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Power Indices and Minimal Winning Coalitions in Simple Games with Externalities

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Abstract: We propose a generalization of simple games to situations with coalitional externalities. The main novelty of our generalization is a monotonicity property that we define for games in partition function form. This property allows us to properly speak about minimal winning embedded coalitions. We propose and characterize two power indices based on these kind of coalitions. We provide methods based on the multilinear extension of the game to compute the indices. Finally, the new indices are used to study the distribution of power in the current Parliament of Andalusia.

JEL Codes: C71.

Keywords: Deegan-Packel Index, Public Good Index, Simple Games, Partition Function Form, Monotonicity, Multilinear Extension.

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1 Introduction

Game theory provides valuable tools to study how power is distributed among the members of a decision making body. Most of the literature identifies the power of an agent with her ability to change the outcome of a ballot. The Shapley-Shubik index (Shapley and Shubik, 1954) is probably the most well known and widely accepted way to measure the power. However, a vast literature in the topic has grown and many different power indices have been proposed so far. The Banzhaf index (Banzhaf, 1964) for instance, can be regarded as the non-efficient relative of the Shapley-Shubik index. Indeed, both of them are probabilistic values (Weber, 1988) based on marginal contributions of an agent to coalitions of other agents. Some authors have claimed that in order to measure the power only minimal winning coalitions should be taken into consideration. A minimal winning coalition is a coalition that can pass a bill but where the participation of every member is necessary. That is, it is a winning coalition, which is minimal with respect to inclusion. Deegan and Packel (1978) proposed a power index based on the assumption that all minimal winning coalitions are equally likely and that members of a given minimal winning coalition are equally important. Some years latter, Holler (1982) introduced the Public Good index assuming that the source of power of an agent is the number of minimal winning coalitions in which she participates. Power indices based on different assumptions can be found in Johnston (1978) or Álvarez-Mozos et al. (2015) (see Alonso-Meijide et al., 2013, for a recent review on the topic).

Usually, the mechanism of a decision making body is based on voting. More precisely, every agent has certain weight and a proposal is approved when the overall weight of the supporters exceeds a given quota. Most of the times, the quota is fixed at simple majority. The classic way to model these processes is by means of simple games in characteristic function form. These games constitute an important subclass of cooperative games in which *(i)* the worth of every coalition is either 0 (losing) or 1 (winning), *(ii)* the grand coalition is winning, and *(iii)* if a coalition grows in size then, its worth cannot decrease.

However, we argue that some procedures of legislatures make use of variable quotas. Consider for instance, situations in which the proclamation of the president is a duty of the parliament and no candidate is supported by a majority of the deputies. In some cases, minority governments emerge with the support of less than half of the chamber. These kind of situations cannot be modeled by games in characteristic function because the worth of a coalition may depend on the coalitional behavior of the remaining agents. In this paper we focus on cooperative games

with coalitional externalities or games in partition function form (Thrall and Lucas, 1963). In a game in partition function form, the worth of a coalition depends on the arrangement of the rest of agents in coalitions, namely on the whole coalition structure. Therefore, the relevant objects are not coalitions but embedded coalitions that consist of a coalition structure and an active coalition whose worth is being evaluated. Myerson (1977) proposed and characterized an extension of the Shapley value for games in partition function form. Recently, the topic has attracted the attention of researchers and important contributions have been made. For instance, Hafalir (2007) has generalized the concepts of superadditivity and convexity. However, most of the recent contributions deal with generalizations of the Shapley value (Macho-Stadler et al., 2007; de Clippel and Serrano, 2008; Dutta et al., 2010). In Álvarez-Mozos and Tejada (2015) the Banzhaf value is generalized to games in partition function form.

In this paper, we first of all tackle this issue and introduce a class of simple games in partition function form. The first two requirements of simple games in characteristic function are easily generalized to the framework of games in partition function form. The interesting part arises when we try to define a monotonicity property in this framework. Actually, it is not clear what does it mean for an embedded coalition to grow in size. Certainly, we would like the active coalition to grow larger. But, what about the remaining coalitions of the structure? At this point several possibilities arise, see for instance Bolger (1986) or Grabisch (2010). Our proposal requires the partition of non-active agents to get finer. This idea of growth fits the investiture situations described above. We would like to point out that the same idea was used in a different setting by Carreras and Magaña (2008). However, to the best of our knowledge, it is the first time in which this family of games is studied.

Notice that other classes of simple games with externalities have been proposed to date. In a series of papers, Bolger (1983, 1986, 1990) studied different generalizations of the Shapley-Shubik and Banzhaf indices for multi-candidate voting games. The main difference between the latter and our simple games is the monotonicity property that we require. Another interesting family of simple games related to ours is the one of games with r -alternatives as introduced by Freixas and Zwicker (2003). The latter deals with ordered partitions. Consequently, they do not lie within the class of games in partition function form.

The Deegan-Packel and Public Good indices are generalized to the new family of simple games. Moreover, we also provide two characterizations for each of the indices by means of four properties. As we will see, the basic properties of efficiency, null player, and symmetry (Felsenthal and Machover, 1998) are easily generalized to our class of games. Then, in order

to single out each of the proposed indices we can use either a merging property (Deegan and Packel, 1978; Holler and Packel, 1983) or a monotonicity property (Lorenzo-Freire et al., 2007; Alonso-Mejjide et al., 2008).

