Parallel characterizations of a generalized Shapley value and a generalized Banzhaf value for cooperative games with levels structure of cooperation

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Abstract

We present parallel characterizations of two different values in the framework of restricted cooperation games. The restrictions are introduced as a finite sequence of partitions defined on the player set, each of them being coarser than the previous one, hence forming a structure of different levels of a priori unions. On the one hand, we consider a value first introduced in , 8), which extends the Shapley value to games with different levels of a priori unions. On the other hand, we introduce another solution for the same type of games, which extends the Banzhaf value in the same manner. We characterize these two values using logically comparable properties.

Keywords: Shapley value, Banzhaf value, levels structure of cooperation

1 Introduction

Transferable utility cooperative games (just games from now on) are used to describe situations in which agents cooperate to obtain some gains, e.g. building a road to connect a number of towns or reaching an agreement to pass a bill. These gains are assumed to be divisible and transferable among players without any loss. Assessing the strength of each player in a given game is a main objective of cooperative game theory. The Shapley value , 5) is the best known concept in this respect. Weighted majority games constitute an important subclass of games and are mainly used to study the distribution of power in voting bodies. The Banzhaf index was proposed as a measure of power in such an environment , 4) and later on extended to the class of all games by , 1).

In the original model there is no restriction to the cooperation, and the game is defined by the worth that any coalition can obtain by its own. However, there are many real situations in which there is a priori information about the behavior of the players or there are environmental restrictions and only partial cooperation occurs. Different approaches have been used to address this type of situations and different models of games with restricted cooperation have been studied. In particular, players may form coalitions and these coalitions may bargain for the division of the worth of the grand coalition. supposes that the restrictions to the cooperation are given by a partition of the set of players. The model with both a game and a partition of the set of players is called a game with a priori unions. For these games, , 2) proposes and characterizes the

Owen value, an extension of the Shapley value , 5) to allocate the gains generated by the grand coalition. Following a similar procedure, in a subsequent paper , 3) defines an extension of the Banzhaf value , 1) known as the Banzhaf-Owen value. The first characterization of this solution concept is presented in . give parallel characterizations of the two aforementioned values which eases the comparison between them.

, 8) takes one step beyond by introducing games with many levels of cooperation, which extends the model of games with a priori unions. He proposes and characterizes an extension of the Owen value for this kind of situations, which we will call the Shapley levels value. As before, players are assumed to be organized in groups to bargain for the division of the worth available (first level of cooperation). Nevertheless, this time the formed unions may again organize themselves in larger groups (second level of cooperation) while they maintain their internal obligations of the first level, and so on and so forth. Hence, this time the restrictions to the cooperation are described by a sequence of partitions of the player set, each of them being coarser than the previous ones. gives an alternative characterization of the Shapley levels value using a balanced contributions property and , 7) implements the Shapley levels value in a subgame perfect equilibrium of a particular bidding mechanism.

In the present paper, we first propose an extension of the Banzhaf-Owen value for games with levels structure of cooperation, which we call the Banzhaf levels value. Levels structures are very reasonable in many voting situations, and hence it seams reasonable to study the extension of the Banzhaf value to a levels structure framework. For the sake of generality, the new value is introduced for the class of all games.

Then, we provide parallel characterizations of both the Shapley levels value and the Banzhaf levels value which reveal differences between both solution concepts. Parallel characterizations of two different values are specially appealing for at least two reasons. In the first place, from a mathematically point of view, characterizing one value using a few independent properties may be more appealing than just giving an explicit formula or procedure to calculate it. In the second place, deciding on whether to use a value or another can be made more easily using a set of comparable properties instead of a formula.

The rest of the paper is organized as follows. Section 2 is mainly devoted to present the model of games with levels structure of cooperation, and in particular the Shapley levels value introduced by , 8). In Section 3 we define the Banzhaf levels value. In Section 4 we introduce and explain some properties that a value for games with levels structure of cooperation might satisfy, and we provide a characterization for each of the two aforementioned values. Section 5 concludes.

2 Preliminaries

An *n-person cooperative game with transferable utility (a game)* is a pair (N, v), where $N = \{1, \ldots, n\}$ is the finite set of players and v, the *characteristic function*, is a real valued function on $2^N = \{S : S \subseteq N\}$ with $v(\emptyset) = 0$. We denote by $\mathcal{G}$ the set of all games. For each $S \subseteq N$ and $i \in N$ we will write $S \cup i$ instead $S \cup \{i\}$ and $S \setminus i$ instead $S \setminus \{i\}$.

Given $(N, v) \in \mathcal{G}$, a player $i \in N$ is a *dummy* if $v(S \cup i) = v(S) + v(i)$ for all $S \subseteq N \setminus i$, that is, if all her *marginal contributions*, $v(S \cup i) - v(S)$, are equal to $v(i)$. Two players $i, j \in N$ are *symmetric* if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, that is, if their marginal contributions to each coalition coincide.

A *value on* $\mathcal{G}$ is a map f that assigns to every game $(N, v) \in \mathcal{G}$ a vector $f(N, v) \in \mathbb{R}^n$. The following definitions provide the explicit expressions of two well-known values in the literature. Throughout the paper, for each finite set given by a capital letter, the corresponding lowercase letter stands for the cardinality of the set. Also, if needed, we use the $|\cdot|$ operator to denote the cardinality of a set.

Definition 2.1. , 5) Given a game (N, v), the *Shapley value*, ϕ, is a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \left[v(S \cup i) - v(S) \right], \text{ for every } i \in N.$$

Definition 2.2. , 1) Given a game (N, v), the *Banzhaf value*, ψ, is a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:

$$\psi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{1}{2^{n-1}} \left[v(S \cup i) - v(S) \right], \text{ for every } i \in N.$$

We denote by $\mathcal{P}(N)$ the set of all partitions of a finite set of players N, and for each $P \in \mathcal{P}(N)$ and each $S \subseteq N$, $P_{|S} \in \mathcal{P}(S)$ is the partition of S induced by P, i.e., $P_{|S} = \{U \cap S : U \in P, U \cap S \neq \emptyset\}$. A *levels structure of cooperation* is a pair $(N, \underline{B})$, where N is the set of players and $\underline{B} = \{B_0, \ldots, B_k\}$ is a sequence of partitions of N such that $B_0 = \{\{i\} : i \in N\}$ and, for each $r \in \{0, \ldots, k-1\}$, B_{r+1} is coarser than B_r. That is to say, for each $r \in \{0, 1, \ldots, k-1\}$ and each $S \in B_{r+1}$, there is $B \subseteq B_r$ such that $S = \cup_{U \in B} U$. Each $U \in B_r$ is called a *union* and B_r is called the *r-th level of* $\underline{\mathrm{B}}$.

