BACHELOR'S DEGREE THESIS

Degree in Statistics

Title: Monte Carlo simulations on the Black-Litterman model with absolute views: a comparison with the Markowitz model and an equal weight asset allocation strategy

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Academic year: 2014/2015

Universitat de Barcelona

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Degree Thesis

Monte Carlo simulations on the Black-Litterman model with absolute views: a comparison with the Markowitz model and an equal weight asset allocation strategy

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Abstract

Keywords: Black-Litterman, Modern Portfolio Theory, Portfolio Management, Exchange Traded Funds

MSC2000: 91B70, 91G10, 91G60, 91G70, 65K10

The focus of this degree thesis is on the Black-Litterman asset allocation model applied to recent popular investment vehicles such as Exchange Traded Funds (ETFs) simulating absolute views generated by Monte Carlo simulations that allow the inclusion of correlations. The sensibility of the scalar τ (which is a measure of the investor's confidence in the prior estimates) contained in the Black-Litterman model will be analyzed over several periods of time and the results obtained compared with the Markowitz model developed by Harry Markowitz and an equal weight asset allocation strategy in order to determine the performance of the model.

The results obtained determine that the Markowitz model and the equal weight asset allocation strategy can be beaten by the Black-Litterman model using investors' views that incorporate information no include in the historical data and using the correct value of τ and the adequate time period of data.

Resumen

Simulaciones de Monte Carlo sobre el modelo de Black-Litterman con vistas absolutas: Una comparativa con el modelo de Markowitz y una estrategia equiponderada de gestión de activos

En este trabajo se estudian los resultados de aplicar el modelo de gestión de carteras Black-Litterman a fondos cotizados en bolsa (ETFs), los cuáles están compuestos por una cesta de valores y cotizan en bolsa al igual que las acciones.

Un ETF se puede considerar una cartera diversificada debido a que esta compuesto de varios valores. Al aplicar un modelo de gestión de carteras sobre ETFs, se puede conseguir un grado de diversificación muy elevado debido a que se crea una cartera compuesta de varias carteras.

El modelo de Black-Litterman a diferencia de otros modelos de gestión de cartera, permite tener en consideración los rendimientos esperados de un inversor usando un enfoque Bayesiano. Usando simulaciones de Monte Carlo que permiten tener en cuenta la correlación de los 12 ETF considerados en este trabajo, se han generado distintas simulaciones de posibles escenarios y se han incorporado en el modelo de Black-Litterman como vistas absolutas (ej. El ETF 1 tendrá un rendimiento del 5%) para terminar obteniendo los rendimientos de las carteras en los distintos periodos considerados.

El parámetro τ en el modelo de Black-Litterman afecta al nivel de confianza que tiene el inversor sobre la distribución a priori. Debido a que ha generado mucha discusión y con el fin de determinar su sensibilidad, se ha calculado el modelo de Black-Litterman usando distintos valores de este parámetro.

Los rendimientos de las carteras obtenidos con las simulaciones del modelo de Black-Litterman se han comparado con el rendimiento de la cartera obtenido usando el modelo de Markowitz, el cuál solo tiene en cuenta información histórica correspondiente a los rendimientos y varianzas de los ETFs y con el rendimiento de las cartera que se obtienen usando una estrategia equiponderada para todos los pesos de la cartera.

Después de haber calculado los rendimientos de las carteras con las distintas técnicas de gestión de carteras mencionadas previamente en distintos períodos de tiempo, se ha obtenido que el valor del parámetro τ puede tener un gran efecto o un efecto mínimo sobre el modelo de Black-Litterman según el rango de datos históricos que es usado. Se ha visto, que los rendimientos de carteras obtenidos mediante el modelo de Black-Litterman con las vistas absolutas simuladas en promedio, superan un número elevado de veces los rendimientos obtenidos con el modelo de Markowitz y un número inferior de veces, pero igualmente significativo los rendimientos derivados de una cartera equiponderada.

Suponiendo que se decidiera invertir una cierta cantidad de dinero utilizando uno de los tres modelos calculados en este proyecto con los ETFs seleccionados, seguramente el modelo que preferirían usar la mayoría de inversores para gestionar su inversión sería el modelo de Black-Litterman debido a

la posibilidad de poder incorporar información que no esta basada tan solo en hechos históricos y poder obtener rendimientos superiores a los obtenidos con el modelo de Markowitz o una estrategia equiponderada.

Acknowledgments

I would like to express my very great appreciation to Dr.José Suárez-Lledó for his valuable and constructive suggestions during the planning and development of this degree thesis. His willingness to give his time so generously has been very much appreciated.

I would also like to thank to Dr.Salvador Torra Porras for his advice and assistance in keeping my progress on schedule.

My grateful thanks are also extended to Professor Miguel Antón Sancho, who has also read this dissertation and gave advice.

Last but not least, I am very grateful to my girlfriend Maria Arroyos for her patience and support while I was working on this thesis.

Contents

Chapter 1
Introduction

The aim of this project is to test whether well-known portfolio management techniques can be applied to recent popular investment vehicles such as Exchange Traded Funds, from now ETFs, (which are available to individual investors) and if the standard results obtained can be implemented. In particular, the focus will lie on the Black-Litterman asset allocation model using absolute views and the Modern Portfolio Theory developed by Harry Markowitz.

Since their appearance in the markets around 1993, ETFs have experienced a strong upward trend in trading volumes and popularity to the extent that nowadays they are one of the favorite investment choices for many individual and private investors. The global ETFs market at the beginning of 2015 was $3.3 trillion, of which about $2.2 trillion belongs to the U.S.A. market, while the global MBS (Mortgage Backed Securities) market, which is one of the global biggest markets, was $7.5 trillion. The fact that the size of the global ETF market is approximately equivalent to 44% of the global MBS market is indicative of their importance.

The main idea of the portfolio management is to diversify an investment among different assets in order to obtain the best assets weights combination so as to build an investment portfolio composed by a number of selected ETFs as a type of assets that can be managed into a portfolio. There are several models that pursue the efficiency of the portfolio management, one of them is the Black-Litterman model which is going to be studied and compared with the Markowitz model and an equal weight asset allocation strategy to determine the performance of the model. The reason for choosing the Black-Litterman model study is due to the fact that through a Bayesian methodology an investor can introduce their views of the assets performance that compose a portfolio. The criteria to determine the best portfolios weights combination for the Markowitz model and the Black-Litterman model (using absolute views) is fixed by the investor profile, being defined in this thesis as the portfolios that maximize the Sharpe ratio, which is a measure of the excess return per unit of risk of an investment.

In order to apply the Black-Litterman model, the main input that is needed is the views of the assets performance and the uncertainty around these views. There are two types of views that can be included in the Black-Litterman model: absolute views and relative views. The difference between them is that with a relative view, the investor compares one asset to another in terms of outperform or underperform (ie. ETF 1 will outperform the ETF 2 by a x% return) while with an absolute view the investor specifies the percentage return of the asset (ie: ETF 1 will have a x% return). In this degree thesis the only type of views that will be include are absolute views.

The views and their uncertainties can be subjective to the personal views of an investor, based on technical methodologies or a combination of both. A technical method that is frequently used in investment is Monte Carlo simulations to generate possible scenarios. A special case of Monte Carlo simulations that allows the inclusion of correlations, is going to be used in order to determine simulations of the absolute views and their uncertainties for several scenarios. Afterwards these simulations will be included in the Black-Litterman model to be able to obtain different Black-Litterman portfolio returns with the information of the real returns obtained for the periods considered.

In order to asses whether the Black-Litterman model is better than other options such as the Markowitz model and an equal weight asset allocation strategy, the portfolio returns will be obtained for the last two portfolio management techniques and be compared with the Black-Litterman portfolio returns simulated. They will be tested and compared over several periods of time according to certain performance criteria.

Chapter 2
Methodology and Data

1. ETFs

1.1. Definition.

The Exchange Traded Funds are a type of investment funds whose equity is traded on the stock market as if they were shares. The characteristic of their investing policy is that they most often replicate a benchmark, either equity, fixed income, raw materials or currencies. ETFs, as well as the traditional investment funds, hold a stock basket (bonds or shares) or have swap contracts which provide the same yield of a basket composed by specific values.

The management performed by the ETF fund manager has passive character, what is to say that it does not decide to overweight or underweight the securities which conform the portfolio, but that the composition replicate the percentages of each of the components of the underlying index.

In the Stock Exchange , the ETF has an approximate value to a fraction of the index, adjusted by parity [1]. This type of investment funds, incorporate in its price the dividends, where appropriate, paid by the companies which are members of the index during the year. For that reason, the differences between the ETF value and its reference index come from the dividends, management fees, and derivatives, among others. The evolution of its price is publicly available on real time just as the price of any stock.

The main difference between an ETF and a traditional investment fund is how the investors buy and sell their shares. While in an investment fund the investors buy the participations to a ETF fund manager and resell them when they want to refund them, in an ETF the investors must buy and sell their shares in a stock market, in the same way they would do if they desire to buy or sell shares of a specific company. For that reason, the individual investors use a broker or the brokerage services of an online platform if they want to buy or sell their ETF shares. As the name suggests, the traded funds quote in a market (as for example London Stock Exchange or Madrid Stock Exchange) throughout the day as any other share. On the contrary, the price of a traditional investment fund is fixed once a day, so the investors have to process their selling or purchase order before cut-off time in order to obtain the price of the day. However, unlike the traditional funds, the ETF investors are able to operate the same way of shares, setting a limit, and so on.

[1] In finance, the similarity between the nominal value and the securities values

1.1.1. *Requirements.*

The Exchange Traded funds must fulfill the following requirements to be admitted to trading:

- Obtain an authorization of the CNMV accordingly to the established proceeding and comply with its rules.
- The requirement of having a minimum of a hundred participants in order to be admitted to trading demanded to the traditional funds is not compulsory.
- The investment policy is to replicate a financial Index whose composition is sufficiently diversified, easy to reproduce, adequate market benchmark or set of securities and having an appropriate dissemination.
- The securities that make up the basket have to be considered as eligible for investment, according to the Collective Investment Institutions regulation. That is to say, trading markets where the assets conforming the index have to gather comply with similar characteristics to the ones demanded in the Spanish financial markets.
- The ETF fund manager determines both the fund composition and the amount of cash susceptible to being exchanged for participations (parity).
- For the purpose of adapting the fund asset value estimated in different moments during the session to the quoting value, entities assuming the compromise of offering buying and selling positions must exist.
- The ETF must be properly disseminated, throughout the Stock market where the composition of the fund quotes and the estimated asset value in the different moments of the session.
- These funds are not subjected to 3% liquidity ratios which required to traditional funds, neither to the transfer of shares procedure of the financial investment funds.

Regarding to the information that is to be available for the shareholders, the ETF fund manager is exempted of freely providing both the simplified prospectus and the last half-yearly report. However, upon request, a full prospectus, the last annual and quarterly published reports must be provided.

1.2. ETFs History.

The Exchange traded funds (ETFs) have their origin in EEUU, during the 80s. In that moment, started the negotiation of financial products in baskets of shares. However, one cannot equalize the first products with the actual ETF as they have changed through time and adapting to the different necessities as well as the economic situation.