To conclude, we also propose a multilinear extension technique that can be used to compute the proposed power indices (Leech, 2003; Alonso-Mejjide et al., 2008). The multilinear extension of a game in characteristic form was first proposed by Owen (1972). It can be regarded a generalization of the game where every agent can participate partially in a given coalition. However, the most interesting feature of the multilinear extension is the fact that it can ease the computation of a value. Here we propose an algorithm that facilitates the computation of our indices when the multilinear extension of the game is known.

The rest of the paper is organized as follows. In Section 2 we introduce the key notion of inclusion. This allows us to define monotonic games in partition function form and minimal winning embedded coalitions. In Section 3, we introduce the DP-externality and PG-externality power indices. Then, we provide two characterizations of each of them by means of four properties. In Section 4 we present two algorithms based on the multilinear extension of a game that yield the DP-externality and PG-externality power indices. Finally, in Section 5 a real example from the political field is used to illustrate the performance of our indices.

2 Simple games in partition function form

Let N be a finite set of players, then the set of partitions of N is denoted by $\mathcal{P}(N)$. For convenience, we assume that the empty set is an element of every partition, i.e., for every $P \in \mathcal{P}(N)$, $\emptyset \in P$. An *embedded coalition of N* is a pair (S, P) where $P \in \mathcal{P}(N)$ and $S \in P$. We will sometimes refer to S as the active coalition in P . The set of embedded coalitions of N is denoted by EC^N , i.e. $EC^N = \{(S, P) : P \in \mathcal{P}(N) \text{ and } S \in P\}$. We may abuse language and say that a player $i \in N$ is contained in an embedded coalition $(S, P) \in EC^N$ if player i belongs to S . We may abuse notation and write $S \cup i$ and $S \setminus i$ instead $S \cup \{i\}$ and $S \setminus \{i\}$, respectively. Given $P \in \mathcal{P}(N)$ and a nonempty coalition $S \in P$, we let $P_{-S} \in \mathcal{P}(N \setminus S)$ denote the partition $P \setminus \{S\}$.

A *game in partition function form or game with externalities* is a pair (N, v) , where N is the finite set of players and v is the partition function that assigns to every embedded coalition a real number, i.e., $v : EC^N \rightarrow \mathbb{R}$, with the convention that for every $P \in \mathcal{P}(N)$, $v(\emptyset, P) = 0$. The real number $v(S, P)$ is to be understood as the worth of coalition S when the rest of the players

is organized according to P . The set of games in partition function form with common player set N is denoted by \mathcal{G}^N and the set of games in partition function form with an arbitrary player set is denoted by \mathcal{G} . It is easy to notice that \mathcal{G}^N is a vector space over \mathbb{R} . Indeed, de Clippel and Serrano (2008) devised a basis of the vector space that generalizes the basis of games in characteristic function that consists of unanimity games (Shapley, 1953). Given $(S, P) \in EC^N$, with $S \neq \emptyset$, let $(N, e_{(S,P)}) \in \mathcal{G}$ be defined for every $(T, Q) \in EC^N$ by

$$e_{(S,P)}(T, Q) = \begin{cases} 1 & \text{if } S \subseteq T \text{ and } \forall T' \in Q_{-T}, \exists S' \in P \text{ such that } T' \subseteq S', \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Then, de Clippel and Serrano (2008) show that $\{(N, e_{(S,P)}) : (S, P) \in EC^N \text{ and } S \neq \emptyset\}$ constitutes a basis of \mathcal{G}^N .

In this paper we are concerned with a subclass of \mathcal{G} that generalizes simple games in characteristic function form as introduced by von Neumann and Morgenstern (1944). For so doing, we develop a concept of monotonicity for games in partition function form. The intuition behind monotonic games is that the enlargement of a coalition cannot cause a decrease in its worth. Therefore, in order to generalize this idea, we use a notion of inclusion for embedded coalitions that will be of key importance for our results and that it is implicitly formulated in Eq. (1).

Definition 2.1. *Let N be a finite set and $(S, P), (T, Q) \in EC^N$. We define the inclusion among embedded coalitions as follows:*

$$(S, P) \subseteq (T, Q) \iff S \subseteq T \text{ and } \forall T' \in Q_{-T}, \exists S' \in P \text{ such that } T' \subseteq S'$$

Note that whenever $S \neq \emptyset$, $(S, P) \subseteq (T, Q)$ if and only if $e_{(S,P)}(T, Q) = 1$. According to the above definition, an embedded coalition (S, P) is a subset of another embedded coalition (T, Q) if the coalition S is a subset of T and moreover, the partition $\{R \setminus T : R \in P\}$ is coarser than Q_{-T} . Notice that both families of coalitions are partitions of $N \setminus T$.

We are now in the position to introduce the class of games under study.

Definition 2.2. *A game in partition function form $(N, v) \in \mathcal{G}$ is said to be a simple game in partition function form if it satisfies the three conditions below:*

- i) For every $(S, P) \in EC^N$, $v(S, P) \in \{0, 1\}$.*
- ii) $v(N, \{\emptyset, N\}) = 1$.*
- iii) If $(S, P), (T, Q) \in EC^N$ such that $(S, P) \subseteq (T, Q)$, then $v(S, P) \leq v(T, Q)$.*

An embedded coalition, $(S, P) \in EC^N$, is said to be winning if $v(S, P) = 1$ and loosing otherwise. A game in partition function form satisfying condition iii) is said to be monotonic. The set of simple games in partition function form with common player set N is denoted by \mathcal{SG}^N and the set of simple games in partition function form with an arbitrary player set is denoted by \mathcal{SG} .