We denote by $\mathcal{L}(N)$ the set of all levels structures of cooperation over the set N. The following example illustrates the above definitions.

Example 2.3. Let $N = \{1, 2, 3, 4, 5, 6\}$ and $\underline{B} = \{B_0, B_1, B_2\}$ be a levels structure of cooperation over N with two levels, where

$$B_2 = \{\{1, 2, 3\}, \{4, 5, 6\}\},$$

$$B_1 = \{\{1, 2\}, \{3\}, \{4\}, \{5, 6\}\}, \text{ and}$$

$$B_0 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}.$$

A *cooperative game with levels structure of cooperation* is a triple $(N, v, \underline{B})$, where $(N, v) \in \mathcal{G}$ and $(N, \underline{B}) \in \mathcal{L}(N)$. We denote by $\mathcal{GL}$ the set of all cooperative games with levels structure of cooperation. Given $(N, \underline{B}) \in \mathcal{L}(N)$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i \in N$, we denote by $(N, \underline{B^{-i}}) \in \mathcal{L}(N)$ the levels structure of cooperation obtained from $(N, \underline{B})$ by isolating player i from the union she belongs to at each level, i.e., $\underline{B^{-i}} = \{B_0^{-i}, \ldots, B_k^{-i}\}$, where, for all $r \in \{0, \ldots, k\}$, $B_r^{-i} = \{U \in B_r : i \notin U\} \cup \{U_r \backslash i, \{i\}\}$ given that $i \in U_r \in B_r$. Note that $B_0^{-i} = B_0$. For each $r \in \{1, \ldots, k\}$ and each $U \in B_r$, $[U]$ denotes U considered as a single player at level r, whereas $[B_r]$ denotes the set of players at level r, i.e, $[B_r] = \{[U] : U \in B_r\}$ and $([B_r], \underline{B_r}) \in \mathcal{L}([B_r])$, where $\underline{B_r} = \{B_r, \ldots, B_k\}$. Given $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$, for each $r \in \{0, \ldots, k\}$ we define the *r-th level game* $([B_r], v^r, \underline{B_r}) \in \mathcal{GL}$ as the game with levels structure of cooperation induced from $(N, v, \underline{B})$ by considering the coalitions of B_r as players, i.e., for each $r \in \{0, \ldots, k\}$, $v^r(\cup_{l=1}^t [U_l]) = v(\cup_{l=1}^t U_l)$ with $U_1, \ldots, U_t \in B_r$ for some $t \geq 0$.

In the framework of games with levels structure of cooperation we assume that players are initially organized into the coalition structure B_k as groups that bargain for the division of $v(N)$. Then, each union of the last level is divided again according to the coalition structure B_{k-1} in order to divide the amount that the unions of the last level have obtained, and so on and so forth until the last level, B_0, is reached.

A *value on* $\mathcal{GL}$ is a map f that assigns to every game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{GL}$ a vector $f(N, v, \underline{B}) \in \mathbb{R}^n$. We denote by $\Pi(N)$ the set of permutations of N, i.e., the set of bijective mappings from N to N.

Next, given a levels structure of cooperation $(N, \underline{B}) \in \mathcal{L}(N)$ with $\underline{B} = \{B_0, \ldots, B_k\}$ let the sets $\Omega(\underline{B}) = \Omega_1(\underline{B}) \subseteq \Omega_2(\underline{B}) \subseteq \cdots \subseteq \Omega_k(\underline{B}) \subseteq \Pi(N)$ be defined as follows. First of all,

$$\Omega_k(\underline{B}) = \{\sigma \in \Pi(N) : \forall S \in B_k,\ \forall i, j \in S \in B_k \text{ and } l \in N, \text{if } \sigma(i) < \sigma(l) < \sigma(j) \text{ then } l \in S\}.$$

Then, for $r \in \{k-1, \ldots, 1\}$ we recursively define

$$\Omega_r(\underline{B}) = \{\sigma \in \Omega_{r+1}(\underline{B}) : \forall i, j \in S \in B_r \text{ and } l \in N,\ \text{if } \sigma(i) < \sigma(l) < \sigma(j) \text{ then } l \in S\}.$$

Observe that $\Omega_r(\underline{B})$ denotes the permutations of $\Omega_{r+1}(\underline{B})$ such that the elements of each union of B_r are consecutive. Let us see an example to illustrate the above definitions.

Example 2.4. For the levels structure of cooperation of Example 2.3, $|\Omega_2(\underline{B})| = 72$, $|\Omega_1(\underline{B})| = 32$, $(1,2,4,3,5,6) \notin \Omega_2(\underline{B})$, $(1,3,2,4,5,6) \in \Omega_2(\underline{B}) \backslash \Omega_1(\underline{B})$ and $(3,2,1,5,6,4) \in \Omega_1(\underline{B})$.

Now we are in the position to recall the definition of the already known solution concept for games with levels structure of cooperation.

Definition 2.5. Given a game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{GL}$, the *Shapley levels value* , 8), Φ, is a vector in $\mathbb{R}^n$ where each coordinate is defined as follows:

$$\Phi_i(N, v, \underline{B}) = \frac{1}{|\Omega(\underline{B})|} \sum_{\sigma \in \Omega(\underline{B})} (v(P_i^\sigma \cup i) - v(P_i^\sigma)),$$

where $P_i^\sigma = \{j \in N : \sigma(j) < \sigma(i)\}$ is the set of predecessors of i at σ.

, 8) proves that the Shapley levels value is the unique value on $\mathcal{GL}$ satisfying *efficiency*, *additivity*, *anonymity*[1], the *null player property* and *coalitional symmetry*. The first four properties are extensions of standard properties in the literature, whereas coalitional symmetry demands that the sum of the payoffs to the players belonging to two unions S and U of some level r be the same whenever $[S]$ and $[U]$ are symmetric players in the r-th level game and they belong to the same union in the next level. It is worthy to mention that the five properties are natural extensions of the properties used in , 2) to characterize the Owen value, which is simply the restriction of Φ to games with levels structure of cooperation with a single level.

