Bachellor Thesis

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PERTURBED INVARIANT MANIFOLDS AND CHAOS

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Chapter 1

Introduction

1.1 Abstract

The goal of this thesis is to determine whether a given deterministic dynamical system can display chaotic behaviour, and if so, under which conditions. However, the complexity of the question forces us to reduce the problem to the study of two-dimensional C^r diffeomorphisms. This work is structured in the following way; first, a preliminary chapter with the intention to familiarize the reader with the background needed. Then, two main blocks, which correspond to the third and forth chapters, where the answer to the question is provided in the first one, and whether these conditions can occur for Poincaré maps associated with periodically perturbed systems is treated in the second one. Last, there is an Appendix about the computation of improper integrals which typically occur in Melnikov's theory by the residue theorem.

In the first block, the Smale-Moser Theorem is the key point for seeing that a two-dimensional map, which possesses a homoclinic point at which the stable and unstable manifold of the hyperbolic fixed point intersect transversally, has chaotic behaviour. In the text, this result is clearly achieved in two parts. The first one, which corresponds to sections 3.1, 3.2 and 3.3 is the study of sufficient conditions for the existence of an invariant Cantor set topologically conjugate to a shift on N symbols. Here, Symbolic Dynamics, which is the method for characterizing the orbit structure through infinite sequences of symbols, takes an important role because it enables us to associate a point in a subset of the unit square with a bi-infinite sequence.
The second one, which covers sections 3.4 and 3.5, is about re-writing the conditions needed, which are purely geometrical, to something more analyt-

ically approachable with the purpose of making them easier to be verified under the hypotheses of the Smale-Moser Theorem.

In the second block, we study Hamiltonian systems that suffer periodic non-autonomous perturbations. The aim of this chapter is to provide criteria, which will let us conclude when the associated Poincaré map has a transversal homoclinic point. Therefore, on account of the results from the third chapter we are able to state, under suitable conditions, that there is a chaotic invariant set. Moreover, the research is generalized to Hamiltonian systems that present either a homoclinic orbit or a heteroclinic one, although no similar conclusions regarding its dynamics will be deduced for the latest. Furthermore, this criteria depends on whether the perturbed invariant manifold coincide, split completely or cross. Thus, the track of the distance between the manifolds is important. As a result, the Melnikov function is introduced with the intention to tell us when the distance between the two manifolds becomes nul.
Seeing that, in sections 4.1, 4.2 and 4.3 we have the description of the phase space geometry for the unperturbed system, and its changes after the periodic perturbation. Later, in sections 4.4, 4.5 and 4.6 the Melnikov function is derived and its properties are discussed. Finally, section 4.7 enables the reader to see the applicability of the theory developed during the thesis with one heteroclinic case and one homoclinic case.

1.2 Motivation

The term “chaos” has always fascinated me. The beauty of the idea that two close points in a phase space can diverge completely with the evolution of the system, even though the system is deterministic, made me wonder how this subject is approached mathematically. Honestly, I must admit that after completing this thesis I have a bittersweet feeling. On one side, I wish I could have been able to get more juice from the theory developed with broader and more exotic types of applications. On the other side, I might see this thesis as an inflection point in my studies due to the fact that I may consider keeping my academic carreer related with this field.
Last but not least, through the development of this thesis I realized about the importance of having a general baggage in mathematics due to situations I encountered where issues from other branches such us Topology arose.

Chapter 2

Preliminary Results

The aim of this chapter is to let the reader familiarize with the necessary background for the development of this thesis. Since we are concerned that an exhaustive review of the most relevant results in differential equations would be tedious and unpractical, some concepts like the definition of a dynamical system associated with a differential equation and its results about existence, uniqueness and regularity are assumed to be known by the reader. Nevertheless, we also take for granted concepts from Mathematical Analysis and Topology such us differentiability, homeomorphical spaces and compactness. What we do provide is definitions and results related to the dynamics of systems, from the definition of the flow of a vector field up to the Stable Manifold Theorem, which will be essential for the study of Melnikov's Theory.

Definition 2.1. Let $f:\ \Omega \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^r-map, $r \geq 1$.
The evolutionary solution associated with the differential equation $\dot{x} = f(t,x)$ with initial conditions $x(t_0) = x_0$ is described by

$$\begin{array}{rrcl} \Phi: & D \subset \mathbb{R} \times \Omega \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & (t; t_0, x_0) & \longmapsto & \Phi(t; t_0, x_0)\ , \end{array}$$

where D and Ω open sets such that

- Φ is of class C^r.
- $\forall (t_0, x_0) \in \Omega, \quad I(t_0, x_0) = \{t \in \mathbb{R} |\quad (t; t_0, x_0) \in D\}$ open set and the map $\Phi(\cdot, t_0, x_0):\ I(t_0, x_0) \longrightarrow \mathbb{R}^n$ is C^{r+1} with respect to t.
- $\forall (t_0, x_0) \in \Omega$

1. $\Phi(t_0; t_0, x_0) = x_0$.
2. If $t_1 \in I(t_0, x_0)$ then $\forall t_2 \in I(t_1, \Phi(t_1; t_0, x_0))$ we have $t_2 \in I(t_0, x_0)$ and $\Phi(t_2; t_1, \Phi(t_1; t_0, x_0)) = \Phi(t_2; t_0, x_0)$.

Definition 2.2. Let $X : U \longrightarrow \mathbb{R}^n$ be a C^r-vector field, $r \geq 1$. Let $\dot{x} = X(x)$ be the autonomus ordinary differential equation induced by X. Then the flow ϕ associated with X corresponds to the evolutionary solution associated with $\dot{x} = X(x)$. Thus, the flow ϕ is given by

$$\begin{array}{rrcl} \phi : & D_0 \subset \mathbb{R} \times U \subset \mathbb{R} \times \mathbb{R}^n & \longrightarrow & U \subset \mathbb{R}^n \\ & (t, x_0) & \longmapsto & \phi(t, x_0) = \Phi(t; x_0) \ , \end{array}$$

with $D_0 = \{(t, x_0) \in \mathbb{R} \times U | \quad t \in I(0, x_0) = I(x_0)\}$ satisfying that ϕ is a C^r-map and

- $\phi(0, x) = x$,
- $\phi(s, \phi(t, x)) = \phi(t + s, x)$.

Observation. The flow associated with X satisfies the requirements for being a dynamical system. The solution for the general Cauchy problem $x(t_0) = x_0$ is $x(t) = \Phi(t - t_0; x_0)$.

In addition, it is seen that for a constant linear vector field X, its flow is

$$\phi(t, x) = e^{tX} x \ .$$

Definition 2.3. Let $\varphi : I \subset \mathbb{R} \longrightarrow U \subset \mathbb{R}^n$ be the solution of a Cauchy problem.
The orbit associated with φ is $\gamma = \varphi(I) = \text{Im}(\varphi)$. For a certain point, the orbit of $x_0 \in U \subset \mathbb{R}^n$ is the orbit associated with the solution

$$\begin{array}{rrcl} \phi_{x_0} : & I(x_0) & \longrightarrow & U \subset \mathbb{R}^n \\ & t & \longmapsto & \phi_{x_0}(t) \equiv \phi(t, x_0) \ . \end{array}$$

Definition 2.4. A linear vector field $X \in \mathscr{L}(\mathbb{R}^n)$ is hyperbolic if the spectrum of X is disjoint from the imaginary axis. The number of eigenvalues of X with negative real part is called index of stability of X.

Proposition 2.5. *If $X \in \mathscr{L}(\mathbb{R}^n)$ is a hyperbolic vector field then there exists a unique splitting of $\mathbb{R}^n$ as a direct sum $\mathbb{R}^n = E^s \oplus E^u$, where E^s and E^u are the stable and the unstable invariant subspaces for X respectively. Moreover*

$$\text{if } \lambda \in \text{ eigenvalues of } X^s = X_{|E^s} \quad \text{then} \quad Re[\lambda] < 0\,,$$

$$\text{if } \lambda \in \text{ eigenvalues of } X^u = X_{|E^u} \quad \text{then} \quad Re[\lambda] > 0\,.$$

Definition 2.6. Let $X: \ U \subset \mathbb{R}^n \ \longrightarrow \ \mathbb{R}^n$ be a C^r-vector field with $r \geq 1$. For $x \in U$ the omega and alpha sets are defined as follows

$$w(x) = \{y \in U| \quad \exists (t_n)_n \to +\infty \quad \text{with} \quad \lim_{n\to\infty} \varphi(t_n, x) = y\}\,,$$
$$\alpha(x) = \{y \in U| \quad \exists (t_n)_n \to -\infty \quad \text{with} \quad \lim_{n\to\infty} \varphi(t_n, x) = y\}\,.$$

Definition 2.7. Let $x = \bar{x}$ be a fixed point of $\dot{x} = X(x), \quad x \in \mathbb{R}^n$. Then $\bar{x}$ is called hyperbolic fixed point if $DX(\bar{x})$ is a hyperbolic linear vector field. Moreover, if $\bar{x}$ is already a hyperbolic fixed point of the vector field $X(x)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n eigenvalues (maybe some of them coincide) for the linearization $DX(\bar{x})$, then $\bar{x}$ is called

- saddle if some, but not all, the eigenvalues have positive real parts,
- stable node or sink if all the eigenvalues have negative real parts,
- unstable node or source if all the eigenvalues have positive real parts.

Finally, for two-dimensional vector fields, if $\bar{x}$ is a fixed point of $X(x)$ such that all the eigenvalues of $DX(\bar{x})$ are purely imaginary then $\bar{x}$ is a center.

Definition 2.8. Let ϕ be the flow associated with the vector field X and $\bar{x} \in \mathbb{R}^n$ be a hyperbolic fixed point of X. The set of points in $\mathbb{R}^n$ that have $\bar{x}$ as ω-limit is called the stable set of $\bar{x}$ and it is denoted by $W^s(\bar{x})$, and the set of points in $\mathbb{R}^n$ that have $\bar{x}$ as α-limit is called the unstable set of $\bar{x}$ and it is denoted by $W^u(\bar{x})$. Moreover, the sets

$$W^s_\delta(\bar{x}) = \{x \in W^s(\bar{x})| \quad \phi(t,x) \in B_\delta(\bar{x}), \quad \forall t \geq 0\}\,,$$

$$W^u_\delta(\bar{x}) = \{x \in W^u(\bar{x})| \quad \phi(x,t) \in B_\delta(\bar{x}), \quad \forall t \leq 0\}\,,$$

are called the local stable and local unstable manifolds of size δ, of the point $\bar{x}$.

At this point, we should comment that many definitions and results for C^r-vector fields we have stated above have analogous counterparts for C^r-diffeomorphisms. For instance, the equivalent definition for a hyperbolic fixed points for C^r-diffeomorphisms is the following.

Definition 2.9. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a C^r-diffeomorphism, $r \geq 1$ and $x = \bar{x} \in \mathbb{R}^n$ be a fixed point of f.
Let $y \mapsto Df(\bar{x})y$ with $y \in \mathbb{R}^n$ be the associated linear map of f.
Then, $\bar{x}$ is a hyperbolic fixed point if the eigenvalues of $Df(\bar{x})$, $\lambda_1, \ldots, \lambda_n$, satisfy

$$|\lambda_i| \neq 1, \quad i = 1, \ldots, n .$$

Moreover, the linear map $y \mapsto Df(\bar{x})y$ has the invariant manifolds given by

$$\begin{aligned} E^s &= \text{span}\{e_1, \ldots, e_s\} , \\ E^u &= \text{span}\{e_{s+1}, \ldots, e_{s+u}\} , \\ E^c &= \text{span}\{e_{s+u+1}, \ldots, e_{s+u+c}\} , \end{aligned}$$

where $s + c + u = n$ and

- $e_1, \ldots, e_s$ are the eigenvectors of $Df(\bar{x})$ corresponding to the eigenvalues of $Df(\bar{x})$ having modulus less than one ,
- $e_{s+1}, \ldots, e_{s+u}$ are the eigenvectors of $Df(\bar{x})$ corresponding to the eigenvalues of $Df(\bar{x})$ having modulus greater than one ,
- $e_{s+u+1}, \ldots, e_n$ are the eigenvectors of $Df(\bar{x})$ corresponding to the eigenvalues of $Df(\bar{x})$ having modulus equal to one .

Now, we present the Stable Manifold Theorem and the rest of the results for diffeomorphisms as it is how it will appear in the thesis.

Theorem 2.10 (The Stable Manifold Theorem for maps).
Let $f \in \text{Diff}^{\,r}(\mathbb{R}^n)$ and $p \in \mathbb{R}^n$ be a hyperbolic fixed point of f.
Let E^s be the stable subspace of Df_p. Then

1) *$W^s(p)$ is a C^r injectively immersed manifold in $\mathbb{R}^n$ and the tangent space to $W^s(p)$ at the point p is E^s.*

2) *Let $D \subset W^s(p)$ be an embedded disc containing the point p. Now, consider a neighbourhood $\mathcal{N} \subset \text{Diff}^{\,r}(\mathbb{R}^n)$ such that each $g \in \mathcal{N}$ has a unique hyperbolic fixed point p_g contained in a certain neighbourhood U of p. Then, given $\epsilon > 0$ there exists a neighbourhood $\tilde{\mathcal{N}} \subset \mathcal{N}$ of*

f such that, for each $g \in \tilde{\mathcal{N}}$, there exists a disc $D_g \subset W^s(p_g)$ that is ϵ C^r-close to D.

Remark. Exactly the same results are found for the unstable set of p, $W^u(p)$, and the unstable subspace of Df_p, E^u. Regarding the first statement of this theorem, from now on, we call the stable and unstable sets of p by the stable and unstable manifolds of p, respectively.

Definition 2.11. Let L_1 and L_2 be two submanifolds of a given manifold M. Then L_1 intersects L_2 transversally at p if $p \in L_1 \cap L_2$ and the sum of the tangent spaces of L_1 and L_2 is the tangent space of M,

$$T_pL_1 + T_pL_2 = T_pM \ .$$

Definition 2.12. Let $f \in \mathrm{Diff}^r(\mathbb{R}^n)$ and $p \in \mathbb{R}^n$ be a hyperbolic fixed point of f. Then, r is a homoclinic point if $r \in W^s(p) \cap W^u(p) \setminus \{p\}$. Moreover, if $W^s(p)$ intersects $W^u(p)$ transversally at r, r is called transverse homoclinic point.

Definition 2.13. Let $f \in \mathrm{Diff}^r(\mathbb{R}^n)$ and $l, m \in \mathbb{R}^n$ be two different hyperbolic fixed points of f. Then, $r \neq l, m$ is a heteroclinic point if $r \in W^s(l) \cap W^u(m)$.

Moreover, we state a lemma concerning the iteration of a curve which intersect transversally a stable invariant manifold. For convienience, we state a simplified version of the lemma where the stable and unstable manifolds coincide with the coordinates axes.

Lemma 2.14 (Lambda lemma)**.**
Let $h \in \mathrm{Diff}^r(\mathbb{R}^2)$ and p be a hyperbolic periodic point of h such that p is centered at the origin and its stable and unstable manifolds correspond to the coordinates axis in a neightbourhood of the origin.
Let $\bar{q} \in W^s(0) - \{0\}$.
Let C be a curve intersecting $W^s(0)$ transversally at $\bar{q}$.
Let C^N be the connected component of $h^N(C) \cap U$ to which $h^N(\bar{q})$ belongs.
Then given $\epsilon > 0$ and U sufficiently small,
$\exists N_0 \in \mathbb{Z}^+$ such that for $N \geq N_0$, C^N is C^1 ϵ-close to $W^u(0) \cap U$ i.e. the tangent vectors on C^N are ϵ-close to the tangent vectors on $W^u(0) \cap U$.

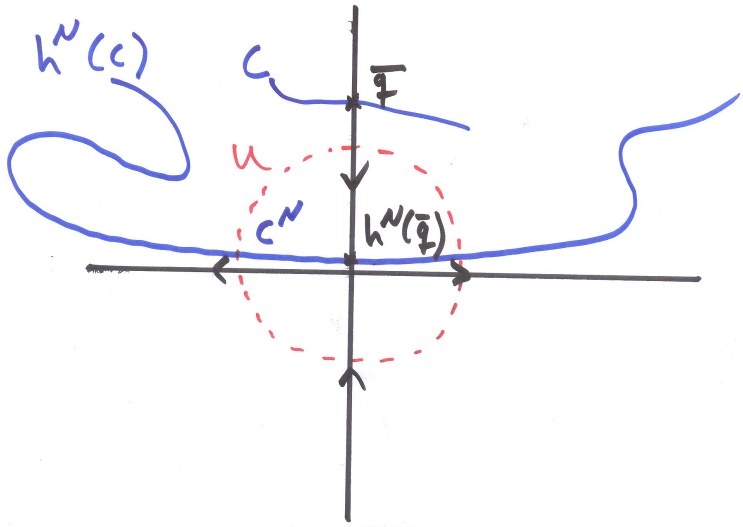

Figure 2.1

Proof. See [Palis and de Melo, 1982, pp. 80-85] . □

Remark. The Lambda Lemma provides information about the stretching of the tangent vectors. Let $z_0 \in h^{-N}(C^N)$ and (ξ_{z_0}, ζ_{z_0}) be a vector tangent to $h^{-N}(C^N)$ at z_0. Then $|\xi_{h^N(z_0)}|$ can be arbitrarily small and $|\zeta_{h^N(z_0)}|$ can be arbitrarily big by taking N large. See Figure 2.1.

Last, we want to recall a lemma which allows us to bound a function under certain conditions.

Lemma 2.15. *Gronwall's Inequality*
Let α, ϕ, Ψ be continuous functions in the interval $[a, b]$ with $a < b$.
Suppose that Ψ in non-negative and α is non-decreasing.
Then, if

$$\phi(t) \leq \alpha(t) + \int_a^t \Psi(s)\phi(s)\, ds, \quad \forall t \in [a, b]\,,$$

we have

$$\phi(t) \leq \alpha(t) e^{\int_a^t \Psi(s)\, ds}, \quad \forall t \in [a, b]\,.$$

Chapter 3

Symbolic Dynamics

3.1 Introduction

The idea to characterise the orbit structure of a dynamical system is not new. It already appears in different context like the study on geodesics on some surfaces on negative curvature, or the work on periodically excited Van der Pol equations. In this chapter we aim to see that C^r-diffeomorphisms with a homoclinic point, at which the stable and unstable manifold intersect transversally, have chaotic dynamics. The result is seen by Theorem 3.13, but part of the work is done by Theorem 3.17 where we see which conditions have to be satisfied by a map of the unit square into itself to be topologically conjugated to a shift on N symbols.

