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Acyclic Classes of Nonrepresentable Homologies

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Introduction

A homology theory is a collection of functors that assign to each topological space a collection of abelian groups and satisfy certain axioms. Since one of these axioms is homotopy invariance, homology theories are useful tools to study the homotopy type of spaces. This project is focused on homology theories defined over CW-complexes, as customary. Most of the authors restrict to a smaller class of homology theories: the representable homology theories. Representable homology theories have many nice properties, one of which is that they commute with directed colimits.

Sometimes it is useful to study for which spaces does a given homology theory vanish, or in other words, which are the acyclic spaces of the homology theory. Thus, one can give the following definition: two homology theories are Bousfield equivalent if they have exactly the same acyclic spaces. This is an equivalence relation between homology theories. Tetsusuke Ohkawa proved in [10] that the collection $\mathbf{B}$ of representable, i.e., having a representable representative, Bousfield classes is a set (thus, not a proper class). The proof used in an essential way that representable homology theories commute with directed colimits. This made us suspect that if we remove the representability hypothesis, what we obtain is a proper class. Moreover, it is natural to expect that we still have a set of Bousfield classes if we replace the condition "commute with directed colimits" by a weaker condition, say "commute with α-directed colimits" for a given cardinal α. In our project, we answer both questions affirmatively, i.e., we prove that:

1. There is a proper class of Bousfield classes of nonrepresentable homologies.
2. If α is a regular cardinal, then the collection $\mathbf{B}_\alpha$ of Bousfield classes of homologies that commute with α-directed colimits is a set.

In the same direction, there naturally arise some (still unanswered) questions:

1. Does there exist a homology theory that does not commute with α-directed colimits for any cardinal α?
2. Which is the cardinality of $\mathbf{B}_\alpha$?

For the representable case (i.e., with $\alpha = \aleph_0$), a bound is given in [5]: the cardinality of $\mathbf{B}$ lies between 2^α and 2^{2^α}. It has been an open problem since then to determine the precise cardinality of $\mathbf{B}$, so it would probably be a more tractable problem to give bounds of the cardinality of $\mathbf{B}_\alpha$ rather than determining its exact cardinality.

Such interactions between algebraic topology and set theory are becoming increasingly relevant in current research. Several problems in algebraic topology have been found to depend on set theoretical concepts and results.

Description of contents

This project is divided into three chapters.

The first chapter is an introduction and motivation for the second and third chapters. We state a collection of well-known results and definitions. First, we introduce the notion of a homology theory and we give some basic examples of homology theories. We recall the dimension axiom, and the coefficient groups of a homology theory. Then we state the Uniqueness Theorem, which says that if there is a natural transformation between two homology theories giving an isomorphism in the coefficient groups, then it yields an isomorphism for any pair of finite CW-complexes.

Later, we introduce a concept equivalent to a homology theory: a reduced homology theory. This concept provides, in fact, a simplification of many statements on homology theory, and, like unreduced homology, reduced homology is characterized by a (different) collection of axioms. This is why, in the work, we assume every homology theory to be reduced. We discuss the Milnor axiom, which imposes on a homology theory the property of preserving arbitrary coproducts, and call a theory additive if it satisfies this axiom. We motivate the use of this axiom by saying that, assuming it as a hypothesis in the Uniqueness Theorem, the natural transformation yields an isomorphism in homology for every (not necessarily finite) pair of CW-complexes. Also, we have an application of the Uniqueness Theorem: if a homology theory h_* satisfies the Milnor axiom and the dimension axiom then there exists a natural equivalence between h_* and ordinary homology $H(\cdot;G)$ with coefficients in some group G.

Later on, we talk about the homology theories that are representable by a spectrum and. We state a theorem that says representable is equivalent to additive. Then, we define Bousfield classes of a homology theory h_*, denoted by $\langle h_*\rangle$, and give and give some examples related to Bousfield classes. Finally, we state Ohkawa's Theorem.

The topic of the second chapter is nonrepresentable homology theories. We start by recalling the first studied example of a nonrepresentable homology theory, which is defined using singular homology theory. We define the support of a sequence and a new concept, which is a key point in proving that there is a proper class of nonrepresentable homology theories: the content of a sequence of elements in a set. Given a cardinal α and a group A, the group A^α of sequences of α elements in A has two subgroups: $A^{<\alpha}$, the sequences whose support has cardinality $< \alpha$, and $A^{\prec\alpha}$, the sequences whose content has cardinality $< \alpha$. Whereas the support tells you in which positions the elements of a sequence are nonzero, the content is the union of all the elements in the sequence. We give a result about the exactness of some induced sequences, which allows us to prove that some constructions, such as the direct sum, the product, or the product quotiented by the direct sum of a family of homology theories is another homology theory. Also this result proves that given a homology theory h_*, the functors

$$h^1_n(X) = (h_n(X))^\alpha/(h_n(X))^{<\alpha}, \quad h^2_n(X) = (h_n(X))^\alpha/(h_n(X))^{\prec\alpha}, \quad n \in \mathbb{Z},$$

define new homology theories. After this, we answer two questions that arose while we were trying to prove that the collection of nonrepresentable Bousfield classes is a proper class:

1. Let h_* and g_* be two Bousfield equivalent homology theories. Is h_* representable if and only if g_* is representable?

2. Does there exist a Bousfield class $\langle h_*\rangle$ all whose representatives are nonrepresentable?

The first question is answered negatively and the second one affirmatively.

To conclude the second chapter, prove that the collection of nonrepresentable Bousfield classes is a proper class. The proof is structured in the same order as we discovered it. First suppose that for each cardinal α we have a homology theory h^α_* such that the CW-complexes with less than α cells are h^α_*-acyclic and there is a CW-complex with α cells which is not h^α_*-acyclic. Then, for cardinals $\alpha \neq \beta$, $\langle h^\alpha_*\rangle \neq \langle h^\beta_*\rangle$, so we obtain a proper class of distinct Bousfield classes.