Note that, simple games, as defined above, are a model to describe the parliamentary situations considered in the introduction. First, in the investiture session of a parliament each embedded coalition is either winning or loosing. Second, the grand coalition is always winning coalition.¹ Third, suppose that $(S, P) \in EC^N$ is a winning embedded coalition, then if coalition S grows or the remaining agents are more divided the resulting embedded coalition remains to be winning.

It is straightforward to check that the games that form the basis of de Clippel and Serrano (2008), see Eq. (1), are instances of simple games in partition function form as defined above. That is, if N is a finite set of players, then for every $(S, P) \in EC^N$, $(N, e_{(S,P)}) \in \mathcal{SG}$.

The monotonicity property considered above allows us to properly speak about minimal winning embedded coalitions. Let $(N, v) \in \mathcal{SG}$, a winning embedded coalition, $(S, P) \in EC^N$, is said to be *minimal* if every proper subset of it is a losing embedded coalition, i.e., if $(T, Q) \subsetneq (S, P)$ implies that $v(T, Q) = 0$.² The set of all minimal winning embedded coalitions of a simple game in partition function form is denoted by $\mathcal{M}(v)$ and the subset of minimal winning embedded coalitions that contain a given player $i \in N$ is denoted by $\mathcal{M}_i(v)$, i.e., $\mathcal{M}_i(v) = \{(S, P) \in \mathcal{M}(v) : i \in S\}$.

A player $i \in N$ is said to be a *null player* in $(N, v) \in \mathcal{SG}$ if she does not participate in any minimal winning embedded coalition, i.e., $\mathcal{M}_i(v) = \emptyset$. Similarly, two players $i, j \in N$ are said to be *symmetric* in $(N, v) \in \mathcal{SG}$ if exchanging the two players does not change the worth of an embedded coalition, i.e., if for every $(S, P) \in EC^N$ such that $S \subseteq N \setminus \{i, j\}$,

$$(S \cup i, P_{-S, P(i)} \cup \{S \cup i, P(i) \setminus i\}) \in \mathcal{M}(v) \Leftrightarrow (S \cup j, P_{-S, P(j)} \cup \{S \cup j, P(j) \setminus j\}) \in \mathcal{M}(v).$$

It is easy to notice, that as a consequence of the monotonicity property, a simple game in partition function form is completely determined by the set of minimal winning embedded coalitions of the game. In a sense, all the relevant information of a simple game in partition function form is condensed in the set of minimal winning embedded coalitions. We state this fact formally in the proposition below.

¹Note that we can talk about the grand coalition because there is a single embedded coalition of the type (N, P) .

²A proper subset, $(T, Q) \subsetneq (S, P)$, is a subset $(T, Q) \subseteq (S, P)$ satisfying $(T, Q) \neq (S, P)$.

Proposition 2.1. *Every simple game in partition function form can be obtained as the maximum of games in a subset of $\{(N, e_{(S,P)}) : (S, P) \in EC^N \text{ and } S \neq \emptyset\}$. Indeed, for every $(N, v) \in \mathcal{SG}$ and every $(T, Q) \in EC^N$ it holds that*

$$v(T, Q) = \max_{(S,P) \in \mathcal{M}(v)} e_{(S,P)}(T, Q).$$

Proof. Let $(N, v) \in \mathcal{SG}$ and $(T, Q) \in EC^N$. On the one hand, if (T, Q) is a winning embedded coalition then, by definition there is $(S, P) \subseteq (T, Q)$ where $(S, P) \in \mathcal{M}(v)$ and $v(T, Q) = \max_{(S,P) \in \mathcal{M}(v)} e_{(S,P)}(T, Q) = 1$. On the other hand, if (T, Q) is a losing embedded coalition then, since (N, v) is monotonic it holds that for every $(S, P) \subseteq (T, Q)$, (S, P) is a losing embedded coalition. In particular there is no minimal winning embedded coalition contained in (T, Q) and thus, $v(T, Q) = \max_{(S,P) \in \mathcal{M}(v)} e_{(S,P)}(T, Q) = 0$. \square

Proposition 2.2. *Let $\mathcal{C} \subseteq EC^N$ be such that for every $(S, P), (T, Q) \in \mathcal{C}$ with $(S, P) \neq (T, Q)$, $(S, P) \not\subseteq (T, Q)$ and $(T, Q) \not\subseteq (S, P)$. Then, there exists a unique simple game in partition function form, (N, v) , such that $\mathcal{M}(v) = \mathcal{C}$.*

Proof. Let $\mathcal{C} \subseteq EC^N$ be such that for every $(S, P), (T, Q) \in \mathcal{C}$ with $(S, P) \neq (T, Q)$, $(S, P) \not\subseteq (T, Q)$ and $(T, Q) \not\subseteq (S, P)$. Define $(N, v) \in \mathcal{SG}$ for every $(T, Q) \in EC^N$ by $v(T, Q) = \max_{(S,P) \in \mathcal{C}} e_{(S,P)}(T, Q)$. Since there is no inclusion relation among the embedded coalitions in \mathcal{C} , it follows that $\mathcal{M}(v) = \mathcal{C}$. Finally, we show that such (N, v) is unique. Indeed, suppose $(N, w) \in \mathcal{SG}$ satisfying $\mathcal{M}(w) = \mathcal{C}$. Then, by Proposition 2.1, $w = \max_{(S,P) \in \mathcal{M}(w)} e_{(S,P)} = \max_{(S,P) \in \mathcal{C}} e_{(S,P)} = v$, which concludes the proof. \square

3 Power indices based on minimal winning embedded coalitions

In this section, we first propose natural generalizations of the Deegan-Packel and Public Good indices (see Deegan and Packel, 1978; Holler, 1982) for simple games in partition function form. Next, we consider a number of properties and characterize the proposed indices.