In 1987, the Index Participation Shares started to quote in the American Stock Exchange, although they did not meet the expectations neither had the expected success, consequently these products have an ephemeral life. Its commercialization ended after a lawsuit from Chicago Mercantile Exchange. Two years later, Toronto Index Participation Shares (TIPS) were launched in the Toronto Stock Exchange,

with an increased acceptation and demand between investors.

In December 1992 the first "SuperTrust",consistent in an index fund of sorts created and designed for institutional investors to give them the ability to buy and sell an entire basket of S&P stocks in one trade on an stock exchange , was launched and had a maturity of three years. After this time, a new "SuperTrust" was to replace the maturing units, but there was no second issue. The detriment of this product, was the large minimum investment size, the complexity and adverse tax rulings. Individual investors had no appeal to the product and only institutional could be interested. The failure of this product leads to the creation of the first ETF.

The launching of the first ETF, as we know it, took place in January 1993, with the Standard&Poors 500 Depository Receipt (SPDR), actually managed by State Street Global Advisors. Since that time, the ETFs development has been progressive. At the end of the millennium, in March 1999, the QQQ ETF was launched on the Nasdaq index and at that stage, the first ETF in Hong Kong and Canada were created. This ETF was launched just before the collapse of the .com bubble. Nowadays it is composed by assets representing $24,000 million.

In Europe, the firsts ETF emerged later, in 2000. However, in Spain, ETFs were not implemented until 2006, shortly before the economical crisis.

To date, the United States does not have a specific regulation of the ETFs. In fact, they are regulated by the Investment Company Act of 1940, which regulates the development of the investment funds. The regulation is nearly fifty years previous from the emergence of the first ETF so it could hardly foresee the development of this type of funds.

At the beginning, ETFs were negotiated by the professional investors in the American stock exchanges, although progressively have been used by all kind of investors.

Some years later, in 2007, just before the worldwide economic crisis, the daily trading volume was exceeding $80,000 million. The most actively traded security worldwide is an ETF that replies S&P 500, the SPY. As it has been said, this ETF was the first one to be traded in the world, and now, more than twenty years later its average daily trading exceeds $37,000 million.

Currently, in Europe, although ETFs did not appear until 2000, more than 400 ETFs are issued by the Managing Agents. At the end of 2007, the assets managed by Europe were valued in more than 90.000 million Euros.

In the last years, the number of ETFs and investors who make this option has increased significantly and it is expected to continue its rising, although they have been charged of causing an increase in the market volatility.

1.3. Legal aspects of ETFs.

I. Spanish Level (35/2003 Act of Collective Investment Institutions)

The Exchange Traded Funds first regulation appeared in 2003 with the Law 35/2003 of 4 November, on collective investment institutions, collected by Royal Decree-Law 1309/2005. However, the regulation that develops the mentioned Law is the Royal decree 1082/2012 of July 13, on collective investment institutions.

Currently, Law 31/2011 is the governing law as the Collective Investment Institutions Act has been rescinded. The aim of this Law is, on one hand to facilitate and enhance the supervisory activity of the CNMV and, on the other hand regarding with the ETFs, it amends the aspects related with the separation right by substitution of the depositary and the concretion of the exercise of the right.

Law 31/2011 defines the Exchange Traded Funds are those whose shares are admitted on stock market trading. It states the differences between the traded funds and the collective investment institutions, mainly four:

- The mandatory liquidation coefficient does not apply, so the investor can only dispose of the 97% of the fund's assets
- The ETFs taxation receives a worse treatment than the other funds
- Selling the participations is not subjected to withholding tax
- The acquisition of participations in ETFs stock exchange is exempted of obligations

II. European Level (UCITS IV)

In the European Union, the main problem is that there is no specific legislation of the Exchange Traded funds on a harmonized basis. For that reason, the member states draw up their own regulation and, as a consequence it seriously hampers the creation of a single European financial common market. However, as a result of the promulgation of European Directive UCITS IV it involves a progress towards a common European regulation.

The interest of the authorities is concentrated on the two following aspects:

- Design of mechanisms in order to improve the protection of the investors
- The field of financial stability

The mentioned aspects will be explained in the corresponding "ETFS advantages and disadvantages" section.

III. Consultation report on principles for the regulation of Exchange Traded Funds (IOSCO)

However, it must be noted that aside from the regulation, it is also relevant the public consultation report on principles for the regulation of ETFs made by IOSCO on March, 2012. In this respect, the Committee published a list of fifteen principles which come to identify different ETFs problems and propose a solution to each one. The four main principles on which the report is distributed are:

a. Classification and transparency (1-8)

What is suggested is that the prospectus identifies the replicated index, as well as the using of derivatives identifying the counterparties and the assets in guarantee and the securities operating activities.

b. Commercialization and selling (9-12)

It is established the need that the broker take all necessary steps to ensure that the fund is suitable for the investor taking into account the experience, the goal of the investment, the risk profile and the capability to assume losses. To this end, the investor, before investing, must know and understand the risks of the product that the broker commercializes.

The broker have to guarantee by written that the commercialized funds are suitable for a determinate investor. This constitutes a relevant fact that contrasts with the traditional funds.

c. Structure (13-14)

An appropriate identification and treatment of the potential conflicts of interest is needed. The report has already pointed out some of this conflicts based on whether the replicating index has been designed for a specific ETF or not or in the case of securities, if the lending agent is not independent, for example. It is also required an accurate treatment of the counterpart risk as well as the management of the collaterals received as guarantees.

d. Global aspects (15)

The aim of IOSCO is to prevent the contagion effect between markets (in this sense it is important to note the "flash crash" which took place in May, 6, 2010 in the EEUU markets and that lead to tense situation produced by sudden movements transferred between markets in contact, derivatives and traded funds).

Such high-energy price volatility of the securities that makes up the reference index complicates the valuation of the fund or even delays the liquidation of the purchase with respect to periods with less volatility. In these circumstances, the difficulty of getting securities is another problem that increases the ETFs liquidation risk, and even the divergences between the theoretical and market value.

1.4. Participating Entities.

The agents or participants operationally involved in this type of Collective investment institution are the issuer and the manager, participants, the specialist and finally the depositary.

- The issuer and ETF fund manager is who sets-up the traded fund and is in charge of the promotion and diffusion, issue and participations refund, administration, management and fund

representation, as well as the calculation of the asset value.

- The Authorized participants (APs) are the stockbroker companies authorized by the ETF fund manager to create and redeem the creation unities representing participation unities. These companies are mainly large institutional investors and, as they are directly related with the creation and redemption, play a critical role in ETF liquidity. In essence, the APs are ETF liquidity providers that have the exclusive right to change the supply of ETF shares on the market but they do not receive any compensation or commission for creation of redemption.

- The participants are the investors, namely natural and legal persons as well as other collective investment companies, who contribute to the investment fund expecting to earn a return. The participants acquire this condition trough the acquisition or participation subscription.

- The specialists are entities obligated to provide liquidity in the market of traded funds.

- The depositary is the entity entrusted of maintaining the securities or assets acquired by the traded fund, as well as the management monitoring of the ETF fund manager. Credit institutions, corporations and stock agencies can be depositories.

It should be noted that Traded Funds are negotiated in the stock market, in a segment known as Continuous market, and institutionally are organized as follows:

- National Stock Market Commission (henceforth termed CNMV), the public institution in charge of the monitoring, supervision and inspection of the stock market, investment funds, management companies, depositories and those who operate in the Collective investment institutions or may be affected by the its regulation. Thus, the CNMV ensures the stock market transparency, the correct price formation and the protection of the investors.

- The Spanish Stock Exchanges and Markets holding (henceforth termed BME) is the company which integrates the equity, fixed income and derivatives market, as well as the negotiation, compensation and liquidation systems in Spain. Included in this group there are markets related with the ETFs quotation, specifically, in its negotiation are involved the following institutions:

  - The Management Companies are in charge of the administration and management of each of the stock market. These are stock companies whose shareholders are, exclusively, securities firms, securities agencies and its credit entities.

  - The Stock Exchanges Company is an institution operating as the responsible body of the SIBE functioning or continuous market management, which is an electronic market connected to the four Spanish stock markets. It is a market segment where the Traded Funds are negotiated.

  - The Company in charge of Register of Securities, and the Clearing and Settlement of all trades (henceforth termed Iberclear): the investment procedure ends with the delivery and reception of the securities and the cash resulting through book entry.

1.5. ETF's Taxation in Spain.

ETFs are subjected to a different taxation to the one applicable to traditional investment funds and this is largely based on the fact that the first one is quoted, wherewith because of being subjected to the shares in which quotes, it is going to be taxed in reference to themselves. However, the taxation of the ETF itself must be clearly distinguished from the one that follows the dividends obtained from the investment. Consequently, first, reference will be made to the taxation applicable to the ETF and then to the one applicable to dividends. It should be noted that the taxation which is going to be exposed below will operate when the investor is a natural person, therefore will be taxed by Personal Income Tax (IRPF).

The ETF will be taxed for the difference between the amount of the sale and the price of the acquisition price. The result will be included in the tax base, specifically in the savings base or in the general one. The obtained result will constitute a capital gain if there has been a positive yield and its integration in the savings or general base depends on the taxable period during which the benefit is generated. Following the Law 26/2014, issued on November 27, amending the Law 35/2006, November 28, about Personal Income Tax Law, which entered into force last 1 January 2015, the differentiation with regard to the antiquity has been deleted. In this regard, the reform has introduced three different tax rates, which will apply all over the common territory, depending on benefit obtained from the selling or transfer of the ETF and which consist in:

- 20% on the first 6.000€
- 22% up to 50.000€
- 24% on any other amount

It is important to note as a differentiating factor versus the traditional investment funds the fact that the ETF's capital gain obtained by the selling or transfer is not subject to the withholding tax. However, the tax exemption contemplated for the transfer of shares does not apply in the ETFs case whereas in the traditional funds it does. For that reason, in the moment that the shares are transferred from one fund to another with the objective of avoiding their selling or refund, there is no deferral in the taxation. In such a way, the capital transfer between funds is not exempted.

On the contrary, dividends obtained from the shares in which the ETF quotes will be taxed as capital gains tax. The dividends received conform the tax base and will be incorporated to the savings base.

It should be noted that, in 2015 the applicable taxation to dividends has changed as a consequence of the amendment introduced in the IRPF Law, thus many exemptions and benefits which so far operated have been deleted. First of all, the exemption applicable to the first 1.500€ dividends gained, has been deleted and will be taxed as the rest of the benefits obtained. In addition, the mentioned percentage steps for both years 2015 and 2016 will also apply to dividends charged as of 2015.

As it has been noted that ETFs are not subjected to withholding tax, in the case of dividends they are and so will be levied with a twenty percent (20%) tax.

1.6. Dividends.

The main idea that has to be taken into account is that not all ETF actually distribute dividends. The fact that these typologies of funds are linked to certain amount of shares does not indicate that automatically every ETF will pay dividends. However, in most cases, the funds that replicate a portfolio of equity will receive dividends. As the ETF is related to the underlying shares, the distribution of dividends depends on the dividend policy adopted by each company and, ultimately, on the dividend policy "imposed" by the ETF funds manager.