3.2 Space of Symbol Sequences

To begin with, we should define properly what is exactly the space of Symbol Sequences. Let's take

$A:$ finite or denumerable set of symbols, called alphabet. Let $A = \{1, 2, \ldots, N\}$ with $N \geq 2$ be our alphabet for simplicity.

Σ : set of bi-infinite sequences with elements from A. Formally, Σ can be built as a bi-infinite Cartesian product of A.

$$\Sigma = \ldots \times A \times A \times A \ldots = \prod_{i=-\infty}^{\infty} A^i, \quad A^i = A.$$

If $s \in \Sigma$ we write $s = \{\dots, s_{-n}, \dots, s_{-1}, s_0, s_1, \dots, s_n, \dots\}$ with $s_n \in A$, $\forall n \in \mathbb{Z}$.

Remark. For convenience we will take $s = \{\dots s_{-n} \dots s_{-1} s_0 . s_1 \dots s_n \dots\}$. The dot '.' is separating the bi-infinite sequence into two parts.

Now, we introduce a distance in A:

$$d(a,b) \equiv |a-b|, \quad \forall a,b \in A.$$

Observation. The set A equipped with the distance $d(a,b) \equiv |a-b|, \quad \forall a,b \in A$, is a compact, totally disconnected metric space.

Similarly, let's introduce a distance in Σ with the aim of obtaining some properties for Σ.

Definition 3.1. For $s, s^* \in \Sigma$ with

$$s = \{\dots s_{-n} \dots s_{-1} s_0 . s_1 \dots s_n \dots\} \qquad \text{and}$$

$$s^* = \{\dots s^*_{-n} \dots s^*_{-1} s^*_0 . s^*_1 \dots s^*_n \dots\} \qquad \text{we define}$$

$$d(s,s^*) = \sum_{i=-\infty}^{\infty} \frac{1}{2^{|i|}} \frac{|s_i - s_i^*|}{|s_i - s_i^*| + 1} \quad .$$

Observation.

$$\text{The map} \qquad \begin{array}{rccc} d: & \Sigma \times \Sigma & \longrightarrow & \mathbb{R} \\ & (s,s^*) & \longmapsto & d(s,s^*) \end{array} \qquad \text{is a distance.}$$

Indeed, for any $s, s^*, \hat{s} \in \Sigma$ we have

1) $d(s,s^*) \geq 0$ as it is a sum of positive terms.

2) $d(s,s^*) = 0 \iff s = s^*$.
The implication $\Leftarrow$ follows by definition.
The implication $\Rightarrow$ follows because if a sum of positive terms is zero, every term has to be zero.

3) $d(s,s^*) = d(s^*,s)$ since $|s_i - s_i^*| = |s_i^* - s_i|$.

4) $d(s, \hat{s}) \leq d(s, s^*) + d(s^*, \hat{s})$.

For every term we have $\frac{|s_i - \hat{s}_i|}{|s_i - \hat{s}_i|+1} \leq \frac{|s_i - s_i^*|}{|s_i - s_i^*|+1} + \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1}$. Indeed

- if $s_i = \hat{s}_i \quad \Rightarrow \quad 0 = \frac{|s_i - \hat{s}_i|}{|s_i - \hat{s}_i|+1} \leq \frac{|s_i - s_i^*|}{|s_i - s_i^*|+1} + \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1} = 2\frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1}$
- if $s_i = s^* \quad \Rightarrow \quad \frac{|s_i - \hat{s}_i|}{|s_i - \hat{s}_i|+1} = \frac{|s_i - s_i^*|}{|s_i - s_i^*|+1} + \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1} = 0 + \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1}$
- if $s_i^* = \hat{s}_i \quad \Rightarrow \quad \frac{|s_i - \hat{s}_i|}{|s_i - \hat{s}_i|+1} = \frac{|s_i - s_i^*|}{|s_i - s_i^*|+1} + \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1} = \frac{|s_i - s_i^*|}{|s_i - s_i^*|+1} + 0$
- if $s_i \neq \hat{s}_i \neq s_i^* \neq s_i \quad \Rightarrow \frac{|s_i - \hat{s}_i|}{|s_i - \hat{s}_i|+1} < 1 = \frac{1}{2} + \frac{1}{2} \leq \frac{|s_i - s_i^*|}{|s_i - s_i^*|+1} + \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1}$,

  since $|s_i - s_i^*|, |s_i - \hat{s}_i| \geq 1$ implies that $\frac{|s_i - s_i^*|}{|s_i - s_i^*|+1}, \frac{|s_i^* - \hat{s}_i|}{|s_i^* - \hat{s}_i|+1} \geq \frac{1}{2}$.

Thus (Σ, d) form a Metric Space.

Observation. The Metric (Σ, d) induces the product topology on Σ. This means that the topology induced by (Σ, d) is the coarsest topology which makes all projections

$$\begin{array}{ccc} \Sigma & \longrightarrow & \mathbb{R} \\ s & \longmapsto & s_i \end{array} \qquad \text{continuous in } \Sigma.$$

Remark. With this metric, two bi-infinite sequences are close if they have the same terms in a big central part of the sequence. To state this idea explicitly we have the following lemma which provides useful tools for the proof of the next proposition.

Lemma 3.2. *For* $s, \hat{s} \in \Sigma$

i) if $\quad d(s, \hat{s}) < \frac{1}{2^{M+1}} \quad \Rightarrow \quad \forall i, |i| \leq M \quad s_i = \hat{s}_i$,

ii) if $\quad \forall i, |i| \leq M \quad s_i = \hat{s}_i \quad \Rightarrow d(s, \hat{s}) \leq \frac{1}{2^{M-1}}$.

Proof. See [Wiggins, 1990, pp. 440-441]. $\square$

Proposition 3.3. *The space Σ with the distance introduced by Definition (3.1) is a metric space such that Σ is*

i) compact.

ii) totally disconnected.

iii) perfect.

Proof.

i) If A is compact, $\Sigma = \ldots \times A \times A \times A \times \ldots$ is compact according to Tychonoff's theorem which states that the product of any collections of compact topological spaces is compact with respect to the product topology.

ii) If A is totally disconnected, Σ is totally disconnected due to the fact that the product of totally disconnected spaces is totally disconnected.

iii) To prove that Σ is perfect we have to see that given an arbitrary $s \in \Sigma$ and $N_\epsilon(s)$ neighbourhood of s, $\exists \hat{s} \in N_\epsilon(s), \hat{s} \in \Sigma$ such that $\hat{s} \neq s$.

Let $s^* \in \Sigma$ and $N_\epsilon(s^*)$ be a neighbourhood of s^*

$$\begin{aligned} N_\epsilon(s^*) &= \{s \in \Sigma| \quad d(s_i, s_i^*) < \epsilon\} \\ &= \{s \in \Sigma| \quad s_i = s_i^*, \quad \forall |i| \leq M, s_i, s_i^* \in A\} \end{aligned}$$

with $\epsilon < \frac{1}{2^{M+1}}$ for some M.
Then the sequence $\hat{s}$ defined as

$$\begin{array}{lll} \text{for} \quad i \neq M+1 & \hat{s}_i = s_i^* & \\ \text{for} \quad i = M+1 & \hat{s}_i = s_i^* + 1 & \text{if} \quad s_i^* < N \\ & \hat{s}_i = s_i^* - 1 & \text{if} \quad s_i^* = N \end{array}$$

belongs to $N_\epsilon(s^*)$, but $\hat{s} \neq s^*$.

□

After defining a topological structure in Σ, let's define the following map, called the shift map:

$$\begin{array}{rccc} \sigma : & \Sigma & \longrightarrow & \Sigma \\ & s & \longmapsto & \sigma(s) \end{array}$$

such that

$$\forall k \in \mathbb{Z}, \quad (\sigma(s))_k = s_{k-1} \,.$$

In other terms,

$$\text{if} \quad s = \{\dots s_{-n} \dots s_{-1} s_0 . s_1 \dots s_n \dots\}$$
$$\text{then} \quad \sigma(s) = \{\dots s_{-n} \dots s_{-1} . s_0 s_1 \dots s_n \dots\} \,.$$

Proposition 3.4. *The shift map σ is continuous.*

Proof. We should see that, given $\epsilon > 0$, $\exists \delta(\epsilon)$ such that,

$$d(s, s^*) < \delta(\epsilon) \quad \Rightarrow \quad d(\sigma(s), \sigma(s^*)) < \epsilon \quad \text{for} \quad s, s^* \in \Sigma \,.$$

For any arbitrary $\epsilon > 0$ let's take $\delta = \frac{1}{2^{M+1}}$ with M such that $\frac{1}{2^{M-2}} < \epsilon$. Then, if $d(s, s^*) < \delta = \frac{1}{2^{M+1}}$ we have that for $|k| \leq M$, $s_k = s_k^*$ by the first part of Lemma (3.2). Thus, for $|k| \leq M - 1$, $\sigma(s)_k = \sigma(s^*)_k$. Finally, applying the second part of Lemma (3.2) we get that $d(\sigma(s), \sigma(s^*)) < \frac{1}{2^{M-2}} < \epsilon$ by the choice of M. □

Proposition 3.5. *The shift map has*

i) a countable infinity of periodic orbits and orbits of all periods,

ii) an uncountable infinity of non-periodic orbits,

iii) dense orbits.

Proof.

i) Orbits of sequences that periodically repeat are periodic under iteration by σ. Precisely, the orbits of σ having period k correspond to the orbits of sequences made up of periodically repeating blocks of elements from the alphabet A of length k.

$$s = \{\overline{s_1 s_2 \dots s_k . s_1 s_2 \dots s_k}\} \quad \text{with} \quad s_i \in A, \quad i = 1, \dots, k \,.$$

The sequence s after k iterations of σ will be itself again

$$\sigma^k(s) = s \,.$$

It is important to point out that for a certain k, the number of sequences having a periodically repeating block of length k is N^k. Hence, for each k we have a finite number, smaller than N^k, of orbits of σ having period 0. Consequently, there is a countable infinity of periodic orbits and orbits of every period.

ii) To see that Σ has an uncountable infinity of non-periodic orbits, we will build a correspondence between Σ and the closed unit interval $[0,1]$ with the aim of showing that Σ is uncountable .
A bi-infinite sequence corresponds to an infinite one by the relation

$$\{\dots s_{-n}\dots s_{-1}s_0.s_1\dots s_n\dots\} \longmapsto \{s_0 s_1 s_{-1}\dots s_n s_{-n}\dots\} \, .$$

Furthermore, every number from the interval $[0,1]$ can be expressed in base N as an infinite sequence. Thus, since $[0,1]$ is uncountable, Σ is uncountable.
Last, subtracting the subset of periodic orbits, which is countable, to an uncountable set like Σ we still have an uncountable subset, which is the one of the non-periodic orbits.

iii) Finally, σ has a dense orbit if $\exists s \in \Sigma$ such that for any $s' \in \Sigma$ and $\epsilon > 0, \quad \exists n \in \mathbb{Z}$ such that $d(\sigma^n(s), s') < \epsilon$.

First, we realize that we can order all the finite sequences with elements from the alphabet A. For $s = \{s_1 \dots s_k\}$, $\bar{s} = \{\bar{s_1}\dots \bar{s_{k'}}\}$

$$\begin{aligned} &s < \bar{s} \quad \text{if} \quad k < k' \\ &s < \bar{s} \quad \text{if} \quad k = k' \quad \text{and} \quad s_i < \bar{s}_i \end{aligned}$$

with i first integer such that $s_i \neq \bar{s}_i$.

Now let's consider our candidate to be a bi-infinite sequence s which contains all possible finite sequences stated above.
Then, for $s' \in \Sigma$ and $\epsilon > 0$ arbitrary, every $s'' \in N_\epsilon(s')$ neighbourhood of s', satisfies that $\exists N \in \mathbb{Z}, \quad s''_i = s'_i \quad \forall |i| \leq N$. Since the sequence $\{s'_{-N}\dots s'_{-1}.s'_0\dots s'_N\}$ is contained in s by definition, it will exists $\hat{N} \in \mathbb{Z}$ such that $\forall |i| \leq N, \quad \sigma^{\hat{N}}(s)_i = s'_i$.

Thus $d(\sigma^{\bar{N}}(s), s') < \epsilon$.

□

3.3 The Conley-Moser Conditions

The goal of this section is to provide sufficient conditions for a two-dimensional invertible map to have an invariant Cantor set, whose dynamics is topologically conjugate to the shift on N symbols. Although none of the lemmas that follow is explicitly proven in this thesis, they are needed for the demonstration of the first Theorem of this chapter.

Definition 3.6. A ν_v-vertical curve is the graph of a function

$$\begin{array}{rccc} v: & [0,1] & \longrightarrow & [0,1] \\ & x & \longmapsto & y = v(x) \end{array} \quad \text{such that}$$

- for $y \in [0,1], \quad 0 \leq v(y) \leq 1,$
- for $0 \leq y_1, y_2 \leq 1, \quad |v(y_1) - v(y_2)| \leq \nu_v |y_1 - y_2|$.

Definition 3.7. A ν_h-horizontal curve is the graph of a function

$$\begin{array}{rccc} u: & [0,1] & \longrightarrow & [0,1] \\ & y & \longmapsto & x = u(y) \end{array} \quad \text{such that}$$

- for $x \in [0,1], \quad 0 \leq u(x) \leq 1,$
- for $0 \leq x_1, x_2 \leq 1, \quad |u(x_1) - u(x_2)| \leq \nu_h |x_1 - x_2|$.

Definition 3.8. Being $v_1(y) < v_2(y)$ two ν_v-vertical curves, we define a ν_v-vertical strip by

$$V = \{(x,y) \in \mathbb{R}^2 | \quad x \in [v_1(y), v_2(y)], \quad y \in (0,1)\}$$

and we call width of V to

$$d(V) = \max_{y \in [0,1]} |v_2(y) - v_1(y)| \, .$$

Definition 3.9. Being $u_1(x) < u_2(x)$ two ν_h-horizontal curves, we define a ν_h-horizontal strip by

$$U = \{(x,y) \in \mathbb{R}^2 | y \in [u_1(x), u_2(x)], \quad x \in (0,1)\}$$

and we call width of U to

$$d(U) = \max_{x \in [0,1]} |u_2(x) - u_1(x)| \, .$$

Lemma 3.10.

1) *If* $V^1 \supset V^2 \supset \cdots \supset V^k \supset \ldots$ *nested sequence of* ν_v*-vertical strips with width* $d(V^k)$ *such that* $\lim_{k\to\infty} d(V^k) = 0$ *then*

$$\bigcap_{k=1}^{\infty} V^k = V^{\infty} \quad \textit{is a } \nu_v\textit{-vertical curve}\,.$$

2) *If* $U^1 \supset U^2 \supset \cdots \supset U^k \supset \ldots$ *nested sequence of* ν_h*-horizontal strips with width* $d(U^k)$ *such that* $\lim_{k\to\infty} d(U^k) = 0$ *then*

$$\bigcap_{k=1}^{\infty} U^k = U^{\infty} \quad \textit{is a } \nu_h\textit{-horizontal curve}\,.$$

Proof. See [Wiggins, 1990, pp. 445-446] . □

Lemma 3.11. *Suppose* $0 \leq \nu_v\nu_h < 1$*, then a* ν_v*-vertical curve and a* ν_h*-horizontal curve intersect in a unique point.*

Proof. See [Wiggins, 1990, pp. 446-447] .

□

Now, we are capable of stating the sufficient conditions mentioned above. To do so, we use some of the formalism introduced in the previous section. Being

- $A = \{1, 2, \ldots, N\}, \quad N \leq 2$ the alphabet,
- V_a, for $a \in A$ set of disjoint ν_v-vertical strips,
- U_a, for $a \in A$ set of disjoint ν_h-horizontal strips,
- $Q = \{(x, y) \in \mathbb{R}^2 | 0 \leq x \leq 1, \quad 0 \leq y \leq 1\}$ the unit square,

let's consider the map

$$\phi : \quad Q \subset \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}^2$$

under the following assumptions.

Assumption 1.

- $0 \leq \nu_v \nu_h \leq 1$ *and* ϕ *maps* V_a *homeomorphically to* U_a *for* $a \in A$, *i.e*

$$\phi(V_a) = U_a, \quad a \in A$$

- ϕ *maps the vertical boundaries of* V_a *to the vertical boudaries of* U_a, *i.e being* $\partial V_{a,1}$, $\partial V_{a,2}$ *the two vertical boundaries of each* V_a *and* $\partial U_{a,1}$, $\partial U_{a,2}$ *the two vertical boundaries of each* U_a *then*

$$\phi(\partial V_{a,i}) = \partial U_{a,j}, \quad i, j \in \{1, 2\}\,.$$

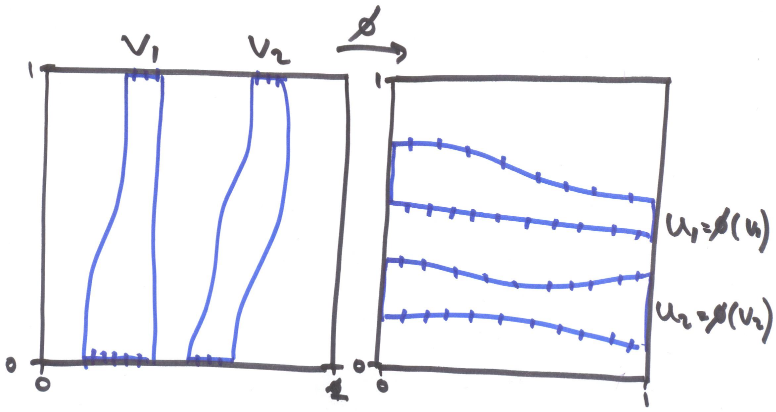

Figure 3.1

Assumption 2.

- *if* V *is a vertical strip in* $\bigcup_{a\in A} V_a$ *then, for every* $a \in A$

$$\hat{V}_a = \phi^{-1}(V) \cap V_a \quad \textit{is a vertical strip}$$

with width $\;d(\hat{V}_a) \leq \nu d(V_a)\;$ *for some fixed* $\nu \in (0, 1)$.

- *if* U *is a horizontal strip in* $\bigcup_{a\in A} U_a$ *then, for an arbitrary* $a \in A$

$$\hat{U}_a = \phi(U) \cap U_a \quad \textit{is a horizontal strip}$$

with width $\;d(\hat{U}_a) \leq \nu d(U_a)\;$ *for some fixed* $\nu \in (0, 1)$.