Finding a homology theory h^α_* satisfying such conditions is not an easy problem, due to the restrictive axiomatic of the definition of a homology theory. However, we noticed that composing a homology theory with an exact functor $\mathbf{Ab} \to \mathbf{Ab}$ from the category of abelian groups to the category of abelian groups yields another homology theory. So, if we find a functor $F_\alpha\colon \mathbf{Ab} \to \mathbf{Ab}$ such that $F_\alpha A = 0$ if A has cardinality $< \alpha$ and $F_\alpha(\bigoplus_{i<\alpha}\mathbb{Z}) \neq 0$, then the composition $F_\alpha \circ \tilde{H}_*$, where H_* is singular homology, will be our h^α_* satisfying the desired properties. The functor that satisfies the desired properties is:

$$\begin{aligned} F_\alpha\colon \mathbf{Ab} &\longrightarrow \mathbf{Ab} \\ A &\longmapsto A^\alpha/A^{\prec\alpha}. \end{aligned}$$

This concludes the proof of the theorem. We are indebted to Fernando Muro for having suggested us this approach. The given proof prompts some questions:

1. Does the homology theory h^α_* commute with α-directed colimits?

2. Let $\mathbf{H}$ be a subclass of the class of all homology theories. How many homology theories can be reached by the composition of an exact functor $\mathbf{Ab} \to \mathbf{Ab}$ with an element of $\mathbf{H}$?

3. For which subclasses $\mathbf{H}$ the collection of all composites of a functor $\mathbf{Ab} \to \mathbf{Ab}$ with an element of $\mathbf{H}$ is the whole class of homology theories?

The third chapter consists of the statement and proof the fact that there is a set of Bousfield classes of homology theories that commute with α-directed colimits, where α is a regular cardinal. What we do in this chapter is generalizing the proof given in [5] to any regular cardinal.

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Chapter 1

Preliminaries

Although algebraic topology studies topological spaces, it is often convenient to focus on CW-complexes or simplicial sets, because the category of topological spaces is too general for many purposes. CW-complexes (also called cellular complexes) are topological spaces which are constructed by adjoining cells (balls of any dimension). The precise definition can be found in [7]. Singular homology defined over CW-complexes (or simplicial sets) is especially suited for calculations.

1.1 Unreduced homology

In order to study homotopy types of spaces (which will be CW-complexes in this work), we consider homotopy invariant functors taking values in the category **Ab** of abelian groups.

Notations 1.1.1.

- We denote by $\mathcal{C}$ the category of CW-complexes and by $\mathcal{C}^2$ the category of pairs (X, A) where X is a CW-complex and A is a subcomplex of X. We identify $X \in \mathcal{C}$ with $(X, \varnothing) \in \mathcal{C}^2$.
- If a collection of functors $h_n : \mathcal{C}^2 \to \mathbf{Ab}$ is given for $n \in \mathbb{Z}$, we write

$$h_*(X) = \bigoplus_{n \in \mathbb{Z}} h_n(X)$$

  and we view it as a graded abelian group. That is, $h_*(X) \cong g_*(X)$ means $h_n(X) \cong g_n(X)$ for each $n \in \mathbb{Z}$.
- We denote by $*$ the one-point space.
- If X is a CW-complex, when we talk about its cardinality we will mean the cardinality of the set of its cells. Thus, a finite CW-complex is a CW-complex with a finite number of cells.

We recall the following basic concept from [6].

Definition 1.1.2. A *homology theory* is a collection of functors $h_n\colon \mathcal{C}^2 \to \mathbf{Ab}$, $n \in \mathbb{Z}$ satisfying the following properties:

1. Homotopy invariance. Two maps $f\colon (X,A) \to (Y,B)$ satisfy $h_*(f) = h_*(g)$ whenever $f \simeq g$.

2. Exactness. For every pair $(X,A) \in \mathcal{C}^2$, there exist connecting morphisms $\partial : h_{n+1}(X,A) \to h_n(A)$ such that

$$\cdots \to h_{n+1}(X,A) \xrightarrow{\partial} h_n(A) \to h_n(X) \to h_n(X,A) \xrightarrow{\partial} \cdots$$

is a long exact sequence, where the unlabelled arrows are the morphisms induced by the inclusions $(A,\emptyset) \hookrightarrow (X,\emptyset) \hookrightarrow (X,A)$.

3. Excision. Let $(X,A) \in \mathcal{C}^2$ and $U \subseteq X$ open such that the closure of U is contained in the interior of A. Then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X,A)$ induces for all n isomorphisms

$$h_n(X \setminus U, A \setminus U) \cong h_n(X,A).$$

Examples 1.1.3. 1. Singular homology H_* is a homology theory and, for each abelian group G, homology $H_*(\cdot, G)$ with coefficients in G is a homology theory too.

2. The functor that sends every space to 0 for all n is a homology theory.

This definition of homology theory was first given by Samuel Eilenberg and Norman Steenrod in [6]. Their axiomatic approach generalized singular homology. In their definition, they gave an additional axiom, the dimension axiom:

Axiom 1.1.4. A homology theory h_* satisfies the dimension axiom if $h_n(*) = 0$ for $n \neq 0$.

The dimension axiom is omitted from the definition of a homology theory because there are many important examples that do not satisfy it, such as K-theory or cobordism.

Definition 1.1.5. The abelian groups $h_n(*)$, $n \in \mathbb{Z}$ are called the *coefficients* of the homology theory h_*. The dimension axiom can be restated by saying that the coefficient groups are concentrated in degree 0.

The following theorem can be found in [12].