Every ETF is ruled by its own dividend policy that receives from the shares which form the fund. The criteria are available in the ETF's issue prospectus and will not be static as they are likely to change through a certain period of time.

Dividends received by an ETF are formed by two main components. The first one is based on the total amount of receivable dividends from the shares forming the ETF and, secondly, by the interests generated, where appropriate.

An important element to bear in mind is the benchmark rate as the ETF is replicated by this index. The investor must know that any change suffered on the shareholder remuneration policy is likely of modifying the composition and the dividend distribution of the ETF. If the rate changes, the composition is modified and so also the ETF.

Some investors may ask themselves about the possibility about paying dividends in shares. It has to be noted that the structure of the ETF ease the payment in shares although an essential factor must be taken into account. As it has been said in the paragraph above, the ETF can be modified by many circumstances, and one of these is the payment of dividends in shares. When choosing this way of remuneration, the ETF will be affected as the benchmark rate in which replicates has been modified. If the rate does not decrease or increase, the structure of the ETF will not be modified.

1.6.1. *Reinvestment of Dividends.*

It is not possible to opt for the reinvestment of the receivable dividends in all the cases. There is no general rule on the cases when the investor chooses for the reinvestment. It will only be possible if the structure of the ETF offers the possibility to do so. Consequently, although not in every case, it is enabled to reinvest the shares or bonds incomes which conforms the ETF's basket in the underlying securities.

Thus, distribution of dividends depends, in first instance, on the ETF fund manager and, secondly, on the dividend policy adopted by each quoted company which shares conform the ETF's basket.

1.7. Creation and Redemption of an ETF.

The process of creation and remption comparing ETF funds with traditional mutual funds works absolutely different. In the case of Exchange Traded Funds, the investor has no contact at all with the

ETF provider when buying or selling shares. On the contrary, in the case of traditional funds, the fund company gets the investor's money and puts it on the open market, and, when the investor sells, the process in completely the opposite as the fund company sells the assets and delivers cash to shareholders. Instead, ETF shares bought or sold by investors are, first of all, created or redeemed exclusively by Authorized participants. The task of these mentioned participants consists in contracting with the ETF sponsors.

APs create groups of ETF shares known as "creation units", which typically consist in 50,000 shares. Creation units are basically large blocks of shares to be sold to investors in smaller lots. These units can be either hold by the authorized participants in a company account, trading to other APs or broke up into individual ETF shares. The investor is enabled to buy individual shares on the public market as common shares.

The fund companies must assume diverse costs derivate from licensing to the index provider, and these costs are part of the expense of an ETF. Thus, with the aim to reduce costs, some ETF companies create their own indexes, although as the SEC require that an outside management company has to manage the ETF it adds costs back in. Moreover, when the Authorized Participants redeem a creation unit, there is a small fee for creating or redeeming it.

Institutionalizing the creation or redemption process enables the process to work on an in-kind basis. In-kind redemption is the reason why ETFs tend to be more tax-efficient than traditional mutual funds. Thus, the ETF issuer can largely avoid creating capital gains. The issuer can also distribute the lowest-cost tax lots during redemptions, further limiting capital gains exposure. However, this in-kind process is far from being perfect, as some ETFs pay capital gains distributions each year due to index changes or other factors.

The mentioned process of creation and redemption presents another advantage which consists in keeping the prices of ETF shares in line with the NAV (Net Asset Value). When underlying securities priced lower than the NAV, the authorized participants will buy it and create shares of the more valuable ETF, and vice versa.

It has been noticed that the most relevant entities involved in creation and redemption process are the Authorized Participants (APs), otherwise known as both market makers and specialists. Creating new funds involves having to sign a participant agreement with independent AP firms.

The custodial bank also plays an important role as it is the entity that holds the securities for the fund. After ensuring everything is correct it checks both the type and quantity of securities and also earns a small fee based on the fund's assets.

1.8. What affects the price of an ETF?.

In the forming of the price of an ETF, many factors intervene. This subsection is an overlook through firstly the modification of the ETF by consequence of the variation in the shares conforming it and, secondly, the two main components of this fund and its variable price and then the two consequences

of the deviation between the index performance and the ETF performance.

Related with the variation of the shares, the price of an ETF depends on the market participant's valuation of the issuing company. Such valuation depends on different factors. The main ones are both the expectations about the future company benefit and the evolution of interest rates. Other factors that influence the price are for example the expectations about various macroeconomic indicators and the investors' confidence. The actual value of these expectations varies constantly, and as a consequence, the volume of bid and demand titles for each price. As a result, prices are modified through day-to-day trading.

The Net Asset Value is on a daily basis calculated based on the closing price of the ETF underlying securities and it calculates the day-to-day performance and compares it with its benchmark. Thus, the NAV is affected by the market tendency and its daily fluctuations as common shares. The official value of the NAV is published only once per day after the markets close.

However, the NAV is not the only value of an ETF as it can be referred to Intraday Value. This last value calculates an estimated price of the ETF based on the last prices of the underlying securities that conforms the fund. The exchanges publish the Intraday Value almost every 15 seconds. Thus, the investor is able to know and compare if the market price of the ETF is close to its holdings price, or if it is actually trading at discount or premium.

But even these values are relevant for the investor, one cannot expect to sell or buy in any of this value, as they are only a reference for the participants.

On the contrary, the prices at which the investors sell or buy shares of an ETF are the market prices. During sessions of significant market volatility, an ETF's market price may vary more widely from its intraday value, so the investor has to separate the value and the market price as this last one is closer to the real value of the ETF and finally, the performance got by the investor is determined by the market price.

Market price is composed by three different elements. Firstly, ETF can trade at a discount or premium price and it is determined by bid and demand. Premium and discount are expressed in a percentage of the intraday value. Secondly, the Closing market price records the price of the daily trading session. This price can be calculated before or after the NAV calculation and so it is usual to be similar to this one. However in the case of high volatile markets, the different between these two values may be high. Finally, the Bid-Ask price is related on one hand with the highest price a investor is willing to pay for buying an ETF, and, on the other hand, the ask price is the lowest price acceptable for a seller during the daily trading.

1.8.1. *Tracking Error.*

Performance is not the only indicator to take into account when investing. It is obvious that the investor is looking for the highest performance when he is investing his money. Regarding to this and to the characteristics of any ETF that have been appointed, one must have already noticed that most of these funds track an index. However it is actually common that the index performance is not the same as the ETF performance, as many factors prevent it from mimicking the tracked index. This

difference between the index performance and the ETF performance is called the "tracking difference".

The tracking error is not related with performance but with volatility difference between the fund performance and its benchmark. Someone can define the "tracking error" as the annualized standard deviation of daily return differences between the total return performance of the fund and the total return performance of its underlying index. Thus, the tracking error basically measures the variability and looks at the volatility in the difference of performance between the fund and its index.

If the tracking error is low, the likelihood that its behavior is similar to the benchmark is higher. And vice versa, the higher the tracking error is, the smaller the chances on following the benchmark. However, tracking error is more than a simple risk indicator. It states the type of management and it is interesting for all the managers, analysts and investors.

1.9. ETFS advantages and disadvantages.

1.9.1. *Advantages.*

Liquidity: ETFs are liquid products. The liquidity of an ETF does not depend on contracted volume. In the shares case, the average volume of the daily operations and the market capitalization reflexes the selling and buyers activity in the Exchange Stock Market. The operations volume constitutes an indicator of the ease for entering or leaving a position, as well as the possible impact of the transaction. The ETF opened structure makes this funds as liquids as its underlying. ETFs offer two different liquidity sources: first the traditional liquidity, measured by the volume of the secondary operations in the Stock Market; secondly, the liquidity provided by the creation process of the underlying securities position which conform the ETF. The volume of the secondary operations does not reflect the real liquidity of the fund, as it can exceed in grand part of its volume or market capitalization. The ETFs creation and disintegration mechanism allows a continuous offer of the ETFs units. This continuous offer means that the authorized participants can create and place additional units of an ETF in the market, obtaining more available units to accomplish with the investors demand.

Transparency: during the trading hours, the stock market calculates and disseminates an estimated asset value which allows the investor to know how the investment evolves. Thus, the investor has real and complete information.

Flexibility: on the contrary to the traditional investment funds in which both the subscription and refund is done at the daily asset value without the opportunity to make the transaction during the session, ETFs can be bought or sold at the actual market price in every moment during the stock session.

Immediacy: the operations are performed at the purchase and selling price offered by the counterparts in each moment. The trader has a great certainty about the selling or purchase price than in the traditional funds, as will be very similar to the last asset value. The sales will be liquidated on the same timescales of the shares.

Diversification: on the contrary to the traditional funds, the exchange traded funds offers the possibility of investing either in a national or international market. The investor does not have the obligation to invest in each of the securities integrating the reference index.

Accessibility: is easier for all kind of investors to access to this type of funds as the demanded amount is reduced. ETFs are highly accessible for all type of investors. Its procurement is identical to national or international shares. The minimum investment amount is a fraction of the index itself, so the minimum theoretical investment is really low, easing the access for all investors. Currently it is possible to get access to hundreds of ETFs through the online financial broker services.

Reduced commissions: the management commissions are fewer than for the rest of investment funds. They are not charged with subscription or refund commissions. However the purchase or selling of ETFs may be charged with financial broker commission.

Dividends: where appropriate, as these funds follow the system of the shares, the investor will obtain a dividend for his investment. The tax rate you pay on equity ETF dividends is not determined by the length of time you hold the fund. On the contrary, is determined from when specific tax lots of stock were accepted into the fund, what is to say, the date the AP turned them in for a creation unit. Once a tax lot of stocks has been held in an ETF for more than 60 days, dividends from that tax lot are taxed at the preferred rate. Taxable investors in ETFs will eventually pay capital gain taxes, but only after they decide to sell shares that are at a profit.

1.9.2. *Disadvantages.*

Risk: Although, ETFs are diversified, the investors assume a market risk, so in bear market situations huge losses can be recorded and, conversely, great gains may be obtained in the bull market.

Divergence between the purchase or selling Price and the liquidity Price: Due to the fact that the specialists introduce purchase or selling orders with a differential on the price. Thus, normally, the participant uses to make the purchase in a higher price than the asset one and, on the contrary, the selling use to be made in a lower price, although in general the difference between them is really reduced.

Information obligations: Although the investor has the right to get the prospectus, it is not compulsory for the brokers to deliver it before the purchase.

Costs: Intermediary cost as well as management and depositary commissions. Although the cost is minor than the traditional funds, the fact that it is a traded funds has the consequence that the investor has to have a securities account. Before investing in this kind of funds, the investor must take into account all the direct and indirect costs as those will have a direct impact on the final profitability.

2. Data Analysis

In this section a full description and analysis of the prices and the returns of the ETFs chosen in the sample period (2011/11-2015/03) is provided.

The twelve ETFs selected are described with their main features including the index that they track.

2.1. ETFs Description.

A portfolio is a set of financial assets such as cash, stocks, bonds, ETFs and mutual funds among others, managed by individual investors or financial professionals. In this project, a portfolio composed only by ETFs is going to be created and managed using the Black-Litterman model.

The composition of the portfolio assets can be specified in sub-asset classes usually classified as industries, sectors and geographies, fixed income, equities and commodities, etc. Using different ETFs, a portfolio manager can diversify the portfolio into the sub-asset classes perviously named.

