequence of W-graded S-modules and homogeneous homomorphisms:

$$\begin{aligned} 0 &\longrightarrow {}^*\operatorname{Hom}_S(S/I_{t+1}, (S/I_t)(-D_{t+1})) \longrightarrow {}^*\operatorname{Hom}_S(S/I_t, (S/I_t)(-D_{t+1})) \\ &\xrightarrow{\cdot y_{t+1}} {}^*\operatorname{Hom}_S((S/I_t)(-D_{t+1}), (S/I_t)(-D_{t+1})) \longrightarrow {}^*\operatorname{Ext}^1_S(S/I_{t+1}, (S/I_t)(-D_{t+1})) \\ &\longrightarrow {}^*\operatorname{Ext}^1_S(S/I_t, (S/I_t)(-D_{t+1})) \xrightarrow{\cdot y_{t+1}} {}^*\operatorname{Ext}^1_S((S/I_t)(-D_{t+1}), (S/I_t)(-D_{t+1})). \end{aligned}$$

Regardless, we have checked in the foregoing parts that ${}^*\operatorname{Hom}_S(S/I_{t+1},(S/I_t)(-D_{t+1}))=0$, ${}^*\operatorname{Hom}_S(S/I_t,(S/I_t)(-D_{t+1}))=(S/I_t)(-D_{t+1})$, and that one also has the equality ${}^*\operatorname{Hom}_S((S/I_t)(-D_{t+1}),(S/I_t)(-D_{t+1}))=S/I_t$. In this way, we can rewrite the previous exact sequence in the next way:

$$\begin{aligned}&0\longrightarrow (S/I_t)(-D_{t+1})\xrightarrow{\cdot y_{t+1}} S/I_t\longrightarrow {}^*\operatorname{Ext}^1_S(S/I_{t+1},(S/I_t)(-D_{t+1}))\\ &\longrightarrow {}^*\operatorname{Ext}^1_S(S/I_t,(S/I_t)(-D_{t+1}))\xrightarrow{\cdot y_{t+1}} {}^*\operatorname{Ext}^1_S((S/I_t)(-D_{t+1}),(S/I_t)(-D_{t+1})).\end{aligned}$$

Moreover, since the cokernel of $(S/I_t)(-D_{t+1})\xrightarrow{\cdot y_{t+1}}S/I_t$ is S/I_{t+1}, it follows that we have the following exact sequence:

$$\begin{aligned}&0\longrightarrow S/I_{t+1}\longrightarrow {}^*\operatorname{Ext}^1_S(S/I_{t+1},(S/I_t)(-D_{t+1}))\longrightarrow {}^*\operatorname{Ext}^1_S(S/I_t,(S/I_t)(-D_{t+1}))\\ &\xrightarrow{\cdot y_{t+1}} {}^*\operatorname{Ext}^1_S((S/I_t)(-D_{t+1}),(S/I_t)(-D_{t+1})).\end{aligned}\tag{4.9}$$

Now, we claim that

$${}^*\operatorname{Ext}^1_S(S/I_t,(S/I_t)(-D_{t+1}))\cong\bigoplus_{j=1}^{t}(S/I_t)(D_j-D_{t+1}).$$

Indeed, it is well known that ${}^*\operatorname{Ext}^1_S(S/I_t,(S/I_t)(-D_{t+1}))$ is the first cohomology group of the complex ${}^*\operatorname{Hom}_S(K_\bullet(y_1,\ldots,y_t),(S/I_t)(-D_{t+1}))$, where $K_\bullet(y_1,\ldots,y_t)$ denotes the homological Koszul resolution of S/I_t. However, since all the spots in the cochain complex ${}^*\operatorname{Hom}_S(K_\bullet(y_1,\ldots,y_t),(S/I_t)(-D_{t+1}))$ are S/I_t-modules and all the differentials ∂^i in such cochain complex are represented by matrices with entries in I_t, it follows that all these differentials are zero and therefore

$${}^*\operatorname{Ext}^1_S(S/I_t,(S/I_t)(-D_{t+1}))=\ker(\partial^1)=\bigoplus_{j=1}^{t}(S/I_t)(D_j-D_{t+1}).$$

In addition, by similar reasons one also has that

$${}^*\operatorname{Ext}^1_S((S/I_t)(-D_{t+1}),(S/I_t)(-D_{t+1}))\cong\bigoplus_{j=1}^{t}(S/I_t)(D_j).$$

In this way, the map

$${}^*\operatorname{Ext}^1_S(S/I_t,(S/I_t)(-D_{t+1}))\xrightarrow{\cdot y_{t+1}}{}^*\operatorname{Ext}^1_S((S/I_t)(-D_{t+1}),(S/I_t)(-D_{t+1}))$$

can be rewritten in the following way:

$$\bigoplus_{j=1}^{t}(S/I_t)(D_j-D_{t+1})\xrightarrow{\cdot y_{t+1}}\bigoplus_{j=1}^{t}(S/I_t)(D_j).$$

But this homomorphism is clearly injective. In this way, combining this fact joint with (4.9) one finally obtains that

$$S/I_{t+1} \cong {}^*\operatorname{Ext}^1_S(S/I_{t+1}, (S/I_t)(-D_{t+1})) = {}^*\operatorname{Ext}^1_S(S/I_{t+1}, S/I_t)(-D_{t+1}),$$

whence part (v) holds too. The reader should notice that the righmost equality is well known (cf. [30, 14.1.10]).

Now, we can deduce part (vi) combining part (v) jointly with Proposition 4.3.28 in the following manner:

$$\begin{aligned} S/I_{t+1} &\cong \left((S/I_{t+1})^\vee\right)^\vee \cong \left({}^*\operatorname{Ext}^1_S(S/I_{t+1}, (S/I_t)(-D_{t+1}))^\vee\right)^\vee \\ &\cong \operatorname{Tor}^S_1(S/I_{t+1}, [(S/I_t)(-D_{t+1})]^\vee)^\vee \cong \operatorname{Tor}^S_1((S/I_t)^\vee(D_{t+1}), S/I_{t+1})^\vee \\ &\cong {}^*\operatorname{Ext}^1_S((S/I_t)^\vee (D_{t+1}), (S/I_{t+1})^\vee). \end{aligned}$$

Finally, the graded local duality theorem (cf. [30, 14.4.1]) implies that $(S/I_t)^\vee(-c) \cong H_t$ and $(S/I_{t+1})^\vee(-c) \cong H_{t+1}$, where $c = c_1 + \ldots + c_d$ and $c_1, \ldots, c_d$ are the columns of matrix W. Whence

$${}^*\operatorname{Ext}^1_S(H_t, H_{t+1}) \cong {}^*\operatorname{Ext}^1_S((S/I_t)^\vee (-c), (S/I_{t+1})^\vee (-c)) \cong {}^*\operatorname{Ext}^1_S((S/I_t)^\vee, (S/I_{t+1})^\vee).$$

But the leftmost term is isomorphic to $(S/I_{t+1})(D_{t+1})$; the proof is therefore completed. □

A generalization of Gräbe's formula

As the title says, the goal of this part is to show how the results obtained in Proposition 4.3.29 (overall, part (vii)) recover and generalize the so-called *Gräbe's formula* obtained by H.-G. Gräbe in [54, Theorem 2].

More precisely, next result, which is just a particular case of Proposition 4.3.29, provides such generalization.

Theorem 4.3.30. *Let $\mathbb{K}$ be a field, and let S be the polynomial ring $\mathbb{K}[x_1, \ldots, x_d]$ graded by a positive type matrix $W \in \mathcal{M}_{m\times d}(\mathbb{Z})$. Moreover, for any $1 \le t \le d$ set F_t as the face ideal of S generated by $x_1, \ldots, x_d$ and $H_t := H^{d-t}_{\mathfrak{m}}(S/F_t)$. Then, there is a canonical isomorphism*

$${}^*\operatorname{Ext}^1_S(H_t, H_{t+1}) \cong (S/F_{t+1})(c_{t+1}),$$

where c_{t+1} denotes the $(t+1)$th column of matrix W; furthermore, such natural isomorphism is induced by multiplication by x_{t+1}.

Remark 4.3.31. When W is the identity matrix of size d, Theorem 4.3.30 is exactly what Gräbe shows in [54, Theorem 2]; on the other hand, E. Miller proved in his thesis (cf. [102, Corollary 6.24] that Gräbe's formula is equivalent to the one obtained by M. Mustaţă in [106, Theorem 2.1 and Corollary 2.2]; keeping in mind this fact, Theorem 4.3.30 can also be regarded as a generalization of Mustaţă's formula.

Bibliographical notes

Generalized local cohomology modules were introduced by J. Herzog in his Habilitationsschrift (cf. [64]). It was, as far as we know, the first attempt to generalize the ordinary local cohomology modules. With time, other generalizations of local cohomology modules have appeared in the literature; the reader is encouraged to consult [14] and [143] as a taste.

As pointed out by C. U. Jensen in [74], the study of the higher derived functors of the inverse limit functors was pionereed by several authors. At the best of our knowledge, chronologically it seems that the right derived functors of the inverse limit were originally studied by J. Milnor, overall with applications in Algebraic Topology in mind (cf. [144]); later on, J. E. Roos and G. Nöbeling introduced independently in [112] and [108] what in this chapter is known as the Roos complex for computing the right derived functors of the inverse limit. It is worth noting that in [108] is also presented a detailed treatment of the left derived functors of the direct limit. Of course, at the same time other authors produced similar results and constructions, such as in [42] and [91]; the interested reader may like to consult [72] for additional information.