3 A new value on $\mathcal{GL}$

In this section we introduce a new value on $\mathcal{GL}$ that coincides with the Banzhaf-Owen value , 3) when the levels structure of cooperation has just one level, i.e., when $\underline{B} = \{B_0, B_1\}$. The idea for defining this new value is to induce, for each player, a partition of the set of players that respects the restrictions of the levels structure of cooperation. In other words, instead of looking at which permutations are feasible for the given levels

[1] In , 8) it is denoted by *individual symmetry*.

structure, as in , 8), for each player we look at which coalitions are feasible for the given levels structure of cooperation.

Given a levels structure of cooperation $(N, \underline{B}) \in \mathcal{L}(N)$, for each player $i \in N$, let $i \in U_0 = \{i\} \subseteq U_1 \subseteq \cdots \subseteq U_k$ such that $U_r \in B_r$ for all $r \in \{0, \ldots, k\}$. Then, the *partition induced by* $\underline{B}$ *on* i is defined as follows,

$$P(i, \underline{B}) = \bigcup_{r=0}^{k} (B_r)_{|U_{r+1} \setminus U_r},$$

where $U_{k+1} = N$ by convenience. Then, $P(i, \underline{B}) \in \mathcal{P}(N \setminus i)$. We denote $|P(i, \underline{B})|$ by m_i, and the unions of the partition induced by $\underline{B}$ on i, by $P(i, \underline{B}) = \{T_1, \ldots, T_{m_i}\}$. Finally the set of indices of the partition induced by $\underline{B}$ on i is denoted by $M_i = \{1, \ldots, m_i\}$ which can be seen as the set of representatives of the unions of $P(i, \underline{B})$.

Example 3.1. For the levels structure of cooperation of Example 2.3 we have, for instance, $P(1, \underline{B}) = \{\{2\}, \{3\}, \{4, 5, 6\}\}$ and $P(3, \underline{B}) = \{\{1, 2\}, \{4, 5, 6\}\}$.

Using the partition induced by the levels structure for each player, we define a new value on $\mathcal{GL}$, namely the Banzhaf levels value, which is built based on the Banzhaf-Owen value for games with a priori unions.

Definition 3.2. Given a cooperative game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{GL}$, the *Banzhaf levels value*, Ψ, is a value on $\mathcal{GL}$ defined, for every $i \in N$, as follows:

$$\Psi_i(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{1}{2^{m_i}} \left[v(T_R \cup i) - v(T_R) \right],$$

where $T_R = \cup_{r \in R} T_r$.

One can easily check that the coalitions considered in each marginal contribution, T_R, are the coalitions for which there exists $\sigma \in \Omega(\underline{B})$ such that $T_R = P_i^{\sigma}$. Therefore, exploiting the link between coalitions of elements of $P(i, \underline{B})$, for each $i \in N$, and permutations of $\Omega(\underline{B})$ the Shapley levels value, Φ, can be written in an alternative way.

Remark 3.3. Given a cooperative game with levels structure of cooperation $(N, v, \underline{B}) \in \mathcal{GL}$,

$$\Phi_i(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{c_R^i}{|\Omega(\underline{B})|} \left[v(T_R \cup i) - v(T_R) \right],$$

where $c_R^i = |\{\sigma \in \Omega(\underline{B}) : P_i^\sigma = T_R\}|$.

Expressions of Φ and Ψ above lead to the Owen , 2) and Banzhaf-Owen , 3) values respectively for levels structure of cooperation with a single level.

4 Two parallel axiomatic characterizations

In this section we characterize both Φ and Ψ based on two different groups of properties. The first group applies only to games with the trivial levels structure $\underline{B} = \{B_0\} = \{\{i\} : i \in N\}$ and points out which value on $\mathcal{G}$ does the value on $\mathcal{GL}$ generalize, either the Shapley value or the Banzhaf value. The second group of properties describes the performance of the values in $\mathcal{GL}$ with respect to the levels structure and they are logically related.

We consider a number of properties that a value on $\mathcal{GL}$, f, might be asked to satisfy. We start with a first set of properties.

EFF A value f on $\mathcal{GL}$ satisfies *efficiency* if for every $(N, v) \in \mathcal{G}$,

$$\sum_{i \in N} f_i(N, v, \{B_0\}) = v(N).$$

2-EFF A value f on $\mathcal{GL}$ satisfies *2-efficiency* if for every $(N,v) \in \mathcal{G}$ and any $i,j \in N$,

$$f_i(N,v,\{B_0\}) + f_j(N,v,\{B_0\}) = f_p(N^{ij}, v^{ij}, \{B_0\}^{ij}),$$

where $(N^{ij}, v^{ij}, \{B_0\}^{ij})$ is the game with levels structure of cooperation such that player i and j have merged into the new player $p \notin N$, i.e., $N^{ij} = (N \setminus \{i,j\}) \cup p$, $\{B_0\}^{ij} = \{\{l\} : l \in N^{ij}\}$, and

$$v^{ij}(S) = \begin{cases} v(S) & \text{if } p \notin S \\ v((S \setminus p) \cup i \cup j) & \text{if } p \in S \end{cases} \quad \text{for every } S \subseteq N^{ij}.$$

DPP A value f on $\mathcal{GL}$ satisfies *the dummy player property* if for every $(N,v) \in \mathcal{G}$, if $i \in N$ is a dummy player on (N,v),

$$f_i(N,v,\{B_0\}) = v(i).$$

SYM A value f on $\mathcal{GL}$ satisfies *symmetry* if for every $(N,v) \in \mathcal{G}$, if $i,j \in N$ are symmetric players in (N,v),

$$f_i(N,v,\{B_0\}) = f_j(N,v,\{B_0\}).$$

EMC A value f on $\mathcal{GL}$ satisfies *equal marginal contributions* if for every $(N,v),(N,w) \in \mathcal{G}$ and every $i \in N$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for all $S \subseteq N \setminus i$,

$$f_i(N,v,\{B_0\}) = f_i(N,w,\{B_0\}).$$

The above properties are standard in the literature for games without restricted cooperation. The EFF property states that the whole worth available is shared among the players. The 2-EFF property is a collusion neutrality property which states that the payoff of two players does not change if they decide to artificially merge in a new player. Properties of this kind are used in many characterizations of the Banzhaf value, see for instance , or , 0). The SYM and DPP properties are clear by themselves. The property of EMC states that if a player's marginal contributions to any coalition in two games coincide, then her payoffs also coincide in the case of the trivial levels structure. Stronger versions of EMC have been used in characterizations of both Shapley and Banzhaf values and they are called *monotonicity* , 9).