In order to create the portfolios, twelve different ETFs available in ING bank have been chosen trying to achieve a high degree of diversification. Two of them track indexes of fixed income assets (corporate bonds and government bonds), while the others track different indexes of equities that contain different sub-asset classes .

The table below provides the ETFs names, the International Securities Identification Numbers (ISIN) and the Bloomberg codes of the ETFs chosen based on the above explanation:

ETF	ISIN	BLOOMBERG CODE
DB X-TRACKERS S&P 500 UCITS ETF	LU0490618542	DXSPX:SM
LYXOR UCITS ETF EURO STOXX 50	FR0007054358	MSE:SM
LYXOR ETF EUROMTS HIGHEST RATED MACRO-WEIGHTED GOVT BOND	FR0010820258	MAA:SM
LYXOR ETF IBEX35	FR0010251744	LYXIB:SM
LYXOR ETF STOXX EUROPE 600 OIL & GAS	FR0010344960	OIL:SM
LYXOR UCITS ETF NEW ENERGY - D-EUR	FR0010524777	ENER:SM
LYXOR ETF MSCI EMU VALUE	FR0010168781	VALU:SM
LYXOR ETF STOXX EUROPE 600 TELECOMMUNICATIONS	FR0010344812	TEL:SM
LYXOR ETF STOXX EUROPE 600 HEALTHCARE	FR0010344879	HLT:SM
LYXOR ETF EURO CORPORATE BOND	FR0010737544	CRP:SM
DB X-TRACKERS MSCI EMERGING MARKETS INDEX UCITS ETF	LU0292107645	DXMEM:SM
LYXOR ETF MSCI EMU GROWTH	FR0010168765	GWT:SM

Table 1. ETFs chosen to create the portfolios

Due to the large names of the ETFs and that the source of their historical data is Bloomberg, all the references to the ETFs selected are going to be by the Bloomberg codes.

Each ETF tracks an index, consequently the twelve ETFs are listed below with a brief explanation of the index that they track:

- The DXSPX:SM is an ETF created 26/03/2010 that tracks the S&P 500 TRN Index, which is a market capitalization weighted benchmark that reflects the performance of the 500 large common stocks most actively traded in the markets of NYSE Euronext and NASDAQ OMX.

- The MSE:SM is an ETF created 21/03/2010 that tracks the Euro Stoxx 50 composed of the blue chip values leaders of the Eurozone and represents the 50 leading companies of all sectors in the Eurozone.

- The MAA:SM is an ETF created 16/11/2009 that tracks the EuroMTS Macro-Weighted AAA Government Bonds All-maturity, which is comprised by bonds issued by Eurozone governments with the highest credit ratings with country weights calculated based on macroeconomic indicators.

- The LYXIB:SM is an ETF created 3/10/2006 that tracks the Ibex 35 Net Return composed of the 35 stocks with highest liquidity negotiates in the Spanish market.

- The OIL:SM is an ETF created 30/06/2006 that tracks the Stoxx Europe 600 Oil&Gas Net Return, which is made up of with the largest stocks of the oil & gas industry in Europe included in the STOXX Europe 600, which comprises 600 of the largest European companies.

- The ENER:SM is an ETF created 10/10/2007 that tracks the World Alternative Energy CW Net Total Return, which is a market capitalization weighted benchmark calculated by Dow Jones and compiled by SAM (Sustainable Asset Management)that reflects the 20 largest stocks operating in the world alternative energy sector in the fields of solar, wind and biomass.

- The VALU:SM is an ETF created 01/04/2005 and tracks the MSCI EMU Value Net Total Return, which covers the full range of developed, emerging and All Country MSCI International Equity Indices across all segments.

- The TEL:SM is an ETF created 25/08/2006 that tracks the Stoxx Europe 600 Telecommunications Net Return, which is made up of the largest stocks of the telecommunications industry in Europe included in the STOXX Europe 600, which comprises 600 of the largest European companies.

- The HLT:SM is an ETF created 18/08/2006 that tracks the Stoxx Europe 600 Healthcare Net Return constituted with the largest stocks of the healthcare industry in Europe included in the STOXX Europe 600, which comprises 600 of the largest European companies.

- The CRP:SM is an ETF created the 04/02/2009 that tracks the Markit iBoxx € Liquid Corporates, which is a subset of the Markit iBoxx EUR Corporates Index and contains 40 investment grade rated financial and non-financial securities.

- The DXMEM:SM is an ETF created 22/06/2007 that tracks the MSCI Total Return Net Emerging Markets Index,which is a market capitalization weighted benchmark designed to measure the evolution of the world emerging markets equity.

- The GWT:SM is an ETF created the 01/04/2005 that tracks the MSCI EMU Growth Net Total Return, which covers the full range of developed, emerging and All Country MSCI International Equity Indices across all segments.

2.2. Analysis.

This subsection intends to show a complete analysis of the twelve ETFs selected for this work in order to outline their basic features. Graphs and numerical statistics such as time series, tables of descriptive statistics and histograms are provided to accomplish the objective.

The time period of the ETFs data used is monthly, that means that the data contains the close prices of each month from November of 2010 until March of 2015, which is the sample period used.

The ETFs prices as well as stock prices change every day as a result of the supply and demand of the market. The time series of the ETFs prices shows the magnitude of the prices and how they change through the sample period defined:

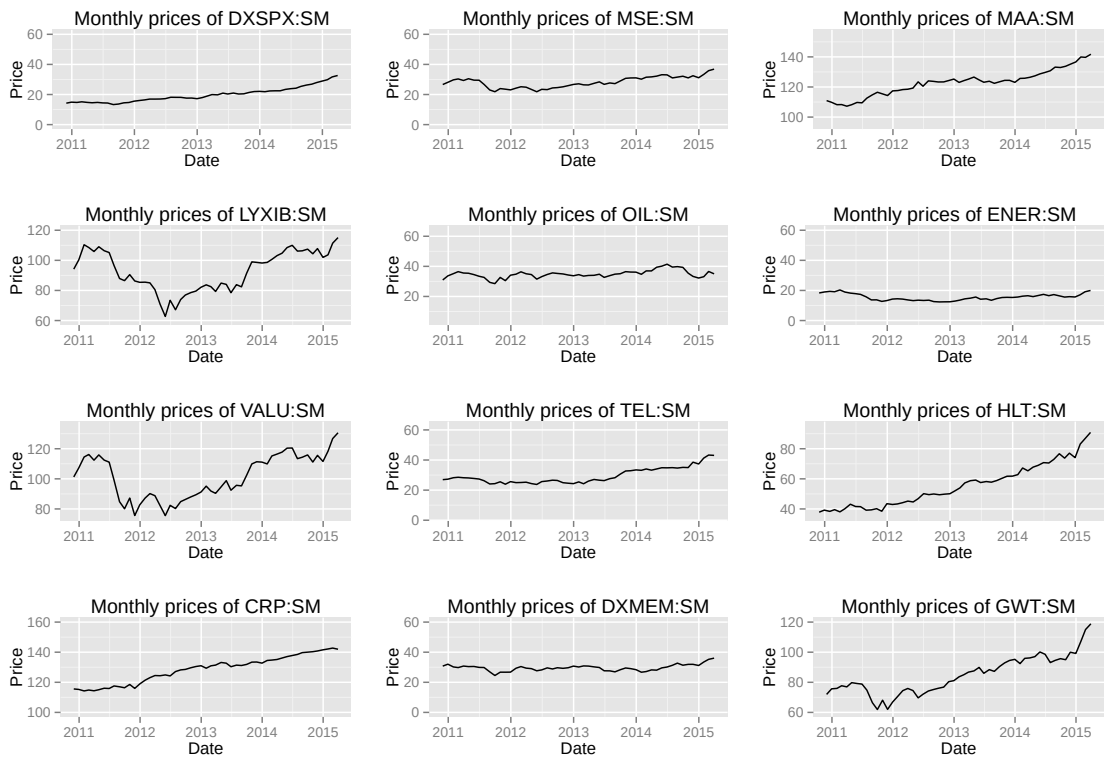

FIGURE 1. Time series of the prices

As can be seen, the ETFs prices differs with each having different prices magnitudes, trends and volatilities. Later on, it will be explained that prices are not used to work in quantitative finance for reasons such as the presence of tendencies on the data and different prices magnitudes among others.

Most statistical methods used in quantitative finance, including the ones that will be used, include the assumption that the sample is drawn from a population where the values follow a normal distribution. Nonetheless, several times this assumption is violated and many investors use parametric models using

non-normal data. The Jarque-Bera test is a goodness-of-fit test of whether the sample data have the skewness and kurtosis corresponding to a normal distribution and it is very popular in investment because skewness and kurtosis are discussed frequently.

This goodness-of-fit test is included with the main descriptive statistics of the ETFs prices in the table below:

	Mean	**Max**	**Median**	**Min**	**SD**	**Skewness**	**Kurtosis**	**Jarque Bera**	**P-Value**
DXSPX.SM	19.74132	32.63000	18.17000	13.31000	4.89458	0.81625	-0.10266	6.23159	0.04434
MSE.SM	28.20811	36.86000	28.12000	21.83000	3.67776	0.12800	-0.86143	1.49125	0.47444
MAA.SM	122.65264	141.76000	123.44000	107.24000	8.58955	0.08640	-0.46038	0.35884	0.83575
LYXIB.SM	92.93472	115.11000	94.15000	62.73000	13.31463	-0.24443	-1.11355	2.94868	0.22893
OIL.SM	34.85349	41.44000	34.77500	28.51500	2.55381	0.23819	0.56537	1.62454	0.44385
ENER.SM	15.54000	20.32000	15.43000	12.36000	2.18349	0.45125	-0.86213	3.24500	0.19740
VALU.SM	101.39528	130.60000	101.25000	75.55000	14.60366	-0.04337	-1.30011	3.38087	0.18444
TEL.SM	29.48877	43.30500	27.36500	23.76500	5.21655	0.97546	-0.00803	8.92517	0.01153
HLT.SM	55.32491	90.85000	52.11000	37.85000	14.18250	0.59533	-0.64319	3.98689	0.13623
CRP.SM	128.45585	142.73000	130.47000	114.26000	9.00102	-0.18044	-1.26334	3.46284	0.17703
DXMEM.SM	29.66094	36.05000	29.67000	24.49000	2.12953	0.46728	0.93481	4.65441	0.09757
GWT.SM	84.59434	118.85000	83.65000	61.90000	12.79728	0.42518	-0.28650	1.76325	0.41411

TABLE 2. Summary of the ETFs prices

Apart from the distinct values of the descriptive statistics of the ETFs prices, it can be seen that two ETFs prices do not follow a normal distribution, which is another reason (as have been commented before) not to include those prices in the analysis.

In the stock market analysis, prices are non-stationary (implying stochastic trends, deterministic trends and heretocedasticity among others), and consequently are unpredictable and cannot be modeled. In order to avoid the non-stationary a transformation is required, leading to the returns.

A return can be defined as the gain or loss of a security in a particular period. The simple returns are the returns that are provided by all the standard sources of information and are adequate to use with monthly data. This returns are going to be used in this work to calculate the ETFs returns, being defined as:

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$$

where R_t is the simple return calculated with P_t being the actual price and P_{t-1} the previous price.