Appendix A

Cartier algebras of Stanley-Reisner rings: computational issues

In Chapter 2, we have studied the Cartier algebra $\mathcal{C}^R$ provided R is a complete Stanley-Reisner ring. It turns out that $\mathcal{C}^R$ can only be principally generated or infinitely generated as R-algebra and that this fact only depends on the minimal primary decomposition of the corresponding Stanley-Reisner ideal. Moreover, in Chapter 2 we have given examples of complete Stanley-Reisner rings with principal Cartier algebra and examples with infinite Cartier algebra.

The main purpose of this appendix is to describe some of the results obtained in Chapter 2 in computational terms. Namely, we provide the pseudo-code of some of the procedures which we wrote in order to study the above-mentioned examples. CoCoA has been used extensively in the implementation of the methods described below. The code is located in [24].

A.1 Theorems which become procedures

A.1.1 Principal generation

We start reminding the main result of Chapter 2 (cf. Theorem 2.3.5) in computational terms.

Theorem A.1.1. *Let $\mathbb{K}$ be an F-finite field, let $d \in \mathbb{N}$, let $S := \mathbb{K}[x_1, \dots, x_d]$ be the polynomial ring in d variables over $\mathbb{K}$, let J be a squarefree monomial ideal of S, set $T := \mathbb{K}[\![x_1, \dots, x_d]\!]$ and $I := JT$. Then, the following statements are equivalent.*

(i) $\mathcal{C}^{T/I}$ *is principally generated.*

(ii) $\left(J^{[2]} :_S J\right) = J^{[2]} + \langle \mathrm{LCM}_J \rangle$, *where* LCM_J *denotes the least common multiple of a minimal monomial squarefree generating set for J.*

In this way, this result can be easily translated into a procedure. It is described in Algorithm 1.

Algorithm 1 IsPrincipal(I)

Input: squarefree monomial ideal I inside a polynomial ring S over any field.
Output: **true** if $\mathcal{C}^{T/I}$ is principally generated and **false**, otherwise.

$K \leftarrow I^{[2]}$
$J \leftarrow (K :_S I)$
$K \leftarrow K + \langle \mathrm{LCM}_I \rangle$
if $J \subseteq K$ **then**
  return true
end if
return false

A.1.2 Gorensteinness of rings

Recall that in Chapter 1 (cf. Proposition 1.4.19) we have proved that if R is Gorenstein then $\mathcal{C}^R$ is principally generated. In this way, it is of some interest for us to have a method for testing when a computable ring is Gorenstein.

We have implemented in our package a well-known criterion in order to determine when a quotient of a polynomial ring is Gorenstein. First of all, we remind this criterion. We skip its proof and refer to [33, Theorem 3.2.10] for further details.

Theorem A.1.2. *Let $\mathbb{K}$ be any field, let $d \in \mathbb{N}$, let $S = \mathbb{K}[x_1, \ldots, x_d]$ and let I be an ideal of S. Then, the following statements are equivalent.*

(i) $A := S/I$ is Gorenstein.

(ii) One has that

$$\dim_{\mathbb{K}} \operatorname{Ext}^i_S(A, \mathbb{K}) = \begin{cases} 1, & \text{if } i = \dim(A), \\ 0, & \text{otherwise.} \end{cases}$$

We shall translate this result into a method for determining the Gorensteinness of a quotient of a polynomial ring. It is described in Algorithm 2.

A.2 Building examples

In this section, we are to introduce procedures which were designed in order to explore concrete examples which later became theoretical results, as it has been described in Chapter 2.

Algorithm 2 IsGorenstein(I)

Input: ideal I inside a polynomial ring S over any field.
Output: **true** if S/I is Gorenstein and **false**, otherwise.

$\mathbb{K} \leftarrow S/\mathfrak{m}$
$A \leftarrow S/I$
$d \leftarrow \dim(A)$
$L \leftarrow \{\mathrm{Ext}^i_S(A,\mathbb{K}) \mid \quad 0 \leq i \leq d\}$
$n \leftarrow \#L - \#\{i \in L \mid \quad \mathrm{Ext}^i_S(A,\mathbb{K}) = 0\}$
if $n > 1$ **then**
  return false
end if
if $\dim_{\mathbb{K}} \mathrm{Ext}^d_S(A,\mathbb{K}) \neq 1$ **then**
  return false
end if
return true

A.2.1 Squarefree Veronese ideals

Throughout this subsection, let $d \in \mathbb{N}$, let $\mathbb{K}$ be a computable field of characteristic zero and set $S := \mathbb{K}[x_1, \ldots, x_d]$.

Definition A.2.1. Let I be an ideal of S. We say that I is a *squarefree Veronese ideal* if there is $k \in \{1, \ldots, d\}$ such that

$$I = \bigcap_{1 \leq i_1 < \ldots < i_k \leq d} \langle x_{i_1}, \ldots, x_{i_k} \rangle.$$

In this case, I will be denoted $I_{k,d}$.

Now, we recall Alexander duality just for squarefree monomial ideals which involve all the variables of our current polynomial ring.

Definition A.2.2. Let I be a squarefree monomial ideal of S and let $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$ be its minimal primary decomposition in terms of face ideals such that $\mathfrak{m} = I_{\alpha_1} + \ldots + I_{\alpha_s}$. Set

$$I^{\mathbf{1}} := \langle \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_s} \rangle$$

and we refer to the ideal $I^{\mathbf{1}}$ to be the the *Alexander dual* of I.

As it was pointed out in Chapter 2,

$$I^{\mathbf{1}}_{k,d} = I_{d-k+1,d}.$$

In this way, we only need to produce $\lceil d/2 \rceil$ squarefree Veronese ideals because the other ones will be produced by Alexander duality.

A function of our package was built in order to produce all the squarefree Veronese ideals of a given polynomial ring over a field of characteristic zero. Indeed, a field of characteristic zero because of our implementation uses Alexander duality, which is computed explicitly using [115].

Now, we write down our algorithm as one may see in Algorithm 3.

Algorithm 3 SqVeroneseIdeals()

Input: polynomial ring over a computable field of characteristic zero.
Output: the list L containing all the squarefree Veronese ideals.

$q \leftarrow \lceil d/2 \rceil$
$L \leftarrow \{\}$
for $i = 1$ **to** q **do**
  $I \leftarrow$ intersection of all face ideals of height i
  $L \leftarrow \{L, I, I^{\mathbf{1}}\}$
end for
return L

A.2.2 Ideals with disjoint variables

We start by introducing the following concept.

Definition A.2.3. Let $\mathbb{K}$ be any field, let $d \in \mathbb{N}$, let $S := \mathbb{K}[x_1, \ldots, x_d]$ and let I be a squarefree monomial ideal of S. We say that I is an ideal *with disjoint variables* if there exists $(t, d_1, \ldots, d_t)$ with $d = d_1 + \ldots + d_t$ and $d_1 \geq d_2 \geq \ldots \geq d_t > 0$ such that

$$I = \bigcap_{i=1}^{t} \langle x_{b_{i-1}+1}, \ldots, x_{b_i} \rangle,$$

where, for each $j \in \{1, \ldots, t\}$,

$$b_0 := 0 \text{ and } b_j := \sum_{l=1}^{j} d_l.$$

It was proved in Chapter 2 (cf. Proposition 2.4.9) that, up to $\langle x_1 \cdots x_d \rangle$ and $\langle x_1, \ldots, x_d \rangle$, the rest of squarefree monomial ideals with disjoint variables have infinitely generated Cartier algebra. Moreover, it was also asked (but not proved) in Chapter 2 as well that M_Δ (cf. Definition 2.4.18) is reached in some ideal with disjoint variables.

The following algorithm produces all the squarefree monomial ideals with disjoint variables. We shall need the following:

Notation A.2.4. Given a list of integers L of length l with $L = [L_1, \ldots, L_l]$ and given $j \in \{1, \ldots, l\}$, set

$$L_{\leq j} := [L_1, \ldots, L_j], \quad L_{\geq j} := [L_j, \ldots, L_l].$$

Our promised procedure is detailed in Algorithm 4.

Algorithm 4 IdealsOfSeparateIndeterminates()

Input: polynomial ring S in d variables over any field.
Output: all the squarefree monomial ideals of S with disjoint variables.

```
L ← Partitions of d
l ← #L
X ← {}
for i = 1 to l do
   P ← L_i    // L = [L_1, ..., L_l]
   J ← ⟨1⟩
   Y ← {x_1, ..., x_d}
   p ← #P
   for j = 1 to p do
      J ← J ∩ ⟨Y_{≤P_j}⟩
      Y ← Y_{≥P_j+1}
   end for
   X ← {X, J}
end for
return X
```

A.3 Some elementary procedures involving test ideals

The following two algorithms are, in fact, results obtained by J. Cowden Vassilev in [137]. In the first one, a computable description of the test ideal of a Stanley-Reisner ring is given (cf. [137, Theorem 3.7]).

Theorem A.3.1. *Let $\mathbb{K}$ be an F-finite field of characteristic p, let $d \in \mathbb{N}$, let $S := \mathbb{K}[x_1, \ldots, x_d]$, let I be a squarefree monomial ideal and let $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$ be its minimal primary decomposition in terms of face ideals. Then,*

$$\tau(S/I) = \sum_{i=1}^{s} \left(I_{\alpha_1} \cap \ldots \cap I_{\alpha_{i-1}} \cap I_{\alpha_{i+1}} \cap I_{\alpha_{i+2}} \cap \ldots \cap I_{\alpha_s} \right).$$

In the second one, a filtration of F-pure ideals of a Stanley-Reisner ring is given (cf. [137, Corollary 3.4]). We shall preserve the assumptions of the previous theorem.

Theorem A.3.2. *There exists a saturated filtration of F-pure ideals*

$$I = J_0 \subseteq J_1 \subseteq J_{l-1} = \sum_{i=1}^{s} I_{\alpha_i} \subseteq J_l = S$$

such that, for each $i \in \{0, \ldots, l-1\}$, $J_{i+1} = \tau(S/J_i)$.