A *power index* is a mapping, f , that assigns to every simple game $(N, v) \in \mathcal{SG}$ a vector $f(N, v) \in \mathbb{R}^N$, where each coordinate $f_i(N, v)$ describes the power of agent $i \in N$.

Definition 3.1. *The DP-externality power index is denoted by DP and is defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by*

$$DP_i(N, v) = \frac{1}{|\mathcal{M}(v)|} \sum_{(S,P) \in \mathcal{M}_i(v)} \frac{1}{|S|}.$$

The motivation behind the above definition emerges from three assumptions on how agents behave in a simple game in partition function form. First, only minimal winning embedded coalitions will be formed. The first assumption is twofold, on the one hand it requires the active coalition to be of a minimal size and on the other hand, it demands the coalitions that may create externalities on the active coalition to be of maximal size. Second, every minimal winning embedded coalition emerges with the same probability. Third, the agents participating in a minimal winning embedded coalition share the power equally.

Definition 3.2. *The PG-externality power index is denoted by PG and is defined for every $(N, v) \in \mathcal{SG}$ and $i \in N$ by*

$$\text{PG}_i(N, v) = \frac{|\mathcal{M}_i(v)|}{\sum_{j \in N} |\mathcal{M}_j(v)|}.$$

Similar to DP, the definition of PG above considers that in order to measure the power of an agent in a simple game in partition function form only minimal winning embedded coalitions should be taken into consideration. The main difference lies in the fact that PG is not sensitive to the sizes of minimal winning embedded coalitions. Instead, it suggests that the power of an agent stems from the number of minimal winning embedded coalitions in which she participates.

Next, we describe a number of properties that a power index may satisfy. The first three are standard axioms in cooperative game theory.

EFF A power index f satisfies *efficiency* if for every $(N, v) \in \mathcal{SG}$,

$$\sum_{i \in N} f_i(N, v) = 1.$$

NPP A power index f satisfies the *null player property* if for every $(N, v) \in \mathcal{SG}$ and every $i \in N$ null player in (N, v) ,

$$f_i(N, v) = 0.$$

SYM A power index f satisfies *symmetry* if for every $(N, v) \in \mathcal{SG}$ and every pair $i, j \in N$ of symmetric players in (N, v) ,

$$f_i(N, v) = f_j(N, v).$$

Little discussion is needed on the first three properties. Indeed, they are trivial adaptations of properties that any sensible power index should satisfy (Felsenthal and Machover, 1998).

In order to single out the DP-externality or the PG-externality power index we need an additional property. As we will see, the two merging properties introduced in Deegan and Packel

(1978) and Holler and Packel (1983) can be easily adapted to our setting with externalities. The common feature of the aforementioned properties is that they require the power in the maximum of certain pairs of simple games to be weighted averages of the powers in the two games. Obviously, the two properties only differ on the weights used in the average. Before we can present the adaptation of the two properties we have to introduce one more concept. It is easy to notice that a minimal winning embedded coalition of the maximum game is a minimal winning embedded coalition in one of the two original games. However, there are cases in which minimal winning embedded coalitions in the original games are not so in the maximum game. These situations are ruled out in the following concept. Two simple games in partition function form $(N, v), (N, w) \in \mathcal{SG}$ are said to be *mergeable* if

$$\forall (S, P) \in \mathcal{M}(v) \text{ and } (T, Q) \in \mathcal{M}(w), \quad (S, P) \not\subseteq (T, Q) \text{ and } (T, Q) \not\subseteq (S, P).$$

It is straightforward to check that the minimal winning embedded coalitions in the maximum of two mergeable games is precisely the union of the minimal winning embedded coalitions in the original simple games in partition function form. Let us denote by \vee the maximum operator, i.e., for every $(N, v), (N, w) \in \mathcal{SG}$, $(N, v \vee w) \in \mathcal{SG}$ is defined for every $(S, P) \in EC^N$ by $(v \vee w)(S, P) = \max \{v(S, P), w(S, P)\}$. The two properties below describe the distribution of power in a simple game in partition function form obtained as the maximum of two mergeable games.