Having the simple returns defined, the time series of the ETFs returns are shown below:

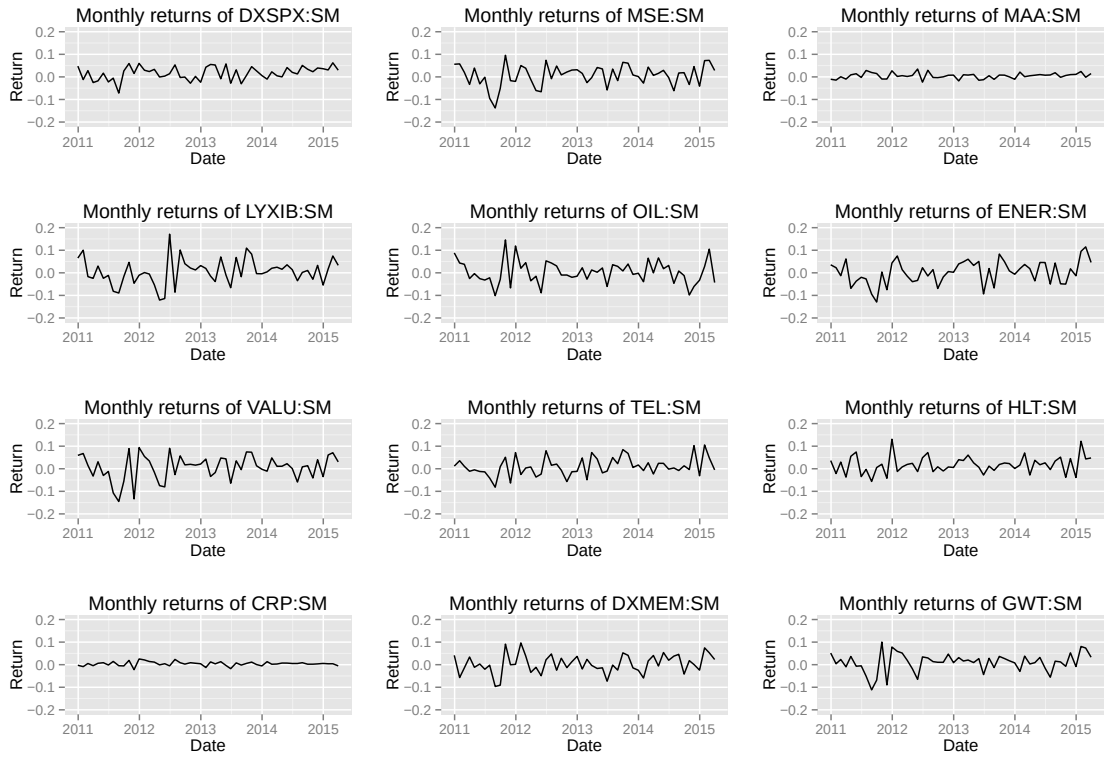

FIGURE 2. Time series of the ETFs returns

The main difference here in comparison with the time series of the ETFs prices illustrated in the figure 1 is the absence of clear trends. For example, the time series of the MAA:SM price had clear upward trend but the time series of their returns does not have any trend.

In order to see the if the ETFs returns follow normal distributions or not, the Jarque-Bera test is presented with the main descriptive statistics in the table below and also the histograms of the historic ETFs returns are shown:

	Mean	**Max**	**Median**	**Min**	**SD**	**Skewness**	**Kurtosis**	**Jarque Bera**	**P-Value**
DXSPX.SM	0.01649	0.06235	0.02125	-0.07183	0.02916	-0.50323	-0.07474	2.33008	0.31191
MSE.SM	0.00736	0.09565	0.01556	-0.13783	0.04681	-0.65851	0.35220	4.49319	0.10576
MAA.SM	0.00480	0.03480	0.00632	-0.02350	0.01270	0.10790	-0.28927	0.17866	0.91454
LYXIB.SM	0.00546	0.17121	0.00303	-0.12073	0.05715	0.20304	0.39572	0.98777	0.61025
OIL.SM	0.00365	0.14571	-0.00103	-0.10129	0.05121	0.36311	0.28672	1.58778	0.45208
ENER.SM	0.00302	0.11494	0.01126	-0.12961	0.05172	-0.30155	-0.33830	0.95290	0.62099
VALU.SM	0.00643	0.09392	0.01420	-0.14516	0.05504	-0.72713	0.19213	5.07699	0.07899
TEL.SM	0.00987	0.10543	0.00455	-0.08195	0.04093	0.33916	-0.13781	1.05801	0.58919
HLT.SM	0.01768	0.13075	0.01820	-0.05637	0.03843	0.56706	0.63658	4.27421	0.11800
CRP.SM	0.00401	0.02545	0.00421	-0.02227	0.00927	-0.24102	0.62751	1.82216	0.40209
DXMEM.SM	0.00391	0.09645	0.00013	-0.09658	0.04170	-0.16317	-0.08826	0.24617	0.88419
GWT.SM	0.01057	0.10016	0.01453	-0.11163	0.04191	-0.64499	0.65779	5.21756	0.07362

TABLE 3. Summary of the ETFs returns

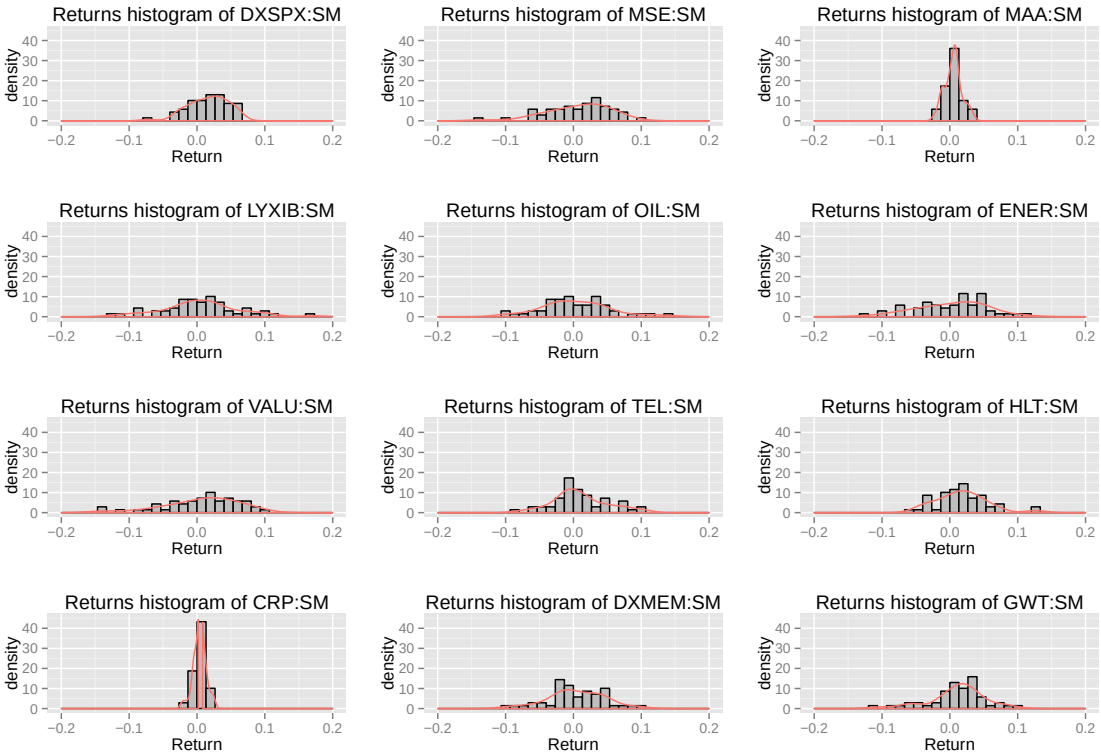

FIGURE 3. ETFs monthly returns histograms

The previous table shows that for all the returns it cannot be said that one of them does not follow a normal distribution with a confidence level of 95% for the reason that all the p-values of the Jarque-Bera test are greater than 0.05 is spite of that the densities and volatilities of them are different as can be seen on the histograms and in the table 3.

All the reasons above-mentioned demonstrate that is better to work with returns instead of prices.

The correlations among the assets of a portfolio are significant statistics to determine whether the portfolio is well-diversified or not. In order to minimize the portfolio risk, an investor prefers uncorrelated assets because he wants to have different asset behaviors. The correlation matrix of the ETFs selected is shown below:

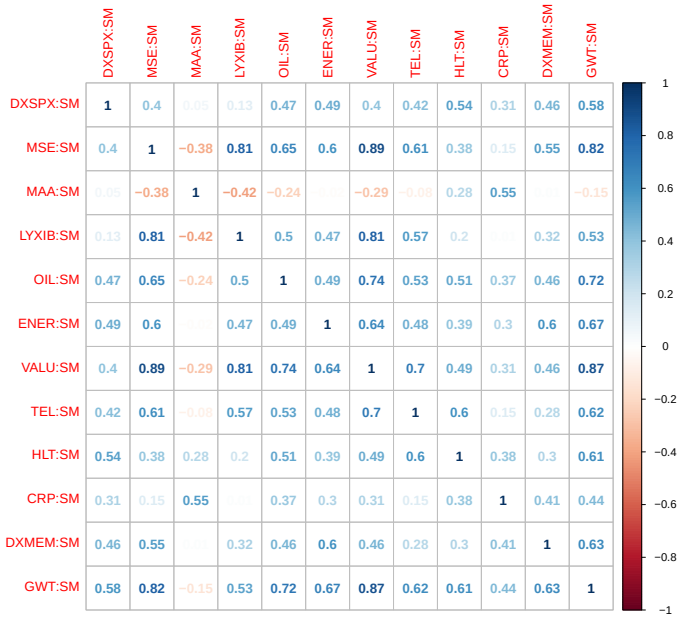

FIGURE 4. Correlation matrix of the ETFs

As can be seen, most of the assets have a medium correlation in respect of other assets, understanding as a medium correlation values about 0.5. Few pairs of assets have strong correlations such as VALU:SM and MSE:SM that have a correlation of 0.89, or negative correlations such as LYXIB:SM and MAA:SM that have a correlation of -0.42.

3. Modern Portfolio Theory

The Modern Portfolio Theory developed by Harry Markowitz and published under the title "Portfolio Selection" in the 1952 Journal of Finance, is based on that the investor does diversify his funds among all those securities which give maximum expected return desiring the minimum variance (volatility).

The Mean-Variance model and the efficient frontier are the main contributions of this theory, being explained in the following subsections.