Remark A.3.3. The fact that J_{l-1} is of the previous form is, in fact, a result due to I. Aberbach and F. Enescu (cf. [1, Proposition 4.10]). It is also worth mentioning that R. Y. Sharp (cf. [122, Corollary 3.8]) obtained a Cowden-type filtration in the complete local case without imposing F-finiteness on the coefficient field.

These two results can be translated into two methods. These procedures are described in Algorithm 5 and Algorithm 6, respectively.

Algorithm 5 TestIdeal(I)

Input: a squarefree monomial ideal I inside a polynomial ring S over any ground field.
Output: the test ideal $\tau(S/I)$.

$L \leftarrow \{I_{\alpha_1}, \ldots, I_{\alpha_s}\}$ // $I = I_{\alpha_1} \cap \ldots \cap I_{\alpha_s}$
$J \leftarrow \langle 0 \rangle$
for $i = 1$ **to** s **do**
  $K \leftarrow I_{\alpha_1} \cap \ldots \cap I_{\alpha_{i-1}} \cap I_{\alpha_{i+1}} \cap I_{\alpha_{i+2}} \cap \ldots \cap I_{\alpha_s}$
  $J \leftarrow J + K$
end for
return J

Algorithm 6 CowdenFiltration(I)

Input: a squarefree monomial ideal I inside a polynomial ring S over any ground field.
Output: the filtration of F-pure ideals given in Theorem A.3.2.

$J \leftarrow I$
$L \leftarrow \{\}$
repeat
  $L \leftarrow \{L, J\}$
  $J \leftarrow$ TestIdeal(J)
until $J = \langle 1 \rangle$
return $\{L, \langle 1 \rangle\}$

A.4 A CoCoA session

In this section, by means of a CoCoA session, we shall explain the usage of our previous procedures.

First of all, we test the function which produces all the squarefree Veronese ideals of a given polynomial ring. We begin fixing a polynomial ring and loading our package.

```
Use R::=QQ[x[1..7]];
Source "testidsq.cpkg";
Alias Test:=$contrib/testidsq;
```

Now, we are ready for computing all the squarefree Veronese ideals of R.

```
L:=Test.SqVeroneseIdeals();
L;
[Ideal(x[1]x[2]x[3]x[4]x[5]x[6]x[7]),
Ideal(x[1], x[2], x[3], x[4], x[5], x[6], x[7]),
Ideal(x[2]x[3]x[4]x[5]x[6]x[7], x[1]x[3]x[4]x[5]x[6]x[7],
x[1]x[2]x[3]x[5]x[6]x[7], x[1]x[2]x[3]x[4]x[5]x[7],
x[1]x[2]x[3]x[4]x[5]x[6], x[1]x[2]x[3]x[4]x[6]x[7], x[1]x[2]x[4]x[5]x[6]x[7]),
Ideal(x[4]x[5], x[3]x[4], x[4]x[6], x[2]x[4], x[4]x[7], x[1]x[4], x[3]x[5],
x[5]x[6], x[2]x[5], x[5]x[7], x[1]x[5], x[3]x[6], x[2]x[3], x[3]x[7],
x[1]x[3], x[2]x[6], x[6]x[7], x[1]x[6], x[1]x[7], x[2]x[7], x[1]x[2]),
Ideal(x[3]x[4]x[5]x[6]x[7], x[2]x[4]x[5]x[6]x[7], x[1]x[4]x[5]x[6]x[7],
x[2]x[3]x[4]x[6]x[7], x[1]x[3]x[4]x[6]x[7], x[2]x[3]x[4]x[5]x[6],
x[1]x[3]x[4]x[5]x[6], x[1]x[2]x[4]x[6]x[7], x[1]x[2]x[4]x[5]x[6],
x[1]x[2]x[3]x[5]x[7], x[1]x[2]x[3]x[4]x[7], x[1]x[2]x[3]x[4]x[5],
x[1]x[2]x[3]x[4]x[6], x[1]x[2]x[3]x[5]x[6], x[1]x[2]x[3]x[6]x[7],
x[1]x[2]x[4]x[5]x[7], x[1]x[2]x[5]x[6]x[7], x[1]x[3]x[4]x[5]x[7],
x[2]x[3]x[4]x[5]x[7], x[1]x[3]x[5]x[6]x[7], x[2]x[3]x[5]x[6]x[7]),
Ideal(x[1]x[2]x[7], x[1]x[6]x[7], x[2]x[6]x[7], x[1]x[2]x[6],
x[2]x[3]x[6], x[3]x[6]x[7], x[1]x[3]x[6], x[1]x[3]x[7], x[2]x[3]x[7],
x[1]x[2]x[3], x[3]x[5]x[6], x[2]x[3]x[5], x[3]x[5]x[7], x[1]x[3]x[5],
x[2]x[5]x[6], x[5]x[6]x[7], x[1]x[5]x[6], x[1]x[5]x[7], x[2]x[5]x[7],
x[1]x[2]x[5], x[3]x[4]x[5], x[4]x[5]x[6], x[2]x[4]x[5], x[4]x[5]x[7],
x[1]x[4]x[5], x[3]x[4]x[6], x[2]x[3]x[4], x[3]x[4]x[7], x[1]x[3]x[4],
x[2]x[4]x[6], x[4]x[6]x[7], x[1]x[4]x[6], x[1]x[4]x[7], x[2]x[4]x[7],
x[1]x[2]x[4]), Ideal(x[4]x[5]x[6]x[7], x[3]x[5]x[6]x[7], x[2]x[5]x[6]x[7],
x[1]x[5]x[6]x[7], x[3]x[4]x[5]x[7], x[2]x[4]x[5]x[7], x[1]x[4]x[5]x[7],
x[2]x[3]x[6]x[7], x[1]x[3]x[6]x[7], x[2]x[3]x[5]x[6], x[1]x[3]x[5]x[6],
x[2]x[3]x[4]x[6], x[1]x[3]x[4]x[6], x[1]x[2]x[6]x[7], x[1]x[2]x[5]x[6],
x[1]x[2]x[4]x[6], x[1]x[2]x[3]x[7], x[1]x[2]x[3]x[5], x[1]x[2]x[3]x[4],
```

```
x[1]x[2]x[3]x[6], x[1]x[2]x[4]x[5], x[1]x[2]x[4]x[7], x[1]x[2]x[5]x[7],
x[1]x[3]x[4]x[5], x[2]x[3]x[4]x[5], x[1]x[3]x[4]x[7], x[2]x[3]x[4]x[7],
x[1]x[3]x[5]x[7], x[2]x[3]x[5]x[7], x[1]x[4]x[5]x[6], x[2]x[4]x[5]x[6],
x[3]x[4]x[5]x[6], x[1]x[4]x[6]x[7], x[2]x[4]x[6]x[7], x[3]x[4]x[6]x[7])]
```

We check Proposition 2.4.7 in this particular case using our package.

```
[Test.IsPrincipal(X)|X In L];
[True, True, True, True, True, True, True]
```

Secondly, we shall show in a particular case that IdealsOfSeparateIndeterminates() produces all the squarefree monomial ideals with disjoint variables of a given polynomial ring.

```
N:=Test.IdealsOfSeparateIndeterminates();
N;
[Ideal(x[7], x[6], x[5], x[4], x[3], x[2], x[1]), Ideal(x[3]x[7], x[2]x[7],
x[1]x[7], x[3]x[6], x[2]x[6], x[1]x[6], x[3]x[5], x[2]x[5], x[1]x[5], x[3]x[4],
x[2]x[4], x[1]x[4]), Ideal(x[2]x[7], x[1]x[7], x[2]x[6], x[1]x[6], x[2]x[5],
x[1]x[5], x[2]x[4], x[1]x[4], x[2]x[3], x[1]x[3]), Ideal(x[1]x[7], x[1]x[6],
x[1]x[5], x[1]x[4], x[1]x[3], x[1]x[2]), Ideal(x[2]x[4]x[7], x[1]x[4]x[7],
x[2]x[3]x[7], x[1]x[3]x[7], x[2]x[4]x[6], x[1]x[4]x[6], x[2]x[3]x[6],
x[1]x[3]x[6], x[2]x[4]x[5], x[1]x[4]x[5], x[2]x[3]x[5], x[1]x[3]x[5]),
Ideal(x[1]x[4]x[7], x[1]x[3]x[7], x[1]x[2]x[7], x[1]x[4]x[6], x[1]x[3]x[6],
x[1]x[2]x[6], x[1]x[4]x[5], x[1]x[3]x[5], x[1]x[2]x[5]),
Ideal(x[1]x[3]x[7], x[1]x[2]x[7], x[1]x[3]x[6], x[1]x[2]x[6], x[1]x[3]x[5],
x[1]x[2]x[5], x[1]x[3]x[4], x[1]x[2]x[4]),
Ideal(x[1]x[2]x[7], x[1]x[2]x[6], x[1]x[2]x[5], x[1]x[2]x[4], x[1]x[2]x[3]),
Ideal(x[1]x[3]x[5]x[7], x[1]x[2]x[5]x[7], x[1]x[3]x[4]x[7], x[1]x[2]x[4]x[7],
x[1]x[3]x[5]x[6], x[1]x[2]x[5]x[6], x[1]x[3]x[4]x[6], x[1]x[2]x[4]x[6]),
Ideal(x[1]x[2]x[4]x[7], x[1]x[2]x[3]x[7], x[1]x[2]x[4]x[6], x[1]x[2]x[3]x[6],
x[1]x[2]x[4]x[5], x[1]x[2]x[3]x[5]),
Ideal(x[1]x[2]x[3]x[7], x[1]x[2]x[3]x[6], x[1]x[2]x[3]x[5], x[1]x[2]x[3]x[4]),
Ideal(x[1]x[2]x[3]x[5]x[7], x[1]x[2]x[3]x[4]x[7], x[1]x[2]x[3]x[5]x[6],
x[1]x[2]x[3]x[4]x[6]),
Ideal(x[1]x[2]x[3]x[4]x[7], x[1]x[2]x[3]x[4]x[6], x[1]x[2]x[3]x[4]x[5]),
Ideal(x[1]x[2]x[3]x[4]x[5]x[7], x[1]x[2]x[3]x[4]x[5]x[6]),
Ideal(x[1]x[2]x[3]x[4]x[5]x[6]x[7])]
```

We can check as well Proposition 2.4.9 in this particular case.

```
[Test.IsPrincipal(X)|X In N];
[True, False, False, False, False, False, False, False, False, False, False,
False, False, False, True]
```

Building from this function we have generated, using once more our package, Table A.1. In this table, d is the number of variables of the current polynomial ring, M_Δ is the numerical function introduced in Chapter 2 (cf. Definition 2.4.18) and I_Δ is the squarefree monomial ideal with disjoint variables in which this upper bound is reached.