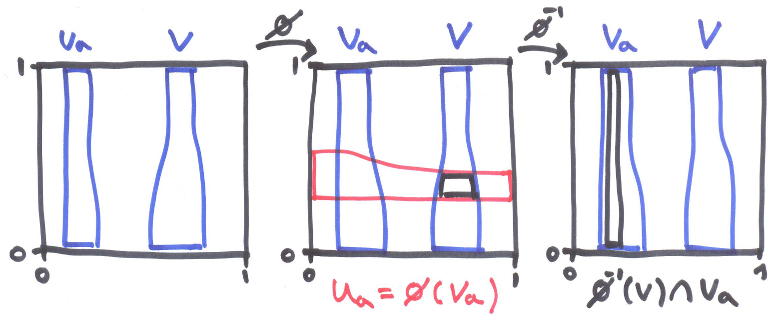

Figure 3.2

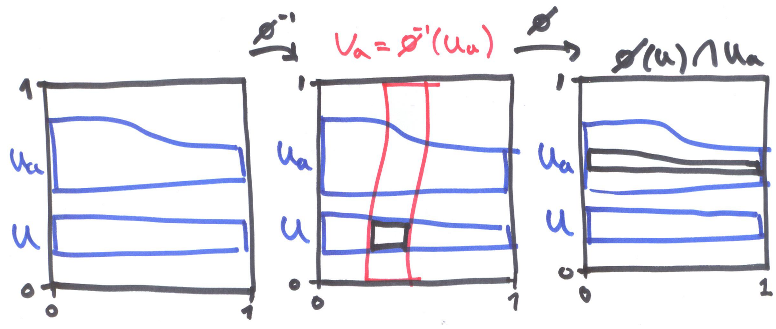

Figure 3.3

Remark. In Figure 3.1 we are able to see how Assumption 1 works for two vertical strips V_1 and V_2. Moreover, Figures 3.2 and 3.3 let us see how a vertical strip and a horizontal strip behave under the maps ϕ^{-1} and ϕ, respectively.

At this point, we just need to introduce the another concept needed for the next theorem. Although a Cantor set has multiple definitions and diferent constructions, we first introduce the formal definition for the ternary set construction and after its generalization.

Definition 3.12. The Ternari Cantor set $\mathcal{C}$ is defined as

$$\mathcal{C} = \bigcap_{n=1}^{\infty} I_n,$$

where I_0 is the closed real interval $[0,1]$ and I_{n+1} is constructed by trisecting I_n and removing the middle third. What is more, the generalization of a Cantor set has the same properties than the Ternari Cantor set, namely, it is non-empty, uncountable, compact, perfect and a totally disconnected metric space.

Last but not least, for higher dimensional generalizations it is good to notice that the one-dimensional case can be extended to the two-dimensional case by substituting the unit interval $[0,1]$ for $[0,1]\times[0,1]$ and similarly for higher dimensions.

Theorem 3.13. *The map ϕ under Assumptions 1 and 2 has an invariant Cantor set Λ, on which it is topologically conjugate to a shift on N symbols, i.e, the following diagram commutes.*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\phi} & \Lambda \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

where τ is a homeomorphism mapping Λ onto Σ .

Proof.

1) **Construction of the invariant set**

First, we construct a set of points, $\Lambda_{+\infty}$, that remains in $\bigcup_{a\in A} V_a$ under all forward iterates. It eventually lead to an uncountable infinity of ν_v-vertical curves. Next we construct a set of points, $\Lambda_{-\infty}$, that remains in $\bigcup_{a\in A} U_a$ under all backward iterates. Similarly, it turns out to be an uncountable infinity of ν_h-horizontal curves. Thus the intersections of this two sets is clearly an invariant set, i.e, a set of points which remains in Q after all iterations by ϕ,

$$\Lambda = \Lambda_{-\infty} \cap \Lambda_{+\infty} \, .$$

To do so, we define inductively for $n \geq 1$

$$V_{s_0 s_{-1}\ldots s_{-n}} = V_{s_0} \cap \phi^{-1}(V_{s_{-1}\ldots s_{-n}}) \quad \text{for} \quad s_{-k} \in A.$$

Observation. Using Assumption 2, let's notice we have inductively defined a sequence of vertical strips

$$\begin{aligned}
&V_{s_{-n}} = V_{s_{-n}} \\
&V_{s_{-n+1}s_{-n}} = V_{s_{-n+1}} \cap \phi^{-1}(V_{s_{-n}}) \\
&\dots \\
&V_{s_{-1}\dots s_{-n}} = V_{s_1} \cap \phi^{-1}(V_{s_{-2}\dots s_{-n}}) \\
&V_{s_0 s_{-1}\dots s_{-n}} = V_{s_0} \cap \phi^{-1}(V_{s_{-1}\dots s_{-n}}) \\
&\dots
\end{aligned}$$

with width

$$d(V_{s_0 s_{-1}\dots s_{-n}}) \leq \nu d(V_{s_{-1}\dots s_{-n}}) \leq \nu^n d(V_{s_{-n}}) \leq \nu^n \, .$$

As formally the sets are

$$V_{s_0 s_{-1}\dots s_{-n}} = \{p \in Q | \quad \phi^k(p) \in V_{s_{-k}}, \quad k = 0, 1, \dots, n\}$$

we have that

$$V_{s_0 s_{-1}\dots s_{-n}} \subset V_{s_0 s_{-1}\dots s_{-n+1}}, \quad \text{for} \quad n \geq 0 \, .$$

Then the intersection

$$V(s) = \bigcap_{n=0}^{\infty} V_{s_0 s_{-1}\dots s_{-n}} = \{p \in Q | \quad \phi^{-k}(p) \in V_{s_k}, \quad k = 0, -1, \dots\}$$

defines a vertical curve owing to Lemma (3.10).

Analogously, a nested sequence of horizontal strips can also be defined

$$U_{s_1 s_2 \dots s_n} = U_{s_1} \cap \phi(U_{s_2 \dots s_n}) \quad \text{for} \quad s_k \in A$$

with width

$$d(U_{s_1\dots s_n}) \leq \nu d(U_{s_2\dots s_n}) \leq \nu^n d(U_{s_n}) \leq \nu^n \, .$$

What is more, using the same reasoning we also see that

$$U_{s_1 s_2\dots s_n} = \{p \in Q | \quad \phi^{-k+1}(p) \in U_{s_k}, \quad k = 0, 1, \dots, n\}$$

we have that

$$U_{s_1 s_2 \dots s_n} \subset V_{s_1 s_2 \dots s_{n-1}}, \quad \text{for} \quad n \geq 1 \, .$$

Then the intersection

$$U(s) = \bigcap_{n=1}^{\infty} U_{s_1 s_2 \ldots s_n} = \{p \in Q | \quad \phi^{-k+1}(p) \in U_{s_k}, \quad k = 1, 2, \ldots\}$$

defines a horizontal curve.

Observation. Recalling Assumption 1, ϕ maps the vertical strips to the horizontal ones, i.e $\phi(V_{s_k}) = U_{s_k}$, we have

$$U(s) = \bigcap_{n=1}^{\infty} U_{s_1 s_2 \ldots s_n} = \{p \in Q | \quad \phi^{-k}(p) \in V_{s_k}, \quad k = 1, 2, \ldots\} .$$

Finally, on account of Lemma (3.11), the intersection

$$V(s) \cap U(s) = \{p \in Q | \quad \phi^{-k}(p) \in V_{s_k} \quad k \in \mathbb{Z}\}$$

is exactly one point p.

Since $s_k \in A$, $k \in \mathbb{Z}$, for every $s_k \in A$ we will have a different sequence of nested vertical and horizontal sequences of strips. Considering all of them we can build the invariant set Λ.

$$\Lambda_{\infty} = \bigcup_{s_k \in A,\, k \in \mathbb{N}} V_{s_0 s_{-1} \ldots s_{-n} \ldots}$$

$$\Lambda_{-\infty} = \bigcup_{s_k \in A,\, k \in \mathbb{N}} U_{s_1 s_2 \ldots s_n \ldots}$$

$$\Lambda = \Lambda_{\infty} \cap \Lambda_{-\infty} \subset \{(\bigcup_{a \in A} V_a) \cap (\bigcup_{a \in A} U_a)\} \subset Q .$$

2) Definition of the map $\boldsymbol{\tau} : \boldsymbol{\Lambda} \longrightarrow \boldsymbol{\Sigma}$

By construction of the invariant set, for any point $p \in \Lambda$ there exist two and only two infinite sequences

$s_0 s_{-1} \ldots s_{-n} \ldots$ associated with the ν_v-vertical curve where p belongs

$s_1 s_2 \ldots s_n \ldots$ associated with the ν_h-horizontal curve where p belongs

for $s_n \in A$, $\quad n \in \mathbb{Z}$, such that

$$p = V_{s_0 s_{-1} \ldots s_{-n} \ldots} \cap U_{s_1 s_2 \ldots s_n \ldots}$$

Then we define

$$\begin{array}{rccl} \tau: & \Lambda & \longrightarrow & \Sigma \\ & p & \longmapsto & \tau(p) = (\ldots s_{-n} \ldots s_{-1} s_0 . s_1 s_2 \ldots s_n \ldots) \end{array}$$

where $\tau(p)$ is the bi-infinite sequence concatenating both infinite sequences. It's clear that τ is well defined by Lemma (3.11).

3) τ is a homeomorphism

Since Λ is a compact set and Σ is Haussdorf, τ just needs to be bijective and continuous for being a homeomorphism.

i) τ bijective

- τ is one-to-one
  For any p, $p' \in \Lambda$, we want to see that

  $$\text{if} \quad \tau(p) = \tau(p') \quad \Rightarrow \quad p = p' \,.$$

  By contradiction, let's assume that $\tau(p) = \tau(p')$ but $p \neq p'$. Being

  $$\tau(p) = \tau(p') = \{\ldots s_{-n} \ldots s_{-1} s_0 . s_1 \ldots s_n \ldots\} \,,$$

  by construction of the invariant set both p and p' lie in a ν_v-vertical cure $V_{s_0 s_{-1} \ldots s_{-n} \ldots}$ and in a ν_h-horizontal curve $U_{s_1 s_2 \ldots s_n \ldots}$. Regarding Lemma (3.11) the intersection of these two curves is just one unique point. Thus $p = p'$!!!

- τ is onto
  For any $s \in \Sigma$, $\quad \exists p \in \Lambda$ such that $\tau(p) = s$.

  Given

  $$s' = \{\ldots s_{-n} \ldots s_{-1} s_0 . s_1 \ldots s_n \ldots\}$$

  $\exists$ the ν_v-vertical curve $V_{s_0 s_{-1} \ldots s_{-n} \ldots} \in \Lambda_\infty$, and
  $\exists$ the ν_h-horizontal curve $U_{s_0 s_1 \ldots s_n \ldots} \in \Lambda_{-\infty}$.
  Then, by Lemma (3.11) $V_{s_0 s_{-1} \ldots s_{-n} \ldots}$ and $U_{s_1 s_2 \ldots s_n \ldots}$ intersect in a unique point p whose associated sequence is $\tau(p) = s$.

ii) τ continuous
We would like to see whether for an arbitrary $p \in \Lambda$ and $\epsilon > 0$ $\exists \delta > 0$ such that

$$|p - p'| < \delta \quad \Rightarrow \quad d(\tau(p), \tau(p')) < \epsilon \, .$$

Let's recall that $\Lambda \subset Q \subset \mathbb{R}^2$ has the Euclidean distance, but Σ has the distance introduced in Definition (3.1).

For $\epsilon > 0$ given, the condition $d(\tau(p), \tau(p')) < \epsilon$ holds as long as $\exists \; N = N(\epsilon)$, actually N satisfying $\epsilon < \frac{1}{2^{N+1}}$ such that, being

$$\tau(p) = \{ \dots s_{-n} \dots s_{-1} s_0 . s_1 \dots s_n \dots \}$$

$$\tau(p') = \{ \dots s'_{-n} \dots s'_{-1} s'_0 . s'_1 \dots s'_n \dots \}$$

then $s_i = s'_i \quad \forall |i| \leq N$.

By construction of Λ, p and p' belong to the vertical strip $V_{s_0 s-1 \dots s_{-n}}$ and the horizontal strip $U_{s_1 \dots s_n}$. Now the goal is to find a δ such that

$$|p - p'| < \delta \, .$$

First, let's denote

- $x = v_1(y)$, $x = v_2(y)$ vertical boundaries for the $V_{s_0 s_{-1} \dots s_{-N}}$
- $y = u_1(x)$, $y = u_2(x)$ horizontal boundaries for $U_{s_1 \dots s_N}$

with intersection points

$$p_{11} = u_1(x) \cap v_1(y) \qquad p_{12} = u_1(x) \cap v_2(y)$$
$$p_{22} = u_2(x) \cap v_2(y) \qquad p_{21} = u_2(x) \cap v_1(y) \, .$$

If $|p_{11} - p_{22}| > |p_{12} - p_{21}|$ we denote $p_{11} = (x_1, y_1)$, $p_{22} = (x_2, y_2)$.
If $|p_{11} - p_{22}| < |p_{12} - p_{21}|$ we denote $p_{12} = (x_1, y_1)$, $p_{21} = (x_2, y_2)$.
Then, we know for sure that

$$|p - p'| \leq |x_1 - x_2| + |y_1 - y_2| \, .$$

Lemma 3.14. *The following relations hold true:*

$$|x_1 - x_2| \leq \frac{1}{1 - \nu_v \nu_h} [\|v_1 - v_2\| + \nu_v \|u_1 - u_2\|]$$

$$|y_1 - y_2| \leq \frac{1}{1 - \nu_v \nu_h} [\|u_1 - u_2\| + \nu_h \|v_1 - v_2\|] \, .$$

Proof. See [Wiggins, 1990, pp. 456-457] . □

Finally, using

$$\|v_1 - v_2\| \equiv \max_{y\in[0,1]} |v_1(y) - v_2(y)| = d(V_{s_0 s_{-1}\dots s_{-N}}) \leq \nu_v^N$$

$$\|u_1 - u_2\| \equiv \max_{x\in[0,1]} |u_1(x) - u_2(x)| = d(V_{s_1\dots s_N}) \leq \nu_h^{N-1}$$

we have

$$|p - p'| \leq \frac{1}{1-\nu_v\nu_h}[(1+\nu_h)\nu_v^N + (1+\nu_v)\nu_h^N] \,.$$

Hence, continuity is checked taking

$$\delta = \frac{1}{1-\nu_v\nu_h}[(1+\nu_h)\nu_v^N + (1+\nu_v)\nu_h^N] \,.$$

4) The diagram commutes $\boldsymbol{\tau\phi = \sigma\tau}$

Last but not least, we see that for any $p \in \Lambda$

$$\text{if} \quad \tau(p) = \{\dots s_{-k}\dots s_{-1}s_0.s_1\dots s_k\}$$

$$\text{then} \quad \sigma\circ\tau(p) = \{\dots s_{-k}\dots s_{-1}.s_0 s_1\dots s_k\} \,.$$

Moreover, since

$$p = V_{s_0 s_{-1}\dots s_{-k}\dots} \cap U_{s_1 s_2\dots s_k\dots} = \{p \in Q| \quad \phi^{-k}(p) \in V_{s_k} \quad k \in \mathbb{Z}\}$$

then $\phi(p) = V_{s_{-1}\dots s_{-k}\dots} \cap U_{s_0 s_1\dots s_k\dots} = \{\phi(p) \in Q| \quad \phi^{-k+1}(p) \in V_{s_k} \quad k \in \mathbb{Z}\}$.

Thus

$$\tau\circ\phi(p) = \{\dots s_{-k}\dots s_{-1}.s_0 s_1\dots s_k\} \,.$$

Remark. The fact that Λ and Σ are homeomorphic allows the map ϕ in Λ to acquire the properties of the shift σ in Σ obtained in the previous chapter. Consequently, we state that ϕ has

- a countable infinity of periodic orbits of all periods,
- an uncuntable infinity of non-periodic orbits,

- a dense orbit.

due to Proposition (3.5). Besides, according to Proposition (3.3) Λ is

- compact,
- totally disconnected,
- perfect.

Moreover, in the development of all these properties we have seen that Σ is uncountable. Thus, Λ is uncountable too.

Therefore, since Λ is uncountable, compact, perfect and totally disconnected, Λ is a Cantor set.

$\square$

3.4 Alternate conditions

The fact that the Conley-Moser conditions are hard to verify forces us to strengthen our requirements on ϕ by assuming that ϕ is a diffeomorphism. The idea is to assume certain conditions about differentiability of ϕ which are easier to verify provided that they imply the original assumptions.

We represent ϕ in coordinates by

$$\begin{cases} x_1 = & h_1(x_0, y_0) \\ y_1 = & h_2(x_0, y_0) \end{cases},$$

where (x_1, y_1) is the image point of (x_0, y_0).

The mapping $d\phi$ is

$$\begin{cases} \xi_1 = & \frac{\partial h_1}{\partial x}\xi_0 + \frac{\partial h_1}{\partial y}\zeta_0 \\ \zeta_1 = & \frac{\partial h_2}{\partial x}\xi_0 + \frac{\partial h_2}{\partial y}\zeta_0 \end{cases},$$

where (ξ_0, ζ_0) is the tangent vector at (x_0, y_0), (ξ_1, ζ_1) the tangent vector at (x_1, y_1), and the partial derivatives are evaluated at (x_0, y_0).

Definition 3.15. For any point $z_0 = (x_0, y_0) \in Q$ we denote a vector emanating from this point by $(\xi_{z_0}, \zeta_{z_0}) \in \mathbb{R}^2$.
Moreover, the stable sector at z_0 is defined by

$$S^s_{z_0} = \{(\xi_{z_0}, \zeta_{z_0}) \in \mathbb{R}^2 || \zeta_{z_0}| \leq \nu_h |\xi_{z_0}|\}$$

and, the unstable sector at z_0 is defined by

$$S^u_{z_0} = \{(\xi_{z_0}, \zeta_{z_0}) \in \mathbb{R}^2 || \xi_{z_0}| \leq \nu_v |\zeta_{z_0}|\} .$$

Now we already have the sufficient tools for stating the alternative assumption.

Assumption 3.