3.1. Mean-Variance Model.

The Mean-Variance model also known as Markowitz model when historic data is used, can be resumed by the following points:

- There is a rate at which the investor can gain expected return by taking on variance, or reduce variance by giving up expected return
- Variance is a well-known measure of dispersion about the expected returns

The mathematic formulation of the Markowitz model that maximize the expected return and do not allow short sells can be expressed as

$$\begin{aligned} \textbf{Max} \quad & \sum_{i=1}^{N} w_i \mu_i \\ \text{s.t.} \quad & V = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma ij \\ & \sum_{i=1}^{N} w_i = 1 \\ & w_i \geq 0 \quad \forall i = 1..N \end{aligned} \tag{1}$$

Where:

- μ_i is the expected return of the security i
- w_i is the percentage of the investor's assets which are allocated to the ith security
- V represents the maximum level of risk that an investor is willing to accept
- σ_{ij} is the covariance between R_i and R_j

Also, the Mean-Variance problem can be defined as a minimization problem that involve minimize the portfolio variance given a rate of return of the portfolio and is expressed as

$$
\begin{aligned}
\textbf{Min} \quad & \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} w_i w_j \sigma_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{N} \mu_i w_i \geq k \\
& \sum_{i=1}^{N} w_i = 1 \\
& w_i \geq 0 \quad \forall i = 1..N
\end{aligned} \tag{2}
$$

Where:

- μ_i is the expected return of the security i
- w_i is the percentage of the investor's assets which are allocated to the ith security
- σ_i^2 is the variance of the asset i
- σ_{ij} is the covariance between R_i and R_j
- k is the minimum rate of return of the portfolio

3.1.1. *The concept of short sells.*

The previous restriction of the optimization problems defined as $w_i \geq 0 \quad \forall i = 1..N$ is imposing that short sells are forbidden.

If an investor believe that an stock will rise and bought the stock, it is say that goes long and investment. Otherwise, if believe that the stock will decrease, the investor can goes short, which means sell a stock that the investor does not own.

The process in order to go short on a stock is the following: The broker lend the stocks to the investor and the shares are sold being the proceeds credited to the account. In order to close the short, the investor have to buy back the same number of shares that where lend from the broker making profit if the stocks drop. The interests of the broker have to be considered because the lend is not at zero interest and that interests are charged to the account.

A common investor normally does not make short sells and for this reason the two types of investment strategies (investment without short sells and investment with short sells) are realized in this work.

3.2. Efficient Frontier.

The set of optimal portfolios that offers the highest expected return for a defined level of risk or the lowest risk for a given level of expected return compose the efficient frontier. This set of portfolios is obtained using the Mean-Variance model including the convenient restrictions such as the forbidden of short sells, that have been seen on the subsection above.

The portfolios that are not integrated in the efficient frontier, and in consequence lie below, are denominated sub-optimal as they do not provide the highest return for the level of risk and tend to be less diversified than the optimal portfolios (which provide the benefit of diversification), with the consequence that a portfolio manager will not be interested in select them to invest.

Considering the set of the optimal portfolios that compose the efficient frontier, a portfolio manager would select the optimal portfolio according to a selection criteria given by a maximum level of risk, an expected return level or a combination of both given a priori. To simplify, the portfolio that is going to be selected to invest, is the one that maximize the Sharpe ratio of all the optimal portfolios that compose the efficient frontier.

The Sharpe ratio is a method to calculate the risk-adjusted return, used by many professional investors as a performance measure of a portfolio , being defined as the average return earned in excess of the risk-free rate [2] per unit of volatility or total risk:

$$\text{Sharpe ratio} = \frac{r_p - r_f}{\sigma_p}$$

The risk-free rate considered for all the period of the data, is the 12 month Euribor rate at April 1, 2015 , which gave an interest of 0.196% and for the reason that the returns used are monthly the risk-free rate used is $0.196\%/12 = 0.0163\%$. The reason of use the 12 month Euribor, is that the ETFs selected in this work are geographically distributed among many countries, mainly in Europe, and is used by many Europe Banks as a reference of the risk-free rate.

Using the 12 ETFs considered in this work and the historical data to estimate the expected returns, variances and covariances, the following graph provided below shows the efficient frontiers forbidding short sells in color black and allowing short sells in color blue and also the maximum Sharpe ratio portfolios in both cases:

[2]interest given by a zero risk asset that use to correspond to government bonds or the euribor among others.

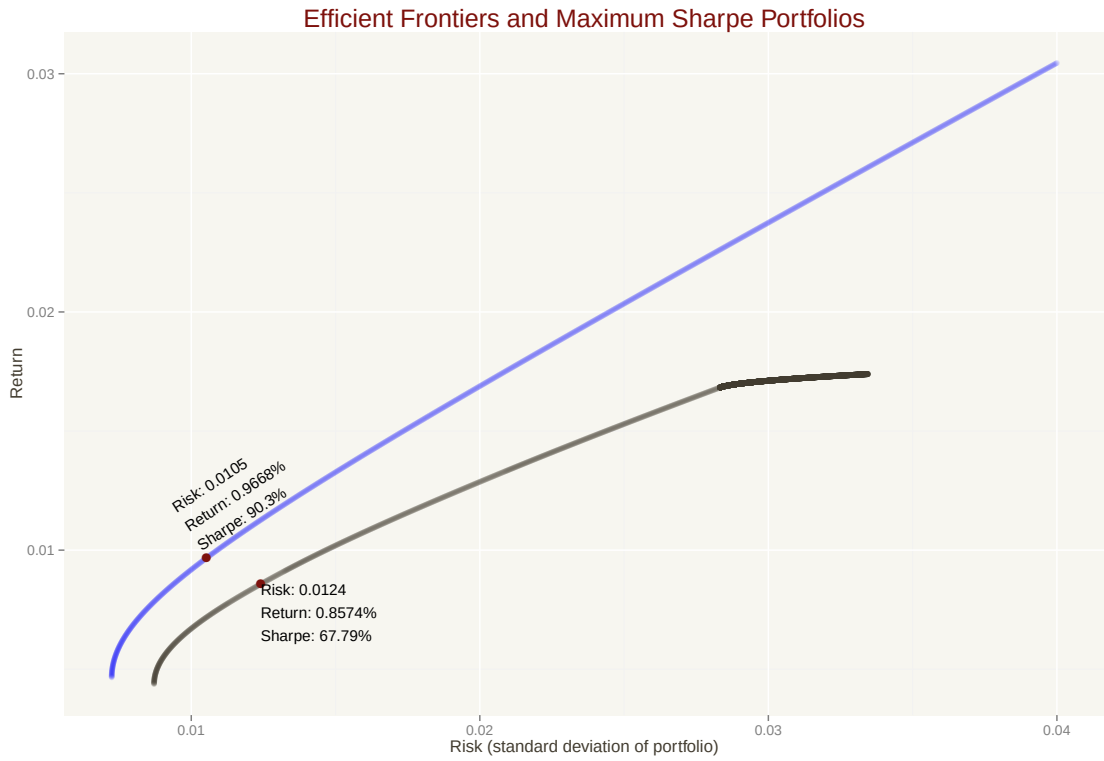

FIGURE 5. Markowitz efficient frontiers forbidding and allowing short sells

There are several differences when short sells are forbidden or allowed as can be seen in the previous graph. The first thing that can be appreciated is the difference between the slope of the two efficient frontiers, where the higher slope of the efficient frontier that allows short sells is interpreted as that given a level of return, the risk assumed is lower than the risk assumed obtained when short sells are forbidden and in consequence the maximum Sharpe ratio obtained is higher when short sells are allowed.

Due to the restriction of forbidden short sells, the corresponding efficient frontier has a maximum risk and the maximum return level that can be achieved (the extreme of the frontier) matches to the ETF with the highest return rate, but if short sells are allowed a maximum level of return or risk does not exist .

The optimal portfolios weights can be positive or negative when short sells are allowed, indicating where the weight is negative that the asset has to be sold with an amount proportional to the corresponding weight and in the case of a positive weight, the asset has to be bought also with an amount proportional to the corresponding weight.

The following table depicts the information of the weights obtained in the maximum Sharpe ratio portfolios of the previous efficient frontiers as well as their volatilities, expected returns and Sharpe

ratios:

	Short sells forbidden	**Short sells allowed**
DXSPX:SM	0.3051	0.2714
MSE:SM	0.0000	0.0994
MAA:SM	0.5579	0.0806
LYXIB:SM	0.0579	0.1934
OIL:SM	0.0000	-0.0696
ENER:SM	0.0000	-0.0575
VALU:SM	0.0000	-0.3130
TEL:SM	0.0000	-0.0044
HLT:SM	0.0169	0.0863
CRP:SM	0.0621	0.6726
DXMEM:SM	0.0000	-0.1277
GWT:SM	0.0000	0.1684
σ	0.0124	0.0105
$E(r)$	0.0086	0.0097
$Sharpe$	0.6779	0.9030

TABLE 4. Weights of the maximum Sharpe ratio portfolios

As have been commented previously the Sharpe ratio, when short sells are allowed, improves highly in comparison to when short sells are forbidden because at the same return level, the level of risk is lower, obtaining a difference of 0.2251 in the Sharpe ratio comparison.

The weights obtained for each asset by the efficient frontiers are completely different, seeing that with short sells allowed only one asset have values close to zero (considering close to zero less than a 5%) while when short sells are forbidden seven assets have zero value and one a value close to zero, indicating the low weight diversification when the Mean-Variance Optimization problem without short sells is applied to the Markowitz model.

4. Black-Litterman Model

The Black-Litterman asset allocation model, created by Fischer Black and Robert Litterman, enables investors to combine the market equilibrium with the investors' expectations of the market.

The model gives a point of reference for the expected initial returns that are in equilibrium as well as a systematic process to express market views and give as results the expected returns of the assets and the optimal portfolio.

4.1. Comparison of the Markowitz model in respect of the Black-Litterman model.

The Mean-Variance model (Markowitz model) without adjustments is not frequently used in practice for many reasons. The Black-Litterman asset allocation model overcomes some of the problems of the Mean-Variance model and for this reason is a great alternative although that also has some problems that will be discussed.

4.1.1. *Problems of the Mean-Variance model.*

The Markowitz model requires an investor to estimate the expected returns, variances and correlations among the assets. This means that for a portfolio of N assets the number of parameters to estimate are $N(N+3)/2$.

One of the main problems of this model is that a slight change in the inputs can result in a large change in the weights obtained by the model which leads to a high sensitive on the inputs, which is fairly undesirable.

Another problem of this model that has been found in the previous section is that can lead to extreme portfolio weights, giving large weights to few assets and weights zero or close to zero to the others.

The fact that an investor cannot quantify the confidence of the model parameters or their views means that, considering the problems of this model, the results obtained are unintuitive and are often unbelievable or hardly accepted.

4.1.2. *Solutions and problems of the Black-Litterman model.*

The Black-Litterman asset allocation model overcomes the problems mentioned about the Markowitz model through a Bayesian approach resulting in a consistent structure of the results and a higher acceptance on the results.

With the Black-Litterman model an investor has reference returns that are the Implied Equilibrium Returns and can quantify the degree of confidence that he has on their views resulting in intuitive results.

The problems of highly-concentrated portfolios and the input-sensitive that results in huge weights fluctuations when the inputs change in the Markowitz model are not produced in this model but there are other problems with no easy solution.

The uncertainty on the views of the investor and the value chosen for the scalar τ, which can be explained as a measure of the investor's confidence in the prior estimates, are not obvious and many experts have wrote about it with different solutions. Through this work these difficulties are going to be explained in more detail with the solution chosen.

4.2. Black-Litterman Theory.

The Black-Litterman model is a two-step process where the first step consists in calculating the vector of implied equilibrium returns of assets using the risk free rate and the assets weights using the reverse optimization. The second step consists in adjusting the implied equilibrium returns according to the investors' views and their uncertainties.