Theorem 1.1.6 (Uniqueness)**.** *Let h_* and g_* be two homology theories such that:*

1. *There is a natural transformation $\eta\colon h_* \to g_*$, that is, a natural transformation $h_n \to g_n$ for each $n \in \mathbb{Z}$ compatible with the connecting morphisms ∂.*

2. *The natural transformation η is an isomorphism in the coefficients*

$$h_*(*) \cong g_*(*).$$

Then η yields an isomorphism $h_*(X,A) \cong g_*(X,A)$ for each pair (X,A) of finite CW-complexes.

1.2 Reduced homology

The *reduced homology* $\tilde{h}_*$ associated with a homology theory h_* is defined as

$$\tilde{h}_n(X) = \ker h_n(p\colon X \to *) \quad \forall n \in \mathbb{Z},$$

where $p\colon X \to *$ is the only map to the one-point space.

Example 1.2.1. The reduced homology of singular homology takes the same values as the unreduced one except for $n = 0$, where $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

Note that a homology theory h_* is a collection of functors $\mathcal{C}^2 \to \mathbf{Ab}$, whereas its reduced homology $\tilde{h}_*$ is more naturally viewed as a collection of functors $\mathcal{C}_0 \to \mathbf{Ab}$, where $\mathcal{C}_0$ is the category of pointed CW-complexes, i.e., pairs (X, x_0) with $x_0 \in X$. Furthermore one can check that $\tilde{h}_*$ satisfies an alternative collection of axioms (see [13]).

1. Homotopy invariance.
2. Exactness. The inclusion $i\colon A \hookrightarrow X$ and the quotient map $q\colon X \to X/A$ induce morphisms $\partial\colon \tilde{h}_{n+1}(X/A) \to \tilde{h}_n(A)$ that yield a long exact sequence
$$\cdots \to \tilde{h}_{n+1}(X/A) \xrightarrow{\partial} \tilde{h}_n(A) \to \tilde{h}_n(X) \to \tilde{h}_n(X/A) \xrightarrow{\partial} \cdots .$$
3. Suspension. For each $n \in \mathbb{Z}$ and each CW-complex X there is a natural isomorphism
$$\tilde{h}_n(X) \cong \tilde{h}_{n+1}(\Sigma X),$$
where ΣX is the reduced suspension of X.

A collection of functors $\mathcal{C}_0 \to \mathbf{Ab}$ satisfying these three axioms is called a *reduced homology theory*.

If $\tilde{h}_*$ is a reduced homology theory, then one can check that if we set

$$\begin{aligned} h_*(X) &= \tilde{h}_*(X \sqcup *), \quad \text{where } \sqcup \text{ denotes the disjoint union,} \\ h_*(X, A) &= \tilde{h}_*(X/A), \quad \text{if } A \neq \varnothing, \end{aligned}$$

what we get is an (unreduced) homology theory. This is the inverse process of taking the reduced homology of a homology theory. So, homology theories and reduced homology theories are essentially equivalent.

For each (unreduced) homology theory h_*, we have

$$h_*(*) \cong \tilde{h}_*(S^0).$$

Thus, the coefficients of a reduced homology theory are defined to be $h_n(S^0)$, $n \in \mathbb{Z}$.

The main advantage of working with reduced homology instead of unreduced homology is that reduced homology leads to simpler formulations of definitions and theorems. Also, the axioms that define reduced homology are easier to manipulate.

For simplicity, from now on we will work with reduced homology. Because of this, when we say "homology theory" we will mean "reduced homology theory". Also, we will remove the tilde from the notation of a reduced homology theory.

1.3 The Milnor axiom

The Uniqueness Theorem 1.1.6 holds for all reduced homology theories and all CW-complexes not necessarily finite, if a new axiom is added to the definition of a homology theory. With the given axioms we cannot deduce that if

$$h_*(X) \cong g_*(X)$$

for each finite CW-complex X, then we have got an isomorphism $h_*(X) \cong g_*(X)$ for each CW-complex. This fact will be true if we impose an additional axiom: the Milnor axiom, or Additivity axiom (see [9] for the proof in the unreduced case).

Axiom 1.3.1. For each wedge sum $X = \bigvee_{i \in I} X_i$ the inclusions $X_i \hookrightarrow X$ yield an isomorphism

$$h_* \left(\bigvee_{i \in I} X_i \right) \cong \bigoplus_{i \in I} h_*(X_i);$$

in other words, h_* preserves coproducts.

This axiom was first introduced in [9] for unreduced homology theories. The unreduced version of this axiom is similar using disjoint unions instead of wedge sums.

Remark 1.3.2. The Milnor axiom for finite wedge sums can be deduced from the other axioms, but when the wedge sum is arbitrary then we cannot. An example of this fact is given in the beginning of the next chapter.

All the important examples of homology theories such as ordinary homology $H_*(\cdot; G)$, K-theory or cobordism turn out to satisfy the Milnor axiom. Thus, many authors decide to include this axiom in the definition of a homology theory. For us, if a homology theory satisfies this axiom, it will be called *representable* or *additive*. The next section justifies the word "representable".

The following theorem, which can be found in [9] for unreduced homologies, is a consequence of the Uniqueness Theorem 1.1.6.

Theorem 1.3.3. *If a homology theory h_* satisfies the dimension axiom and the Milnor axiom, then there exists a group G and a natural equivalence*

$$\eta\colon h_* \longrightarrow \tilde{H}_*(\cdot; G).$$

Note that, in the Uniqueness theorem, the natural transformation is part of the hypotheses, whereas in this theorem the natural equivalence is part of the conclusions. The Milnor axiom is a key point in the hypotheses. If we remove this axiom from the hypotheses, then the theorem is not true anymore. In Example 2.2.3 from Chapter 2, we give an example of a nonadditive homology theory satisfying the dimension axiom and not naturally equivalent to ordinary homology with coefficients. That theorem is another justification of the use of the Milnor axiom.