In Theorem 4.1 (resp. Theorem 4.2) we use, together with other properties that are presented below, EFF, SYM and EMC to characterize the Shapley levels value Φ (resp. 2-EFF, DPP, SYM and EMC to characterize the Banzhaf levels value Ψ). Although all these properties are presented in a weak form, in the sense that they only concern the trivial levels structure, it can be checked that both Φ and Ψ satisfy stronger versions of the corresponding properties. These latter strong versions are stated in the same manner as their weak counterparts but they concern any levels structure, either trivial or not, with the condition that whenever two players are involved they are asked to belong to the same union at each level.

Let us now consider another set of properties.

LGP A value f on $\mathcal{GL}$ satisfies the *level game property* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and every $U \in B_r$ with $r \in \{1, \cdots, k\}$,

$$\sum_{i \in U} f_i(N, v, \underline{B}) = f_{[U]}([B_r], v^r, \underline{B_r}).$$

SLGP A value f on $\mathcal{GL}$ satisfies the *singleton level game property* if for every $(N, v, \underline{B}) \in$

$\mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and every $U \in B_r$ with $r \in \{1, \cdots, k\}$, such that $U = \{i\}$ for some $i \in N$,

$$f_i(N, v, \underline{B}) = f_{[U]}([B_r], v^r, \underline{B_r}).$$

LBC A value f on $\mathcal{GL}$ satisfies *level balanced contributions* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i, j \in U \in B_1$,

$$f_i(N, v, \underline{B}) - f_i(N, v, \underline{B^{-j}}) = f_j(N, v, \underline{B}) - f_j(N, v, \underline{B^{-i}}).$$

LNID A value f on $\mathcal{GL}$ satisfies *level neutrality under individual desertion* if for every $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i, j \in U \in B_1$,

$$f_i(N, v, \underline{B}) = f_i(N, v, \underline{B^{-j}}).$$

The LGP is based on a property used in , 2) to characterize the Owen value. It states that the total payoff obtained by the members of a union in a given level equals the payoff obtained by the union when considering it as a player in the corresponding level game. The SLGP is a weaker version of LGP, which states that any union which is composed of a single player gets the same payoff in the original game and in the corresponding level game when considering the union as a player. The idea behind SLGP is also used in and more recently in .

The LBC property is a reciprocity property that states that the isolation of a player from the levels structure affects the players in her same union of the first level in the same amount as if it happens the other way around. This property has been used in the context of games with a priori unions, e.g. , 6) and . The LNID property is a stronger version of LBC and states that the isolation of a player from the levels structure does not affect the payoffs of the players which are in her same union in all the levels. LNID is introduced in and also used in to characterize extensions of the Banzhaf value to different classes of games.

Next we state and prove the two characterization results, one for the Shapley levels value (Theorem 4.1) and one for the Banzhaf levels value (Theorem 4.2). We start characterizing the Shapley levels value.

Theorem 4.1. *The Shapley levels value, Φ, is the unique value on $\mathcal{GL}$ satisfying* EFF*,* SYM*,* EMC*,* LGP*, and* LBC*.*

Proof. First we show that Φ satisfies the properties and then we prove that it is the only value on $\mathcal{GL}$ satisfying them.

(1) *Existence.* Note that, by definition, for every $(N, v) \in \mathcal{G}$, $\Phi(N, v, \{B_0\}) = \phi(N, v)$. Hence, from , 9) we have that Φ satisfies EFF, SYM, and EMC.

In the case of LGP, let $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$, and consider some $U \subseteq N$ such that $U \in B_r$ with $r \in \{1, \cdots, k\}$. We prove that Φ satisfies LGP by induction over r. If $r = 1$, from the definition of the induced partition, $P(i, \underline{B}) \setminus \{\{j\} : j \in U \setminus i\}$ is the same partition for each $i \in U$. Moreover, for any $i \in U$ it holds that $P([U], \underline{B_1}) = P(i, \underline{B}) \setminus \{\{j\} : j \in U \setminus i\}$. For each $i \in U$, $R \subseteq M_{[U]}$, and $S \subseteq U \setminus i$, let $c^i_{R+S} = |\{\sigma \in \Omega(\underline{B}) : P^\sigma_i = T_R \cup S\}|$. By the way $\Omega(\underline{B})$ is constructed, given $R \subseteq M_{[U]}$ and $i \in U$, c^i_{R+S} is the same for any $S \subseteq U \setminus i$ with a given cardinality s, and hence it can be denoted by c^i_{R+s}. Moreover, $c^i_{R+s} = c^j_{R+s}$ for every $i, j \in U$, $R \subseteq M_{[U]}$, and $S \subseteq U \setminus i$, and thus c^i_{R+s} can be further denoted simply by c_{R+s}. Recall that $c_R^{[U]} = |\{\sigma \in \Omega(\underline{B_1}) : P_{[U]}^{\sigma} = T_R\}|$. Then, by Remark 3.3,

$$\begin{aligned}
&\sum_{i\in U} \Phi_i(N, v, \underline{B}) \\
&= \frac{1}{|\Omega(\underline{B})|} \sum_{i\in U} \sum_{R\subseteq M_{[U]}} \sum_{S\subseteq U\setminus i} c_{R+s} \cdot (v(T_R \cup S \cup i) - v(T_R \cup S)) \\
&= \frac{1}{|\Omega(\underline{B})|} \sum_{R\subseteq M_{[U]}} \sum_{i\in U} \sum_{S\subseteq U\setminus i} c_{R+s} \cdot (v(T_R \cup S \cup i) - v(T_R \cup S)) \\
&= \frac{1}{|\Omega(\underline{B})|} \sum_{R\subseteq M_{[U]}} \left[\begin{array}{c} u \cdot c_{R+(u-1)} \cdot v(T_R \cup U) - u \cdot c_{R+0} \cdot v(T_R) \\ + \sum_{\emptyset \neq S \subsetneq U} \left[s \cdot c_{R+(s-1)} - (u-s) c_{R+s}\right] \cdot v(T_R \cup S) \end{array} \right] \qquad (1) \\
&= \sum_{R\subseteq M_{[U]}} \left[u \cdot \frac{c_{R+(u-1)}}{|\Omega(\underline{B})|} \cdot v(T_R \cup U) + u \cdot \frac{c_{R+0}}{|\Omega(\underline{B})|} \cdot v(T_R) \right] \\
&= \frac{1}{|\Omega(\underline{B_1})|} \sum_{R\subseteq M_{[U]}} c_R^{[U]} \cdot \left[\left(v^1(T_R \cup [U]) - v^1(T_R) \right) \right] = \Phi_{[U]}([B_1], v^1, \underline{B_1}),
\end{aligned}$$

where the third equality is obtained by rearranging the terms of the summation, the fourth equality holds since $\frac{c_{R+s}}{c_{R+(s-1)}} = \frac{s}{u-s}$ for all $1 \leq s \leq u-1$ and the fifth equality holds since

$$c_{R+(u-1)} = c_{R+0} = \frac{c_R^{[u]}}{u} \cdot \frac{\Omega(\underline{B})}{\Omega(\underline{B_1})},$$

which completes the first step of the induction.