Table A.1: Table involving ideals with disjoint variables

d	M_Δ	I_Δ
3	2	$\langle x_1\rangle \cap \langle x_2, x_3\rangle$
4	4	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle$
5	8	$\langle x_1\rangle \cap \langle x_2, x_3\rangle \cap \langle x_4, x_5\rangle$
6	18	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle$
7	26	$\langle x_1\rangle \cap \langle x_2, x_3\rangle \cap \langle x_4, x_5\rangle \cap \langle x_6, x_7\rangle$
8	64	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle \cap \langle x_7, x_8\rangle$
9	83	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle \cap \langle x_7, x_8, x_9\rangle$
10	210	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle \cap \langle x_7, x_8\rangle \cap \langle x_9, x_{10}\rangle$
11	275	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle \cap \langle x_7, x_8\rangle \cap \langle x_9, x_{10}, x_{11}\rangle$
12	664	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle \cap \langle x_7, x_8\rangle \cap \langle x_9, x_{10}\rangle \cap \langle x_{11}, x_{12}\rangle$
13	875	$\langle x_1, x_2\rangle \cap \langle x_3, x_4\rangle \cap \langle x_5, x_6\rangle \cap \langle x_7, x_8\rangle \cap \langle x_9, x_{10}\rangle \cap \langle x_{11}, x_{12}, x_{13}\rangle$

Now, we want to mention the following question.

Question A.4.1. Does the length of the Cowden filtration determine the principal generation of the Cartier algebra?

In other words, one might ask whether the generation (principal or infinite) of the Cartier algebra is reflected in Cowden's filtration. Our first intuition was that if the length of Cowden filtration was 1, then the Cartier algebra would be principally generated.

As the following examples illustrate, our intuition was wrong.

```
Use R::=ZZ/(2)[x,y,z];
I:=Ideal(xy,yz);
J:=Ideal(xy,xz,yz);
K:=Ideal(xyz);
L:=[I,J,K];
[Test.IsPrincipal(X)|X In L];
[False, True, True]
M:=[Test.CowdenFiltration(X)|X In L];
M[1];
M[2];
M[3];
[Ideal(xy, yz), Ideal(y, z, x), Ideal(1)]
-------------------------------
```

```
[Ideal(xy, xz, yz), Ideal(z, y, x), Ideal(1)]
-------------------------------
[Ideal(xyz), Ideal(xy, xz, yz), Ideal(z, y, x), Ideal(1)]
-------------------------------
```

A.4.1 A couple of topological examples

We end this appendix with two final examples. More precisely, we determine whether the Cartier algebra $\mathcal{C}^R$ is principally generated or not, where

$$R := \mathbb{F}_2[\![x_1, \dots, x_d]\!]/I_\Delta$$

and I_Δ is a squarefree monomial ideal which stems from the triangulation of some non-trivial topological spaces.

The first one is the real projective plane $\mathbb{P}^2_{\mathbb{R}}$.

Example A.4.2. Consider the complete, Stanley-Reisner ring

$$\mathbb{F}_2[\![x_1, x_2, x_3, x_4, x_5, x_6]\!]/I_\Delta,$$

where

$$I_\Delta := \langle x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6\rangle.$$

Using once more CoCoA, we have checked that it is a Cohen-Macaulay, 3-dimensional ring. It is not Gorenstein because its type is 6. Now, we determine whether its Cartier algebra is principally generated or infinitely generated using our package.

```
Use S::=ZZ/(2)[x[1..6]];
I:=Ideal(x[1]x[2]x[3],x[1]x[2]x[4],x[1]x[3]x[5],x[1]x[4]x[6],x[1]x[5]x[6],
x[2]x[3]x[6],x[2]x[4]x[5],x[2]x[5]x[6],x[3]x[4]x[5],x[3]x[4]x[6]);
Test.IsPrincipal(I);
True
-------------------------------
```

Whence such Cartier algebra is principally generated.

The second one is the so-called 4-*fold dunce cap*.

Example A.4.3. Consider the complete, Stanley-Reisner ring

$$\mathbb{F}_2[\![x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]\!]/I,$$

where I will be described below. We have checked, using once more CoCoA, that it is a non Cohen-Macaulay, 3-dimensional ring with depth 2. We proceed to verify whether its Cartier algebra is principally generated or infinitely generated.

```
Use S::=ZZ/(2)[x[1..9]];
I:=Ideal(x[1]x[2]x[3], x[1]x[2]x[5], x[1]x[2]x[8], x[1]x[3]x[6], x[1]x[3]x[9],
x[1]x[4]x[5], x[1]x[4]x[6], x[1]x[4]x[7], x[1]x[4]x[8], x[1]x[4]x[9],
x[1]x[5]x[7], x[1]x[5]x[8], x[1]x[5]x[9], x[1]x[6]x[7], x[1]x[6]x[8],
x[1]x[6]x[9], x[1]x[7]x[8], x[1]x[7]x[9], x[1]x[8]x[9], x[2]x[3]x[4],
x[2]x[3]x[7], x[2]x[4]x[6], x[2]x[4]x[7], x[2]x[4]x[8], x[2]x[4]x[9],
x[2]x[5]x[6], x[2]x[5]x[7], x[2]x[5]x[8], x[2]x[5]x[9], x[2]x[6]x[7],
x[2]x[6]x[8], x[2]x[6]x[9], x[2]x[7]x[9], x[3]x[4]x[5], x[3]x[4]x[6],
x[3]x[4]x[7], x[3]x[4]x[8], x[3]x[4]x[9], x[3]x[5]x[6], x[3]x[5]x[7],
x[3]x[5]x[8], x[3]x[5]x[9], x[3]x[6]x[8], x[3]x[6]x[9], x[3]x[7]x[8],
x[3]x[7]x[9], x[3]x[8]x[9], x[4]x[5]x[7], x[4]x[5]x[8], x[4]x[5]x[9],
x[4]x[6]x[7], x[4]x[6]x[9], x[4]x[7]x[8], x[4]x[7]x[9], x[5]x[6]x[7],
x[5]x[6]x[8], x[5]x[6]x[9], x[5]x[7]x[8], x[5]x[7]x[9], x[5]x[8]x[9],
x[6]x[8]x[9], x[7]x[8]x[9]);
Test.IsPrincipal(I);
False
-------------------------------
```

Whence such Cartier algebra is infinitely generated.

We conclude this appendix with the following:

Remark A.4.4. Our motivation to study these previous topological examples comes from the study carried out by A. K. Singh and U. Walther in [127]. On the other hand, regarding the main result obtained by J. Àlvarez Montaner and K. Yanagawa in [7], from the previous computation follows that, whereas the real projective plane has a triangulation which admits an *elementary collapse* (cf. [100, Definition 3.3.1] for unexplained terminology), the triangulation of the 4-fold dunce hat does not admit such an ellementary collapse.

Appendix B

Computing F-pure ideals

In Chapter 3, we have established a procedure in order to describe the so-called *F-pure ideals* of a polynomial ring $\mathbb{K}[x_1, \ldots, x_d]$ provided $\mathbb{K}$ is F-finite. Furthermore, we have as well given theoretical evidence that this method might be of some help to tackle the same problem dropping the F-finiteness assumption over the ground field $\mathbb{K}$.

The aim of this appendix is twofold. Firstly, we shall explain with detail and write down the pseudocode of our procedure in case $\mathbb{K} = \mathbb{F}_p$. Finally, by means of a Macaulay2 session, we are to illustrate the usage of our computational packages in specific examples. It is worth mentioning that Macaulay2 has been used extensively both in constructing and exploring examples, as well as implementing the method described both in Chapter 3 and herein. The code is located in [25].

B.1 Basic constructions

We start introducing the building blocks of our algorithm. As a convention in what follows, unless otherwise is specified, we shall denote by S the polynomial ring $\mathbb{F}_p[x_1, \ldots, x_d]$.

B.1.1 The infinity norm of a polynomial

We recall the following well-known notion which has been already appeared previously in this mimeograph (cf. Definition 1.4.8).

Definition B.1.1. Let $g \in S$ and write

$$g = \sum_{\alpha \in \mathbb{N}^d} g_\alpha \mathbf{x}^\alpha,$$

with $g_\alpha \in \mathbb{F}_p$ and $g_\alpha = 0$ up to a finite number of terms.

(i) We define the *support* of g (which will be denoted $\text{supp}(g)$) as

$$\text{supp}(g) := \left\{ \alpha \in \mathbb{N}^d \mid \quad g_\alpha \neq 0 \right\}.$$

(ii) We define the *infinity norm* of g (which will be denoted $||g||_\infty$) as

$$||g||_\infty := \max_{\alpha \in \text{supp}(g)} ||\alpha||_\infty,$$

where

$$||\alpha||_\infty := \max\{a_1, \ldots, a_d\}.$$

and $\alpha := (a_1, \ldots, a_d)$.