1.4 Representability

For each spectrum X we can define an additive homology theory h_* by setting $h_n(Y) = \pi_n(Y \wedge X)$, where π_n is the n-th homotopy group and $Y \wedge X$ is the smash product of Y and X (see [1] for details). Then we say that X *represents* h_*, or we say that h_* is *representable*. Every representable homology theory is additive. The converse is true (see [11]):

Theorem 1.4.1 (Adams representability)**.** *If a homology theory is additive, then it is representable by a spectrum.*

1.5 Bousfield classes

The following concepts were introduced in [2] only for spectra (i.e., for additive homologies). We generalize the definitions to nonrepresentable homologies.

Definition 1.5.1. Let h_* be a homology theory.

1. A space X is called h_**-acyclic* if $h_n(X) = 0$ for all n.

2. Two homology theories h_* and g_* are *Bousfield equivalent* if they have the same acyclic spaces. This is an equivalence relation and yields the so-called *Bousfield classes*, which are denoted by $\langle h_* \rangle$.

We have defined a Bousfield class as an equivalence class of homology theories. On the other hand, we could have also defined the equivalence class of a homology theory h_* to be the class of h_*-acyclic spaces

$$\langle h_* \rangle = \{X \mid h_*(X) = 0\}.$$

Both points of view of Bousfield classes will be used.

Remark 1.5.2. Let h_* be a homology theory. In general, there are no conditions characterizing the collection of h_*-acyclic spaces, even for the most studied theories such as singular homology or K-theory. Using the the exactness axiom with $X = *$ and $A = *$, one can conclude that for every homology theory h_*, the one-point space is h_*-acyclic. However, there are many more h_*-acyclic spaces than the contractible ones (even for singular homology).

Example 1.5.3. Let $h_* = 0$ be the zero homology theory. Then $\langle h_* \rangle$ is the collection of all CW-complexes.

Example 1.5.4. Bousfield equivalence does not imply natural equivalence. Let p be a prime number. Then we have that $\tilde{H}_*(\cdot, \mathbb{Z}/p)$ and $\tilde{H}_*(\cdot, \mathbb{Z}/p^2)$ are not naturally equivalent but

$$\langle \tilde{H}_*(\cdot, \mathbb{Z}/p) \rangle = \langle \tilde{H}_*(\cdot, \mathbb{Z}/p^2) \rangle.$$

In [3], Bousfield and Kan determined when $\langle \tilde{H}_*(X; A) \rangle = \langle \tilde{H}_*(X; B) \rangle$ for abelian groups A and B. According to their result, Bousfield classes of ordinary homology theories are classified by sets of primes.

1.6 Ohkawa's Theorem

The following theorem is due to Tetsusuke Ohkawa.

Theorem 1.6.1 (Ohkawa). *The collection of Bousfield classes $\langle h_* \rangle$, where $\langle h_* \rangle$ is additive form a set.*

Ohkawa this theorem in [10] for homologies defined over spectra. His proof was not easy, so ten years later William G. Dwyer and John H. Palmieri gave a simplification of the proof in [5]; in their article, they also proved that the cardinality of the set of representable Bousfield classes was bounded between $2^{\aleph_0}$ and $2^{2^{\aleph_0}}$. After ten years, Carles Casacuberta, Javier J. Gutiérrez and Jiri Rosický gave in [4] a generalization of the theorem to any combinatorial model category.

Chapter 2

Nonrepresentable homology theories

2.1 Examples

The first example of a nonrepresentable homology theory was studied by James and Whitehead in [8].

Define a collection of functors $\mathcal{C}_0 \longrightarrow \mathbf{Ab}$, $k \in \mathbb{Z}$, as follows:

$$\mathrm{JW}_k(X) = \frac{\prod_{i=0}^{\infty} \tilde{H}_i(X)}{\bigoplus_{i=0}^{\infty} \tilde{H}_i(X)}.$$

These functors define in fact a homology theory. Homotopy invariance is obvious, since we are using homotopy invariant objects to define it. Suspension is also trivial because if $A_i \cong B_i$, $i \in \mathbb{N}$, then $\frac{\prod_{i\in\mathbb{N}} A_i}{\bigoplus_{i\in\mathbb{N}} A_i} \cong \frac{\prod_{i\in\mathbb{N}} B_i}{\bigoplus_{i\in\mathbb{N}} B_i}$. We apply Proposition 2.1.2 to the collection of sequences

$$B_*^i = \cdots \to H_{n+1+i}(X,A) \to H_{n+i}(A) \to H_{n+i}(X) \to H_{n+i}(X,A) \to \ldots,$$

for $i \in \mathbb{Z}$, to deduce the exactness axiom.

Now we check that this functor does not satisfy the Milnor axiom. Consider $X = \bigvee_{j=0}^{\infty} S^j$. Then

$$\mathrm{JW}_k(X) = \frac{\prod_{i=0}^{\infty} \tilde{H}_i(X)}{\bigoplus_{i=0}^{\infty} \tilde{H}_i(X)} = \frac{\prod_{i=0}^{\infty} \bigoplus_{j=0}^{\infty} \tilde{H}_i(S^j)}{\bigoplus_{i=0}^{\infty} \bigoplus_{j=0}^{\infty} \tilde{H}_i(S^j)} = \frac{\prod_{i=0}^{\infty} \mathbb{Z}}{\bigoplus_{i=0}^{\infty} \mathbb{Z}}$$

while

$$\mathrm{JW}_k(S^j) = \frac{\prod_{i=0}^{\infty} \tilde{H}_i(S^j)}{\bigoplus_{i=0}^{\infty} \tilde{H}_i(S^j)} = \mathbb{Z}/\mathbb{Z} = 0,$$

so that JW_* does not preserve coproducts:

$$\mathrm{JW}_k\left(\bigvee_{j=0}^{\infty} S^j\right) = \frac{\prod_{i=0}^{\infty} \mathbb{Z}}{\bigoplus_{i=0}^{\infty} \mathbb{Z}} \neq 0 = \bigoplus_{j=0}^{\infty} \mathrm{JW}_k(S^j).$$