- *The stable sector for* $z_0 \in \bigcup_{a \in A} V_a$ *is mapped into itself by* $d\phi$, *i.e.*

  $$\text{being} \quad S^s = \bigcup_{z_0 \in \bigcup_{a \in A} V_a} S^s_{z_0} \quad \text{it holds that} \quad d\phi(S^s) \subset S^s .$$

  Moreover, there exists $\mu \in (0, 1)$ *such that if* $(\xi_0, \zeta_0) \in S^s$ *and* $(\xi_1, \zeta_1) \in S^s$ *its image point, then*

  $$|\xi_1| \geq \mu^{-1} |\xi_0| .$$

- *Similarly, the unstable sector is mapped into itself by* $d\phi^{-1}$, *i.e.*

  $$\text{being} \quad S^u = \bigcup_{z_0 \in \bigcup_{a \in A} U_a} S^u_{z_0} \quad \text{it holds that} \quad d\phi^{-1}(S^u) \subset S^u .$$

  In addition, if $(\xi_1, \zeta_1) \in S^u$ *and* $(\xi_0, \zeta_0) \in S^u$ *its pre-image, then*

  $$|\zeta_0| \geq \mu^{-1} |\zeta_1| .$$

Observation. The conditions in Assumption 3 somehow show the instability of the mapping under iteration. As we can see, the horizontal components of a tangent vector increases at least by a factor of μ^{-n} under $d\phi^n$ for $n \geq 1$, and also the vertical component by a factor of μ^{-n} under $d\phi^{-n}$. See Figure 3.4 .

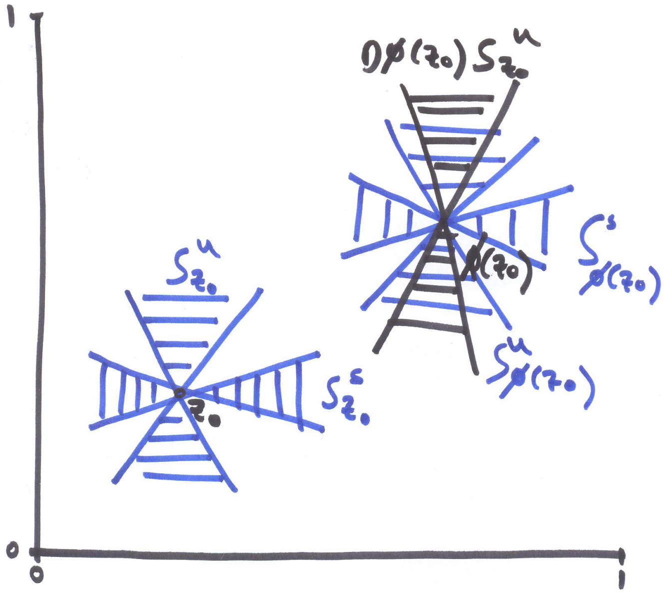

Figure 3.4

Theorem 3.16. *If ϕ is a continuously differentiable mapping satisfying Assumption 1, from the previous section, and Assumption 3, just stated above with $0 < \mu < \frac{1}{2}$, then the conditions from Assumption 2 holds with $\nu = \frac{\mu}{1-\mu}$.*

Proof. Let γ be a vertical curve in an arbitrary vertical strip V_a. Let U_a be the horizontal strip, corresponding to the image of V_a by ϕ, $U_a = \phi(V_a)$. Notice that γ will intersect its boundaries. Let $\hat{\gamma} = \gamma \cap U_a$ be the segment of the curve γ which connects the horizontal boundaries of U_a. See Figure 3.5.

By Assumption 1, $\phi^{-1}(\hat{\gamma})$ connects the horizontal boundaries of $\phi^{-1}(U_a) = V_a$, but we don't know whether $\phi^{-1}(\hat{\gamma}) = \phi^{-1}(\gamma) \cap V_a$ is a vertical curve yet. As $d\phi^{-1}$ maps S^u into S^u it follows by the application of the mean value theorem that for any pair of points $(x_1, y_1), (x_2, y_2) \in \phi^{-1}(\hat{\gamma})$ it holds $|x_1 - x_2| \leq \mu|y_1 - y_2|$.
Thus, $\phi^{-1}(\hat{\gamma})$ is the graph of a vertical curve $x = v(y)$.

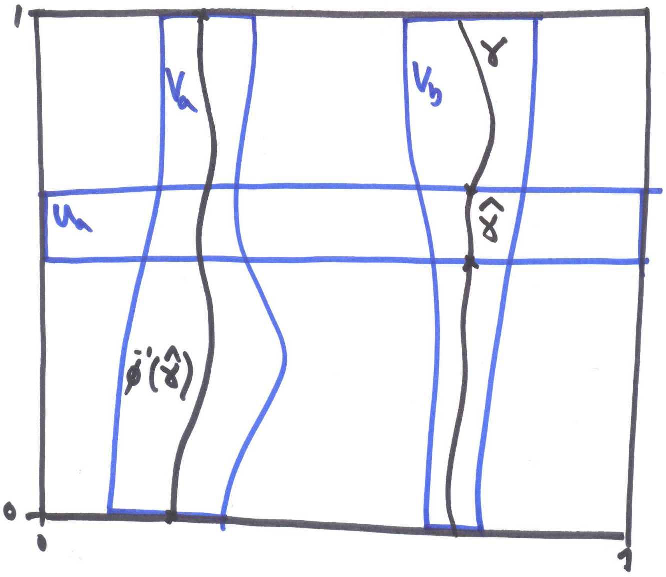

Figure 3.5

Now, what we do first is to apply this result to the boundaries of a vertical strip $V \subset V_a$ and deduce that $\phi^{-1}(\hat{V}) = \phi^{-1}(V) \cap U_a$ is a vertical strip.

Second, we have to verify that

$$d(\phi^{-1}(\hat{V})) \leq \nu d(\hat{V}) \quad \text{for} \quad 0 < \mu < \frac{1}{2} \quad \text{and} \quad \nu = \frac{\mu}{1-\mu} < 1 \,.$$

Let p_1, p_2 be points on the vertical boundaries of $\phi^{-1}(\hat{V})$ with the same y-coordinates such that

$$d(\phi^{-1}(\hat{V})) = |p_1 - p_2| \,.$$

Being $p(t) = (1-t)p_1 + tp_2$ the parameterisation of the segment connecting the two points and $z(t) = \phi(p(t))$ its image curve. Since $p(t)$ is parallel to the x-axis, $\dot{p} \in S^s$. Thus, $\dot{z} = d\phi(\dot{p}) \in S^s$ by Assumption 3.
Therefore, $z(0), z(1)$ lie on a horizontal curve and on two vertical lines at a distance $d(\hat{V})$. By Lemma (3.11)

$$|z(0) - z(1)| \leq \frac{1}{1-\mu} d(\hat{V}) \,.$$

Finally, using the second condition in Assumption 3 we have that for $z(t) = (x(t), y(t))$

$$|\dot{x}| \geq \mu^{-1} |\dot{p}| > 0 \,.$$

Hence, it holds that

$$
\begin{aligned}
d(\phi^{-1}(\hat{V})) = |p_1 - p_2| = \int_0^1 |\dot{p}| \, dt \leq \mu \int_0^1 |\dot{x}| \, dt = \mu |x(1) - x(0)| \\
\leq \mu |z(1) - z(0)| \leq \frac{\mu}{1-\mu} d(\hat{V}) \, .
\end{aligned}
$$

□

3.5 Dynamics Near Saddle Points of two-dimensional Maps

What we want to show in this section is the fact that the existence of certain orbits of a two-dimensional map implies that in a neighbourhood small enough the conditions given in the previous section hold. Hence, we will have sufficient conditions for a two-dimensional map to posses an invariant Cantor set on which is topologically conjugate to a full shift of N symbols. The study deeply cover the case where the two-dimensional map has just one hyperbolic periodic point. Nevertheless, a qualitative discussion will be done for other scenarios with more than one hyperbolic periodic point.

Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a C^r-diffeomorphism and p a hyperbolic periodic point of h. Without loss of generality we can suppose that the periodic point p is a fixed point. Due to the periodicity of p there is $n \in \mathbb{Z}$ such that $h^n(p) = p$, so the further development could be applied to h^n. Now, we assume that p is a saddle, so the eigenvalues λ_1 and λ_2 of dh at p hold $0 < \lambda_1 < 1 < \lambda_2$. Moreover, let's denote by (x_k, y_k) the image point of (x_0, y_0) under h^k and (ξ_k, ζ_k) the image point of (ξ_0, ζ_0) under dh^k. For $k = 1$ we have

$$
\begin{cases}
x_1 = f(x_0, y_0) \\
y_1 = g(x_0, y_0)
\end{cases}
$$

$$
\begin{cases}
\xi_1 &= f_x \xi_0 + f_y \zeta_0 \\
\zeta_1 &= g_x \xi_0 + g_y \zeta_0
\end{cases}
$$

The Smale-Moser Theorem will show sufficient conditions for h to possess an invariant Cantor set in this particular case.

Theorem 3.17 (Smale-Moser)**.**
If a C^∞-diffeomorphism h possesses a homoclinic point r, at which the curves $W^s(p)$ and $W^u(p)$ of a hyperbolic fixed point p intersect transversally, then in any neighbourhood of r there exists a transversal map $\tilde{h}$ (related to h and defined in the proof) of a quadrilateral R which possesses an invariant subset I homeomorphic to Σ, the space of sequences of N symbols and the dynamics of $\tilde{h}$ in I is topologically conjugated to the shift of Bernouilli in Σ.

Proof. To prove this theorem will try to construct a set of vertical strips and a set of horizontal strips in a certain region of the plane such that the alternate conditions from the previous section hold. Then the statement of the theorem will be already reduced to Theorem (3.13) which will lead us to the existence of a Cantor set on which $\tilde{h}$ is homeomorphically conjugated with a shift of N symbols.

1) Local study of the diffeomorphism h near p
To begin with, we want to work with h using the most convenient coordinates near p. To do so, as p is a saddle there will be the two invariant curves $W^u(p)$ and $W^s(p)$. The idea is to introduce a certain local coordinates that let us put p in the origin and the invariant curves to the coordinate axis.
Let U be a neighbourhood of $(0,0)$, where f and g are continuously differentiable in U and

$$f(0,y_0) = g(x_0,0) = 0\ ,$$

$$f_x(0,0) = \lambda_2\ ,$$

$$g_y(0,0) = \lambda_1\ .$$

Hence, the stable invariant curve $W^s(0)$ lays on $x = 0$ and the unstable invariant curve $W^u(0)$ lays on $y = 0$, in the neighbourhood U.

2) Global consequences of a homoclinic orbit
Next, we construct the quadrilateral R at the homoclinic point r such that two of whose sides lay on $W^s(p)$ and $W^u(p)$. This is possible due to the transversality of the intersection of the invariant curves.
Since r is a homoclinic point of h, $\exists\, n_0, n_1 \in \mathbb{Z}$ such that $h^{n_0}(r) \in U$ and $h^{-n_1}(r) \in U$.

What is more, we make R small enough for $h^{n_0}(R) \in U$ and $h^{-n_1}(R) \in U$ as long as $h^{n_0}(R) \cap h^{-n_1}(R) = \emptyset$. Then the following domains can be defined

$$A_0 = h^{n_0}(R) \,,$$
$$A_1 = h^{-n_1}(R) \,.$$

It's interesting to realise that one of the sides of A_0 lays on $x = 0$ and one of the sides of A_1 lays on $y = 0$. See Figure 3.6. Moreover, let's consider the adjacent sides of A_0 by the curves C and C', and two points on these curves $z_0 \in C$ and $z_0' \in C'$, respectively. The Lambda Lemma states that $\exists N_0 \in \mathbb{Z}^+$ such that for $N \geq N_0$, $|\zeta_{h^N(z_0)}|$ can be arbitrarly small, where $(\xi_{h^N(z_0)}, \zeta_{h^N(z_0)})$ is the vector tangent to $h^N(C) \cap U$ at $h^N(z_0)$. The same reasoning is used for the other adjacent side C'. Thus, we have seen that C and C' are ν_h-horizontal curves. Furthermore, if we applied the Lambda Lemma to the inverse map h^{-1} we would also see that the two adjacent sides of A_1 to $y = 0$ are ν_v-vertical curves.

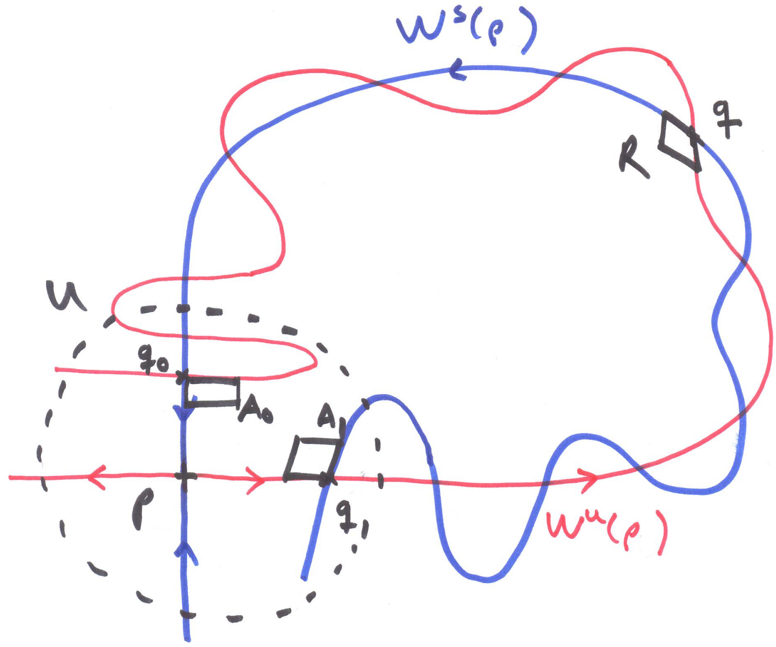

Figure 3.6

3) **Construction of the transversal map ψ**
After the choice of R, A_0 and A_1 the transversal map ψ from A_0 into A_1 is a map
$$\begin{array}{rrcl} \psi: & D(\psi) \subset A_0 & \longrightarrow & A_1 \\ & q & \longmapsto & \psi(q) \end{array}$$
such that a point q belongs to the domain $D(\psi)$ if

- $q \in A_0$,
- $h(q), h^2(q), \ldots, h^{k-1}(q) \in U$,
- $h^k(q) \in A_1$.

Then, the transversal map is defined by $\psi(q) = h^k(q)$ with possitive integer such that $q \in D(\psi)$.

Finally, the transversal map $\tilde{h}$ from R into R comes naturally with the following diagram
$$\begin{array}{ccc} R & \xrightarrow{\tilde{h}} & R \\ {\scriptstyle h^{n_0}}\big\downarrow & & \big\downarrow{\scriptstyle h^{n_1}} \\ A_0 & \xrightarrow{\psi} & A_1 \end{array}$$
satisfying that $\tilde{h}(q) = h^{\tilde{k}}(q)$ for $q \in D(\tilde{h})$, with $\tilde{k} = n_0 + k + n_1$.

4) **Construction of horizontal and vertical strips**
Now if we are able to build a set of vertical strips in A_0 and a set of horizontal strips in A_1 such that the transversal map connects them homeomorphically, we will be able to do the same construction in R with its corresponding transversal map.

First, we start by choosing a set of ν_v-vertical strips in A_0. Then, once we apply the Lambda Lemma (2.14) to the horizontal boundaries of A_0 we see that $\exists N_0 \in \mathbb{Z}^+$ such that for $N \geq N_0$ both horizontal boundaries of the component $h^N(A_0) \cap U$ intersect with the two vertical boundaries of A_1 due to the stretching of the tangent vectors.
Next, if we apply the lambda lemma to $\widetilde{U_N} = h^N(A_0) \cap U$ for h^{-1} we get that for N_0 sufficiently large $\widetilde{V_N} = h^{-N}(\widetilde{U_N})$ is a ν_v-vertical strip with their horizontal boundaries contained in the horizontal boundaries of A_0. Hence, each ν_v-vertical strip in A_0 is map homeomorphically to a ν_h-horizontal strip in A_1. Now we need to proof that the Assumption

1 and Assumption 3 from section (2.2) hold. See Figure 3.7.

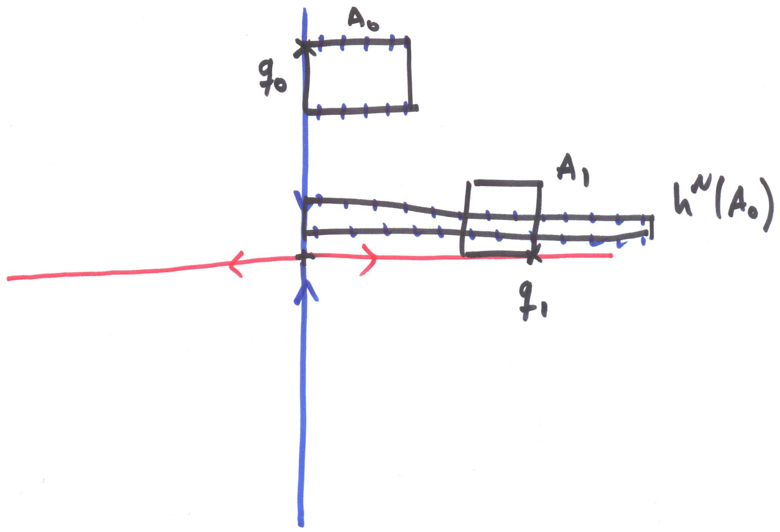

Figure 3.7

5) Alternate conditions

For convenience let's take

$$Q = \{(x, y) \in \mathbb{R}^2 | \quad 0 \leq x \leq a, \quad 0 \leq y \leq a\}$$

$$Q_0 = \{(x, y) \in \mathbb{R}^2 | \quad 0 \leq x \leq \delta, \quad b \leq y \leq a\}$$

$$Q_1 = \{(x, y) \in \mathbb{R}^2 | \quad c \leq x \leq a, \quad 0 \leq y \leq \delta\}$$

where Q is the neighbourhood of p, and Q_0, Q_1 two regions in Q such that $A_0 \subset Q_0$ and $A_1 \subset Q_1$.

Considering the adjacent sides of A_0 by $y = u(x)$ and the adjacent sides of A_1 by $x = v(y)$, the lambda lemma implies that the images under h^k of the two curves $y = u(x)$ bounding A_0, intersect the domain Q_1 in a curve connecting $x = 0$ and $x = a$, for large k. However, what we need is additional information about the derivatives of the boundary curve $y = u(x)$. The following lemma, which comes from applying the Lambda Lemma for our case of study, is a useful tool for acquiring it.