4.2.1. *Reverse optimization.*

The Black-Litterman model begins with the hypothesis that the Global Capital Market portfolio is the optimal mean-variance of risky assets, considering it optimal with the assumption that in this situation the market is in equilibrium, which means that the market value of the assets reflects the homogeneous expectations about the performance of the assets.

The equilibrium returns are the neutral starting point of the model, which are derived using a reverse optimization method and are calculated from the Capital Asset Pricing Model (CAPM) equation:

$$\pi = \lambda \sum w_{mkt} \tag{3}$$

where

- π = Implied Excess Equilibrium Vector
- $\sum$ = Covariance Matrix of Excess Returns
- w_{mkt} = Market Capitalization Weights
- λ = Risk Aversion Coefficient

The covariance matrix of excess returns, is the covariance matrix of the returns minus the risk free rate, which form the excess returns. The market capitalization weights is the vector of market capitalization weights corresponding to the Global Capital Market portfolio. The risk aversion coefficient is a parameter that measures the degree to which the investor is averse to taking risks and in this case acts as a scaling factor, being estimated by dividing the portfolio expected excess return by the portfolio variance.

4.2.2. *Combining implied excess equilibrium returns with the investor views.*

Once the implied excess equilibrium vector is calculated using the reverse optimization, the next step is to combine the equilibrium excess returns with the investors' views resulting in a weighted averaged of the implied equilibrium excess vector and the investor views, where the relative weights depend on the uncertainty of the views and the scalar τ.

The Bayesian statistics allow the mix, combining the Prior Equilibrium Distribution assumed as a Gaussian distribution with the Implied Excess Equilibrium Returns as mean and the covariance matrix of excess returns multiplied by the scalar τ as variance ($N \sim (\pi, \tau \sum)$).

As has been said before, one of the main problems of the Black-Litterman model resides in the complexity of the definition of the scalar τ because its specification is not easy, and the easiest way to calibrate the Black-Litterman model is to make an assumption about the value of the parameter. There are some ways to define the scalar, which are explained in the next section.

Having the Prior Equilibrium Distribution the investor has to define their views and their uncertainties. The Black-Litterman asset allocation model does not require that investors specify views on all assets, if the investor does not have views or does not have views that differ from the Implied Equilibrium Return Vector (π), the investor should hold the market portfolio.

An investor can have two types of views, absolute and relative views. In an absolute view the investor believes that an asset will have an specified expected return and have an uncertainty of this expected return. In a relative view the investor believes that an asset will outperform or underperform one or more assets with a confidence level.

The matrix Q (1 column and V rows corresponding to V views) contains the expected return of the absolute and relative views matched to specific assets contained in the matrix P (N columns corresponding to N assets and V rows corresponding to N views). In the case of an absolute view the row associate to the matrix P will sum 1 and in the case of a relative view the row will sum 0 and the assets outperforming will receive positive weights and the assets underperforming negative weights. The percentage assigned to each asset in a row of the matrix P corresponding to one relative view can vary, but normally it is defined proportional to 1 divided by the number of the assets involving the views and it is the method used in this work.

With matrices Q and P being defined, the variance of each view of the investor can be calculate as $p_k \sum p_k^{'}$, where p_k is a single 1xN row vector from matrix P corresponding to the kth view and $\sum$ is the covariance matrix of excess returns. The uncertainty of the views results in a random, unknown, independent, normally-distributed Error Term Vector (ε) with a men 0 and covariance matrix Ω. ε does not directly enter the Black-Litterman formula.
The variance of the error terms (ω) form Ω, where Ω is a diagonal covariance matrix with 0's in all of the off-diagonal positions because the model assumes that one view is independent of the another one, in consequence ω represent the uncertainty of the views. The confidence levels of the views will make the return vector obtained by Black-Litterman closer or not to the Implied Equilibrium Return Vector, if the confidences levels are close to zero the new return vector will be closer to the Implied Equilibrium Return Vector.

The variance of the views (ω) form Ω, where Ω is a diagonal covariance matrix with 0's in all of the off-diagonal positions because the model assumes that one view is independent of the another one, and in consequence ω represent the uncertainty of the views.

In order to calibrate the Black-Litterman model, the easiest way is calibrate the confidence of a view as in He and Litterman (1999) so that the ratio of ω/τ (where ω is the confidence level of a view) is

equal to the variance of the view portfolio ($p_k \sum p_k^{'}$).

After defining the Prior Equilibrium Distribution ($N \sim (\pi, \tau \sum)$) and the View Distribution ($N \sim (Q, \Omega)$), the New Combined Return Distribution can be obtained.

The mean of the New Combined Return Distribution is the new expected returns being calculated as $E[R] = [(\tau \sum)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau \sum)^{-1}\pi + P'\Omega^{-1}Q]$ and the variance as $[(\tau \sum)^{-1} + (P'\Omega^{-1}P)]^{-1}$

The new weights provided using the New Combined Return Distribution are the solution to the unconstrained maximization problem: max $w'\mu - \lambda w' \sum w/2$, resulting in the formula $w = (\lambda \sum)^{-1}\mu$.

The following conceptual diagram summarizes the previous explanation:

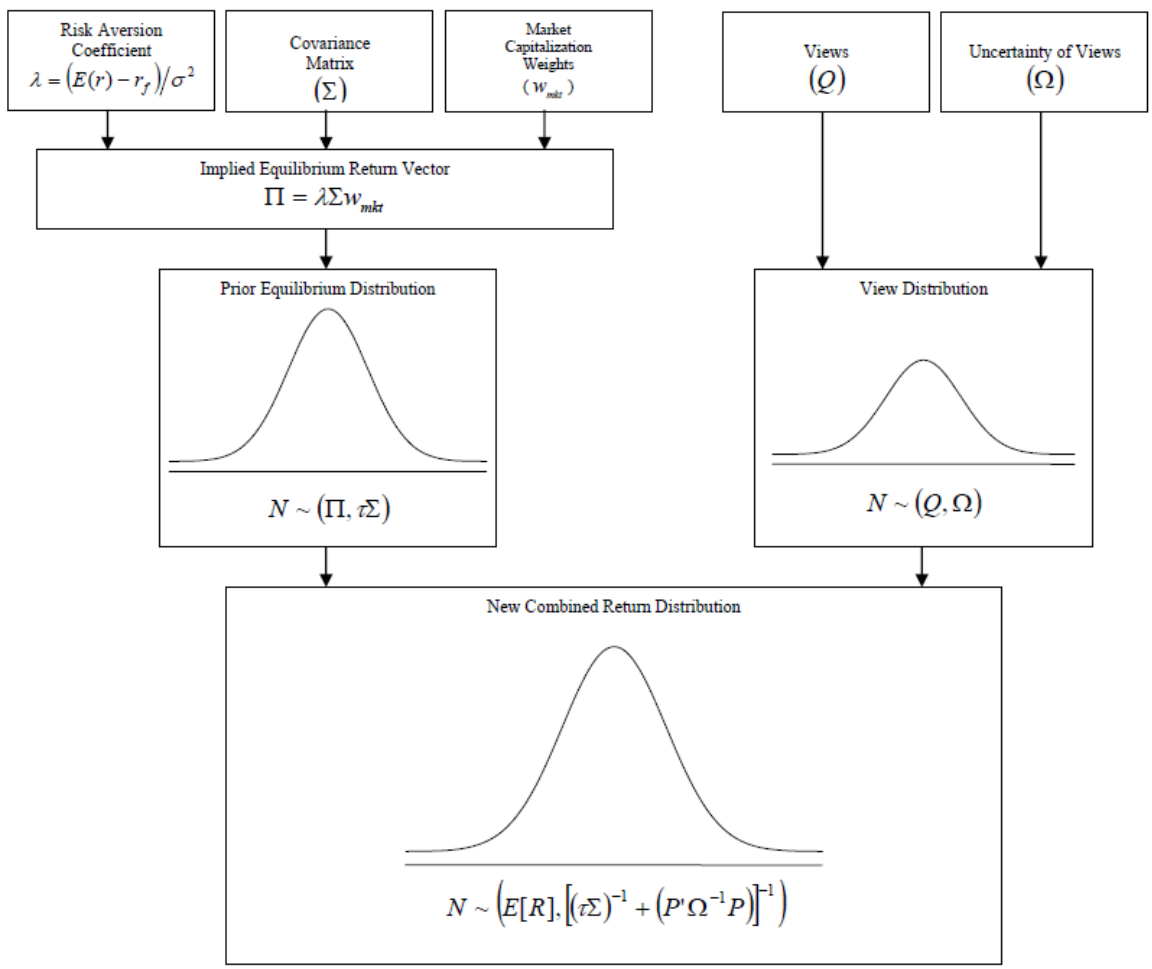

FIGURE 6. Diagram of the Black-Litterman model, from Thomas M. Idzorek, *A STEP-BY-STEP GUIDE TO THE BLACK-LITTERMAN MODEL* (January, 2002)

4.2.3. *The value of the scalar τ.*

The scalar τ is the most difficult parameter of the Black-Litterman model and although some information about how to specify this parameter have been published, there is always a subjective factor that has to be considered by the investor because basically it is a measure of the investor's confidence in the prior estimates.

Black and Litterman (1992) propose that the τ value has to be close to zero because the uncertainty in the mean is much smaller than the uncertainty in the return itself.

Then, He and Litterman (1999) sets the value of τ as 1/t, the ratio of the sampling variance to the distribution variance, that can be related with the previous assumption when t is higher than 20 due to the value will be less than 0.05 and a value lowest than 0.05 is considered close to zero by Black and Litterman (1992).

Blamont and Firoozye (2003) propose τ as approximately 1 divided by the number of observations considering $\tau \sum$ as the standard error of estimate of the Implied Equilibrium Returns.

On the other hand, Satchell and Scowcroft (2000) adjusted the value of τ to 1, being completely opposed with the previous comments.

In this work, taking into account that the number of observations and t are the same, the standard τ assumed will be calculated using the references of He and Litterman (1999) and Blamont and Firoozye (2003).

4.3. Prior Equilibrium Distribution Calculation.

The purpose of this subsection is to explain step-by-step the calculation of the prior equilibrium distribution that will be used in the subsections 1.1 and 1.2 where the Black-Litterman model has to be calculated for the period $T + 1$ with ETFs data from November 2010 to March 2015.

As can be seen on the conceptual diagram of figure 6, the prior equilibrium distribution follows a normal distribution with the implied returns as mean and covariance matrix $\tau \sum$.

The implied equilibrium returns are used in the Black-Litterman model as a neutral starting point, being defined as the set of returns that clear the market [3]. The reverse optimization is used in order to calculate the implied equilibrium returns through the formula:

$$\pi = \lambda \sum w_{mkt}$$

where

[3]Market clearing is achieved when the market is in balance through the equality between the demand price and supply price

- $\sum$ = Covariance Matrix of Excess Returns
- w_{mkt} = Market Capitalization Weights
- λ = Risk Aversion Coefficient

Covariance matrix of the excess returns

The covariance matrix of the excess returns is the covariance matrix calculated with the monthly returns of all the ETFs selected to create the portfolio for the period of data considered relevant for the analysis net of the risk free rate. The period considered is from November 2010 to March 2015 (analyzed in subsection 2.2) and the risk free rate is assumed as $\frac{0.00196}{12}$, which is the monthly Euribor 12 months at March 31, 2015.