DP-MER A power index f satisfies *DP-mergeability* if for every pair of mergeable simple games in partition function form, $(N, v), (N, w) \in \mathcal{SG}$,

$$f(N, v \vee w) = \frac{|\mathcal{M}(v)| \cdot f(N, v) + |\mathcal{M}(w)| \cdot f(N, w)}{|\mathcal{M}(v \vee w)|}.$$

PG-MER A power index f satisfies *PG-mergeability* if for every pair of mergeable simple games in partition function form $(N, v), (N, w) \in \mathcal{SG}$ and every $i \in N$,

$$f_i(N, v \vee w) = \frac{\sum_{j \in N} |\mathcal{M}_j(v)| \cdot f_i(N, v) + \sum_{j \in N} |\mathcal{M}_j(w)| \cdot f_i(N, w)}{\sum_{j \in N} |\mathcal{M}_j(v \vee w)|}.$$

Both DP-MER and PG-MER state that the power in the maximum of two mergeable simple games in partition function form is a weighted average of the power in each of the original games. According to DP-MER the weights used when taking the average are given by the ratio of the number of minimal winning embedded coalitions in the original game and in the maximum

game. On the contrary, PG-MER uses weights given by the ratio of the sum of cardinalities of minimal winning embedded coalitions in which every agent participates both in the original game and in the maximum game.

Theorem 3.1. *The DP-externality power index is the only power index satisfying EFF, NPP, SYM, and DP-MER.*

Proof. Using the definition, it is straightforward to check that DP satisfies EFF, NPP, and SYM. To see that it also satisfies DP-MER, it is enough to take into account that for every pair of mergeable simple games in partition function form, $(N, v), (N, w) \in \mathcal{SG}$, it holds that $\mathcal{M}(v \vee w) = \mathcal{M}(v) \cup \mathcal{M}(w)$.

Next, we show that DP is indeed the only power index satisfying EFF, NPP, SYM, and DP-MER by induction on the number of minimal winning embedded coalitions. Let f be a power index satisfying the four properties. First, let $(N, v) \in \mathcal{SG}$ be such that $|\mathcal{M}(v)| = 1$. Then, $\mathcal{M}(v) = \{(S, P)\}$ for some $(S, P) \in EC^N$ and $v = e_{(S, P)}$. It is immediate to check that every $i \notin S$ is a null player in $(N, e_{(S, P)})$. Then, by NPP, $f_i(N, e_{(S, P)}) = 0$. Similarly, every pair of players in S are symmetric in $(N, e_{(S, P)})$. Then, by SYM they get the same payoff and by EFF we conclude that for every $i \in S$, $f_i(N, e_{(S, P)}) = \frac{1}{|S|}$. Second, suppose that f is uniquely determined for every $(N, v) \in \mathcal{SG}$ with $|\mathcal{M}(v)| < r$. Let $(N, v) \in \mathcal{SG}$ with $\mathcal{M}(v) = \{(S_1, P^1), \dots, (S_r, P^r)\}$. Next, by Proposition 2.1 we have that for every $(T, Q) \in EC^N$,

$$v(T, Q) = \max_{(S, P) \in \mathcal{M}(v)} e_{(S, P)}(T, Q) = \max \{w(T, Q), e_{(S_r, P^r)}(T, Q)\},$$

where $w(T, Q) = \max_{k \in \{1, \dots, r-1\}} e_{(S_k, P^k)}(T, Q)$. Since $\{(S_1, P^1), \dots, (S_r, P^r)\}$ is the set of all minimal winning embedded coalitions of (N, v) then, for every $k \in \{1, \dots, r-1\}$, $(S_k, P^k) \not\subseteq (S_r, P^r)$ and $(S_r, P^r) \not\subseteq (S_k, P^k)$. Then, (N, w) and $(N, e_{(S_r, P^r)})$ are mergeable games and by DP-MER for every $i \in N$

$$f(N, v) = f(N, w \vee e_{(S_r, P^r)}) = \frac{(r-1) \cdot f(N, w) + f(N, e_{(S_r, P^r)})}{r}. \quad (2)$$

Finally, the two payoffs in the right hand side of the equation above are uniquely determined by the induction hypothesis and the proof concludes. \square

Theorem 3.2. *The PG-externality power index is the only power index satisfying EFF, NPP, SYM, and PG-MER.*

Proof. It is straightforward to check that PG satisfies EFF, NPP, and SYM. By definition, for every $(N, v) \in \mathcal{SG}$ and $i \in N$, $\text{PG}_i(N, v) \sum_{j \in N} |\mathcal{M}_j(v)| = |\mathcal{M}_i(v)|$ and it follows that

PG satisfies PG-MER. The uniqueness part follows the same lines as we did in the proof of Theorem 3.1. The only difference is in the last step where instead of DP-MER we now apply PG-MER and instead of Eq. (2) we obtain

$$f_i(N, w \vee e_{(S_r, P_r)}) = \frac{\sum_{j \in N} |\mathcal{M}_j(w)| \cdot f_i(N, w) + \sum_{j \in N} |\mathcal{M}_j(e_{(S_r, P_r)})| \cdot f_i(N, e_{(S_r, P_r)})}{\sum_{j \in N} |\mathcal{M}_j(v)|},$$

for every $i \in N$. As in the proof of Theorem 3.1, the uniqueness follows by induction. \square

Next, we introduce two monotonicity properties that give rise to two alternative characterizations of DP and PG. Monotonicity properties describe how the payoffs of a given player changes when her position in the game is strengthened. Lorenzo-Freire et al. (2007) and Alonso-Mejide et al. (2008) introduced two monotonicity properties in the context of games in characteristic function that can be easily adapted to our setting with externalities. Indeed, consider the following two properties that a power index can be asked to satisfy.