Lemma 3.18. *For sufficiently small $a > 0$ and any sequence of iterates (x_k, y_k), $k = 0, 1, \ldots, n$, in the interior of Q, the inequality*

$$|\zeta_0| \le \sqrt{\frac{y_0}{x_0}}|\xi_0|$$

implies

$$|\zeta_k| \le \sqrt{\frac{y_k}{x_k}}|\xi_k| \quad \textit{for} \quad k = 1, 2, \ldots, n\,.$$

Moreover, under these assumptions one has

$$|\xi_k| \ge \sqrt{\frac{x_k}{x_0}}|\zeta_0|\,.$$

Proof. See [Moser, 2001, pp. 182-183] . □

Since for $(x_0, y_0) \in A_0 \subset Q_0$ we have that $0 \le x_0 \le \delta, \quad b \le y \le a$, the condition for the tangent vectors satisfies

$$|\zeta_0| \le \sqrt{\frac{y_0}{x_0}}|\xi_0| \le \sqrt{\frac{b}{\delta}}|\xi_0|$$

mainly because δ can be as small as needed. Again, as $(x_k, y_k) \in A_1 \subset Q_1$ we have $c \le x_k \le a, \quad 0 \le y_k \le \delta$. Thus, the lemma implies that

$$|\zeta_k| \le \sqrt{\frac{y_k}{x_k}}|\xi_k| \le \sqrt{\frac{\delta}{c}}|\xi_k|\,.$$

As a result, the strips connect the opposite sides of A_1 due to the arbitrarily smallness of δ. What is more, if $y = h(x)$ the images of the boundary curves $y = u(x)$ we have seen that

$$\frac{dh}{dx} \le \sqrt{\frac{\delta}{c}}$$

which basically means that the strip formed by the two curves is a horizontal strip. Now by the same reasoning the pre-image of this horizontal strip is a vertical strip, so the first assumption holds.

$$h^k(\tilde{V}_k) = \tilde{U}_k\,.$$

Then, we have to see that Assumption 3 from the previous section holds. Directly from the lemma we get that for Q sufficiently small the stable sector bundle

$$|\zeta| \leq \sqrt{\frac{y}{x}}|\xi|$$

is mapped into itself under dh and the unstable sector bundle

$$|\xi| \leq \sqrt{\frac{x}{y}}|\zeta|$$

is mapped into itself under dh^{-1}.

Last but not least, setting

$$V_k = h^{-n_0}(\tilde{V_k}), \quad U_k = h^{n_1}(\tilde{U_k})$$

we get the same construction of vertical and horizontal strips in R such that

$$\tilde{h}(V_k) = \tilde{h} \circ h^{-n_0}(\tilde{V_k}) = h^{n_1} \circ \psi(\tilde{V_k}) = U_k \,.$$

$\square$

To conclude this chapter we ask ourselves how the dynamics near the saddle hyperbolic point would be without the hypothesis of the existence of a homoclinic point. Let $h \in \mathrm{Diff}^r(\mathbb{R}^2)$ having two hyperbolic fixed points l_0 and m_0. Suppose $q \in W^s(p_0) \cap W^u(m_0)$ transverse heteroclinic point.
Here, the existence of a transverse heteroclinic point does not imply the existence of a Cantor set on which some iterate of h is topologically conjugate to a shift on N symbols. Although the Lambda Lemma can still be applied to curves intersecting $W^s(l_0)$, it is pointless because we have no information about how $W^u(l_0)$ behaves globally with just these hypotheses. As we see in Figure (3.8), the Lambda Lemma just give us information in a neighbourhood of $W^u(l_0)$.

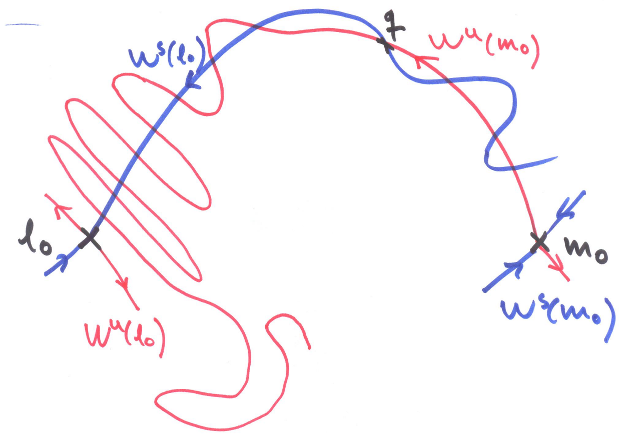

Figure 3.8

Chapter 4

Melnikov Method for Heteroclinic Orbits

4.1 Introduction

Let's consider two-dimensional systems that are periodic in t and their unperturbed vector field is Hamiltonian and autonomous.

$$\begin{cases} \dot{x} = f_1(x,y) + \epsilon g_1(x,y,t,\epsilon) \\ \dot{y} = f_2(x,y) + \epsilon g_2(x,y,t,\epsilon) \end{cases} \tag{4.1}$$

where f and g are C^r functions, $r \geq 2$.

$$\begin{aligned} &f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ &g : U \times \mathbb{R} \times [0, \epsilon_0) \longrightarrow \mathbb{R}^2 \end{aligned}$$

with $f_1(x,y) = \frac{\partial}{\partial y} H(x,y)$, $f_2(x,y) = -\frac{\partial}{\partial x} H(x,y)$ for a C^{r+1} scalar valued function $H(x,y)$ and g periodic in t with period $T = \frac{2\pi}{w}$,

$$g(x,y,t,\epsilon) = g(x,y,t+T,\epsilon) \; .$$

In vector form we have

$$\dot{q} := h(q,t,\epsilon) = JDH(q) + \epsilon g(q,t,\epsilon) \; , \tag{4.2}$$

where $q = (x,y)$, $DH = \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix}$, $g = (g_1, g_2)$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Let us denote the solution of the perturbed system (4.1) by $V(t, x, y, \epsilon)$ where $V(0, x, y, \epsilon) = (x, y)$ and the flow of the unperturbed Hamiltonian system by $\phi_t(x, y) = V(t, x, y, 0)$.

Before we go further, let's introduce a couple of assumptions to restrict our study for heteroclinic orbits.

Assumption 1. *$\exists l_0 m_0 \in U$, two hyperbolic saddle points in the unperturbed system, connected by an heteroclinic orbit, $q_0(t) \equiv (x_0(t), y_0(t))$.*

Assumption 2. *Let $\Gamma_{l_0,m_0} = \{q \in \mathbb{R}^2 | q = q_0(t), t \in \mathbb{R}\} = W^s(l_0) \cap W^u(m_0) \cup \{l_0\} \cup \{m_0\}$ be the set of points that belong to the heteroclinic orbit. Being $W^s(l_0)$ and $W^u(m_0)$ the stable manifold of l_0 and the unstable manifold of m_0, respectively.*

4.2 Phase Space Geometry for the Unperturbed Vector Field

Now let's describe the two-dimensional non-autonomous system (4.1) as an autonomous three-dimensional system

$$\begin{cases} \dot{x} = \frac{\partial}{\partial y} H(x, y) + \epsilon g_1(x, y, \phi, \epsilon) \\ \dot{y} = -\frac{\partial}{\partial x} H(x, y) + \epsilon g_2(x, y, \phi, \epsilon) \\ \dot{\phi} = w \end{cases} \tag{4.3}$$

where $\phi = wt + \phi_0$.

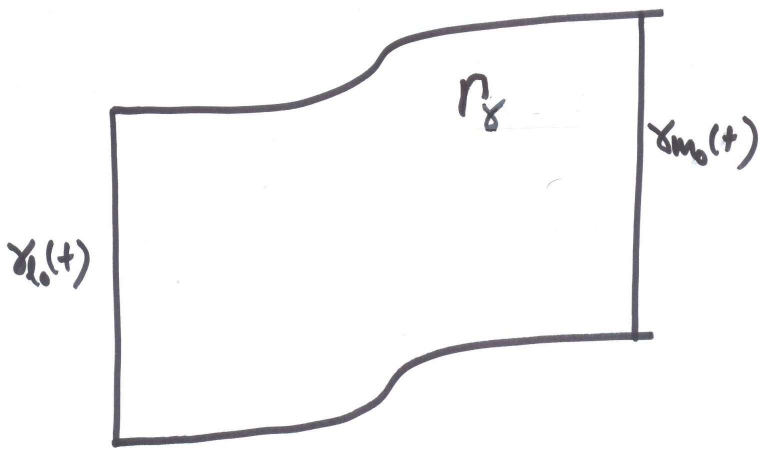

Figure 4.1

Analysing the phase space for the unperturbed system, Figure (4.1), we realize that

- The hyperbolic fixed points l_0, m_0 become the periodic orbits $\gamma_{l_0}(t) = (l_0, \phi(t) = wt + \phi_0)$ and $\gamma_{m_0}(t) = (m_0, \phi(t) = wt + \phi_0)$ respectively.
- According to Assumption 1 above, $W^s(\gamma_{l_0}(t))$ and $W^u(\gamma_{m_0}(t))$, which are the stable and unstable manifolds of the periodic orbits $\gamma_{l_0}(t)$ and $\gamma_{m_0}(t)$ respectively, coincide along a two-dimensional heteroclinic manifold Γ_γ.

The parameterisation of Γ_γ in the heteroclinic coordinates is

$$\Gamma_\gamma = \{(q, \phi) \in \mathbb{R}^2 \times S^1 | \quad q = q_0(-t_0), t_0 \in \mathbb{R}, \quad \phi \in (0, 2\pi]\},$$

where t_0 is the time of flight from $q_0(-t_0)$ till q(0) along the heterocinic orbit $q_0(t)$. Moreover, we also define the normal vector to Γ_γ at each point $p \in \Gamma_\gamma$

$$\Pi_p = \left(\frac{\partial H}{\partial x}(x_0(-t_0), y_0(-t_0)), +\frac{\partial H}{\partial y}(x_0(-t_0), y_0(-t_0)), 0\right).$$

Let's notice that varying t_0 and ϕ_0 let us move Π_p to all the points in Γ_γ.

4.3 Phase Space Geometry for the Perturbed Vector Field

Now we will focus on the study of how Γ_γ is affected by a perturbation.
To begin with, let's see how $\gamma_{l_0}(t)$ and $\gamma_{m_0}(t)$ behave along with $W^s(\gamma_{l_0}(t))$ and $W^u(\gamma_{m_0}(t))$ respectively.

Proposition 4.1. *For ϵ small enough, the periodic orbit of the unperturbed vector field $\gamma_{l_0}(t)$ remain a periodic orbit of the perturbed vector field, $\gamma^\epsilon_{l_0}(t) = \gamma_{l_0}(t) + \mathcal{O}(\epsilon)$, depending on ϵ in a C^r way. In addition, the stability type is preserved and $W^s_\delta(\gamma_{l_0}(t))$ is C^r close to $W^s_\delta(\gamma^\epsilon_{l_0}(t))$.*

Proof. To prove this Theorem we need to apply the Stable Manifold Theorem to the Poincaré map of a cross-section Σ_{ϕ_0} of the phase space $\mathbb{R}^2 \times S^1$. See [Wiggins, 1990, p. 488] . □

Remark. With the aim of clarifying the meaning for the localy stable manifold, let's recall its definition.
$W^s_\delta(\gamma^\epsilon_{l_0}(t)) = \{(q,\phi) \in W^s(\gamma^\epsilon_{l_0}(t))| \quad \phi(t,q,\phi) \in B(0,\delta), \quad \forall t \geq 0 \quad \}$
where ϕ is the flow of the perturbed system. Since the nature of p_0 is hyperbolic the fact that $\phi(t,q,\phi) \in B(0,\delta) \quad \forall t \geq 0$ for an arbitrary δ implies $\lim\limits_{t\to+\infty} \phi(t,q,\phi) \in \gamma^\epsilon_{l_0}$.
Thus, for ϵ_0 small, $\exists N(\epsilon_0)$ neighbourhood in $\mathbb{R}^2 \times S^1$ containing γ_{l_0} such that
$W^s_\delta(\gamma_{l_0}(t)) = W^s(\gamma_{l_0}(t)) \cap N(\epsilon_0)$ and
$W^s_\delta(\gamma^\epsilon_{l_0}(t)) = W^s(\gamma^\epsilon_{l_0}(t)) \cap N(\epsilon_0) \quad \forall t \geq t_0$ for t_0 big enough.

Finally, let's point out that this proposition could also suit the other hyperbolic point in our study, m_0, and its unstable manifold $W^u(m_0)$. The same reasoning is used although $t \to -\infty$ in this case.

After that, let's consider the following cross-section of the phase space.

$$\Sigma^{\phi_0} = \{(q,\phi) \in \mathbb{R}^2 \times S^1 | \phi = \phi_0\} \, .$$

Notice that

- $\gamma_{l_0}(t) \cap \Sigma^{\phi_0} = l_0$,
- $\gamma_{m_0}(t) \cap \Sigma^{\phi_0} = m_0$,
- $\Gamma_\gamma(t) \cap \Sigma^{\phi_0} = \{(q,\phi) \in \mathbb{R}^2 \times S^1| \quad q = q_0(t_0), t_0 \in \mathbb{R} \quad \phi = \phi_0 \in (0, 2\pi]\}$.

Let $(q(t), \phi(t))$ and $(q^\epsilon(t), \phi(t))$ be trajectories of the unperturbed and perturbed vector fields respectively. Their projections onto Σ^{ϕ_0} are given by $(q_0(t), \phi_0)$ and $(q^\epsilon(t), \phi_0)$.

Observation. It is good to notice that $q_\epsilon(t)$ does depend on ϕ_0 but does not $q(t)$. Consequently, $(q_\epsilon(t), \phi_0)$ can be a very complicated curve in Σ^{ϕ_0} intersecting itself multiple times.

Next, we will define the splitting of $W^s(\gamma^\epsilon_{l_0}(t))$ and $W^u(\gamma^\epsilon_{m_0}(t))$, see Figures 4.2 and 4.3. To do so, let's recall that for $p \in \Gamma_\gamma$, $W^s(\gamma_{l_0}(t))$ and $W^u(\gamma_{m_0}(t))$ intersect Π_p transversally at p. Thus, using

- persistence of transversal intersections ,
- $W^s(\gamma^\epsilon_{l_0}(t))$, $W^u(\gamma^\epsilon_{m_0}(t))$ are C^r in ϵ ,

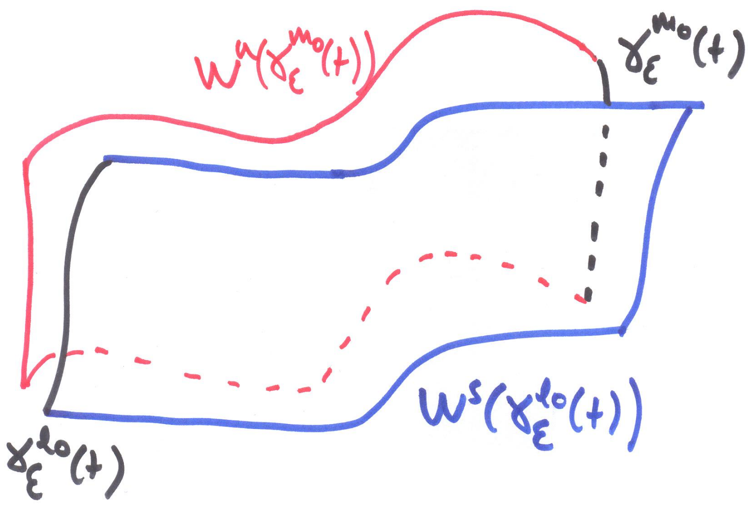

Figure 4.2

we have that for ϵ sufficiently small, $W^s(\gamma^\epsilon_{l_0}(t))$ and $W^u(\gamma^\epsilon_{m_0}(t))$ also intersect Π_p transversally at points p^s_ϵ and p^u_ϵ, respectively.
Hence, the distance between $W^s(\gamma^\epsilon_{l_0}(t))$ and $W^u(\gamma^\epsilon_{m_0}(t))$ at the point p is

$$d(p,\epsilon) \equiv p^u_\epsilon - p^s_\epsilon = \frac{(p^u_\epsilon - p^s_\epsilon)(DH(q_0(-t_0)),0)}{\|DH(q_0(-t_0)),0)\|} \tag{4.4}$$

where $\|(DH(q_0(-t_0)),0)\| = \sqrt{(\frac{\partial}{\partial x}H(q_0(-t_0)))^2 + (\frac{\partial}{\partial y}H(q_0(-t_0)))^2}$. Here we have used that $p^u_\epsilon - p^s_\epsilon$ is parallel to $DH(q_0(-t_0))$.

Remark. The distance introduced above is a distance with sign.

Since p^u_ϵ and p^s_ϵ lie on Π_p, both p^u_ϵ, p^s_ϵ have the same value for ϕ. $p^u_\epsilon = (q^u_\epsilon, \phi_0)$ and $p^s_\epsilon = (q^s_\epsilon, \phi_0)$. What is more, the fact that every $p \in \Gamma_\gamma$ can be uniquely represented by (t_0, ϕ_0), $p = (q_0(-t_0), \phi_0)$, with $t_0 \in \mathbb{R}$, $\phi_0 \in (0, 2\pi]$ allow us to redefine the distance depending just on t_0, ϕ_0 and ϵ .

$$d(p,\epsilon) = d(t_0,\phi_0,\epsilon) = \frac{(q^u_\epsilon - q^s_\epsilon)(DH(q_0(-t_0)),0)}{\|DH(q_0(-t_0)),0)\|} \,. \tag{4.5}$$

However, there are no restrictions about how many times these manifolds could intersect Π_p.

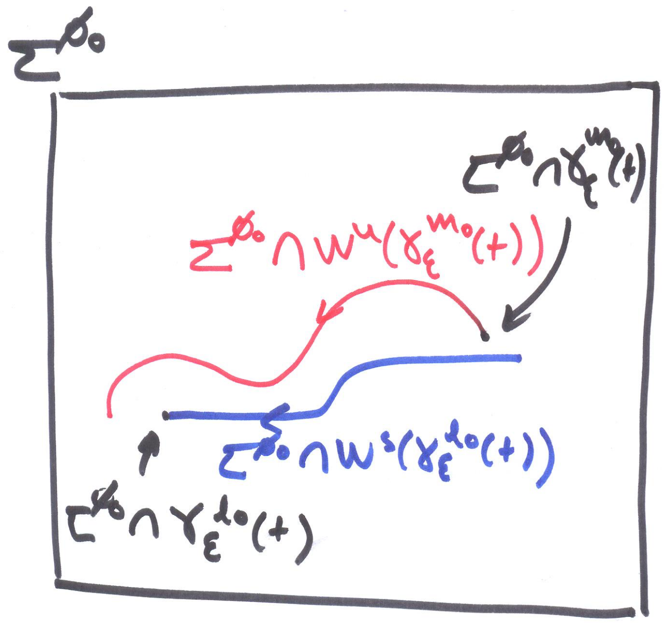

Figure 4.3

Definition 4.2. Let $p^s_{\epsilon,i} \in W^s(\gamma^\epsilon_{p_0}(t)) \cap \Pi_p$ for $i \in I$ where I some index set, be a point of intersection between the stable manifold and Π_p. Let $(q^s_{\epsilon,i}(t), \phi(t)) \in W^s(\gamma^\epsilon_{p_0}(t))$ be the orbit of the perturbed vector field with $(q^s_{\epsilon,i}(0), \phi(0)) = p^s_{\epsilon,i}$. Then for some $i = \bar{i} \in I$, $p^s_{\epsilon,\bar{i}}$ is the point in $W^s(\gamma^\epsilon_{l_0}(t)) \in \Pi_p$ closest to $\gamma^\epsilon_{l_0}(t)$ if

$$(q^s_{\epsilon,\bar{i}}(t), \phi_0) \cap \Pi_p = \emptyset, \qquad \forall t > 0\,.$$

Analogously, the point in $W^u(\gamma^\epsilon_{m_0}(t)) \in \Pi_p$ closest to $\gamma^\epsilon_{m_0}(t)$ can also be defined.