Now suppose that, for some $r \in \{2, ..., k\}$ and for any $S \in B_{r-1}$, $\sum_{i\in S} \Phi_i(N, v, \underline{B}) = \Phi_{[S]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}})$. Let $U \in B_r$. Then

$$\sum_{i\in U} \Phi_i(N, v, \underline{B}) = \sum_{\substack{S\in B_{r-1} \\ S\subseteq U}} \sum_{i\in S} \Phi_i(N, v, \underline{B}) = \sum_{\substack{S\in B_{r-1} \\ S\subseteq U}} \Phi_{[S]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}})$$

by the induction hypothesis. We can follow the argument from eq. (1) with $\left([B_{r-1}], v^{r-1}, \underline{B_{r-1}}\right)$

instead of $(N, v, \underline{B})$ and $[U] \in [B_{r-1}]$ instead of $i \in N$ to obtain

$$\sum_{\substack{S \in B_{r-1} \\ S \subseteq U}} \Phi_{[S]}([B_{r-1}], v^{r-1}, \underline{B_{r-1}}) = \Phi_{[U]}([B_r], v^r, \underline{B_r}),$$

which completes the induction.

In the case of LBC, let $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$ and $i, j \in U \in B_1$. Then, it is easy to check that $P(i, \underline{B}) \setminus j = P(j, \underline{B}) \setminus i$. Hence, we define $P(ij, \underline{B}) = P(i, \underline{B}) \setminus j = P(j, \underline{B}) \setminus i$, $m_{ij} = |P(ij, \underline{B})|$, and $M_{ij} = \{1, \ldots, m_{ij}\}$. Then,

$$\begin{aligned}
&\Phi_i(N, v, \underline{B}) - \Phi_i(N, v, \underline{B^{-j}}) \\
&\quad = \sum_{R \subseteq M_{ij}} \left[\frac{c^i_{R+j}}{|\Omega(\underline{B})|} \left(v(T_R \cup j \cup i) - v(T_R \cup j)\right) + \frac{c^i_R}{|\Omega(\underline{B})|} \left(v(T_R \cup i) - v(T_R)\right) \right] \\
&\quad - \sum_{R \subseteq M_{ij}} \left[\frac{c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|} \left(v(T_R \cup j \cup i) - v(T_R \cup j)\right) + \frac{c^{i,-j}_R}{|\Omega(\underline{B^{-j}})|} \left(v(T_R \cup i) - v(T_R)\right) \right] \\
&\qquad = \sum_{R \subseteq M_{ij}} \left[\left(\frac{c^i_{R+j}}{|\Omega(\underline{B})|} - \frac{c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|} \right) \left(v(T_R \cup j \cup i) - v(T_R \cup j)\right) \right. \\
&\qquad\qquad \left. + \left(\frac{c^i_R}{|\Omega(\underline{B})|} - \frac{c^{i,-j}_R}{|\Omega(\underline{B^{-j}})|} \right) \left(v(T_R \cup i) - v(T_R)\right) \right],
\end{aligned}$$

where for each $R \subseteq M_{ij}$, $c^{i,-j}_R = |\{\sigma \in \Omega(\underline{B^{-j}}) : P^\sigma_i = T_R\}|$ and $c^{i,-j}_{R+j} = |\{\sigma \in \Omega(\underline{B^{-j}}) : P^\sigma_i = T_R \cup j\}|$. Note that, by definition, $c^i_R = c^j_R$, $c^i_{R+j} = c^j_{R+i}$, $c^{i,-j}_R = c^{j,-i}_R$, and $c^{i,-j}_{R+j} = c^{j,-i}_{R+i}$. We additionally claim (see the proof in the Appendix) that

$$\frac{c^i_R + c^i_{R+j}}{|\Omega(\underline{B})|} = \frac{c^{i,-j}_R + c^{i,-j}_{R+j}}{|\Omega(\underline{B^{-j}})|}. \tag{2}$$

Then $\Phi_i(N, v, \underline{B}) - \Phi_i(N, v, \underline{B^{-j}})$ depends on i in the same way it depends on j, which concludes the proof.

(2) *Uniqueness.* In , 9) it is proved that any value on $\mathcal{GL}$ that satisfies EFF, SYM, and EMC is unique for games with the trivial levels structure of cooperation. In other

words, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying EFF, SYM, and EMC, then

$$f^1(N,v,\{B_0\}) = f^2(N,v,\{B_0\}) = \phi(N,v) \quad \text{, for any } (N,v) \in \mathcal{G}.$$

Hence, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying LGP and LBC such that $f^1(N,v,\{B_0\}) = f^2(N,v,\{B_0\})$ for all $(N,v) \in \mathcal{G}$. We prove that for any $(N,v,\underline{B}) \in \mathcal{GL}$, with $\underline{B} = \{B_0,\ldots,B_k\}$, $f^1(N,v,\underline{B}) = f^2(N,v,\underline{B})$ by induction on the number k of levels of $\underline{B}$. The case $k=1$ is proved in , 6). Now suppose that $f^1(N,v,\underline{B}) = f^2(N,v,\underline{B})$ for any $(N,v,\underline{B}) \in \mathcal{GL}$ such that $|\underline{B}| \leq k$ and let $(N,v,\underline{B}) \in \mathcal{GL}$ with $|\underline{B}| = k+1$. Let also $i \in N$. We prove that $f^1_i(N,v,\underline{B}) = f^2_i(N,v,\underline{B})$ by a second induction on $u = |U|$, where $i \in U \in B_1 \in \underline{B}$. If $u=1$, i.e. $U=\{i\}$, since f^1 and f^2 satisfy LGP, we have

$$f^1_i(N,v,\underline{B}) = f^1_{[U]}([B_1],v^1,\underline{B_1}) = f^2_{[U]}([B_1],v^1,\underline{B_1}) = f^2_i(N,v,\underline{B}),$$

where the second equality holds by the first induction hypothesis. Hence, suppose that $f^1_l(N,v,\underline{B}) = f^2_l(N,v,\underline{B})$ for any $(N,v,\underline{B}) \in \mathcal{GL}$, with $|\underline{B}| = k+1$ and any $l \in U \in B_1$ that satisfies $|U| \leq u$. Now suppose that $|U| = u+1$ and let $j \in U \setminus i$. Since f^1 and f^2 satisfy LBC, we have