DP-MON A power index f satisfies *DP-monotonicity* if for every pair of simple games in partition function form, $(N, v), (N, w) \in \mathcal{SG}$ and every $i \in N$ with $\mathcal{M}_i(v) \subseteq \mathcal{M}_i(w)$ we have

$$f_i(N, w)|\mathcal{M}(w)| \geq f_i(N, v)|\mathcal{M}(v)|.$$

PG-MON A power index f satisfies *PG-monotonicity* if for every pair of simple games in partition function form, $(N, v), (N, w) \in \mathcal{SG}$ and every $i \in N$ with $\mathcal{M}_i(v) \subseteq \mathcal{M}_i(w)$ we have

$$f_i(N, w) \sum_{j \in N} |\mathcal{M}_j(w)| \geq f_i(N, v) \sum_{j \in N} |\mathcal{M}_j(v)|.$$

Theorem 3.3. *The DP-externality power index is the only power index satisfying EFF, NPP, SYM, and DP-MON.*

Proof. At this point it is clear that DP satisfies EFF, NPP, and SYM. Let $(N, v), (N, w) \in \mathcal{SG}$ and $i \in N$ with $\mathcal{M}_i(v) \subseteq \mathcal{M}_i(w)$, we have

$$DP_i(N, w)|\mathcal{M}(w)| = |\mathcal{M}_i(w)| \geq |\mathcal{M}_i(v)| = DP_i(N, v)|\mathcal{M}(v)|.$$

Then, DP satisfies DP-MON.

Next, we prove the uniqueness by induction on $|\mathcal{M}(v)|$. Let f be a power index satisfying EFF, NPP, SYM, and DP-MON. Note that the reasoning in the proof of Theorem 3.1 can be repeated for the base case of the induction. Let us assume that the result is true for all $(N, v) \in \mathcal{SG}$ with $|\mathcal{M}(v)| < r$, for some $r > 1$. Let $(N, v) \in \mathcal{SG}$ such that $\mathcal{M}(v) = \{(S_1, P^1), \dots, (S_r, P^r)\}$

with $r > 1$ and $R = S_1 \cap \dots \cap S_r$. Note that $R \neq N$ because $r > 1$. Then, consider $i \notin R$. If $i \notin S_l$ for every $l = 1, \dots, r$, then i is a null player and $f_i(N, v) = 0$ by NPP. If there is some $l \in \{1, \dots, r\}$ such that $i \in S_l \setminus R$ then, define $(N, w) \in \mathcal{SG}$ with $\mathcal{M}(w) = \mathcal{M}_i(v)$. Since the embedded coalitions in $\mathcal{M}_i(v)$ are minimal winning in (N, v) , there is no inclusion relation among them. Then, by Proposition 2.2, $(N, w) \in \mathcal{SG}$ is unique and well defined. Next, taking into account that $\mathcal{M}_i(w) = \mathcal{M}_i(v)$ and applying the DP-MON property twice,

$$f_i(N, v) |\mathcal{M}(v)| = f_i(N, w) |\mathcal{M}(w)|. \quad (3)$$

Note that since $i \notin R$, $|\mathcal{M}(w)| < |\mathcal{M}(v)|$. Then, the right hand side of Eq (3) is unique by induction. Finally, let $i \in R$. By EFF $\sum_{j \in R} f_j(N, v)$ is uniquely determined. Suppose that $|R| > 1$, otherwise the proof concludes. Notice that every pair of players $j, k \in R$ are symmetric in $(N, v) \in \mathcal{SG}$. Then by SYM, $f_i(N, v)$ is also unique. \square

In a similar way we can characterize the PG-externality power index using the PG-MON property.

Theorem 3.4. *The PG-externality power index is the only power index satisfying EFF, NPP, SYM, and PG-MON.*

Proof. On the one hand, we already now that PG satisfies EFF, NPP, and SYM. By definition, for every $(N, v) \in \mathcal{SG}$ and $i \in N$, $\text{PG}_i(N, v) \sum_{j \in N} |\mathcal{M}_j(v)| = |\mathcal{M}_i(v)|$ and it follows that PG satisfies PG-MON. On the other hand, for the uniqueness we follow the same lines as in Theorem 3.3. The only difference is in the last step where instead of DP-MON we now apply PG-MON and instead of Eq. (3) we obtain

$$f_i(N, v) \sum_{j \in N} |\mathcal{M}_j(v)| = f_i(N, w) \sum_{j \in N} |\mathcal{M}_j(w)|,$$

for $i \notin R$. Thus, the uniqueness follows by induction. The reasoning for $i \in R$ is analogous. \square

4 Multilinear extension

In this section we provide algorithms for computing the two power indices introduced in Section 3. These methods are based on the multilinear extension of a game as introduced by Owen (1972) (see also Owen, 1975; Alonso-Meijide et al., 2008). The multilinear extension of an n -person game without externalities is a function defined on the cube $[0, 1]^n$ which is linear in each of its variables and coincides with the game at the corners of the cube. Using this function

several methods have been devised for the computation of power indices in different frameworks. It is well known that the calculation of power indices is computationally costly, especially when the number of players is large. This has been a major factor limiting the study of power in real institutions. It is worth pointing out that even though the original method (Owen, 1972) has exponential time complexity, it has motivated more efficient methods that yield approximations of power indices (Leech, 2003).

Notice that the class of games under study is much more complex than that of simple games without externalities. Indeed, there are much more embedded coalitions than coalitions in a n -person set. For instance, there are 5, 15, 52, 203, and 877 embedded coalitions in sets of 2, 3, 4, 5, and 6 agents, respectively. However, for big enough sets of agents there are more permutations than embedded coalitions.