For a sequence of elements $(a_i)_{i\in I}$ contained in a group G, we define its *support* and its *content* as follows:

$$\mathrm{supp}(a_i) = \{i \in I \mid a_i \neq 0\}, \quad \mathrm{cont}(a_i) = \bigcup_{i\in I}\{a_i\}.$$

Example 2.1.1. Let $(1, 1, 1, \dots) = (1)_{n\in\mathbb{N}}$ be a sequence of integers. Then

$$\mathrm{supp}(1)_{n\in\mathbb{N}} = \mathbb{N}, \quad \mathrm{cont}(1)_{n\in\mathbb{N}} = \{1\}.$$

Given a cardinal α and an abelian group A, define

$$A^\alpha = \prod_{i<\alpha} A,$$

$$A^{<\alpha} = \{(a_i) \in A^\alpha \mid \#\mathrm{supp}(a_i) < \alpha\},$$
$$A^{\prec\alpha} = \{(a_i) \in A^\alpha \mid \#\mathrm{cont}(a_i) < \alpha\}.$$

Observe that both $A^{<\alpha}$ and $A^{\prec\alpha}$ are subgroups of A^α. Indeed, if two elements $(x_i)_i$ and $(y_i)_i$ of A^α have support (resp. content) of cardinality $< \alpha$, then their sum $(x_i + y_i)_i$ also has support (resp. content) of cardinality $< \alpha$. Also observe that for an element $(x_i)_i \in A^\alpha$, its content has cardinality less than or equal to its support. Hence, $A^{\prec\alpha}$ is a subgroup of $A^{<\alpha}$.

Proposition 2.1.2.

1. *Given a collection of short exact sequences*

$$0 \to A_i \to B_i \to C_i \to 0,$$

$i \in I$, the induced sequences

$$0 \to \prod_{i\in I} A_i \to \prod_{i\in I} B_i \to \prod_{i\in I} C_i \to 0,$$

$$0 \to \bigoplus_{i\in I} A_i \to \bigoplus_{i\in I} B_i \to \bigoplus_{i\in I} C_i \to 0,$$

$$0 \to \frac{\prod_{i\in I} A_i}{\bigoplus_{i\in I} A_i} \to \frac{\prod_{i\in I} B_i}{\bigoplus_{i\in I} B_i} \to \frac{\prod_{i\in I} C_i}{\bigoplus_{i\in I} C_i} \to 0,$$

are exact.

2. *Given a cardinal α and an exact sequence $0 \to A \to B \to C \to 0$, the induced sequences*

$$0 \to A^\alpha/A^{<\alpha} \to B^\alpha/B^{<\alpha} \to C^\alpha/C^{<\alpha} \to 0,$$
$$0 \to A^\alpha/A^{\prec\alpha} \to B^\alpha/B^{\prec\alpha} \to C^\alpha/C^{\prec\alpha} \to 0$$

are exact.

Proof. 1) For a collection of group morphisms $f_i : G_i \longrightarrow H_i$, $i \in J$, we define

$$\begin{aligned} F_1 : \prod_{i\in J} G_i &\longrightarrow \prod_{i\in J} H_i \\ (g_i)_i &\longmapsto (f_i(g_i))_i. \end{aligned}$$

Observe that F_1 takes $\bigoplus_{i\in J} G_i$ into $\bigoplus_{i\in J} H_i$. Denote by F_2 the restriction of F_1 to $\bigoplus_{i\in J} G_i$. Then clearly $\ker F_1 = \prod_{i\in J} \ker f_i$, $\operatorname{im} F_1 = \prod_{i\in J} \operatorname{im} f_i$ and $\ker F_2 = \bigoplus_{i\in J} \ker f_i$, $\operatorname{im} F_1 = \bigoplus_{i\in J} \operatorname{im} f_i$. Applying these remarks and the exactness of each sequence $0 \to A_i \to B_i \to C_i \to 0$, we conclude that the first two sequences are exact.

For the third sequence, observe that the reduction F of the map F_1 to the quotient is well-defined. If $0 = F(\overline{(g_i)_i}) = \overline{(f_i(g_i))_i}$, then $\overline{(g_i)_i} = \overline{(g_i')_i}$, where

$$g_i' =: \begin{cases} g_i & \text{if } f_i(g_i) = 0, \\ 0 & \text{if } f_i(g_i) \neq 0. \end{cases}$$

Therefore,

$$\ker F = \frac{\prod_i \ker f_i}{\bigoplus_i G_i}.$$

Also,

$$\operatorname{im} F = \{\overline{(f_i(g_i))_i} \mid g_i \in G_i\} = \frac{\prod_i \operatorname{im} f_i}{\bigoplus_i H_i}.$$

Again, we apply these remarks and the exactness of each sequence to prove that the third sequence is exact.

2) Let $f : G \longrightarrow H$ be a group morphism. Define another group morphism

$$\begin{aligned} F_1 \colon G^\alpha &\longrightarrow H^\alpha \\ (g_i)_i &\longmapsto (f(g_i))_i. \end{aligned}$$

Note that F_1 takes $G^{<\alpha}$ to $H^{<\alpha}$ and $G^{\prec\alpha}$ to $H^{\prec\alpha}$. Therefore the reductions F and F' of the morphism F_1 to the quotients $A^\alpha/A^{<\alpha}$ and $A^\alpha/A^{\prec\alpha}$ are well-defined.