Observation. Somehow we can forget about the other non-closest points because when we restrict our study in a sufficiently small compact domain for ϕ_0 and t_0, there are just the p^s_ϵ and p^u_ϵ closest to $\gamma^\epsilon_{l_0}$ and $\gamma^\epsilon_{m_0}$ respectively.

4.4 Derivation of the Melnikov Function

Since we are studying the behaviour of the system (4.3) for ϵ small, a Taylor expansion around $\epsilon = 0$ for the distance is permitted.

$$d(t_0, \phi_0, \epsilon) = d(t_0, \phi_0, 0) + \epsilon \frac{\partial}{\partial \epsilon} d(t_0, \phi_0, 0) + O(\epsilon^2)$$

where

- $d(t_0, \phi_0, 0) = 0$
- $\frac{\partial}{\partial\epsilon} d(t_0, \phi_0, 0) = \frac{DH(q_0(-t_0))\left(\frac{\partial q_\epsilon^u}{\partial\epsilon}|_{\epsilon=0} - \frac{\partial q_\epsilon^s}{\partial\epsilon}|_{\epsilon=0}\right)}{\|DH(q_0(-t_0))\|}$.

The Melnikov function is defined as the lowest order non-zero term in the Taylor expansion, up to a normalization factor, for the distance between $W^s(\gamma_{l_0}^\epsilon(t))$ and $W^u(\gamma_{m_0}^\epsilon(t))$ at the point p.

$$M(t_0, \phi_0) \equiv DH(q_0(-t_0))\Big(\frac{\partial q_\epsilon^u}{\partial\epsilon}|_{\epsilon=0} - \frac{\partial q_\epsilon^s}{\partial\epsilon}|_{\epsilon=0}\Big) . \tag{4.6}$$

Since $DH(q_0(-t_0))$ is not zero for t_0 finite we have that

$$M(t_0, \phi_0) = 0 \Longrightarrow \frac{\partial}{\partial\epsilon} d(t_0, \phi_0) = 0 .$$

In addition, a time dependent Melnikov function is also introduced

$$M(t; t_0, \phi_0) \equiv DH(q_0(t - t_0))\Big(\frac{\partial q_\epsilon^u(t)}{\partial\epsilon}|_{\epsilon=0} - \frac{\partial q_\epsilon^s(t)}{\partial\epsilon}|_{\epsilon=0}\Big) . \tag{4.7}$$

Notice that for $t = 0$, $M(0; t_0, \phi_0) = M(t_0, \phi_0)$.
What is next, we will work with this time-dependent Melnikov function with the aim of obtaining a variational equation for ϵ. Its solution with the constrain of the time-dependent Melnikov's function for $t \to \pm\infty$ will give us another way for understanding the original Melnikov function.
In a more compact notation, $M(t; t_0, \phi_0) \equiv \Delta^u(t) - \Delta^s(t)$, with

$$\Delta^{u,s}(t) = DH(q_0(t - t_0))\Big(\frac{\partial}{\partial\epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) . \tag{4.8}$$

On one hand, the term $q_\epsilon^{u,s}(t)$ solves

$$\frac{d}{dt} q_\epsilon^{u,s}(t) = JDH(q_\epsilon^{u,s}(t)) + \epsilon g(q_\epsilon^{u,s}(t), \phi(t), \epsilon)$$

with $\phi(t) = wt + \phi_0$.

Regarding the fact that, in a vector field $\dot{x} = f(x, t, \nu)$ with solution $x(t; t_0, x_0, \nu)$, if $f(x, t, \nu)$ is C^r in $U \subset \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^p$ with $r \geq 1$, then $x(t; t_0, x_0, \nu)$ is a C^r function with respect to t, t_0, x_0 and ν. We see that, the solution $q_\epsilon^{u,s}(t)$ is

C^r with respect to t, ϵ.
Consequently, after differentiating with respect to ϵ and evaluating for $\epsilon = 0$

$$\frac{d}{dt}\Big(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) = JD^2H(q_0(t-t_0))\Big(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) + g(q_0(t-t_0), \phi(t), 0) \tag{4.9}$$

which is the first variational equation.

Remark. $\frac{\partial}{\partial \epsilon} q_\epsilon^u(t)$ is solution of (4.8) for $t \in (-\infty, 0]$ and $\frac{\partial}{\partial \epsilon} q_\epsilon^s(t)$ for $t \in [0, +\infty)$.

On the other hand, differentiating (4.8) with respect to t

$$\frac{d}{dt}\Delta^{u,s}(t) = \frac{d}{dt}[DH(q_0(t-t_0))]\Big(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) + DH(q_0(t-t_0))\frac{d}{dt}(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}). \tag{4.10}$$

and substituting (4.9) into (4.10) gives

$$\begin{aligned} \frac{d}{dt}\Delta^{u,s}(t) =& \frac{d}{dt}[DH(q_0(t-t_0))](\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}) \\ &+ DH(q_0(t-t_0))JD^2H(q_0(t-t_0))\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0} \\ &+ DH(q_0(t-t_0))g(q_0(t-t_0), \phi(t), 0)\,. \end{aligned} \tag{4.11}$$

Lemma 4.3. *We have*

$$\frac{d}{dt}DH(q_0(t-t_0))\Big(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) + DH(q_0(t-t_0))JD^2H(q_0(t-t_0))\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0} = 0.$$

Proof. First, let's notice that

$$\begin{aligned} \frac{d}{dt}(DH(q_0(t-t_0))) &= D^2H(q_0(t-t_0))\dot{q_0}(t-t_0) \\ &= D^2H(q_0(t-t_0))JDH(q_0(t-t_0)) \\ &= (D^2H)(JDH)(q_0(t-t_0))\,. \end{aligned} \tag{4.12}$$

Second, we have

$$DH^\top JDH = \begin{pmatrix} \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix} = 0\,.$$

Third, differentiating the above expression with respect to $q = (x, y)$

$$D^2 HJDH + (DH)(JD^2 H) = 0\,.$$

Finally, we have that

$$\frac{d}{dt} DH(q_0(t-t_0))\Big(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) + DH(q_0(t-t_0))JD^2H(q_0(t-t_0))\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0} =$$
$$= ((D^2H)(JDH) + (DH)(JD^2H))(q_0(t-t_0))\Big(\frac{\partial}{\partial \epsilon} q_\epsilon^{u,s}(t)|_{\epsilon=0}\Big) = 0\,.$$

□

Therefore,

$$\frac{d}{dt}\Delta^{u,s}(t) = DH(q_0(t-t_0))g(q_0(t-t_0), \phi(t), 0) \tag{4.13}$$

where we consider $\Delta^u(t)$ for $t \in (-\infty, 0]$ and $\Delta^s(t)$ for $t \in [0, +\infty)$.
Given $\tau > 0$, we integrate (4.13) from $-\tau$ to 0 for the unstable part and 0 to $+\tau$ obtaining

$$\Delta^u(0) - \Delta^u(-\tau) = \int_{-\tau}^0 DH(q_0(t-t_0))g(q_0(t-t_0), wt + \phi_0, 0)\, dt$$
$$\Delta^s(\tau) - \Delta^s(0) = \int_0^\tau DH(q_0(t-t_0))g(q_0(t-t_0), wt + \phi_0, 0)\, dt\,.$$

Then the Melnikov function can be represented by

$$M(t_0, \phi_0) = M(0; t_0, \phi_0) = \Delta^u(0) - \Delta^s(0)$$
$$= \int_{-\tau}^\tau DH(q_0(t-t_0))g(q_0(t-t_0), wt + \phi_0, 0)\, dt + \Delta^s(\tau) - \Delta^u(-\tau)\,.$$

Lemma 4.4. *Let $(q_\epsilon^s(t), \phi(t)) \in W^s(\gamma_{p_0}^\epsilon)$ be the orbit of the perturbed vector field such that $(q_\epsilon^s(0), \phi(0)) = p_\epsilon^s$. Let $(q_\epsilon^u(t), \phi(t)) \in W^u(\gamma_{l_0}^\epsilon)$ be the orbit of the perturbed vector field such that $(q_\epsilon^u(0), \phi(0)) = p_\epsilon^u$.*
If ϵ is sufficiently small, then

$$q_\epsilon^s(t) = q_0(t-t_0) + \epsilon \frac{\partial}{\partial \epsilon} q_\epsilon^s(t)|_{\epsilon=0} + \mathcal{O}(\epsilon^2), \quad t \geq 0$$

$$q_\epsilon^u(t) = q_0(t-t_0) + \epsilon \frac{\partial}{\partial \epsilon} q_\epsilon^u(t)|_{\epsilon=0} + \mathcal{O}(\epsilon^2), \quad t \leq 0$$

where the functions $\frac{\partial}{\partial \epsilon} q_\epsilon^s(t)|_{\epsilon=0} : \ (0,\infty) \ \rightarrow \ \mathbb{R}^2$ *and* $\frac{\partial}{\partial \epsilon} q_\epsilon^u(t)|_{\epsilon=0} : \ (-\infty, 0) \ \rightarrow \ \mathbb{R}^2$ *are bounded.*

Proof. We will prove the Lemma for orbits on the stable manifold. First, with t_0 and ϕ fixed, let's recall that q_ϵ^s is solution of the differential equation

$$\dot{q} = h(q,t,\epsilon) = f(q) + \epsilon g(q,t,\epsilon)$$

and $q_0(t-t_0)$ is solution of the differential equation

$$\dot{q} = h(q,t,0) = f(q) \, .$$

Integrating we have

$$\begin{aligned} q_\epsilon^s(t,\epsilon) - q_\epsilon^s(0) &= \int_0^t h(q_\epsilon^s(t',\epsilon),t',\epsilon)\, dt' \\ q_0(t-t_0) - q_0(-t_0) &= \int_0^t h(q_0(t'-t_0),t',0)\, dt' \, . \end{aligned}$$

By the smoothness of the function h, $\exists C_1 > 0$ such that for arbitrary q_1, q_2 $\in U$

$$|h(q_1,t,\epsilon) - h(q_2,t,\epsilon)| \leq C_1(|q_1 - q_2| + |\epsilon|) \, .$$

Again, by smoothness of the stable manifold with respect to ϵ, $\exists C_2 > 0$ such that for ϵ small enough

$$|q_\epsilon^s(0) - q_0(-t_0)| \leq \epsilon C_2 \, .$$

Thus

$$\begin{aligned} |q_\epsilon^s(t,\epsilon) - q_0(t-t_0)| &= |q_\epsilon^s(0) - q_0(-t_0) + \int_0^t (h(q_\epsilon^s(t',\epsilon),t',\epsilon) - h(q_0(t'-t_0),t',\epsilon))\, dt'| \\ &\leq |q_\epsilon^s(0) - q_0(-t_0)| + \int_0^t |h(q_\epsilon^s(t',\epsilon),t',\epsilon) - h(q_0(t'-t_0),t',\epsilon)|\, dt' \\ &\leq C_2\epsilon + C_1 \int_0^t (|q_\epsilon^s(t',\epsilon) - q_0(t'-t_0)| + |\epsilon|)\, dt' \\ &\leq C_2\epsilon + C_1\epsilon t + C_1 \int_0^t |q_\epsilon^s(t',\epsilon) - q_0(t'-t_0)|\, dt' \, . \end{aligned}$$

Then, applying Gronwall's inequality

$$|q^s_\epsilon(t,\epsilon) - q_0(t-t_0)| \leq \epsilon(C_2 + C_1 t)e^{C_1 T} \quad \text{for} \quad 0 \leq t \leq T \, .$$

We also need a bound for $t > T$. On account of Proposition (4.1) we get $l_0^\epsilon = l_0 + \mathcal{O}(\epsilon)$. Then, since the solutions in the inequality belong to the stable manifold of l_0^ϵ and the stable manifold of l_0, respectively, $\exists C_3 > 0$, $T > 0$ such that if $t > T$

$$|q^s_\epsilon(t,\epsilon) - q_0(t-t_0)| \leq \epsilon C_3 \, .$$

All in all, for ϵ sufficiently small

$$|q^s_\epsilon(t,\epsilon) - q_0(t-t_0)| \leq \begin{cases} \epsilon(C_2 + C_1 t) & \text{for} \quad 0 \leq t \leq T \\ \epsilon C_3 & \text{for} \quad t > T \, . \end{cases}$$

Thus, $\exists C > 0$ such that

$$|q^s_\epsilon(t,\epsilon) - q_0(t-t_0)| \leq \epsilon C \quad \forall t > 0$$

and

$$q^s_\epsilon(t,\epsilon) = q_0(t-t_0) + \epsilon \frac{\partial}{\partial \epsilon} q^s_\epsilon(t,0) + \epsilon^2 \mathcal{O}(\epsilon) \, .$$

Consequently

$$\epsilon |\frac{\partial}{\partial \epsilon} q^s_\epsilon(t,0) + \epsilon \mathcal{O}(\epsilon)| \leq \epsilon C$$

and hence

$$|\frac{\partial}{\partial \epsilon} q^s_\epsilon(t,0)| \leq C \, .$$

□

Lemma 4.5. *Under the previous conditions we have*

$$\lim_{\tau \to +\infty} \Delta^s(\tau) = \lim_{\tau \to -\infty} \Delta^u(\tau) = 0 \, .$$

Proof. On one side, $DH(q_0(t-t_0))$ goes to zero exponentially fast when $q_0(t-t_0)$ tends to the hyperbolic fiexed points l_0 for $t \to +\infty$ and m_0 for $t \to -\infty$

The idea behind this is based on a Taylor expansion around l_0 or m_0. Let's show it for l_0.

$$DH(q_0(t-t_0)) = DH(l_0) + D^2H(l_0)(q(t-t_0) - l_0) + \mathcal{O}((q(t-t_0) - l_0)^2)$$

Since $DH(l_0) = 0$ the dominating term $(q(t-t_0) - l_0)$ is the one responsible for the exponential convergence.

On the other side, by Lemma (4.4), $\frac{\partial}{\partial \epsilon} q^s_\epsilon(t)|_{\epsilon=0}$ and $\frac{\partial}{\partial \epsilon} q^u_\epsilon(t)|_{\epsilon=0}$ are bounded.

□

Finally, for $\tau \to \infty$, the Melnikov function is

$$M(t_0, \phi_0) = \int_{-\infty}^{+\infty} DH(q_0(t - t_0))g(q_0(t - t_0), wt + \phi_0, 0)\, dt\ . \tag{4.14}$$

$M(t_0, \phi_0)$ converges absolutely due to the fact that $g(q_0(t - t_0), wt + \phi_0, 0)$ is bounded $\forall t$ and $DH(q_0(t - t_0)$ converges exponentially to zero as we have seen in the proof of the previous Lemma.

4.5 Properties of the Melnikov function

Lemma 4.6. *We have*

$$\frac{\partial}{\partial \phi_0} M(t_0, \phi_0) = \frac{1}{w}\frac{\partial}{\partial t_0} M(t_0, \phi - 0)\ .$$

Proof. Let's observe that the Melnikov function after the transformation $t \mapsto t + t_0$ remains

$$M(t_0, \phi_0) = \int_{-\infty}^{+\infty} DH(q_0(t))g(q_0(t), wt + wt_0 + \phi_0, 0)\, dt\ .$$

On account of the structure of $g(q, \cdot, 0)$ we have that

$$\frac{\partial}{\partial \phi_0} M(t_0, \phi_0) = \frac{1}{w}\frac{\partial}{\partial t_0} M(t_0, \phi_0)\ .$$

$\square$

Therefore,

$$\frac{\partial}{\partial t_0} M(t_0, \phi_0) = 0 \iff \frac{\partial}{\partial \phi_0} M(t_0, \phi_0) = 0$$

$$\frac{\partial}{\partial t_0} M(t_0, \phi_0) \neq 0 \iff \frac{\partial}{\partial \phi_0} M(t_0, \phi_0) \neq 0\ .$$

Theorem 4.7.

(1) Suppose we have a point $(t_0, \phi_0) = (\bar{t_0}, \bar{\phi_0})$ *such that*

- $M(\bar{t_0}, \bar{\phi_0}) = 0$
- $\frac{\partial}{\partial t_0} M(t_0, \phi_0)|_{(\bar{t_0}, \bar{\phi_0})} \neq 0$.

Then, for ϵ small enough $W^s(\gamma^\epsilon_{l_0})$ and $W^u(\gamma^\epsilon_{m_0})$ intersect transversally at $(q_0(-\bar{t_0}) + \mathcal{O}(\epsilon), \bar{\phi_0})$.

(2) If $\forall (t_0, \phi_0) \in \mathbb{R} \times S$ $\quad M(t_0, \phi_0) \neq 0$ then, for ϵ small enough $\quad W^s(\gamma^\epsilon_{l_0}(t)) \cap W^u(\gamma^\epsilon_{m_0}(t)) = \emptyset$.

Proof.

1) From the Taylor expansion of the distance between $W^s(\gamma^\epsilon_{l_0}(t))$ and $W^u(\gamma^\epsilon_{m_0}(t))$ around $\epsilon = 0$, we have

$$\begin{aligned} d(t_0, \phi_0, \epsilon) &= \epsilon \frac{M(t_0, \phi_0)}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon^2) \\ &= \epsilon\left(\frac{M(t_0, \phi_0)}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon)\right) \\ &= \epsilon \hat{d}(t_0, \phi_0, \epsilon) \ . \end{aligned}$$

First, the new distance introduced above

$$\hat{d}(t_0, \phi_0, \epsilon) = \frac{M(t_0, \phi_0)}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon)$$

satisfies that $\hat{d}(t_0, \phi_0, \epsilon) = 0 \Rightarrow d(t_0, \phi_0, \epsilon) = 0$. Consequently, we are able to work with the former for our purposes.