In this way, the computation of the infinity norm of a polynomial is described in Algorithm 7.

Algorithm 7 Gauge(g)

Input: polynomial $g \in S$.
Output: the infinity norm $||g||_\infty$ of g.

$L \leftarrow \text{supp}(g)$.
$X \leftarrow \{\}$.
for all $Y \in L$ **do**
  $X \leftarrow \{X, \max(Y)\}$
end for
return $\max(X)$

B.1.2 The eth root ideal

We recall that in Chapter 3 we have defined, for any ideal J of S, a new ideal $I_e(J)$ which is characterized as the smallest one (namely, K) such that $J \subseteq K^{[p^e]}$.

In this section, we present a method in order to compute $I_e(J)$. It is worth mentioning that such procedure was already described in [82, Section 6] and implemented in [83].

First of all, we recall from Chapter 3 the necessary theoretical background (cf. Proposition 3.1.7).

Proposition B.1.2. *The following statements hold.*

(i) If $J_1, \ldots, J_r$ are ideals of S, then

$$I_e(J_1 + \ldots + J_r) = \sum_{l=1}^{r} I_e(J_l).$$

The reader should notice that this fact implies that it is enough to know how to calculate $I_e(J)$ when J is a principal ideal.

(ii) Let $g \in S$. If

$$g = \sum_{0 \leq ||\alpha||_\infty \leq p^e - 1} g_\alpha^{p^e} \mathbf{x}^\alpha,$$

then $I_e(g)$ is the ideal of S generated by all the g_α's.

Therefore, we can turn Proposition B.1.2 into an effective method for computing $I_e(J)$ as follows.

Algorithm B.1.3 (Effective computation of the eth root ideal)**.** We shall assume, for the sake of simplicity and without loss of generality, that our input ideal J is generated by a single element $g \in S$.

(a) Consider the polynomial ring $S_1 := S[y_1, \ldots, y_d] = \mathbb{F}_p[x_1, \ldots, x_d, y_1, \ldots, y_d]$ and put on S_1 any term ordering such that, in this order, the indeterminates $x_1, \ldots, x_d$ are greater than the indeterminates $y_1, \ldots, y_d$.

(b) Consider the list $G := \{y_1 - x_1^{p^e}, \ldots, y_d - x_d^{p^e}\}$.

(c) Reduce the input g with respect to the list G using the division algorithm in S_1 given by the previous term ordering. Save the remainder of the result in a new list; namely, H.

(d) Regarding the elements of H as polynomials in the variables $x_1, \ldots, x_d$, set I' as the ideal generated by the coefficients of the elements of H. Whence I' is an ideal of $\mathbb{F}_p[y_1, \ldots, y_d]$.

(e) Output the ideal $I'S$, where $I'S$ is the extended ideal with respect to the following trivial homomorphism of $\mathbb{F}_p$-algebras:

$$\mathbb{F}_p[y_1, \ldots, y_d] \longrightarrow \mathbb{F}_p[x_1, \ldots, x_d] = S$$
$$y_1 \longmapsto x_1, \ldots, y_d \longmapsto x_d.$$

Hereafter, EthRootIdeal(e, J) is to denote the method which, given any ideal J of S, returns as output $I_e(J)$.

B.1.3 The hash operation

Now, we shall exhibit an algorithm for computing $J^{\#e}$ (cf. Definition 3.2.2 of Chapter 3), where J is any ideal of S and $e \in \mathbb{N}$. Such procedure is described in Algorithm 8.

Algorithm 8 HashOperation(e, J, u)

Input: a non-negative integer e, an ideal J of S and a fixed element u of S.
Output: the ideal $J^{\#e}$ which contains all the $u\Phi_e$-fixed ideals contained in J.

$q \leftarrow p^e$
$D_e \leftarrow \lceil \text{Gauge}(u)/(q-1) \rceil$
$K \leftarrow J$
repeat
  $H \leftarrow K$
  $K \leftarrow H \cap \left(H^{[q]} :_S H\right) \cap \text{EthRootIdeal}(e, uH)$
  $K \leftarrow \langle K \cap S_{D_e} \rangle$
until $H = K$
return K

B.1.4 Some Linear Algebra over finite fields

We recall that one crucial step in the algorithm described in Chapter 3 involves the calculation of

$$\left\{ \mathfrak{m}K \subseteq V \subseteq K \mid \quad \dim_{\mathbb{F}_p} V/\mathfrak{m}K = 1 \right\},$$

where K is any ideal of S. This problem is equivalent to compute all the $\mathbb{F}_p$-vector subspaces of $\mathbb{F}_p^t$ of codimension 1, where t denotes the minimum number of generators of K.

In our package, this is achieved after making the following steps.

(a) Generate all the matrices of size $1 \times t$ over $\mathbb{F}_p$.

(b) Compute the kernel of such matrices. For every row matrix, such kernel can be generated by $t-1$ vectors. These $t-1$ vectors can be regarded as rows of a matrix with $t-1$ rows and t columns. In this way, we have produced all the matrices of size $(t-1) \times t$ over $\mathbb{F}_p$ of maximal rank.

(c) Use the matrices which have been built in part (b) in order to compute the foregoing set.

We shall denote by Preimages(K) the procedure that, given any ideal K of S, returns as output the previous set. It is noteworthy that the cardinal of such set is

$$\binom{t}{t-1}_p = \frac{(p^t-1)(p^{t-1}-1)\cdots(p^2-1)}{(p^{t-1}-1)\cdots(p^2-1)(p-1)} = \frac{p^t-1}{p-1} = 1 + p + \ldots + p^{t-1}.$$

B.2 The algorithm

Thus, we are ready to describe a recursive procedure for producing all the $u\Phi_e$-fixed ideals of S. It is described in Algorithm 9.

Algorithm 9 FPureAlgorithm(e, I, G, L, u)

Input: $e \in \mathbb{N}$ and $u \in F^e_* S$. In the first step, $I = \langle 1 \rangle^{\#_e}$, $L = \{\}$ and $G = \{\}$.
Output: L will be the list of all $u\Phi_e$-fixed ideals of S.

```
G ← {G, I}
if I = EthRootIdeal(e, uI) and I ∉ L then
  L ← {L, I}
end if
M ← I/𝔪I
t ← minimum number of generators of M
if t > 0 then
  if t > 1 then
    H ← Preimages(I)
    for all X ∈ H do
      J ← HashOperation(e, X, u)
      if J = EthRootIdeal(e, uJ) and J ∉ L then
        L ← {L, J}
      end if
      if J ∉ G then
        L ← {L, FPureAlgorithm(e, J, G, L, u)}
      end if
    end for
  else
    K ← HashOperation(e, 𝔪I, u)
    if K ∉ G then
      L ← {L, FPureAlgorithm(e, K, G, L, u)}
    end if
  end if
end if
return L
```

Remark B.2.1. It is worth mentioning that the list G which appears in Algorithm 9 must be regarded as a list of exclusion in order to avoid superflous calculations. Actually, in our experiments with Macaulay2 we have verified that the introduction of such list is necessary in order to guarantee that the program terminates.

B.2.1 A variant for a Frobenius splitting

The aim of this subsection is to propose a variant of our previously introduced algorithm in case $u\Phi_e$ is a Frobenius splitting (cf. Definition 3.1.4). It turns out that this fact leads to introduce some simplifications in our original method.

First of all, we introduce a procedure in order to determine whether $u\Phi_e$ is a Frobenius splitting. It is described in Algorithm 10.

Algorithm 10 IsSplitting(e, u)

Input: $e \in \mathbb{N}$ and $u \in F_*^e S$.
Output: **true** if $u\Phi_e$ defines a Frobenius splitting and **false**, otherwise.

$q \leftarrow p^e$
$f \leftarrow \mathbf{x}^{(q-1)\mathbb{1}}$ // $u = gf + r$
if $g = 1$ **then**
  return **true**
end if
return **false**

Now, we introduce a recursive method to compute all the $u\Phi_e$-fixed ideals of S under the additional assumption that $u\Phi_e$ defines a Frobenius splitting. As the reader may easily point out, this new procedure is just a mild modification of Algorithm 9.

Such method is described in Algorithm 11. We have to remind that, in the below method, $u\Phi_e$ is assumed to define a Frobenius splitting. This fact is straightforward to check using Algorithm 10.

We end this subsection with the following:

Remark B.2.2. The unique difference between Algorithm 9 and Algorithm 11 is that, once we know that $u\Phi_e$ is a Frobenius splitting, this fact implies, for any ideal I of S, that $I^{\#_e}$ is $u\Phi_e$-fixed. Indeed, by the definition of the hash operation (cf. Definition 3.2.2) $I^{\#_e}$ is a $u\Phi_e$-compatible ideal, whence it is $u\Phi_e$-fixed provided $u\Phi_e$ is a Frobenius splitting (cf. Lemma 3.1.5).

B.3 A Macaulay2 session

In this section, by means of a Macaulay2 session, we shall illustrate the algorithm described in this appendix.

Algorithm 11 SplittingAlgorithm(e, I, L, u)

Input: $e \in \mathbb{N}$ and $u \in F^e_* S$. In the first step, $I = \langle 1 \rangle^{\#_e}$, $L = \{\}$ and $G = \{\}$.
Output: L will be the list of all $u\Phi_e$-fixed ideals of S.

```
G ← {G, I}
if I ∉ L then
  L ← {L, I}
end if
M ← I/𝔪I
t ← minimum number of generators of M
if t > 0 then
  if t > 1 then
    H ← Preimages(I)
    for all X ∈ H do
      J ← HashOperation(e, X, u)
      if J ∉ L then
        L ← {L, J}
      end if
      if J ∉ G then
        L ← {L, FPureAlgorithm(e, J, G, L, u)}
      end if
    end for
  else
    K ← HashOperation(e, 𝔪I, u)
    if K ∉ G then
      L ← {L, FPureAlgorithm(e, K, G, L, u)}
    end if
  end if
end if
return L
```

We begin clearing the previous input and loading our scripts.

```
clearAll;
load "~/FPureAlgorithm.m2";
```

We start computing all the fixed ideals of a polynomial ring in three variables with respect to the element $u = xyz$.

```
p=2;
F=ZZ/p;
R=F[x,y,z];
L=first entries vars(R);
u=product(L);
time L=FPureIdeals(u);
-- used 7.93653 seconds
#L
20
```

Indeed, we obtain the twenty squarefree monomial ideals in the indeterminates x, y, z.