If $0 = F(\overline{(g_i)_i}) = \overline{(f(g_i))_i}$ in $A^\alpha/A^{<\alpha}$ (resp. in $A^\alpha/A^{\prec\alpha}$), then $\overline{(g_i)_i} = \overline{(g_i')_i}$ in $A^\alpha/A^{<\alpha}$ (resp. in $A^\alpha/A^{\prec\alpha}$), where

$$g_i' =: \begin{cases} g_i & \text{if } f(g_i) = 0, \\ 0 & \text{if } f(g_i) \neq 0. \end{cases}$$

Therefore,

$$\ker F = \frac{(\ker f)^\alpha}{G^{<\alpha}} \quad \text{and} \quad \ker F' = \frac{(\ker f)^\alpha}{G^{\prec\alpha}}.$$

Also,

$$\operatorname{im} F = \frac{(\operatorname{im} f)^\alpha}{H^{<\alpha}} \quad \text{and} \quad \operatorname{im} F' = \frac{(\operatorname{im} f)^\alpha}{H^{\prec\alpha}}.$$

These remarks and the exactness of the sequence $0 \to A \to B \to C \to 0$ prove the exactness of the two sequences. $\square$

Remark 2.1.3. The proposition we have just proved allows us to construct new homology theories from given ones:

- For a collection of homology theories $\{h_*^i\}_{i\in I}$, we can define new homology theories:
$$H_n^1(X)=\prod_{i\in I} h_n^i(X),\quad H_n^2(X)=\bigoplus_{i\in I} h_n^i(X),\quad H_n^3(X)=\frac{\prod_{i\in I} h_n(X)}{\bigoplus_{i\in I} h_n(X)^i}.$$
- Given a cardinal α and a homology theory h_*, we can define new homology theories
$$h_n^1(X)=(h_n(X))^\alpha/(h_n(X))^{<\alpha},\quad h_n^2(X)=(h_n(X))^\alpha/(h_n(X))^{\prec\alpha}.$$

Indeed, applying proposition 2.1.2, all these new homology theories satisfy the exactness axiom. The other axioms are obviously satisfied.

2.2 A proper class of nonrepresentable Bousfield classes

One of the aims of this project was to prove that the collection of Bousfield classes of nonrepresentable homologies forms a proper class. Around this problem, two questions arose:

Question 2.2.1. Let h_* and g_* be two Bousfield equivalent homology theories. Is it true that h_* is representable if and only if g_* is representable?

Question 2.2.2. Are there Bousfield classes $\langle h_*\rangle$ such that there is no representable homology theory g_* such that $\langle h_*\rangle=\langle g_*\rangle$?

The answer of the first question is no. Next we give a counterexample.

Example 2.2.3. Define $h_*=\tilde{H}_*$ and $g_*=\tilde{H}_*\oplus \mathrm{JW}_*$, where H_* denotes singular homology. First observe that the acyclic spaces of JW_* are
$$\langle\tilde{H}_*\rangle\cup\{X\mid \tilde{H}_k(X)\neq 0 \text{ for at most a finite number of } k\in\mathbb{N}\}.$$
Now,
$$\langle g_*\rangle=\langle\tilde{H}_*\rangle\cap\langle \mathrm{JW}_*\rangle=\langle\tilde{H}_*\rangle=\langle h_*\rangle.$$
The theory h_* is representable. However,
$$\begin{aligned} g_1\left(\bigvee_{j\in\mathbb{N}} S^j\right) &= H_1\left(\bigvee_{j\in\mathbb{N}} S^j\right)\oplus \mathrm{JW}_1\left(\bigvee_{j\in\mathbb{N}} S^j\right)\\ &=\bigoplus_{j\in\mathbb{N}} H_1(S^j)\oplus\frac{\prod_{j\in\mathbb{N}}\mathbb{Z}}{\bigoplus_{j\in\mathbb{N}}\mathbb{Z}}\\ &=\mathbb{Z}\oplus\frac{\prod_{j\in\mathbb{N}}\mathbb{Z}}{\bigoplus_{j\in\mathbb{N}}\mathbb{Z}}\end{aligned}$$

and

$$g_1(S^j) = H_1(S^j) \oplus \mathrm{JW}_1(S^j) = \begin{cases} \mathbb{Z} & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\bigoplus_{j\in\mathbb{N}} g_1(S^j) = \mathbb{Z} \ncong g_1\left(\bigvee_{j\in\mathbb{N}} S^j\right),$$

which proves that g_* is nonrepresentable.

The answer of the second question is yes. We have got two proofs of this fact:

1. An example is provided by JW_*. Indeed, if we had $\langle \mathrm{JW}_* \rangle = \langle g_* \rangle$, for a representable homology theory g_*, then the collection of JW_*-acyclic spaces would be closed under arbitrary wedge sums. But S^i is JW_*-acyclic for $i \in \mathbb{N}$ and $\bigvee_{i\in\mathbb{N}} S^i$ is not JW_*-acyclic.

2. We can obtain it as a corollary of the theorem proved in the next section: there is a proper class of distinct Bousfield classes of nonrepresentable homology theories. Combining this theorem with Ohkawa's theorem, we answer affirmatively the question.

Our main result in this chapter is the following.

Theorem 2.2.4. *There is a proper class of Bousfield classes of nonrepresentable homology theories.*

This theorem is a consequence of the following result:

Theorem 2.2.5. *For each infinite cardinal α there is a reduced homology theory h_* such that:*

1. *The CW-complexes of cardinality $< \alpha$ are h_*-acyclic.*

2. *There exists a CW-complex X of cardinality α which is not h_*-acyclic.*

Proof of Theorem 2.2.4. If 2.2.5 is true, we can find a collection of pairwise nonequivalent homology theories indexed by the collection of all cardinals. Since the collection of all cardinals form a proper class, we find a proper class of Bousfield classes of nonrepresentable homology theories. □

Theorem 2.2.5 will arise as a corollary of the next two statements:

Theorem 2.2.6. *For each infinite cardinal α there exists an exact functor F_α :* **Ab** $\longrightarrow$ **Ab** *such that*

1. $F_\alpha A = 0$ *if* $\#A < \alpha$.