$$\begin{aligned} f^1_i(N,v,\underline{B}) - f^1_j(N,v,\underline{B}) &= f^1_i(N,v,\underline{B^{-j}}) - f^1_j(N,v,\underline{B^{-i}}) \\ &= f^2_i(N,v,\underline{B^{-j}}) - f^2_j(N,v,\underline{B^{-i}}) = f^2_i(N,v,\underline{B}) - f^2_j(N,v,\underline{B}), \end{aligned} \quad (3)$$

where the second equality follows from the second induction hypothesis, since $i \in U \setminus j \in B_1^{-j}$ and $j \in U \setminus i \in B_1^{-i}$ with $|U \setminus j| = |U \setminus i| = u$, where $|\underline{B}^{-j}| = |\underline{B}^{-i}| = k+1$. Now, adding up eq. (3) for each $j \in U \setminus i$, we have

$$(u+1)f^1_i(N,v,\underline{B}) - \sum_{j\in U} f^1_j(N,v,\underline{B}) = (u+1)f^2_i(N,v,\underline{B}) - \sum_{j\in U} f^2_j(N,v,\underline{B}). \quad (4)$$

Finally, since f^1 and f^2 satisfy LGP, we have that

$$\sum_{j\in U} f_j^1(N,v,\underline{B}) = f_{[U]}^1([B_1],v^r,\underline{B_1}) = f_{[U]}^2([B_1],v^r,\underline{B_1}) = \sum_{j\in U} f_j^2(N,v,\underline{B}), \qquad (5)$$

where the second equality holds by the first induction hypothesis since $|\underline{B_1}| = k$. Combining eq. (4) and (5) we obtain $f_i^1(N,v,\underline{B}) = f_i^2(N,v,\underline{B})$, which completes the proof. $\square$

In the next theorem we characterize the Banzhaf levels value with a set of six properties, four properties that characterize the Banzhaf value, ψ, and two additional properties that describe the way in which the Banzhaf levels value, Ψ, deals with the levels structure of cooperation. Recall that the last two properties are logically related to those used to characterize the Shapley levels value.

Theorem 4.2. *The Banzhaf levels value, Ψ, is the unique value on $\mathcal{GL}$ satisfying* 2-EFF, DPP, SYM, EMC, SLGP, *and* LNID.

Proof. As in the previous theorem, we first show that Ψ satisfies the properties and then we prove that it is the only value satisfying them.

(1) *Existence.* Note that, by definition, for every $(N,v) \in \mathcal{G}$, $\Psi(N,v,\{B_0\}) = \psi(N,v)$. Hence, from , 0) we have that Ψ satisfies 2-EFF, DPP, SYM, and EMC.

In the case of SLGP, the proof follows immediately taking into account the fact that, for any $(N,v,\underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0,\ldots,B_k\}$ and $U = \{i\} \in B_r$ for some $r \in \{1,\cdots,k\}$, $P(i,\underline{B}) = P([U],\underline{B_r})$.

In the case of LNID, we only need to check that for any $(N,v,\underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0,\ldots,B_k\}$, and any $i,j \in U \in B_1$, $P(i,\underline{B}) = P(i,\underline{B^{-j}})$, which follows from the definition of the partition induced by $\underline{B}$.

(2) *Uniqueness.* From the characterization in , 0), we have that any value on $\mathcal{GL}$ that satisfies 2-EFF, DPP, SYM, and EMC is unique for games with the trivial levels structure of cooperation. In other words, let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying 2-EFF, DPP, SYM, and EMC, then

$$f^1(N, v, \{B_0\}) = f^2(N, v, \{B_0\}) = \psi(N, v) \quad \text{, for any } (N, v) \in \mathcal{G}.$$

Now let f^1 and f^2 be two values on $\mathcal{GL}$ satisfying SLGP and LNID such that $f^1(N, v, \{B_0\}) = f^2(N, v, \{B_0\})$ for all $(N, v) \in \mathcal{G}$. We prove that for any $(N, v, \underline{B}) \in \mathcal{GL}$, with $\underline{B} = \{B_0, \ldots, B_k\}$, $f^1(N, v, \underline{B}) = f^2(N, v, \underline{B})$ by induction on the number k of levels of $\underline{B}$. The case $k = 1$ is proved in . Hence suppose that $f^1(N, v, \underline{B}) = f^2(N, v, \underline{B})$ for any $(N, v, \underline{B}) \in \mathcal{GL}$ such that $|\underline{B}| \leq k$ and let $(N, v, \underline{B}) \in \mathcal{GL}$ such that $|\underline{B}| = k + 1$. Let $i \in U \in B_1$ be an arbitrary player that belongs to an arbitrary union of the first level. We prove that $f_i^1(N, v, \underline{B}) = f_i^2(N, v, \underline{B})$ by a second induction on $u = |U|$. If $u = 1$, i.e. $U = \{i\}$, since f^1 and f^2 satisfy SLGP, we have

$$f_i^1(N, v, \underline{B}) = f_{[U]}^1([B_1], v^1, \underline{B_1}) = f_{[U]}^2([B_1], v^1, \underline{B_1}) = f_i^2(N, v, \underline{B}),$$

where the second equality holds by the first induction hypothesis since $([B_1], \underline{B_1})$ is a levels structure with k levels. Now suppose that $f_l^1(N, v, \underline{B}) = f_l^2(N, v, \underline{B})$ for any $(N, v, \underline{B})$ such that $|\underline{B}| = k + 1$ and any $l \in U \in B_1$ where $|U| \leq u$. Next, suppose that $|U| = u + 1$ and let $j \in U \setminus i$. Since f^1 and f^2 satisfy LNID, we have

$$f_i^1(N, v, \underline{B}) = f_i^1(N, v, \underline{B^{-j}}) = f_i^2(N, v, \underline{B^{-j}}) = f_i^2(N, v, \underline{B}),$$

where the second equality holds by the second induction hypothesis since $i \in U \setminus j \in B_1^{-j} \in \underline{B^{-j}}$, $\underline{B^{-j}}$ has $k+1$ levels of cooperation, and $|U \setminus j| = u$, which concludes the proof. $\square$

Finally, we check that the proposed properties are independent axioms, and hence we cannot drop any of them from the characterizations. We start examining the properties used for the characterization of the Shapley levels value, Φ.