Finally, we conclude with the following example, which is interesting because $u\Phi_1$ is far from being a Frobenius splitting. Moreover, in this case, the characteristic of our ground field is greater than two.

```
p=5;
F=ZZ/p;
R=F[x,y,z];
N=p-1;
L=first entries vars(R);
L=apply(i->i^N);
u=sum toList L;
u=u^N;
L=FPureIdeals(u);
-- used 5254.22 seconds
#L
66
```

In this case, we recover the ideals described in Example 3.3.10 of Chapter 3.

Appendix C

A Koszul-type resolution over the Frobenius-Ore extension ring

The Frobenius-Ore extension ring has played a central role throughout this mimeograph; indeed, given a local ring R which verifies Serre's condition S_2 we saw in Chapter 1 (cf. Example 1.4.18) that $R[\Theta; F]$ turns out to be the Frobenius algebra of operators attached to the top local cohomology module $H_{\mathfrak{m}}^{\dim(R)}(R)$. Moreover, we also showed in Chapter 2 (cf. Theorem 2.3.5) that the Frobenius algebra associated to the injective hull of the residue field of a complete, Stanley-Reisner ring is of the form $B[u\Theta; F]$ whenever such Frobenius algebra is principally generated, where $B = \mathbb{K}[\![x_1, \dots, x_d]\!]/I_\Delta$ and $u = (x_1 \cdots x_d)^{p-1}$.

The purpose of this Appendix is to exhibit an explicit finite free resolution in the category of left $A[\Theta; F]$-modules, where A is a commutative Noetherian ring of prime characteristic p; more precisely, given $y_1, \dots, y_n$ elements of A, we shall produce a chain complex (namely, $CK_\bullet(y_1, \dots, y_n)$) which, in case $y_1, \dots, y_n$ forms an A-regular sequence, produces a finite free resolution of $A/I_n^{[p]}$ in the category of left $A[\Theta; F]$-modules, where I_n is the ideal generated by the y_i's. From this point of view, under the regularity assumptions, $CK_\bullet(y_1, \dots, y_n)$ may be regarded as a sort of Koszul resolution in the category of left $A[\Theta; F]$-modules.

Unless otherwise is specified, from now on S denotes the polynomial ring $\mathbb{Z}/p\mathbb{Z}[x_1, \dots, x_n]$. Our strategy, slightly loosely speaking, will consist on firstly construct the announced resolution in the category of left $S[\Theta; F]$, prove that it defines a free resolution, and then transport such construction to an arbitrary ring A of prime characteristic p under the homomorphism of $\mathbb{Z}/p\mathbb{Z}$-algebras $S \longrightarrow A$ which maps each x_i into y_i.

C.1 The Cartier-Koszul chain complex

Definition C.1.1. We define the *Cartier-Koszul chain complex* with respect to $x_1, \ldots, x_n$ as the chain complex

$$0 \longrightarrow CK_{n+1} \xrightarrow{\partial_{n+1}} CK_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_1} CK_0 \longrightarrow 0.$$

Here, for each $0 \leq l \leq n+1$,

$$CK_l := \bigoplus_{1 \leq i_1 < \ldots < i_l \leq n} S[\Theta; F](\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}) \oplus \bigoplus_{1 \leq j_1 < \ldots < j_{l-1} \leq n} S[\Theta; F](\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u),$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ corresponds respectively to $x_1, \ldots, x_n$ and u corresponds to $\Theta - 1$. Here, we are adopting the convention that $CK_0 := S[\Theta; F]$.

Moreover, one defines $CK_l \xrightarrow{\partial_l} CK_{l-1}$ as the unique homomorphism of left $S[\Theta; F]$-modules which, on basic elements, acts in the following manner; on one hand, given $1 \leq i_1 < \ldots < i_l \leq n$, set

$$\partial_l (\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}) := \sum_{r=1}^{l} (-1)^{r-1} x_{i_r} \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{r-1}} \wedge \mathbf{e}_{i_{r+1}} \wedge \mathbf{e}_{i_{r+2}} \wedge \ldots \wedge \mathbf{e}_{i_l} \right).$$

On the other hand, given $1 \leq j_1 < \ldots < j_{l-1} \leq n$, set

$$\partial_l \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u \right) := (-1)^{l-1} \left(\Theta - \left(x_{j_1} \cdots x_{j_{l-1}} \right)^{p-1} \right) \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \right) + \sum_{r=1}^{l-1} (-1)^{r-1} x_{j_r}^p \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u \right).$$

Henceforth, we shall denote such chain complex by $CK_\bullet(x_1, \ldots, x_n)$.

Before going on, we make the following useful:

Discussion C.1.2. Given a free, finitely generated left S-module M (whence M is abstractly isomorphic to $S^{\oplus r}$ for some $r \in \mathbb{N}$), we denote by $S[\Theta; F] \otimes_S S^{\oplus r} \xrightarrow{\lambda_M} S[\Theta; F]^{\oplus r}$ the natural isomorphism of left $S[\Theta; F]$-modules given by the assignment $s \otimes m \longmapsto sm$; moreover, we denote by

$$0 \longrightarrow K_n \xrightarrow{d_n} K_{n-1} \longrightarrow \ldots \longrightarrow K_2 \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \longrightarrow 0$$

the Koszul chain complex of S with respect to $x_1, \ldots, x_n$ regarded as a chain complex in the category of left S-modules. The reader should notice that, for each $0 \leq l \leq n$, we have the following commutative diagram:

$$\begin{array}{ccc} S[\Theta; F] \otimes_S K_{l+1} & \xrightarrow{\mathbb{1}_{S[\Theta;F]} \otimes d_{l+1}} & S[\Theta; F] \otimes_S K_l \\ \downarrow \lambda_{K_{l+1}} & & \downarrow \lambda_{K_l} \\ \bigoplus_{1 \leq i_1 < \ldots < i_{l+1} \leq n} S[\Theta; F](\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}}) & \xrightarrow{d'_{l+1}} & \bigoplus_{1 \leq i_1 < \ldots < i_l \leq n} S[\Theta; F](\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}). \end{array}$$

Here, d'_{l+1} denotes the map ∂_{l+1} restricted to the direct summand

$$\bigoplus_{1 \leq i_1 < \ldots < i_{l+1} \leq n} S[\Theta; F](\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}}).$$

As we shall see quickly, this fact turns out to be very useful in what follows.

The first thing we have to check out is that $CK_\bullet(x_1, \ldots, x_n)$ defines a chain complex in the category of left $S[\Theta; F]$-modules. This fact follows from the next:

Proposition C.1.3. *For any $0 \leq l \leq n$, one has that $\partial_l \partial_{l+1} = 0$.*

Proof. Regarding the very definition of the ∂'s, we only have to distinguish two cases.

On one hand, given $1 \leq i_1 < \ldots < i_{l+1} \leq n$ one has, keeping in mind Discussion C.1.2, that

$$\begin{aligned} &\partial_l \left(\partial_{l+1} \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}} \right) \right) = d'_l \left(d'_{l+1} \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}} \right) \right) \\ &= \left(\lambda_{K_{l-1}} \circ \left(\mathbb{1}_{S[\Theta;F]} \otimes d_{l-1} \right) \circ \lambda_{K_l}^{-1} \right) \circ \left(\lambda_{K_l} \circ \left(\mathbb{1}_{S[\Theta;F]} \otimes d_l \right) \circ \lambda_{K_{l+1}}^{-1} \right) \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}} \right) \\ &= \left(\lambda_{K_{l-1}} \circ \left(\mathbb{1}_{S[\Theta;F]} \otimes (d_l \circ d_{l+1}) \right) \circ \lambda_{K_{l+1}}^{-1} \right) \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}} \right) \\ &= \left(\lambda_{K_{l-1}} \circ \left(\mathbb{1}_{S[\Theta;F]} \otimes 0 \right) \circ \lambda_{K_{l+1}}^{-1} \right) \left(\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{l+1}} \right) = 0; \end{aligned}$$

indeed, the reader should notice that $d_l d_{l+1} = 0$ because they are the usual chain differentials in the Koszul chain complex of S with respect to $x_1, \ldots, x_n$.