Since , there is $\hat{d} : \ A \times B \ \longrightarrow \ \mathbb{R}$ with $A \subset \mathbb{R}$, $B \in (0, 2\pi] \times \mathbb{R}$ with $x = t_0$, $y = (\phi_0, \epsilon)$ satisfying that

- $\exists (a, b) = (\bar{t_0}, \bar{\phi_0}, 0) \in A \times B$ such that $\hat{d}(\bar{t_0}, \bar{\phi_0}, 0) = 0$

  $\hat{d}(\bar{t_0}, \bar{\phi_0}, 0) = \frac{M(\bar{t_0}, \bar{\phi_0})}{\|DH(q_0(-\bar{t_0}))\|} = 0$ since $M(\bar{t_0}, \bar{\phi_0}) = 0$.

- $\hat{d} \in C^r(A \times B)$ for $r \geq 0$.
- $\det(D_{t_0}\hat{d}(\hat{t_0}, \hat{\phi_0}, 0)) \neq 0$, because

$$\begin{aligned} \det(D_{t_0}\hat{d}(\bar{t_0}, \bar{\phi_0}, 0)) &= \Big| \frac{\partial}{\partial t_0} \hat{d}(\bar{t_0}, \bar{\phi_0}, 0) \Big| \\ &= \Big| \frac{\frac{\partial}{\partial t_0} M(\bar{t_0}, \bar{\phi_0}, 0) \|DH(q_0(-\bar{t_0}))\| + M(\bar{t_0}, \bar{\phi_0}, 0) \frac{\partial}{\partial t_0} \|DH(q_0(-\bar{t_0}))\|}{\|DH(q_0(-\bar{t_0}))\|^2} \Big| \\ &= \Big| \frac{\frac{\partial}{\partial t_0} M(\bar{t_0}, \bar{\phi_0}, 0)}{\|DH(q_0(-\bar{t_0}))\|} \Big| \neq 0 \end{aligned}$$

since $\quad \frac{\partial}{\partial t_0}M(\bar{t_0},\bar{\phi_0}) \neq 0$ and $M(\bar{t_0},\bar{\phi_0},0) = 0$.

Using the Implicit function theorem we have that

- $\exists$ neighbourhoods $U \subset A$ and $V \subset B$ containing a and b respectively.
  $\exists\ {t_0}^* : \quad V \quad \longrightarrow \quad U \quad$ such that $\forall t_0 \in U$, $(\phi_0, \epsilon) \in V$
  $$ {t_0}^* = t_0(\phi_0,\epsilon) \iff \hat{d}(t_0,\phi_0,\epsilon) = 0 $$
- ${t_0}^* \in C^r(U)$
- $\forall t_0 \in U$
  $$ D_{(\phi_0,\epsilon)}{t_0}^*(\phi_0,\epsilon) = -((D_{t_0}\hat{d}({t_0}^*(\phi_0,\epsilon),\phi_0,\epsilon))^{-1}D_{(\phi_0,\epsilon)}\hat{d}({t_0}^*(\phi_0,\epsilon),\phi_0,\epsilon)). $$

Focusing on the first statement, in other words, it says that
for $|\phi - \phi_0|, \quad \epsilon$ small enough, $\exists t_0 = {t_0}^*(\phi_0,\epsilon)$ such that

$$ \hat{d}({t_0}^*(\phi_0,\epsilon),\phi_0,\epsilon) = 0 \quad \Longrightarrow \quad d({t_0}^*(\phi_0,\epsilon),\phi_0,\epsilon) = 0\ . $$

Which ensures that $W^s(\gamma^{\epsilon}_{l_0}(t))$ and $W^u(\gamma^{\epsilon}_{m_0}(t))$ intersect $\mathcal{O}(\epsilon)$ close to $q_0(-t_0,\phi_0)$.

Second, we want to see that the manifolds intersect transversally. Thus we need that

$$ T_pW^s(\gamma^{\epsilon}_{l_0}(t)) + T_pW^u(\gamma^{\epsilon}_{m_0}(t)) = \mathbb{R}^3\ . $$

For ϵ small enough, the points in $W^s(\gamma^{\epsilon}_{l_0}(t))$ and $W^u(\gamma^{\epsilon}_{m_0}(t))$ that are closest to $\gamma^{\epsilon}_{l_0}(t)$ and $\gamma^{\epsilon}_{m_0}(t)$ respectively, can be parametrisized by t_0 and ϕ_0 due to the fact that the stable and the unstable manifolds intersect at Π_p. Thus

$$ (\frac{\partial q^u_\epsilon}{\partial t_0},\frac{\partial q^u_\epsilon}{\partial \phi_0}) \quad \text{is a basis for} \quad T_pW^u(\gamma^{\epsilon}_{m_0}(t)) $$

$$ (\frac{\partial q^s_\epsilon}{\partial t_0},\frac{\partial q^u_\epsilon s}{\partial \phi_0}) \quad \text{is a basis for} \quad T_pW^s(\gamma^{\epsilon}_{l_0}(t))\ . $$

Then we can figure out that $T_pW^s(\gamma^{\epsilon}_{l_0}(t))$ and $T_pW^u(\gamma^{\epsilon}_{m_0}(t))$ won't be tangent at p if

$$ \frac{\partial q^u_\epsilon}{\partial t_0},\quad \frac{\partial q^s_\epsilon}{\partial t_0},\quad \frac{\partial q^s_\epsilon}{\partial \phi_0} \quad \text{are linearly independent.} $$

Lastly, since

$$\frac{\partial}{\partial t_0} d(\bar{t_0}, \bar{\phi_0}, 0) = \epsilon \frac{\partial}{\partial t_0} \hat{d}(\bar{t_0}, \bar{\phi_0}, 0) \neq 0$$

the condition

$$\Big(\frac{\partial q_\epsilon^u}{\partial t_0} - \frac{\partial q_\epsilon^s}{\partial t_0}\Big) \neq 0$$

holds because

$$\frac{\partial}{\partial t_0} d(\bar{t_0}, \bar{\phi_0}, 0) = \frac{DH(q_0(-\bar{t_0}))\big(\frac{\partial q_\epsilon^u}{\partial t_0} - \frac{\partial q_\epsilon^s}{\partial t_0}\big)}{\|DH(q_0(-\bar{t_0})\|} \, .$$

(2) Last but not least, we want to see whether $W^s(\gamma_{l_0}^\epsilon(t)) \cap W^u(\gamma_{m_0}^\epsilon(t)) = \emptyset$ holds, provided that $\forall (t_0, \phi_0) \in \mathbb{R} \times S^1 \quad M(t_0, \phi_0) \neq 0$

As the distance is

$$\begin{aligned} d(t_0, \phi_0, \epsilon) &= d(t_0, \phi_0, 0) + \epsilon \frac{\partial}{\partial \epsilon} d(t_0, \phi_0, 0) + \mathcal{O}(\epsilon^2) \\ &= \epsilon \frac{M(t_0, \phi_0)}{\|DH(q_0(-t_0))\|} + \mathcal{O}(\epsilon^2) \\ &= \epsilon (\frac{M}{\|DH\|} + \mathcal{O}(\epsilon)) \, , \end{aligned}$$

by the triangle inequality we have

$$\Big| d(t_0, \phi_0, \epsilon) \Big| = \Big| \epsilon (\frac{M}{\|DH\|} + \mathcal{O}(\epsilon)) \Big| \geq \epsilon (\Big| \frac{M}{\|DH\|} \Big| - |\mathcal{O}(\epsilon)|) \neq 0 \, .$$

□

4.6 Melnikov Method for an Autonomous Perturbation

Now we consider the system of equations (4.1), with $g(q, \epsilon)$ independent of time.

$$\begin{cases} \dot{x} = \frac{\partial}{\partial y} H(x, y) + \epsilon g_1(x, y, \epsilon) \\ \dot{y} = -\frac{\partial}{\partial x} H(x, y) + \epsilon g_2(x, y, \epsilon) \end{cases} \tag{4.15}$$

Suppose the unperturbed system, $\epsilon = 0$, satisfies Assumption 1 and Assumption 2 from the previous sections. All in all, the unperturbed system will have

- l_0, m_0 hyperbolic saddle points with a heteroclinic orbit $q_0(t) = (x_0(t), y_0(t))$.
- The stable and unstable manifolds, $W^s(l_0)$ and $W^u(m_0)$, are trajectories and coincide with $q_0(t)$.
- The normal vector to $q_0(t)$ is $\Pi_p = (\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y})$.

Then, using Proposition (4.1) we have similar results for the perturbed autonomous case.

- The fixed points l_0, m_0 remain equilibrium points slightly perturbed: $l_0^\epsilon = l_0 + O(\epsilon)$, $m_0^\epsilon = m_0 + O(\epsilon)$ and the hyperbolicity is preserved.
- The perturbed trajectories satisfy that $W^s_{loc}(l_0^\epsilon)$ and $W^u_{loc}(p_0^\epsilon)$ are C^r ϵ-close to $W^s_{loc}(l_0)$ and $W^u_{loc}(m_0)$.

As before, the distance between the stable and the unstable trajectories will be

$$d(t_0, \epsilon) = \frac{DH(q_0(-t_0))(q_\epsilon^u - q_\epsilon^s)}{\|DH(q_0(-t_0))\|}$$

for any $p \in q_0(t)$, being q_ϵ^s, q_ϵ^u points of intersection between Π_p with $W_\epsilon^s(l_0)$ and $W_\epsilon^u(m_0)$ respectively.

Finally the Melnikov function will be

$$M(t_0) = \int_{-\infty}^{+\infty} DH(q_0(t - t_0))g(q_0(t - t_0), 0)\, dt$$

and making the change $t \to t + t_0$ we obtain

$$M(t_0) = \int_{-\infty}^{+\infty} DH(q_0(t))g(q_0(t), 0)\, dt\, .$$

4.7 Poincaré Maps of a Cross-section to the Phase Space

Before the last chapter about the applications of the Melnikov theory, there remains a topic we would like to discuss. At this section, we suppose the fixed points l_0 and m_0 are the same point and we denote it by p_0. Thus, the periodic orbits $\gamma_{l_0}(t)$ and $\gamma_{m_0}(t)$ coincide in one periodic orbit denoted by $\gamma_{p_0}(t)$ and, analogously, the perturbed periodic orbits $\gamma_{l_0}^{\epsilon}(t)$ and $\gamma_{m_0}^{\epsilon}(t)$ also coincide in one periodic orbit denoted by $\gamma_{p_0}^{\epsilon}(t)$. Moreover, the two-dimensional heteroclinic manifold Γ_γ becomes a two-dimensional homoclinic manifold.

Let Σ_{ϕ_0} be the cross-section of the phase space $\mathbb{R}^2 \times S^1$

$$\Sigma_{\phi_0} = \{(q, \phi) \in \mathbb{R}^2 \times S^1 | \quad \phi = \phi_0\} ,$$

Let p_0^{ϵ,ϕ_0} be the intersection point between the periodic orbit $\gamma_{p_0}^{\epsilon}(t)$ and the cross-section Σ_{ϕ_0}

$$p_0^{\epsilon,\phi_0} = \gamma_{p_0}^{\epsilon}(t) \cap \Sigma_{\phi_0} .$$

Then the Poincaré map of Σ_{ϕ_0} into itself is defined by

$$\begin{array}{rccc} P_\epsilon : & \Sigma_{\phi_0} & \longrightarrow & \Sigma_{\phi_0} \\ & q_\epsilon(0) & \longmapsto & q_\epsilon(T) \end{array} \tag{4.16}$$

where $q_\epsilon(t)$ is the first component of the flow generated by the perturbed vector field $(q_\epsilon(t), \phi(t) = wt + \phi_0)$. In addition, p_0^{ϵ,ϕ_0} is a hyperbolic fixed point for the Poincaré map such that

$$W^s(p_0^{\epsilon,\phi_0}) \equiv W^s(\gamma_{p_0}^{\epsilon}(t)) \cap \Sigma_{\phi_0}$$

and

$$W^u(p_0^{\epsilon,\phi_0}) \equiv W^u(\gamma_{p_0}^{\epsilon}(t)) \cap \Sigma_{\phi_0}$$

are, respectively, its stable manifold and its unstable manifold.

Given a fixed ϕ_0, if $\exists \bar{t_0} \in \mathbb{R}$ such that $M(\bar{t_0}, \phi_0) = 0$ and $\frac{\partial}{\partial t_0} M(t_0, \phi_0)|_{(\bar{t_0},\phi_0)} \neq 0$, then for ϵ small enough $W^s(p_0^{\epsilon,\phi_0})$ and $W^u(p_0^{\epsilon,\phi_0})$ intersect transversally at a certain point p by Theorem 4.7. Thus, the point p correspond to a homoclinic point for the Poincaré map P_ϵ.
Finally, the Smale-Moser Theorem can be applied to deduce that the Poincaré map P_ϵ displays chaotic dynamics.

4.8 Application of the Melnikov theory

In this final section we provide a couple of well known cases with the aim to illustrate the utility of the Melnikov function. We compare the periodically forced pendulum with the dumped, forced duffing oscillator because their corresponding Melnikov function have resemblance, even though they are quite different.

First, let us consider the periodically forced pendulum

$$\ddot{x} = -\sin x + \epsilon a \sin(\Omega t + \phi_0)\,.$$

On the phase cylinder, this is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x + \epsilon a \sin\phi \\ \dot{\phi} = \Omega \end{cases} \tag{4.17}$$

To begin with, we study the unperturbed system which is Hamiltonian

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} = y \\ \dot{y} = -\dfrac{\partial H}{\partial x} = -\sin x \end{cases} \tag{4.18}$$

with Hamiltonian $H = \frac{y^2}{2} - \cos x$. Next, we shall find out its fixed points and their nature. To do so, we linearize the system and look for the eigenvalues of the matrix associated with the linearized vector field at the fixed points.

Imposing $\begin{cases} 0 = \dot{x} = y \\ 0 = \dot{y} = -\sin x \end{cases}$ the fixed points are $p_k = (\pi k, 0), \quad k \in \mathbb{Z}$

Computing the energy for each fixed point we get that there are two different eneregies associated with the fixed points.

$$H(p_k) = \begin{cases} -1 & \text{k even} \\ +1 & \text{k odd} \end{cases}$$

In addition, the linearized vector field is given by

$$Dh = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix}\,.$$

Consequently, the eigenvalues associated with the fixed points with even k are $\lambda^e_{1,2} = \pm i$, and the eigenvalues associated with the fixed points with odd k are $\lambda^o_{1,2} = \pm 1$, so they are centers and hyperbolic saddle points respectively. Since the perturbed system (4.17) is 2π-periodic respect to x we just work with the domain constrained to $x \in [-\pi, \pi]$. As a result, we just consider the fixed points $(-\pi, 0)$, $(0, 0)$ and $(\pi, 0)$.

What is more, we should check whether the two saddles with the same energy are connected by a heteroclinic orbit.

Imposing $\quad H(-\pi, 0) = H(\pi, 0) = 1 = H(x, y) \quad$ we get $\quad y = \pm 2\cos\dfrac{x}{2}$.

However, we would like to have x and y parameterized by t. As $\dot{x} = y$ we have a differential equation for x which can be integrated

$$\int \frac{1}{2\cos\frac{x}{2}}\, dx = \pm \int dt\,. \tag{4.19}$$

The left side of the equation (4.19) is a trigonometric integral which can be computed

$$\int \frac{1}{2\cos\frac{x}{2}}\, dx = \frac{1}{2}\ln\frac{1+\sin\frac{x}{2}}{1-\sin\frac{x}{2}} + C = \operatorname{arctanh}(\sin\frac{x}{2}) + C$$

where C is a constant. Thus, we have the following equation

$$x(t) = 2\arcsin\left(\tanh\left(\pm t + C\right)\right).$$

For convenience, since C is an arbitrary constant, we take $C = 0$. In addition, due to the fact that the function $\arcsin(\tanh x)$ is odd, let us write

$$x(t) = \pm 2\arcsin\left(\tanh t\right).$$
$$y(t) = \pm 2\cos\left(\frac{2\arcsin\left(\tanh t\right.}{2}\right)) = \pm 2\sqrt{1-\tanh^2 t} = \pm 2\ \mathrm{sech} t\,.$$

Hence, we obtain two heteroclinic orbits which are denoted by

$$\begin{cases} x_0^{\pm}(t) = \pm 2\arcsin\left(\tanh t\right) \\ y_0^{\pm}(t) = \pm 2\ \mathrm{sech} t \end{cases} \tag{4.20}$$

Observation. Since

$$\lim_{t\to+\infty} x_0^+(t) = \pi\,, \qquad \lim_{t\to-\infty} x_0^+(t) = -\pi$$
$$\lim_{t\to+\infty} y_0^+(t) = \lim_{t\to-\infty} y_0^+(t) = 0$$

and

$$\lim_{t\to+\infty} x_0^-(t) = \pi\,, \qquad \lim_{t\to-\infty} x_0^-(t) = -\pi$$
$$\lim_{t\to+\infty} y_0^-(t) = \lim_{t\to-\infty} y_0^-(t) = 0$$

the heteroclinic orbit denoted by $(x_0^+(t), y_0^+(t))$ corresponds to $W^s((\pi,0))$ which coincides with $W^u((-\pi,0))$ and the heteroclinic orbit denoted by $(x_0^-(t), y_0^-(t))$ corresponds to $W^s((-\pi,0))$ or $W^u((\pi,0))$ which coincide.