Remark 4.3. Independence of properties for Theorem 4.1

(i) The value on $\mathcal{GL}$, g, given by $g(N, v, \underline{B}) = 0$ for all $(N, v, \underline{B}) \in \mathcal{GL}$ satisfies SYM, EMC, LGP, LBC but not EFF.

(ii) Let g be the value on $\mathcal{GL}$ defined as follows:

- If $N = \{i, j\}$ and $\underline{B} = \{\{i\}, \{j\}\}$,

$$\begin{aligned} g_i(N, v, \underline{B}) &= \frac{3}{4}\left(v(N) - v(j)\right) + \frac{1}{4}v(i) \quad \text{and} \\ g_j(N, v, \underline{B}) &= \frac{1}{4}\left(v(N) - v(i)\right) + \frac{3}{4}v(j). \end{aligned}$$

- Otherwise, $g(N, v, \underline{B}) = \Phi(N, v, \underline{B})$.

  Thus, g satisfies EFF, EMC, LGP, LBC, but not SYM.

(iii) Consider the value on $\mathcal{GL}$, g, given by

$$g(N, v, \underline{B}) = \begin{cases} \Phi(N, v, \underline{B}) & \text{if } (N, v, \underline{B}) \notin \mathcal{C} \\ a_{i(N,v)} \mathbf{1}_{i(N,v)} & \text{if } (N, v, \underline{B}) \in \mathcal{C} \end{cases}$$

where

$\mathcal{C} = \{(N, v, \underline{B}) \in \mathcal{GL} : v = b_i \tau_i + (a_i - b_i)\delta_N$, for some $i = i(N, v) \in N$ and $0 \leq b_i < a_i\}$

such that, for every $S \subseteq N$, and $\mathbf{1}_k \in \mathbb{R}^n$ is such that $\mathbf{1}_k(l) = 1$ if $k = l$ and $\mathbf{1}_k(l) =$

$$\tau_i(S) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_N(S) = \begin{cases} 1 & \text{if } S = N \\ 0 & \text{otherwise} \end{cases},$$

0 if $k \neq l$. Then g satisfies EFF, SYM, LGP, LBC, but not EMC.

(iv) The value on $\mathcal{GL}$, g, given by $g(N, v, \underline{B}) = \phi(N, v)$ for all $(N, v, \underline{B}) \in \mathcal{GL}$ satisfies EFF, SYM, EMC, LBC, but not LGP.

(v) Let g be the value on $\mathcal{GL}$ defined as follows:

- If $N = \{i, j\}$ and $\underline{B} = \{\{\{i\}, \{j\}\}, N\}$, $g(N, v, \underline{B}) = (\frac{v(N)}{2}, \frac{v(N)}{2})$.
- Otherwise, $g(N, v, \underline{B}) = \Phi(N, v, \underline{B})$.

  Thus, g satisfies EFF, SYM, EMC, LGP but not LBC.

Lastly, we examine the properties used for the characterization of the Banzhaf levels value, Ψ.

Remark 4.4. Independence of axioms for Theorem 4.2

(i) The value on $\mathcal{GL}$, g, given by

$$g_i(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{|R|!(m_i - |R| - 1)!}{m_i!} \left(v(T_R \cup i) - v(T_R) \right),$$

satisfies DPP, SYM, EMC, SLGP, LNID but not 2-EFF.

(ii) The value on $\mathcal{GL}$, g, given by $g(N, v, \underline{B}) = 0$ for all $(N, v, \underline{B}) \in \mathcal{GL}$ satisfies 2-EFF, SYM, EMC, SLGP, LNID, but not DPP.

(iii) Let g be the value on $\mathcal{GL}$ defined as follows:

- If $N = \{i, j\}$ and $\underline{B} = \{\{i\}, \{j\}\}$,

$$\begin{aligned} g_i(N, v, \underline{B}) &= \frac{3}{4}\left(v(N) - v(j)\right) + \frac{1}{4}v(i) \quad \text{and} \\ g_j(N, v, \underline{B}) &= \frac{1}{4}\left(v(N) - v(i)\right) + \frac{3}{4}v(j). \end{aligned}$$

- Otherwise, $g(N, v, \underline{B}) = \Psi(N, v, \underline{B})$.

Thus, g satisfies 2-EFF, DPP, EMC, SLGP, LNID, but not SYM.

(iv) The value on $\mathcal{GL}$, g, given by

$$g(N, v, \underline{B}) = \begin{cases} \Psi(N, v, \underline{B}) & \text{if } (N, v, \underline{B}) \notin \mathcal{C} \\ 0 & \text{if } (N, v, \underline{B}) \in \mathcal{C} \end{cases}$$

where $\mathcal{C} = \{(N, v, \underline{B}) \in \mathcal{GL} : v = a_S \delta_S, \text{ for some } S \subseteq N\}$, satisfies 2-EFF, DPP, SYM, SLGP, LNID, but not EMC.

(v) The value on $\mathcal{GL}$, g, given by $g(N, v, \underline{B}) = \psi(N, v)$ for all $(N, v, \underline{B}) \in \mathcal{GL}$ satisfies 2-EFF, DPP, SYM, EMC, LNID, but not SLGP.

(vi) The value on $\mathcal{GL}$, g, given by

$$g_i(N, v, \underline{B}) = \sum_{R \subseteq M_i} \frac{1}{2^{m_i - |T_R \cap U_k|}} \cdot \frac{|T_R \cap U_k|!(|U_k \setminus T_R| - 1)!}{|U_k|!} \left(v(T_R \cup i) - v(T_R)\right),$$

satisfies 2-EFF, DPP, SYM, EMC, SLGP, but not LNID, where recall that U_k is the union of the k-th level to which player i belongs.

5 Conclusions

In the present paper we have proposed a new value for games with levels structure of cooperation and we have provided characterizations of this new value, the Banzhaf levels value, and the Shapley levels value. It should be pointed out that, in both theorems, the group of properties that apply only for the trivial levels structure can be replaced by any other group of properties that characterize either the Shapley or the Banzhaf value. The remaining properties, LGP and LBC in Theorem 4.1 and SLGP and LNID in Theorem 4.2, describe the behavior of the solutions with respect to the levels structure of cooperation. Moreover, since these latter properties are logically comparable, our paper serves in the purpose of deciding which value to use in a particular framework of restricted cooperation given by a sequence of union levels.

6 Appendix

Proof of the claim in the Proof of Theorem 4.1.