2. $F_\alpha(\bigoplus_{i<\alpha} \mathbb{Z}) \neq 0$.

Theorem 2.2.7. *For every homology theory h_* and every exact functor*

$$F\colon \mathbf{Ab} \to \mathbf{Ab},$$

the composition $F \circ h_*$ is another homology theory.

Proof of Theorem 2.2.5. Given an infinite cardinal α, define

$$h_* := F_\alpha \circ \tilde{H}_*.$$

By Theorem 2.2.7, this is a homology theory.

If a CW-complex X has $\gamma < \alpha$ cells, since $H_*(X)$ is computed using cellular homology, it is equal to the quotient of a subgroup of $\bigoplus_{i<\gamma} \mathbb{Z}$ by a smaller subgroup. Since $\gamma < \alpha$ and α is infinite, $\bigoplus_{i<\gamma} \mathbb{Z}$ has cardinality $< \alpha$, and hence

$$\#\tilde{H}_*(X) \leq \#H_*(X) < \alpha,$$

so applying 2.2.6 we conclude that $h_*(X) = F_\alpha(\tilde{H}_*(X)) = 0$.

Let $X = \bigvee_{i<\alpha} S^1$; this space has α cells. Then $\tilde{H}_1(\bigvee_{i<\alpha} S^1) = \bigoplus_{i<\alpha} \mathbb{Z}$, so by 2.2.6,

$$h_1(X) = F_\alpha\left(\bigoplus_{i<\alpha} \mathbb{Z}\right) \neq 0.$$

□

It only remains to prove 2.2.6 and 2.2.7.

Proof of Theorem 2.2.6. Define, for each α, a functor

$$\begin{aligned} F_\alpha\colon \mathbf{Ab} &\longrightarrow \mathbf{Ab} \\ A &\longmapsto A^\alpha / A^{\prec\alpha}. \end{aligned}$$

The image of the morphisms by the functor are the obvious ones. Now, if $\#A < \alpha$, then for each sequence $(x_i)_i \in A^\alpha$ we have

$$\#\mathrm{cont}(x_i)_i = \#\bigcup_{i<\alpha}\{x_i\} \leq \#A < \alpha.$$

Therefore, $A^\alpha = A^{\prec\alpha}$, so $F_\alpha A = 0$.

Now, for $A = \bigoplus_{i<\alpha} \mathbb{Z} = \bigoplus_{i<\alpha} \mathbb{Z}e_i$, take $(x_i)_i = (e_i)_i \in A^\alpha$. Then

$$\#\mathrm{cont}(x_i)_i = \alpha,$$

which implies that $\overline{(x_i)_i} \neq 0$ in $A^\alpha/A^{\prec\alpha}$, so $A^\alpha/A^{\prec\alpha} \neq 0$. □

Proof of Theorem 2.2.7. We have to check homotopy invariance and the exactness and suspension axioms. Homotopy invariance is trivial and suspension is trivial too because functors preserve isomorphisms.

By definition, if a functor is exact, then it preserves short exact sequences, which implies that it also preserves long exact sequences. This proves exactness. □

Chapter 3

A generalization of Ohkawa's Theorem to any cardinality

Notations 3.0.8. Let α be a cardinal.

1. An α-CW-complex is a CW-complex having less than α cells.
2. An α-homology theory is a homology theory that commutes with α-directed colimits.
3. Denote by $\mathbf{B}_\alpha$ the collection of Bousfield classes $\langle h_* \rangle$ such that $\langle h_* \rangle = \langle g_* \rangle$ for some α-homology theory g_*.

Recall that a cardinal α is called *regular* if it cannot be written as a sum of $< \alpha$ cardinals smaller than α.

Theorem 3.0.9 (Generalized Ohkawa theorem)**.** *Given a regular cardinal α, the collection $\mathbf{B}_\alpha$ is a set.*

The proof of this fact will consist of an adaptation of the proof given in the article [5]. Note that the proof in [5] is for homology theories defined over spectra, whereas we work with CW-complexes.

In order to prove 3.0.9, we are going to define the α-Okhawa class $\langle\langle h_* \rangle\rangle$ of a homology theory. If we denote by $\mathbf{O}_\alpha$ the collection of all α-Ohkawa classes $\langle\langle h_* \rangle\rangle$ such that $\langle\langle h_* \rangle\rangle = \langle\langle g_* \rangle\rangle$ for some α-homology theory g_*, we will prove that $\mathbf{O}_\alpha$ is a set. Then we will give a surjection $\mathbf{O}_\alpha \twoheadrightarrow \mathbf{B}_\alpha$, which will complete the proof of the theorem.

Definition 3.0.10. Let $\mathcal{F}_\alpha$ be the homotopy category of α-CW-complexes and let $\overline{\mathcal{F}}_\alpha$ be the set of isomorphism classes of objects of $\mathcal{F}_\alpha$.

Observe that if α is regular, then every CW-complex can be written as an α-directed colimit of its α-subcomplexes. This colimit is partially ordered by inclusion.

A *left ideal* I in the category $\mathcal{F}_\alpha$ is a set of maps between α-CW-complexes, which is closed under left composition: if $f : A \to B$ belongs to I then $g \circ f$ is in I for any map $g \colon B \to C$ between α-CW-complexes. We say that I is *based* at A if the domain of every map in I is A.

Definition 3.0.11 (α-Ohkawa class). Let h_* be an α-homology theory, A an α-CW-complex and $x \in h_*(A)$. We define the *annihilator ideal* of x as

$$\mathrm{ann}_A(x) = \{f \in [A, B] : [B] \in \overline{\mathcal{F}}_\alpha, (h_* f)(x) = 0\},$$

which is a left ideal based at A. Then the α-Ohkawa class of h_* is defined to be

$$\langle\langle h_* \rangle\rangle = \{\mathrm{ann}_A(x) : [A] \in \overline{\mathcal{F}}_\alpha, x \in h_*(A)\},$$

and the collection of all of them is denoted by $\mathbf{O}_\alpha$.