On the other hand, given $1 \leq j_1 < \ldots < j_l \leq n$, one has that

$$\partial_l \left(\partial_{l+1} \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_l} \wedge u\right)\right) = \partial_l \Bigg((-1)^l \left(\Theta - (x_{j_1} \cdots x_{j_l})^{p-1}\right) (\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_l}) +$$

$$\sum_{r=1}^{l} (-1)^{r-1} x_{j_r}^p \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_l} \wedge u\right)\Bigg) =$$

$$\sum_{r=1}^{l} (-1)^{r+l-1} \left(x_{j_r}^p \Theta - (x_{j_1} \cdots x_{j_l})^{p-1} x_{j_r}\right) \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_l}\right) +$$

$$\sum_{r=1}^{l} \sum_{k=1}^{r-1} (-1)^{r+k-2} x_{j_k}^p x_{j_r}^p \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{k-1}} \wedge \mathbf{e}_{j_{k+1}} \wedge \mathbf{e}_{j_{k+2}} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_l} \wedge u\right) +$$

$$\sum_{r=1}^{l} \sum_{k=r}^{l} (-1)^{r+k-2} x_{j_k}^p x_{j_r}^p \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_{k-1}} \wedge \mathbf{e}_{j_{k+1}} \wedge \mathbf{e}_{j_{k+2}} \wedge \ldots \wedge \mathbf{e}_{j_l} \wedge u\right) +$$

$$\sum_{r=1}^{l} (-1)^{l+r-2} \left(x_{j_r}^p \Theta - \left(x_{j_1} \cdots x_{j_{r-1}} x_{j_{r+1}} x_{j_{r+2}} \cdots x_{j_l}\right)^{p-1} x_{j_r}^p\right) \left(\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{r-1}} \wedge \mathbf{e}_{j_{r+1}} \wedge \mathbf{e}_{j_{r+2}} \wedge \ldots \wedge \mathbf{e}_{j_l}\right).$$

Regardless, the reader should notice that, starting from the top, the first summand (respectively, the second summand) cancels out the fourth summand (respectively, the third summand) and therefore the whole expression vanishes, just what we finally wanted to check. □

Now, we are to describe $CK_\bullet(x_1, \ldots, x_n)$ for some small values of n for the convenience of the reader.

(i) When $n = 1$, $CK_\bullet(x_1)$ turns out to be the chain complex

$$0 \longrightarrow S[\Theta; F] \xrightarrow{\partial_2} S[\Theta; F]^{\oplus 2} \xrightarrow{\partial_1} S[\Theta; F] \longrightarrow 0,$$

where ∂_2 is represented by right multiplication by matrix $\begin{pmatrix} x_1^{p-1} - \Theta & x_1^p \end{pmatrix}$ and ∂_1 is represented by right multiplication by matrix $\begin{pmatrix} x_1 & \Theta - 1 \end{pmatrix}^T$.

(ii) When $n = 2$, $CK_\bullet(x_1, x_2)$ boils down to the chain complex

$$0 \longrightarrow S[\Theta; F] \xrightarrow{\partial_3} S[\Theta; F]^{\oplus 3} \xrightarrow{\partial_2} S[\Theta; F]^{\oplus 3} \xrightarrow{\partial_1} S[\Theta; F] \longrightarrow 0,$$

where ∂_3 is given by right multiplication by matrix $\begin{pmatrix} \Theta - (x_1 x_2)^{p-1} & (-1)^p x_2^p & x_1^p \end{pmatrix}$, ∂_2 is given by right multiplication by matrix

$$\begin{pmatrix} -x_2 & x_1 & 0 \\ x_1^{p-1} - \Theta & 0 & x_1^p \\ 0 & x_2^{p-1} - \Theta & x_2^p \end{pmatrix}$$

and ∂_1 is given by right multiplication by matrix $\begin{pmatrix} x_1 & x_2 & \Theta - 1 \end{pmatrix}^T$.

Before showing some basic properties of the Cartier-Koszul chain complex, we want to establish a certain technical fact, which is interesting in its own right; actually, one should regard this result as a non-obvious consequence of Kunz's Theorem (cf. Theorem 1.4.1).

Proposition C.1.4. *$S[\Theta; F]$ is a flat right S-module.*

Proof. By its very definition (cf. Subsection 1.4.3),

$$S[\Theta; F] = \bigoplus_{e \geq 0} S\Theta^e.$$

Moreover, it is known that tensor product commute with filtered colimits (in particular, with filtered direct sums). In this way, combining these facts it is enough to check out that, for any $e \geq 0$, $S\Theta^e$ is a flat right S-module.

Fix $e \geq 0$. Firstly, albeit the notation $S\Theta^e$ might suggest that it is just a left S-module, this is not the case because, since S is regular, $S\Theta^e$ can be identified with $\Theta^e S^{1/p^e}$, where S^{1/p^e} denotes the ring of p^e-roots of S; from this point of view, it is clear that $S\Theta^e$ may be also regarded as a right S-module. Therefore, keeping in mind the previous identification one has that the map $\Theta^e S^{1/p^e} \longrightarrow S^{1/p^e}$ given by the assignment $\Theta^e s^{1/p^e} \longmapsto s^{1/p^e}$ defines an abstract isomorphism of right S-modules, whence $S\Theta^e$ is (abstractly) isomorphic to S^{1/p^e} in the category of right S-modules and then the result follows from the fact that S^{1/p^e} is a flat right S-module because of Kunz's Theorem (cf. Theorem 1.4.1); the proof is therefore completed. □

Next result provides some useful properties of $CK_\bullet(x_1, \ldots, x_n)$.

Proposition C.1.5. *Let*

$$0 \longrightarrow K_n \xrightarrow{d_n} K_{n-1} \longrightarrow \ldots \longrightarrow K_2 \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0 \longrightarrow 0$$

be the Koszul chain complex of S with respect to $x_1, \ldots, x_n$ (regarded as a chain complex in the category of left S-modules) and suppose that each differential d_l is represented by right multiplication by matrix M_l; moreover, for each $l \geq 0$ $M_l^{[p]}$ denotes the matrix obtained by raising all the entries of M_l to its pth power. Then, the following statements hold.

(a) ∂_1 is represented by right multiplication by matrix $\begin{pmatrix} x_1 & \ldots & x_n & \Theta - 1 \end{pmatrix}^T$.

(b) ∂_{n+1} is represented by right multiplication by matrix $\begin{pmatrix} (-1)^n \left(\Theta - (x_1 \cdots x_n)^{p-1}\right) & M_n^{[p]} \end{pmatrix}$.

(c) *For each* $1 \leq l \leq n-1$, ∂_{l+1} *is represented by right multiplication by matrix*

$$\left(\begin{array}{c|c} M_{l+1} & \mathbf{0} \\ \hline (-1)^l D_l & M_l^{[p]} \end{array}\right),$$

where D_l *is a diagonal matrix with non-zero entries* $\Theta - (x_{i_1} \cdots x_{i_l})^{p-1}$, *where* $1 \leq i_1 < \ldots < i_l \leq n$. *It is worth noting that we are adopting the convention that* $M_{n+1} = 0 = M_0$.

(d) $H_0(CK_\bullet(x_1, \ldots, x_n)) \cong S/I_n^{[p]}$ *as left* $S[\Theta; F]$*-modules, where the reader should remind that* $I_n = Sx_1 + \ldots + Sx_n$.

(e) $H_{n+1}(CK_\bullet(x_1, \ldots, x_n)) = 0$.

Proof. First of all, parts (a), (b) and (c) follows from the very definition of $CK_\bullet(x_1, \ldots, x_n)$. On the other hand, using part (a) it follows that

$$H_0(FK_\bullet(x_1, \ldots, x_n)) = \frac{S[\Theta; F]}{\operatorname{Im}(\partial_1)} \cong \frac{S[\Theta; F]}{S[\Theta; F]I_n + S[\Theta; F](\Theta - 1)} \cong S/I_n^{[p]}.$$

Therefore, part (d) also holds. So, it only remains to check part (e).

Consider the composition

$$S[\Theta; F] \xrightarrow{\partial_{n+1}} S[\Theta; F] \oplus S[\Theta; F]^{\oplus n} \xrightarrow{\pi} S[\Theta; F]^{\oplus n},$$

where π denotes the corresponding projection. In this way, the reader should notice that $\pi\partial_{n+1}$ turns out to be, up to isomorphisms, $\mathbb{1}_{S[\Theta;F]} \otimes d_n^{[p]}$ (indeed, this fact follows directly from the commutative square established in Discussion C.1.2); regardless, since $d_n^{[p]}$ is an injective homomorphism between free left S-modules, and $S[\Theta; F]$ is a flat right S-module (cf. Proposition C.1.4), one has that $\mathbb{1}_{S[\Theta;F]} \otimes d_n^{[p]}$ is an injective homomorphism between free left $S[\Theta; F]$-modules. Therefore, $\pi\partial_{n+1}$ is also an injective homomorphism, whence ∂_{n+1} is so. This fact concludes the proof. □

Now, we want to single out the following technical fact because it will play a key role during the proof of the main result of this part.

Lemma C.1.6. *Preserving the notations introduced in Proposition C.1.5, one has, for any* $0 \leq l \leq n$, *that* $M_l^{[p]} D_{l-1} = D_l M_l$.

Proof. Fix $0 \leq l \leq n$. Proposition C.1.3 implies that $\partial_l \partial_{l+1} = 0$; moreover, according to part (c) of Proposition C.1.5, ∂_l and ∂_{l+1} are represented respectively by right multiplication by matrices

$$\left(\begin{array}{c|c} M_l & \mathbf{0} \\ \hline (-1)^{l-1} D_{l-1} & M_{l-1}^{[p]} \end{array}\right), \text{ and } \left(\begin{array}{c|c} M_{l+1} & \mathbf{0} \\ \hline (-1)^l D_l & M_l^{[p]} \end{array}\right).$$

In this way, combining these two facts it follows that

$$\left(\begin{array}{c|c} M_{l+1} & \mathbf{0} \\ \hline (-1)^l D_l & M_l^{[p]} \end{array}\right) \left(\begin{array}{c|c} M_l & \mathbf{0} \\ \hline (-1)^{l-1} D_{l-1} & M_{l-1}^{[p]} \end{array}\right) = \mathbf{0};$$

in particular, we must have $(-1)^l D_l M_l + (-1)^{l-1} M_l^{[p]} D_{l-1} = \mathbf{0}$, which is equivalent to say that

$$(-1)^l \left(D_l M_l - M_l^{[p]} D_{l-1} \right) = \mathbf{0}.$$

Whence $M_l^{[p]} D_{l-1} = D_l M_l$, just what we finally wanted to show. □

C.1.1 Main result

Now, we state and prove the first main result of this Appendix, which is the following:

Theorem C.1.7. *$CK_\bullet(x_1, \ldots, x_n)$ provides a free resolution of $S/I_n^{[p]}$ in the category of left $S[\Theta; F]$-modules.*

Proof. Proposition C.1.5 implies that it is enough to check out, for any $1 \leq l \leq n$, that $H_l\left(CK_\bullet(x_1, \ldots, x_n)\right) = 0$.