Let $(N, v, \underline{B}) \in \mathcal{GL}$ with $\underline{B} = \{B_0, \ldots, B_k\}$, $i, j \in U_1 \subseteq ... \subseteq U_k$ with $U_r \in B_r$ for each $r \in \{1, \cdots, k\}$, and $R \subseteq M_{ij}$. Let us define, for $r \in \{1, \cdots, k\}$,

$$\begin{aligned}
\lambda_R^r &= |\{\sigma \in \Omega_r(\underline{B}) : P_i^\sigma = T_R\}| + |\{\sigma \in \Omega_r(\underline{B}) : P_i^\sigma = T_R \cup j\}| \quad \text{, and} \\
\lambda_R^{-r} &= |\{\sigma \in \Omega_r(\underline{B^{-j}}) : P_i^\sigma = T_R\}| + |\{\sigma \in \Omega_r(\underline{B^{-j}}) : P_i^\sigma = T_R \cup j\}|.
\end{aligned}$$

Observe that $\lambda_R^1 = c_R^i + c_{R+j}^i$ and $\lambda_R^{-1} = c_R^{i,-j} + c_{R+j}^{i.-j}$. We prove that $\frac{\lambda_R^r}{|\Omega_r(\underline{B})|} = \frac{\lambda_R^{-r}}{|\Omega_r(\underline{B^{-j}})|}$ for all $r \in \{1, \cdots, k\}$ by backward induction on r. For each $r \in \{1, \cdots, k\}$, let $b_r = |B_r|$, $u_r = |U_r|$, $A^r = |\{U \in B_r \setminus U_r : U \subseteq U_{r+1} \text{ and } U \cap T_R = \emptyset\}|$, and $B^r = |\{U \in B_r \setminus U_r : U \subseteq U_{r+1} \text{ and } U \subseteq T_R\}|$. Recall that by convenience $U_{k+1} = N$. Observe that $A^k + B^k + 1 = b_k$ and that, for each $r \in \{1, \cdots, k\}$, $|U_r \cap T_R| + |U_r \setminus T_R| = u_r$.

We start proving the case $r = k$. Recall that $U_k \in B_k$ is such that $i, j \in U_k$. In particular, $i, j \in U_k \setminus T_R$ and thus $|U_k \setminus T_R| \geq 2$. By definition of λ_R^r,

$$\lambda_R^k = \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 1)!$$
$$+ \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R| + 1)! \cdot (|U_k \setminus T_R| - 2)!$$
$$= \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \cdot u_k.$$

Similarly, by definition of λ_R^{-k},

$$\lambda_R^{-k} = \prod_{S \in B_k \setminus U_k} |S|! \cdot (A^k + 1)! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)!$$
$$+ \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot (B^k + 1)! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)!$$
$$= \prod_{S \in B_k \setminus U_k} |S|! \cdot A^k! \cdot B^k! \cdot (|U_k \cap T_R|)! \cdot (|U_k \setminus T_R| - 2)! \cdot (b_k + 1).$$

Hence, for every $R \subseteq M_{ij}$, $\frac{\lambda_R^k}{\lambda_R^{-k}} = \frac{u_k}{b_k+1}$. To conclude with the first step of the induction one can easily check that $\frac{\Omega_k(\underline{B})}{\Omega_k(\underline{B^{-j}})} = \frac{u_k}{b_k+1}$.

Now suppose that for every $R \subseteq M_{ij}$, $\frac{|\Omega_{r+1}(\underline{B})|}{|\Omega_{r+1}(\underline{B^{-j}})|} = \frac{\lambda_R^{r+1}}{\lambda_R^{-,r+1}}$, for some $r \in \{2, \ldots, k\}$.

By definition of λ_R^k,

$$\frac{\lambda_R^r}{\lambda_R^{r+1}} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|!$$
$$\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 1)! + (|U_r \cap T_R| + 1)! \cdot (|U_r \setminus T_R| - 1)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 1)! + (|U_{r+1} \cap T_R| + 1)! \cdot (|U_{r+1} \setminus T_R| - 2)!}$$
$$= \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|!$$
$$\cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 2)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 2)!} \cdot \frac{u_{r+1}}{u_r},$$

where $h(S) = |\{S' \in B_r : S' \subseteq S\}|$ for each $S \in B_{r+1}$. Similarly, by definition of λ_R^{-k},

$$\frac{\lambda_R^{-,r}}{\lambda_R^{-,r+1}} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot A^r! \cdot B^r! \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq U_{r+1} \setminus U_r}} |S'|! \\ \cdot \frac{(|U_r \cap T_R|)! \cdot (|U_r \setminus T_R| - 2)!}{(|U_{r+1} \cap T_R|)! \cdot (|U_{r+1} \setminus T_R| - 2)!}.$$

Combining the two above expressions we obtain

$$\frac{\lambda_R^r}{\lambda_R^{-,r}} = \frac{\lambda_R^{r+1}}{\lambda_R^{-,r+1}} \cdot \frac{u_r}{u_{r+1}}. \tag{6}$$

Furthermore,

$$\frac{|\Omega_r(\underline{B})|}{|\Omega_{r+1}(\underline{B})|} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot \frac{h(U_{r+1})!}{u_{r+1}!} \cdot \left(\prod_{\substack{S' \in B_r \setminus U_r \\ S' \subseteq U_{r+1}}} |S'|! \right) \cdot u_r!,$$

and

$$\frac{|\Omega_r(\underline{B^{-j}})|}{|\Omega_{r+1}(\underline{B^{-j}})|} = \prod_{S \in B_{r+1} \setminus U_{r+1}} \left(\frac{h(S)!}{|S|!} \cdot \prod_{\substack{S' \in B_r \\ S' \subseteq S}} |S'|! \right) \cdot \frac{h(U_{r+1})!}{(u_{r+1} - 1)!} \cdot \left(\prod_{\substack{S' \in B_r \setminus U_r \\ S' \subseteq U_{r+1}}} |S'|! \right) \cdot (u_r - 1)!.$$

Thus

$$\frac{|\Omega_r(\underline{B})|}{|\Omega_r(\underline{B^{-j}})|} = \frac{|\Omega_{r+1}(\underline{B})|}{|\Omega_{r+1}(\underline{B^{-j}})|} \cdot \frac{u_r}{u_{r+1}}. \tag{7}$$

Hence, from eq. (6) and (7), using the induction hypothesis we obtain

$$\frac{\lambda_R^r}{|\Omega_r(\underline{B})|} = \frac{\lambda_R^{-,r}}{|\Omega_r(\underline{B^{-j}})|},$$

which concludes the proof. $\square$

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