Before we can present the two procedures we define the multilinear extension of a simple game with externalities.³ Let $(N, v) \in \mathcal{SG}$. Given an embedded coalition $(S, P) \in EC^N$, we identify (S, P) with the ordered partition $P' = (P'_1, \dots, P'_{m(P)})$ where $m(P) = |P|$, $P'_1 = S$, and for every $l \in \{2, \dots, m(P)\}$ there exist $P_l \in P$, such that $P'_l = P_l$, and for every pair of different numbers $l, m \in \{2, \dots, m(P)\}$, if $l < m$ then $\min\{k : k \in P'_l\} < \min\{k : k \in P'_m\}$. Let $x_i, y_k^{(j)} \in [0, 1]$, for every $i, k \in \{1, 2, \dots, n\}$ and $j \in \{2, \dots, n\}$. We define the *multilinear extension* of (N, v) as:

$$f\left(x_1, \dots, x_n, (y_i^{(2)})_{i \in N}, \dots, (y_i^{(n)})_{i \in N}\right) = \sum_{P' \equiv (S, P) \in EC^N} \prod_{i \in S} x_i \prod_{j=2}^{m(P)} \prod_{k \in P_j} y_k^{(j)} v(S, P).$$

Theorem 4.1. *Let $(N, v) \in \mathcal{SG}$ and $i \in N$. The DP-externality power index can be obtained from the following procedure.⁴*

1. Let f be the multilinear extension of (N, v) .
2. Consider all the monomials of f with minimum degree.
3. Define a new function, g , that consists of all the monomials obtained in the previous step. Let d be the degree of these monomials.
4. While $d \leq n - 1$, do:

³Note that the same expression can be used to define the multilinear extension of an arbitrary game in partition function form.

⁴For the sake of exposition, every time we say degree of a monomial we mean the degree of the monomial with respect to the variables x_i .

(a) For every pair of monomials of g of degree d , $a = \prod_{i \in S} x_i \prod_{j=2}^{m(P)} \prod_{l \in P_j} y_l^{(j)}$ and $b = \prod_{i \in S} x_i \prod_{j=2}^{m(Q)} \prod_{l \in Q_j} y_l^{(j)}$, check if for every $j \in \{2, \dots, m(P)\}$ there is some $r \in \{2, \dots, m(Q)\}$ such that the monomial $\prod_{l \in P_j} y_l$ is divisible by $\prod_{l \in Q_r} y_l$. In that case delete the monomial b from g .

(b) $d = d + 1$.

(c) For every monomial of f of degree d , $c = \prod_{i \in T} x_i \prod_{j=2}^{m(Q)} \prod_{l \in Q_j} y_l^{(j)}$ check if there is a monomial in g , $a = \prod_{i \in S} x_i \prod_{j=2}^{m(P)} \prod_{l \in P_j} y_l^{(j)}$ such that $\prod_{i \in T} x_i$ is divisible by $\prod_{i \in S} x_i$. In the negative, add c to the function g . In the affirmative, check if for every $j \in \{2, \dots, m(P)\}$ there is some $r \in \{2, \dots, m(Q)\}$ such that the monomial $\prod_{l \in P_j} y_l$ is divisible by $\prod_{l \in Q_r} y_l$. In the negative, add c to the function g .

5. Consider the function $h(x_1, \dots, x_n) = g(x_1, \dots, x_n, 1, \dots, 1)$

6. Then,

$$DP_i(N, v) = \frac{1}{h(1, \dots, 1)} \int_0^1 \frac{\partial h}{\partial x_i}(r, \dots, r) dr.$$

Proof. Let $(N, v) \in \mathcal{SG}$ and $i \in N$. In Step 1, we build the multilinear extension of (N, v) . In Step 2, we obtain the addends corresponding to embedded coalitions (S, P) with minimal cardinality of S . In Step 4a) we obtain the minimal winning embedded coalitions (S, P) with $|S| = d$. In Step 4b) we increase the size of the coalitions in one unit, $d + 1$. In Step 4c), we incorporate possible minimal winning embedded coalitions of size $d + 1$ to the function g . At the end of Step 4, each monomial correspond to a minimal winning embedded coalition, i.e.,

$$g\left(x_1, \dots, x_n, (y_i^{(2)})_{i \in N}, \dots, (y_i^{(n)})_{i \in N}\right) = \sum_{P' \equiv (S, P) \in \mathcal{M}(v)} \prod_{i \in S} x_i \prod_{j=2}^{m(P)} \prod_{k \in P_j} y_k^{(j)} v(S, P).$$

Then, $h(1, \dots, 1) = |\mathcal{M}(v)|$. Moreover, note that

$$\int_0^1 \frac{\partial h}{\partial x_i}(r, \dots, r) dr = \int_0^1 \sum_{(S, P) \in \mathcal{M}_i(v)} r^{|S|-1} dr,$$

which concludes the proof. \square

Theorem 4.2. Let $(N, v) \in \mathcal{SG}$ and $i \in N$. The PG-externality power index can be obtained from the following procedure:

1, 2, 3, and 4 as in Theorem 4.1.