Lemma 3.0.12. $\mathbf{O}_\alpha$ *is a set.*

Proof. We will distinguish two cases:
1) Suppose $\alpha > 2^{\aleph_0}$.

Claim 1: For all α-CW-complexes A and B, $\#[A, B] < 2^\alpha$.

The cardinality of spheres of any (finite) dimension is $2^{\aleph_0} < \alpha$. Therefore if A is an α-CW-complex, its cardinality as a set is $< \alpha \cdot 2^{\aleph_0} = \alpha$. Then, for α-CW-complexes A and B,

$$\begin{aligned}
\#[A, B] &\leq \#\{f \colon A \longrightarrow B \mid f \text{ continuous map}\} \\
&\leq \#\{f \colon A \longrightarrow B \mid f \text{ map}\} \\
&\leq \#B^{\#A} \\
&< \alpha^\alpha \\
&= 2^\alpha.
\end{aligned}$$

Claim 2: $\overline{\mathcal{F}}_\alpha$ is a set.

Every α-CW-complex is constructed by adjoining $< \alpha$ cells, each of which occurs in some finite dimension. So, the $< \alpha$ cells are distributed among $\aleph_0$ positions. An α-CW-complex is determined by one of these distributions and $< \alpha$ attaching maps, one for each position of the distribution. For each position, we can choose

$$[S^i, A] < 2^\alpha$$

attaching maps, where $i \in \mathbb{N}$ and A is an α-CW-complex.

Let J be the set of possible distributions of α elements in $\aleph_0$ positions. If $\#J = \beta$, then we have

$$\#\overline{\mathcal{F}}_\alpha \leq \beta^{(2^{\alpha^\alpha})}.$$

J is a subset of the set L of the distributions among $\aleph_0$ positions with $\leq \alpha$ elements in each position. If I is a set with cardinality α, then the cardinality of L is equal to

$$\#\mathcal{P}(I^{\aleph_0}) = 2^{(\alpha^{\aleph_0})}.$$

Now that the claims are proved, we can conclude that the set of all left ideals has cardinality

$$\leq \#\overline{\mathcal{F}}_\alpha \cdot \#\overline{\mathcal{F}}_\alpha \cdot 2^\alpha.$$

Since α-Ohkawa classes are sets of annihilator ideals, the collection $\mathbf{B}_\alpha$ of all them is a set.
2) If $\alpha \leq 2^{\aleph_0}$, then $\overline{\mathcal{F}}_\alpha$ is a subcategory of $\overline{\mathcal{F}}_{\alpha'}$ for $\alpha' > 2^{\aleph_0}$. Hence, the two claims are true for α, and arguing in the same way as the first case we conclude the result. □

Now we will prove that if α is regular, then the map

$$\begin{aligned} \phi : \mathbf{O}_\alpha &\longrightarrow \mathbf{B}_\alpha \\ \langle\langle h_* \rangle\rangle &\longmapsto \langle h_* \rangle \end{aligned}$$

is well-defined and hence surjective. More precisely, both $\mathbf{B}_\alpha$ and $\mathbf{O}_\alpha$ are posets and we will show that the map is a map of posets, and this will imply that ϕ is well-defined.

The partial ordering on $\mathbf{B}_\alpha$ is defined by reverse inclusion; that is, we say that $\langle h_* \rangle \geq \langle g_* \rangle$ if h_*-acyclic implies g_* acyclic. The partial ordering on $\mathbf{O}_\alpha$ is defined by inclusion. In other words, $\langle\langle h_* \rangle\rangle \geq \langle\langle g_* \rangle\rangle$ if for each $\mathrm{ann}_A(x) \in \langle\langle g_* \rangle\rangle$ there exists $y \in h_*(A)$ such that $\mathrm{ann}_A(x) = \mathrm{ann}_A(y)$.

Lemma 3.0.13. *If $\langle\langle h_* \rangle\rangle \geq \langle\langle g_* \rangle\rangle$, then $\langle h_* \rangle \geq \langle g_* \rangle$, i.e., ϕ is a map of posets.*

Proof. Suppose that $\langle\langle h_* \rangle\rangle \geq \langle\langle g_* \rangle\rangle$. If $h_*(X) = 0$, we want to see that $g_*(X) = 0$. Write X as an α-directed colimit of its α-CW-subcomplexes: $X = \mathrm{colim}_i X_i$. Since h_* is an α-homology theory, it commutes with α-directed colimits. Hence, it suffices to show that for each $x \in g_*(X_k)$, x is 0 in $g_*(X)$.

For such a x we have $\mathrm{ann}_A(x) = \mathrm{ann}_A(y)$ for some $y \in h_*(X_\alpha)$, because $\langle\langle h_* \rangle\rangle \geq \langle\langle g_* \rangle\rangle$. Since $h_*(X) = 0$, y is 0 in $h_*(X)$ and then we have $i_{kl}(y) = 0$ for all large enough l, where $i_{kl} \colon X_k \to X_l$ is the inclusion. Therefore $i_{kl} \in \mathrm{ann}_A(y) = \mathrm{ann}_A(x)$ for any large enough l, which implies that x is 0 in $g_*(X)$. Therefore $g_*(X) = 0$. □

Corollary 3.0.14. *The map*

$$\begin{aligned} \phi : \mathbf{O}_\alpha &\longrightarrow \mathbf{B}_\alpha \\ \langle\langle h_* \rangle\rangle &\longmapsto \langle h_* \rangle \end{aligned}$$

is well-defined, surjective and order-preserving.

This finishes the proof of our generalization of Ohkawa's theorem.

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