So, fix $1 \leq l \leq n$. Our goal is to show that $\ker(\partial_l) \subseteq \operatorname{Im}(\partial_{l+1})$; in other words, we have to prove that the chain complex $CK_{l+1} \xrightarrow{\partial_{l+1}} CK_l \xrightarrow{\partial_l} CK_{l-1}$ is midterm exact. First of all, the reader should remind that $CK_l = K'_l \oplus K''_l$, where

$$\begin{aligned} K'_l &:= \bigoplus_{1 \leq i_1 < \ldots < i_l \leq n} S[\Theta; F](\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}), \text{ and} \\ K''_l &:= \bigoplus_{1 \leq j_1 < \ldots < j_{l-1} \leq n} S[\Theta; F](\mathbf{e}_{j_1} \wedge \ldots \wedge \mathbf{e}_{j_{l-1}} \wedge u). \end{aligned}$$

Furthermore, Discussion C.1.2 implies that the chain complexes

$$K'_\bullet : \quad 0 \longrightarrow K'_n \xrightarrow{d'_n} K'_{n-1} \longrightarrow \ldots \longrightarrow K'_2 \xrightarrow{d'_2} K'_1 \xrightarrow{d'_1} K'_0 \longrightarrow 0$$

and

$$(K'_\bullet)^{[p]} : \quad 0 \longrightarrow K'_n \xrightarrow{(d'_n)^{[p]}} K'_{n-1} \longrightarrow \ldots \longrightarrow K'_2 \xrightarrow{(d'_2)^{[p]}} K'_1 \xrightarrow{(d'_1)^{[p]}} K'_0 \longrightarrow 0$$

are respectively canonically isomorphic to $S[\Theta; F] \otimes_S K_\bullet$ and $S[\Theta; F] \otimes_S K_\bullet^{[p]}$; in particular, since $S[\Theta; F]$ is a flat right S-module (cf. Proposition C.1.4), $K'_\bullet$ and $(K'_\bullet)^{[p]}$ are both acyclic chain complexes in the category of left $S[\Theta; F]$-modules. On the other hand, we also have to keep in mind that d'_l and $(d'_l)^{[p]}$ are respectively represented by right multiplication by matrix M_l and $M_l^{[p]}$ (cf. Proposition C.1.5 and its corresponding notation).

Now, let $P \in \ker(\partial_l) \subseteq FK_l$. Since $FK_l = K'_l \oplus K''_l$, we may write $P = (P', P'')$ for certain $P' \in K'_l$ and $P'' \in K''_l$; in this way, as $P \in \ker(\partial_l)$ it follows that

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} P' & P'' \end{pmatrix} \left(\begin{array}{c|c} M_l & \mathbf{0} \\ \hline (-1)^{l-1} D_{l-1} & M_{l-1}^{[p]} \end{array} \right) = \begin{pmatrix} P'M_l + P''(-1)^{l-1}D_{l-1} & P''M_{l-1}^{[p]} \end{pmatrix},$$

which leads to the following system of equations:

$$P'M_l + P''(-1)^{l-1}D_{l-1} = \mathbf{0}, \; P''M_{l-1}^{[p]} = \mathbf{0}.$$

In particular, since $P''M_{l-1}^{[p]} = \mathbf{0}$ one has that $P'' \in \ker((d'_{l-1})^{[p]}) = \operatorname{Im}((d'_l)^{[p]})$; therefore, there is $Q'' \in K''_l$ such that $Q''M_l^{[p]} = P''$. Using this fact, it follows that

$$P'M_l + Q''(-1)^{l-1}M_l^{[p]}D_{l-1} = 0.$$

Regardless, Lemma C.1.6 tells us that $M_l^{[p]}D_{l-1} = D_l M_l$, whence

$$P'M_l + Q''(-1)^{l-1}D_l M_l = 0,$$

which is equivalent to say that $\left(P' + Q''(-1)^{l-1}D_l\right) M_l = 0$. In this way, one has that

$$P' + Q''(-1)^{l-1}D_l \in \ker(d'_l) = \operatorname{Im}(d'_{l+1})$$

and therefore there exists $Q' \in K'_{l+1}$ such that $Q'M_{l+1} = P' + Q''(-1)^{l-1}D_l$.

Summing up, setting $Q := (Q', Q'') \in FK_{l+1}$, it follows that

$$\begin{aligned} \begin{pmatrix} Q' & Q'' \end{pmatrix} \left(\begin{array}{c|c} M_{l+1} & \mathbf{0} \\ \hline (-1)^l D_l & M_l^{[p]} \end{array} \right) &= \begin{pmatrix} Q'M_{l+1} + Q''(-1)^l D_l & Q''M_l^{[p]} \end{pmatrix} \\ &= \begin{pmatrix} P' + Q''(-1)^{l-1}D_l + Q''(-1)^l D_l & P'' \end{pmatrix} = \begin{pmatrix} P' & P'' \end{pmatrix} \end{aligned}$$

and therefore we can conclude that $P \in \operatorname{Im}(\partial_{l+1})$, which is exactly what we wanted to show. □

C.2 The Cartier-Koszul chain complex in full generality

Our next aim is to define the Cartier-Koszul chain complex over an arbitrary commutative Noetherian ring of prime characteristic p, and explore some specific situations on which we can ensure that defines a finite free resolution.

In this way, unless otherwise is specified, hereafter A denotes a commutative Noetherian ring of prime characteristic p, and $y_1, \ldots, y_n$ denote arbitrary elements of A; moreover, we regard A as an $S = \mathbb{F}_p[x_1, \ldots, x_n]$-algebra under the natural homomorphism $S \longrightarrow A$ of $\mathbb{F}_p$-algebras which sends each x_i to y_i.

Definition C.2.1. We define the *Cartier-Koszul chain complex* of A with respect to $y_1, \ldots, y_n$ as the chain complex

$$CK_\bullet(y_1, \ldots, y_n; A) := A \otimes_S CK_\bullet(x_1, \ldots, x_n).$$

Keeping in mind the previous definition, next result is a direct consequence of Theorem C.1.7.

Theorem C.2.2. *We suppose that the natural map $S \longrightarrow A$ is flat. Then, one has that $CK_\bullet(y_1, \ldots, y_n; A)$ defines a finite free resolution of $A/I_n^{[p]}$ in the category of left $A[\Theta; F]$-modules, where I_n is the ideal of A generated by the y_i's.*

One possible application of Theorem C.2.2 is given in the next:

Theorem C.2.3. *Let A be a commutative Noetherian ring containing a field of prime characteristic p, and let $y_1, \ldots, y_n$ be an A-regular sequence which is contained in the Jacobson radical of A. Then, $CK_\bullet(y_1, \ldots, y_n; A)$ defines a finite free resolution of $A/I_n^{[p]}$ in the category of left $A[\Theta; F]$-modules, where I_n is the ideal of A generated by the y_i's.*

Proof. It is known, as a consequence of results obtained independetly by I. Kaplansky and R. Hartshorne (cf. [116, Theorem 2.1] and the references therein) that, under these assumptions, the natural map $S \longrightarrow A$ is flat; whence the result follows directly from Theorem C.2.2. □

Actually, we know that, under slightly different assumptions, $CK_\bullet(y_1, \ldots, y_n; A)$ also defines a finite free resolution; before establishing such result, we have to review the following:

Definition C.2.4. Let R be an arbitrary ring, let $s \in \mathbb{N}$, and let $f_1, \ldots, f_s$ be a sequence of elements in R. It is said that $f_1, \ldots, f_s$ is a *Koszul regular sequence* provided the Koszul chain complex $K_\bullet(f_1, \ldots, f_s; R)$ provides a free resolution of R/I_s, where I_s is the ideal generated by the f's.

Keeping in mind the previous notion, we are ready for proving the next:

Theorem C.2.5. *Let A be a commutative Noetherian ring of prime characteristic p, and let $y_1, \ldots, y_n$ be an A-Koszul regular sequence. Then, $CK_\bullet(y_1, \ldots, y_n; A)$ defines a finite free resolution of $A/I_n^{[p]}$ in the category of left $A[\Theta; F]$-modules, where I_n is the ideal generated by the y_i's.*

Proof. Mutatis mutandi the same proof of Theorem C.1.7 replacing S by A and $x_1, \ldots, x_n$ by $y_1, \ldots, y_n$; indeed, a simple inspection of the proof of Theorem C.1.7 reveals that we only used there the regularity of S (in order to apply Kunz's theorem) and the fact that the Koszul chain complex $K_\bullet(x_1, \ldots, x_n)$ defines a finite free resolution of the coefficient field of S; the proof is therefore completed. □

In this way, combining Theorem C.2.3 and Theorem C.2.5 we obtain the second main result of this Appendix, which turns out to be the final result of this mimeograph.

Theorem C.2.6. *Let A be a commutative Noetherian ring of prime characteristic p, and let $y_1, \ldots, y_n$ be a sequence of elements in A. Moreover, we assume that one of the following statements hold.*

(i) $y_1, \ldots, y_n$ *is an A-regular sequence contained in the Jacobson radical of A.*

(ii) A *is regular and* $y_1, \ldots, y_n$ *is an A-Koszul regular sequence.*

Then, $CK_\bullet(y_1, \ldots, y_n; A)$ *defines a finite free resolution of* $A/I_n^{[p]}$ *in the category of left* $A[\Theta; F]$*-modules, where* I_n *is the ideal generated by the* y_i*'s.*

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