5. For every $j \in N$, consider the functions $h_i(x_i) = g(1, \dots, x_i, 1, \dots, 1)$.

6. Finally, compute the derivatives, $h'_i(x_i)$, of the above functions. Then,

$$\text{PG}_i(N, v) = \frac{h'_i(x_i)}{\sum_{j \in N} h'_j(x_j)}$$

Proof. Let $(N, v) \in \mathcal{SG}$ and $i \in N$. From the proof of Theorem 4.1 we know that the function g at the end of Step 4 contains the monomials associated with minimal winning embedded coalitions. Then, for every $j \in N$,

$$h_j(x_j) = |\mathcal{M}_j(v)|x_j,$$

and the proof concludes. □

5 A political example

In this section, we illustrate the performance of the new power indices and compare the distribution of power of the usual simple game with that of the simple game with externalities. The Parliament of Andalusia, one of Spain's seventeen regions, is constituted by 109 members. Since elections in 2015, the parliament consists of 47 members of the social-democratic party PSOE (*PS*), 33 members of the conservative party PP (*PP*), 15 members of the young left-wing party Podemos (*PO*), 9 members of the young liberal party Ciudadanos (*CI*), and 5 members of the traditional left-wing party IU (*IU*). After a long negotiation process, the candidate of *PS* was invested president of the regional government with the votes of *PS* and *CI*. It is important to point out that we abstract from the ideological and political strategies of the parties. In order to account for these considerations there are more sophisticated models that could be used (see for instance Alonso-Mejjide and Bowles, 2005; Alonso-Mejjide et al., 2009).

In the first place, we describe the situation without externalities. That is, we consider the most common decision rule in the Parliament which is the simple majority. Indeed, this is the rule used to pass most of the bills in the parliament. More precisely, let $N = \{PS, PP, PO, CI, IU\}$ and $w = (w_{PS}, w_{PP}, w_{PO}, w_{CI}, w_{IU}) = (47, 33, 15, 9, 5)$. Then for every $S \subseteq N$, $v(S) = 1$ if and only if $\sum_{i \in N} w_i \geq 55$. In this simple game in characteristic function there are only 4 minimal winning coalitions, namely $\{PS, PP\}$, $\{PS, PO\}$, $\{PS, CI\}$, and $\{PP, PO, CI\}$. From an

inspection of the minimal winning coalition we conclude that IU is a null player and PP , PO , and CI are symmetric players.

In the second place, we consider a voting mechanism in which the quota is not fixed a priori. To be precise, suppose that parties organize themselves in a coalition structure and a coalition is winning whenever it receives as many votes as any other coalition of the structure. Formally, consider the simple game with externalities, $(N, v) \in \mathcal{SG}$, defined for every $(S, P) \in EC^N$ by

$$v(S, P) = 1 \iff w(S) \geq w(T) \quad \forall T \in P.$$

Notice that our voting game allows for draws. In the example under consideration there is a coalition structure, namely $P = \{\{PS\}, \{PP, CI, IU\}, \{PO\}\}$ in which the coalitions $\{PS\}$ and $\{PP, CI, IU\}$ get the same votes. Therefore, $v(\{PS\}, P) = v(\{PP, CI, IU\}, P) = 1$. In Table 1 we list the 10 minimal winning embedded coalitions of the simple game in partition function form.

Active coalition	Coalition structure
$\{PS\}$	$\{PS\}, \{PP, CI, IU\}, \{PO\}$
$\{PS\}$	$\{PS\}, \{PP\}, \{PO, CI, IU\}$
$\{PS\}$	$\{PS\}, \{PP, IU\}, \{PO, CI\}$
$\{PS\}$	$\{PS\}, \{PP, CI\}, \{PO, IU\}$
$\{PS, CI\}$	$\{PS, CI\}, \{PP, PO, IU\}$
$\{PS, PO\}$	$\{PS, PO\}, \{PP, CI, IU\}$
$\{PS, IU\}$	$\{PS, IU\}, \{PP, PO\}, \{CI\}$
$\{PP, PO\}$	$\{PP, PO\}, \{PS\}, \{CI, IU\}$
$\{PP, PO, CI\}$	$\{PP, PO, CI\}, \{PS, IU\}$
$\{PP, CI, IU\}$	$\{PP, CI, IU\}, \{PS\}, \{PO\}$

Table 1: Minimal winning embedded coalitions.

Note that IU is not null when we consider a variable quota because it participates in two minimal winning embedded coalitions (seventh and tenth rows). Another difference that can be observed from the list above is that PP , PO , and CI are not symmetric anymore.

In Table 2 we depict the Deegan-Packel and Public Good power indices of the parties in the Parliament of Andalusia both when we consider the simple game in characteristic function form (fixed quota) and in partition function form (variable quota).

	Seats	Charac. function		Partition function	
		D-P	P-G	DP	PG
PS	47	.375	.33	.550	.389
PP	33	.208	.22	.117	.167
PO	15	.208	.22	.133	.167
CI	9	.208	.22	.117	.167
IU	5	0	0	.083	.111

Table 2: The Deegan-Packel and Public Good indices both without and with externalities.

Several comments are in order. First, when we move from a fixed quota to a variable quota the strongest party becomes more powerful. Second, the three parties that were symmetric with the fixed quota decrease their power. Third, and probably the most surprising observation is that according to the Deegan-Packel index, *PO* has more power than *PP* even though it has less seats. This is an evidence of the complexity of the family of games and indices considered in this paper.

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