Next, the Melnikov function is

$$\begin{aligned} M^{\pm}(t_0,\phi_0) &= \int_{-\infty}^{\infty} \begin{pmatrix} \sin(x_0(t)) \\ y_0(t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ a\sin(\Omega t + \Omega t_0 + \phi_0) \end{pmatrix} dt \\ &= \pm 2a \int_{-\infty}^{\infty} \operatorname{sech} t \sin(\Omega t + \Omega t_0 + \Omega_0)\, dt \\ &= \pm 2a \sin(\Omega t_0 + \phi_0) \int_{-\infty}^{\infty} \operatorname{sech} t \cos(\Omega t) \end{aligned} \tag{4.21}$$

where we have used that $\sin(\Omega t + \Omega t_0 + \phi_0) = \sin(\Omega t)\cos(\Omega t_0 + \Omega_0) + \cos(\Omega t)\sin(\Omega t_0 + \Omega_0)$ and since $\operatorname{sech} t \sin(\Omega t)$ is odd its integral vanishes. After the evaluation of the integral $I_1 = \int_{-\infty}^{\infty} \operatorname{sech} t \cos(\Omega t)$ (see Appendix A) the Melnikov function and its partial derivative with respect to t_0 easily follows.

$$M^{\pm}(t_0,\phi_0) = -2a\pi \operatorname{sech}\left(\frac{\pi\Omega}{2}\right) \sin(\Omega t_0 + \phi_0)\,, \tag{4.22}$$

$$\frac{\partial}{\partial t_0} M^{\pm}(t_0,\phi_0) = -2a\pi\Omega \operatorname{sech}\left(\frac{\pi\Omega}{2}\right) \cos(\Omega t_0 + \phi_0)\,. \tag{4.23}$$

Let's observe that for $(\bar{t_0}, \bar{\phi_0}) \in \{(t_0,\phi_0) \mid \quad \Omega t_0 + \phi_0 = \pi k, \quad k \in \mathbb{Z}\}$ it holds that

- $M^{\pm}(\bar{t_0}, \bar{\phi_0}) = 0$
- $\frac{\partial}{\partial t_0} M^{\pm}(t_0,\phi_0)|_{(\bar{t_0},\bar{\phi_0})} \neq 0$

As a result, by Theorem 4.7, $W^s(\gamma^{\epsilon}_{(\pi,0)}))$ and $W^u(\gamma^{\epsilon}_{(-\pi,0)}))$ intersect transversally at $r \equiv (x_0^+(-\bar{t_0}) + \mathcal{O}(\epsilon), y_0^+(-\bar{t_0}) + \mathcal{O}(\epsilon), \bar{\phi_0})$. Thus, $W^s(\gamma^{\epsilon}_{(\pi,0)}))$ and $W^u(\gamma^{\epsilon}_{(-\pi,0)}))$ intersect transverselly infinitely many times. However, since the transversal intersection points in $W^s(\gamma^{\epsilon}_{(\pi,0)})) \cap W^u(\gamma^{\epsilon}_{(-\pi,0)}))$ correspond to heteroclinic points for the Poincaré map of Σ_{ϕ_0}, the Smale-Moser Theorem can not be applied.

Observation. The Melnikov function $M^+(t_0, x_0)$ is related to the separation of the invariant manifolds $W^s(\gamma^{\epsilon}_{(\pi,0)}))$ and $W^u(\gamma^{\epsilon}_{(-\pi,0)}))$ and the Melnikov function $M^-(t_0, x_0)$ is related to the separation of the invariant manifolds $W^s(\gamma^{\epsilon}_{(-\pi,0)}))$ and $W^u(\gamma^{\epsilon}_{(\pi,0)}))$. Hence, the same reasoning can be applied to the former ones.

Now, let's take into account the dumped, forced duffing oscillator

$$\ddot{x} = x - x^3 + \epsilon(\gamma \cos(wt + \phi_0) - \delta\dot{x})$$

on the phase cylinder, which is equivalent to the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + \epsilon(\gamma\cos(\phi) - \delta y) \\ \dot{\phi} = w \end{cases} \tag{4.24}$$

Here, the Hamiltonian associated with the unperturbed system is

$$H = \frac{y^2}{2} - \frac{1}{2}x^2 + \frac{1}{4}x^4$$

and its equations of motion are

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} = y \\ \dot{y} = -\dfrac{\partial H}{\partial x} = x - x^3 \end{cases} \tag{4.25}$$

Imposing $\begin{cases} 0 = \dot{x} = y \\ 0 = \dot{y} = x - x^3 \end{cases}$ we obtain the fixed points $\begin{cases} p_1 = (-1, 0) \\ p_2 = (\ 0, 0) \\ p_3 = (+1, 0) \end{cases}$

Moreover, the linearized vector field is given by

$$Dh = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{pmatrix} .$$

Seeing that, if we evaluate Dh at the fixed points we get that p_1 and p_3 have eigenvalues $\lambda^{1,3}_{1,2} = \pm\sqrt{2}i$ and p_2 have eigenvalues $\lambda^2_{1,2} = \pm 1$, that is, p_1 and p_3 are centers and p_2 is a hyperbolic saddle point.

$$\text{Imposing} \qquad H(p_2) = 1 = H(x,y) \qquad \text{we get} \qquad y = \pm x\sqrt{1-\frac{1}{2}x^2} .$$

Since we need to have x and y parameterized by t we solve the differential equation given by $\dot{x} = y = \pm x\sqrt{1-\frac{1}{2}x^2}$, integrating in both sides

$$\int \frac{1}{x\sqrt{1-\frac{1}{2}x^2}}\, dx = \pm\int dt . \tag{4.26}$$

The left side of the equation (4.26) is an irrational integral which can be computed

$$\int \frac{1}{x\sqrt{1-\frac{1}{2}x^2}}\, dx = \ln\sqrt{2} + \ln\left[\frac{(1+\sqrt{1-(\frac{x}{\sqrt{2}})^2}}{\frac{x}{\sqrt{2}}}\right] + D = \operatorname{arcsech}\big(\pm\frac{x}{\sqrt{2}}\big) + D'$$

where D and D' are constants. Thus, we have the following equation

$$x(t) = \pm\sqrt{2}\,\operatorname{sech}(\pm t + D') .$$

For convenience, since D' is also an arbitrary constant, we take $D' = 0$. Moreover, since the function $\operatorname{sech} t$ is even we write

$$\begin{aligned} x(t) &= \pm\sqrt{2}\,\operatorname{sech} t \\ y(t) &= \pm\sqrt{2}\,\operatorname{sech} t\sqrt{1-\frac{1}{2}(\sqrt{2}\,\operatorname{sech} t)^2} = \pm\sqrt{2}\,\operatorname{sech} t\tanh t \end{aligned}$$

Hence, we obtain two homoclinic orbits denoted by

$$\begin{cases} x_0^{\pm}(t) = \pm\sqrt{2}\,\operatorname{sech} t \\ y_0^{\pm}(t) = \pm\sqrt{2}\,\operatorname{sech} t\tanh t \end{cases} \tag{4.27}$$

Observation. Again, since

$$\lim_{t\to+\infty} x_0^+(t) = \lim_{t\to-\infty} x_0^+(t) = 0$$
$$\lim_{t\to+\infty} y_0^+(t) = \lim_{t\to-\infty} y_0^+(t) = 0$$

,

$$\lim_{t\to+\infty} x_0^-(t) = \lim_{t\to-\infty} x_0^-(t) = 0$$
$$\lim_{t\to+\infty} y_0^-(t) = \lim_{t\to-\infty} y_0^-(t) = 0$$

and the function $\operatorname{sech} t$ is defined positive $\forall t \in \mathbb{R}$, the homoclinic orbit denoted by $(x_0^+(t), y_0^+(t))$ corresponds to $W_+^s(p_2)$ which coincides with $W_+^u(p_2)$ and the homoclinic orbit denoted by $(x_0^-(t), y_0^-(t))$ corresponds to $W_-^s(p_2)$ or $W_-^u(p_2)$ which coincide. Here, the subindices $+$ and $-$ denote which of the two homoclinic orbits we refer. All in all, $+$ refers to the homoclnic loop for $x > 0$ and $-$ refers to the homoclinic loop for $y < 0$.

Therefore, the Melnikov function is

$$\begin{aligned} M^{\pm}(t_0, \phi_0) &= \int_{-\infty}^{\infty} \begin{pmatrix} -x_0(t) + {x_0}^3(t) \\ y_0(t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \gamma \cos(wt + wt_0 + \phi_0) - \delta y \end{pmatrix} dt \\ &= \pm\gamma\sqrt{2} \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \cos(wt + wt_0 + \phi_0)\, dt - \delta \int_{-\infty}^{\infty} {y_0}^2(t)\, dt \\ &= \mp\gamma\sqrt{2} \sin(wt_0 + \phi_0) \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \sin(wt)\, dt - \delta\frac{4}{3} \end{aligned}$$

where we have used that $\cos(wt + wt_0 + \phi_0) = \cos(wt)\cos(wt_0 + \phi_0) - \sin(wt)\sin(wt_0 + \phi_0)$. Since the term $\operatorname{sech} t \tanh t \cos(wt)$ is odd its integral vanishes. Once we compute the integral $I_2 = \int_{-\infty}^{\infty} \operatorname{sech} t \tanh t \sin(wt)\, dt$ (see Appendix A) the Melnikov function and its partial derivative with t_0 easily follows.

$$M^{\pm}(t_0, \phi_0) = \mp\gamma\sqrt{2}\pi w \operatorname{sech}\left(\frac{\pi w}{2}\right) \sin(wt_0 + \phi_0) - \delta\frac{4}{3}\,, \tag{4.28}$$

$$\frac{\partial}{\partial t_0} M^{\pm}(t_0, \phi_0) = \mp\gamma\sqrt{2}\pi w^2 \operatorname{sech}\left(\frac{\pi w}{2}\right) \cos(wt_0 + \phi_0)\,. \tag{4.29}$$

Considering the extra factor depending on δ in equation (4.28), we get the following constrain of the parameters (γ, w, δ) for the stable and unstable manifolds $W_+^s(\gamma_{p2}^\epsilon(t))$, $W_+^u(\gamma_{p2}^\epsilon(t))$ to intersect

$$\delta < \left(\frac{3\pi w \operatorname{sech}(\frac{\pi w}{2})}{2\sqrt{2}}\right)\gamma\,. \tag{4.30}$$

Thus, if the inequality (4.30) is satisfied, for $(\bar{t_0}, \bar{\phi_0}) \in \{(t_0, \phi_0)| \quad wt_0 + \phi_0 = \pi k, \quad k \in \mathbb{Z}\}$ it holds that

- $M(\bar{t_0}, \bar{\phi_0}) = 0$
- $\frac{\partial}{\partial t_0} M(t_0, \phi_0)|_{(\bar{t_0}, \bar{\phi_0})} \neq 0$.

As a result, by Theorem 4.7, $W^s_+(\gamma^\epsilon_{p_2}))$ and $W^u_+(\gamma^\epsilon_{p_2}))$ intersect transversally at $(x_0^+(-\bar{t_0}) + \mathcal{O}(\epsilon), y_0^+(-\bar{t_0}) + \mathcal{O}(\epsilon), \bar{\phi_0})$. Thus, $W^s_+(\gamma^\epsilon_{p_2}))$ and $W^u_+(\gamma^\epsilon_{p_2}))$ intersect transverselly infinitely many times.
Contrary to what has happaned for the pendulum case, here the transversal intersection points in $W^s_+(\gamma^\epsilon_{p_2})) \cap W^u_+(\gamma^\epsilon_{p_2}))$ correspond to homoclinic points for the Poincaré map of Σ_{ϕ_0}. Hence, the Smale-Moser Theorem can be used and we can see that the Poincaré map P_ϵ has chaotic dynamics.

Last but not least, it is worth to point out that if we identify the plane $x = \pi$ with the plane $x = -\pi$ in the phase space $\mathbb{R}^2 \times S^1$ of the pendulum case, the forced pendulum is equivelent to the undumped forced duffing oscillator ($\delta = 0$).

Appendix A

Contour Integration

Due to the Euler identity we have the following relation for a certain type of improper integrals

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx \quad .$$

Thus,

$$I_1 = \int_{-\infty}^{+\infty} \frac{\cos(\Omega t)}{\cosh t}\,dt = \mathrm{Re}[\int_{-\infty}^{+\infty} \frac{e^{i\Omega t}}{\cosh t}\,dt]$$
$$I_2 = \int_{-\infty}^{\infty} \frac{\sinh t \sin(wt)}{\cosh^2 t}\,dt = \mathrm{Re}[\int_{-\infty}^{\infty} \frac{\sinh t\, e^{iwt}}{\cosh^2 t}\,dt]$$

We proceed to integrate I_1 and I_2 using contour integration. To start with, we define the close curve C in the complex plane, as it is shown in Figure A.1, by a rectangle with vertices $(R, 0)$, $(R, i\pi)$, $(-R, i\pi)$ and $(-R, 0)$.

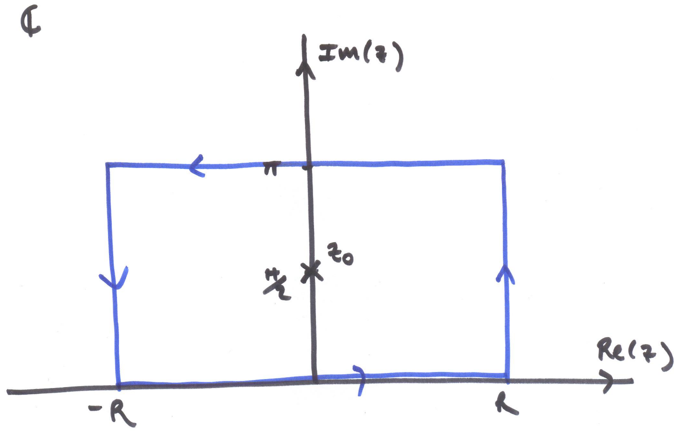

Figure A.1

Computation of I_1

Let

$$f(z) = \frac{e^{i\Omega z}}{\cosh z} \, .$$

It is analitic on and inside C except at $z_0 = i\frac{\pi}{2}$. By the residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \mathrm{Res}_{z_0} f(z) \, .$$

Since $\mathrm{Res}_{z_0} f(z) = -ie^{-\Omega\frac{\pi}{2}}$ we get

$$\oint_C f(z)\, dz = 2\pi e^{-\Omega\frac{\pi}{2}} \, .$$

Next, we split the integral along C in four different integrals, where $z = x+iy$

$$\oint_C f(z)\, dz = \underbrace{\int_{-R}^{R} \frac{e^{i\Omega x}}{\cosh x}\, dx}_{I_1^1} + \underbrace{\int_0^{\pi} \frac{e^{i\Omega (R+iy)}}{\cosh{(R+iy)}} i\, dy}_{I_1^2}$$
$$+ \underbrace{\int_{R}^{-R} \frac{e^{i\Omega (x+\pi i)}}{\cosh{(x+\pi i)}}\, dx}_{I_1^3} + \underbrace{\int_{\pi}^{0} \frac{e^{i\Omega (-R+iy)}}{\cosh{(-R+iy)}} i\, dy}_{I_1^4}$$

In the first place, we see that I_1^2 and I_1^4 tend to 0 for $R \to \infty$. Let's show it for I_1^2.

$$|\int_0^{\pi} \frac{e^{i\Omega (R+iy)}}{\cosh{(R+iy)}} i\, dy| \leq \int_0^{\pi} \frac{e^{-\Omega y}}{\sinh R}\, dy = \frac{(e^{-\Omega \pi} - 1)}{\Omega \sinh R} \overset{R\to\infty}{\longrightarrow} 0$$

owing to

- $|\cosh{(R+iy)}| \geq \frac{|e^{(R+iy)}| - |e^{-(R+iy)}|}{2} = \frac{(e^R - e^{-R})}{2} = \sinh R$
- $|ie^{i\Omega(R+iy)}| = |e^{i\frac{\pi}{2}} e^{i\Omega R} e^{-\Omega y}| = |e^{i(\frac{\pi}{2} + \Omega R)} e^{-\Omega y}| = e^{-\Omega y}$.

In the second place, we realize that I_1 and I_3 are the same integral up to a factor owing to $\cosh{(x+\pi i)} = -\cosh x$.

Finally, if we let $R \longrightarrow \infty$

$$\oint_C f(z)\, dz = \int_{-\infty}^{\infty} \frac{e^{i\Omega x}(1+e^{-\pi\Omega})}{\cosh x}\, dx = 2\pi e^{-\Omega\frac{\pi}{2}}$$

and

$$I_1 = \mathrm{Re}\Big[\int_{-\infty}^{\infty} \frac{e^{i\Omega x}}{\cosh x}\,dx\Big] = \mathrm{Re}\Big[\frac{2\pi e^{-\Omega\frac{\pi}{2}}}{(1+e^{-\pi\Omega})}\Big] = \pi\ \mathrm{sech}(\frac{\Omega\pi}{2}) = \pi\ \mathrm{sech}(\frac{\Omega\pi}{2})$$

Computation of I_2

Let

$$g(z) = \frac{\sinh z\, e^{iwz}}{\cosh^2 z}\ .$$

It is also analitic on and inside C except at $z_0 = i\frac{i\pi}{2}$ where we have a second order pole. As a consequence

$$\begin{aligned}
\mathrm{Res}_{z_0} g(z) &= \lim_{z\to z_0} \frac{d}{dz}((z-z_0)^2 g(z)) \\
&= \lim_{z\to z_0} (2(z-z_0)g(z) + (z-z_0)^2 \frac{d}{dz} g(z)) \\
&= 2we^{-w\frac{\pi}{2}} - we^{-w\frac{\pi}{2}} = we^{-w\frac{\pi}{2}}\ ,
\end{aligned}$$

where the limits have been calculated using Hopital's rule. Therefore,

$$\oint_C g(z)\,dz = 2\pi i\, we^{-w\frac{\pi}{2}}$$

Again, we split the integral along C in the same way we have proceeded above.

$$\begin{aligned}
\oint_C g(z)\,dz = &\underbrace{\int_{-R}^{R} \frac{\sinh x\, e^{iwx}}{\cosh^2 x}\,dx}_{I_2^1} + \underbrace{\int_0^{\pi} \frac{\sinh(R+iy)\, e^{iw(R+iy)}}{\cosh^2(R+iy)} i\,dy}_{I_2^2} \\
&+ \underbrace{\int_{R}^{-R} \frac{\sinh(x+\pi i)\, e^{iw(x+\pi i)}}{\cosh^2(x+\pi i)}\,dx}_{I_2^3} + \underbrace{\int_{\pi}^{0} \frac{\sinh(-R+iy)\, e^{iw(-R+iy)}}{\cosh^2(-R+iy)} i\,dy}_{I_2^4}
\end{aligned}$$

With a similar reasoning as before, we deduce that I_2^2, $I_2^4 \overset{R\to\infty}{\longrightarrow} 0$ and I_2^1, I_2^3 are the same integral up to a factor. For $R \longrightarrow \infty$

$$\oint_C g(z)\,dz = \int_{-\infty}^{\infty} \frac{\sinh x\, e^{iwx}\,(1+e^{-\pi w})}{\cosh^2 x}\,dx = 2\pi i\, we^{-w\frac{\pi}{2}}$$

and

$$I_2 = \mathrm{Im}\Big[\int_{-\infty}^{\infty} \frac{\sinh x \sin(wt)}{\cosh^2 x}\,dx\Big] = \mathrm{Im}\Big[\frac{2\pi i\, we^{-w\frac{\pi}{2}}}{(1+e^{-\pi w})}\Big] = \frac{2\pi\, we^{-w\frac{\pi}{2}}}{(1+e^{-\pi w})} = \pi w\ \mathrm{sech}(\frac{w\pi}{2}) \quad .$$

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