The covariance of the excess returns obtained is:

$$\sum = \begin{pmatrix} 0.00085 & 0.00055 & 0.00002 & 0.00022 & 0.00070 & 0.00074 & 0.00065 & 0.00051 & 0.00061 & 0.00008 & 0.00056 & 0.00071 \\ 0.00055 & 0.00219 & -0.00023 & 0.00217 & 0.00156 & 0.00146 & 0.00231 & 0.00118 & 0.00068 & 0.00007 & 0.00108 & 0.00160 \\ 0.00002 & -0.00023 & 0.00016 & -0.00030 & -0.00016 & -0.00001 & -0.00020 & -0.00004 & 0.00014 & 0.00007 & 0.00000 & -0.00008 \\ 0.00022 & 0.00217 & -0.00030 & 0.00327 & 0.00148 & 0.00138 & 0.00254 & 0.00134 & 0.00044 & 0.00001 & 0.00076 & 0.00128 \\ 0.00070 & 0.00156 & -0.00016 & 0.00148 & 0.00262 & 0.00130 & 0.00208 & 0.00111 & 0.00101 & 0.00017 & 0.00099 & 0.00154 \\ 0.00074 & 0.00146 & -0.00001 & 0.00138 & 0.00130 & 0.00267 & 0.00183 & 0.00101 & 0.00078 & 0.00014 & 0.00129 & 0.00146 \\ 0.00065 & 0.00231 & -0.00020 & 0.00254 & 0.00208 & 0.00183 & 0.00303 & 0.00157 & 0.00104 & 0.00016 & 0.00106 & 0.00201 \\ 0.00051 & 0.00118 & -0.00004 & 0.00134 & 0.00111 & 0.00101 & 0.00157 & 0.00167 & 0.00094 & 0.00006 & 0.00048 & 0.00106 \\ 0.00061 & 0.00068 & 0.00014 & 0.00044 & 0.00101 & 0.00078 & 0.00104 & 0.00094 & 0.00148 & 0.00013 & 0.00048 & 0.00098 \\ 0.00008 & 0.00007 & 0.00007 & 0.00001 & 0.00017 & 0.00014 & 0.00016 & 0.00006 & 0.00013 & 0.00009 & 0.00016 & 0.00017 \\ 0.00056 & 0.00108 & 0.00000 & 0.00076 & 0.00099 & 0.00129 & 0.00106 & 0.00048 & 0.00048 & 0.00016 & 0.00174 & 0.00111 \\ 0.00071 & 0.00160 & -0.00008 & 0.00128 & 0.00154 & 0.00146 & 0.00201 & 0.00106 & 0.00098 & 0.00017 & 0.00111 & 0.00176 \end{pmatrix}$$

Market Capitalization Weights

The market capitalization weights used in the reverse optimization are used to obtain the weights proportionals to the market capitalization of each ETF. Using Bloomberg as source of data at 31st March, the market capitalizations and the market capitalization weights of the ETFs considered are shown below:

ETF	**Market Capitalization**	**Weight (w_{mkt})**
DXSPX:SM	1.000.600.000.000	18%
MSE:SM	2.072.000.000.000	37%
MAA:SM	1.415.000.000.000	25%
LYXIB:SM	48.700.000	0%
OIL:SM	22.960.000	0%
ENER:SM	221.100.000	0%
VALU:SM	1.088.000.000.000	20%
TEL:SM	176.200.000	0%
HLT:SM	254.400.000	0%
CRP:SM	245.400.000	0%
DXMEM:SM	129.500.000	0%
GWT:SM	319.700.000	0%

TABLE 5. Market capitalizations and weights

The previous table demonstrates the large variations on the amount of capital invested in the ETFs selected. Due to the large differences among the ETFs market capitalizations, only four of the twelve ETFs have weights that differ from zero, resulting in a clear unbalance of the market capitalization weights.

Risk aversion coefficient

The risk aversion coefficient is a parameter that measures the degree to which the investor is averse to taking risks. In the implied returns calculation it acts as a scaling factor for the reverse optimization estimate of excess returns.

The risk aversion coefficient λ is estimated by dividing the portfolio expected excess return by the portfolio variance (Grinold and Kahn (1999)):

$$\lambda = \frac{E(r) - r_f}{\sigma^2}$$

The portfolio expected excess return $(E(r) - r_f)$ in the risk aversion coefficient is defined as the expected returns of the portfolio over the period considered with the market capitalization weights net of the risk free rate:

$$E(r) - r_f = \sum_{i=1}^{N} w_{mkt}\mu_i - r_f = 0.008179 - (0.00196/12) = 0.008016$$

The portfolio variance in the risk aversion coefficient is calculated over the period defined using the market capitalization weights and the covariance matrix net of the risk free rate:

$$\sigma^2 = \sum_{i=1}^{N} w_i^2\sigma_i^2 + \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} = 0.00086$$

Having the portfolio expected excess return and the portfolio variance calculated, the risk aversion is:

$$\lambda = \frac{0.008179 - (0.00196/12)}{0.00086} = 9.3270$$

Implied equilibrium returns

The implied equilibrium returns obtained after the reverse optimization are shown in the following table:

	Implied Equilibrium Returns
DXSPX:SM	0.0046
MSE:SM	0.0123
MAA:SM	-0.0007
LYXIB:SM	0.0119
OIL:SM	0.0101
ENER:SM	0.0097
VALU:SM	0.0142
TEL:SM	0.0077
HLT:SM	0.0056
CRP:SM	0.0008
DXMEM:SM	0.0067
GWT:SM	0.0103

TABLE 6. Implied Equilibrium Returns

The MAA:SM is the unique ETF with a negative implied return although their return is close to zero due to its return is -0.0007. The highest implied equilibrium return obtained is 0.0119 resulted of the LYXIB:ETF, so all the other ETFs have a return positive but lower than the 1.19%

5. Simulations

5.1. Asset Correlated Paths.

This subsection intends to explain how to obtain the views that an investor has to define in the Black-Litterman model using simulations. The views defined by the matrix Q on the Black-Litterman model can be absolutes or relatives, depending on whether more than one asset is involved in the view,in this case being a relative view, or whether one asset is involved being defined as an absolute view.

The corresponding value on the matrix Q for an absolute view will be the expected return of the asset considered in the view, while for a relative view the value will be how much it will outperform or underperform a determinate number of assets in respect to other assets.

Regardless of the type of view selected, it is clear that what is needed is the expected returns (absolutes or relatives) of the views that want to be incorporated in the model. It is at this point that where the simulations are used, through the use of a particular type of a Monte Carlo simulation the expected returns at the time $T+t$ (being T the last time observed) can be simulated in order to include several scenarios of the views in the model.

5.1.1. *Monte Carlo simulation.*

The Monte Carlo simulation is a technique widely used for dealing with uncertainty in many aspects, with the main objective of making predictions based on how the range of estimates are created in order to have an idea of the results that could be obtained, commonly used in the financial area to be used for option pricing and scenarios simulation of the future assets prices.

Supposing that an investor has 12 assets in his portfolio and wants to predict their future prices. In a Monte Carlo simulation, 12 random values are generated from a normal distribution with $\mu = 0$ and $\sigma = 1$ corresponding to each asset. The model that generates the future prices of the assets calculates the output based on the random values generated. The results obtained are stored and the process is repeated the number of times defined by the investor in order to obtain the desired number of simulations. When the Monte Carlo simulation is finished, the investor has a large number of scenarios that can be used in order to obtain the probabilities of the future prices among others uses.

The standard Monte Carlo simulation does not take into account one important measure that is usually considered in finance such as the correlation and for this reason this method is modified to include this measure and the Asset Correlated Paths is introduced to do the simulations of the Black-Litterman views.

5.1.2. *Asset Correlated Paths.*

In the previous subsection the Monte Carlo simulation has been explained leading to the conclusion that it is not a method to be considered without modifications if the correlation among the assets have to be considered. A Monte Carlo simulation assumes that the N assets that the investor wants to forecast are independent amongst themselves and in consequence assumes that they are uncorrelated when, in general, the assets of a portfolio are correlated.

The Asset Correlated Paths consists of a Monte Carlo simulation modified to incorporate the correlation among the assets by the use of the Cholesky factorization.

Assuming that there are N assets in the portfolio of an investor consequently N correlated simulation paths have to be generated with N correlated random numbers generated by a normal distribution with the corresponding μ and σ that are selected for each asset.

The correlated numbers are calculated using the Cholesky factorization, that says that every symmetric positive definite matrix M has a unique factorization $M = LL*$ where L is a lower triangular matrix and $L*$ is its conjugate transpose.

For a portfolio of N assets the correlation matrix is symmetric and positive definite and it may be factorized as $RR*$ where R is a lower triangular matrix. Then the correlated random numbers can be calculated as the multiplication of R and the random numbers generated previously by a normal distribution with $\mu = 1$ and $\sigma = 0$.

There are some models to generate future prices, the model that generates the future prices in this work is assumed as a Geometric Brownian Motion model.

<u>Geometric Brownian Motion model</u>

A Geometric Brownian Motion Model S_t is a stochastic process that satisfies the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where W_t is a Brownian Motion, μ is a constant called the percentage drift and σ is a constant called the percentage volatility.

On the right hand of the equation, the term $\mu S_t dt$ defines the trend of the Brownian Motion trajectory and the term $\sigma S_t dW_t$ is the random noise of the trajectory.

Using the technique of separation of variables, the solution of the differential equation is obtained in few steps:

- Taking the integration of both sides: $\int \frac{dS_t}{S_t} = \int (\mu d_t + \sigma W_t) dt$
- Applying the $It\bar{o}$ calculus: $In\left(\frac{dS_t}{S_t}\right) = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t$
- The analytical solution is obtained taking the exponential of both sides and plugging the initial condition S_0:

$$S_t = S(0) exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right)$$

With the constants μ and σ defined and with the substitution of the term σW_t by $\left(\sigma\sqrt{t}\right)\varepsilon$, where ε correspond to the random numbers obtained by the use of the normal distribution correlated using the Cholesky factorization, a Geometric Brownian Motion solution can be produced with the final equation:

$$S_t = S(0)exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \left(\sigma\sqrt{t}\right)\varepsilon\right)$$

Chapter 3
Results

1. Black-Litterman simulations using absolute views in the out-of-sample period

This section shows the results of applying the Black-Litterman model with different values of τ using absolute views in the out-of-sample period $T + 1$, which corresponds to the next month of the sample period, April 2015.

The objective of generating the Black-Litterman model simulations is to obtain portfolios in several scenarios that could happen. In order to do this, the first thing that is needed is to simulate the future returns of the next month through the use of the Asset Correlated Path algorithm explained in the previous chapter, which will be used to define the absolute views of the Black-Litterman model (matrix Q). The number of simulations to generate in this work is determined in 100, which means that 100 of expected returns will be obtained for each ETF through the use of the Asset Path Correlated.

After finishing the Asset Correlated Paths 100 of expected returns are obtained for each ETF, the next graph shows the comparison of the simulated expected returns with the real return obtained that month represented with a black vertical line:

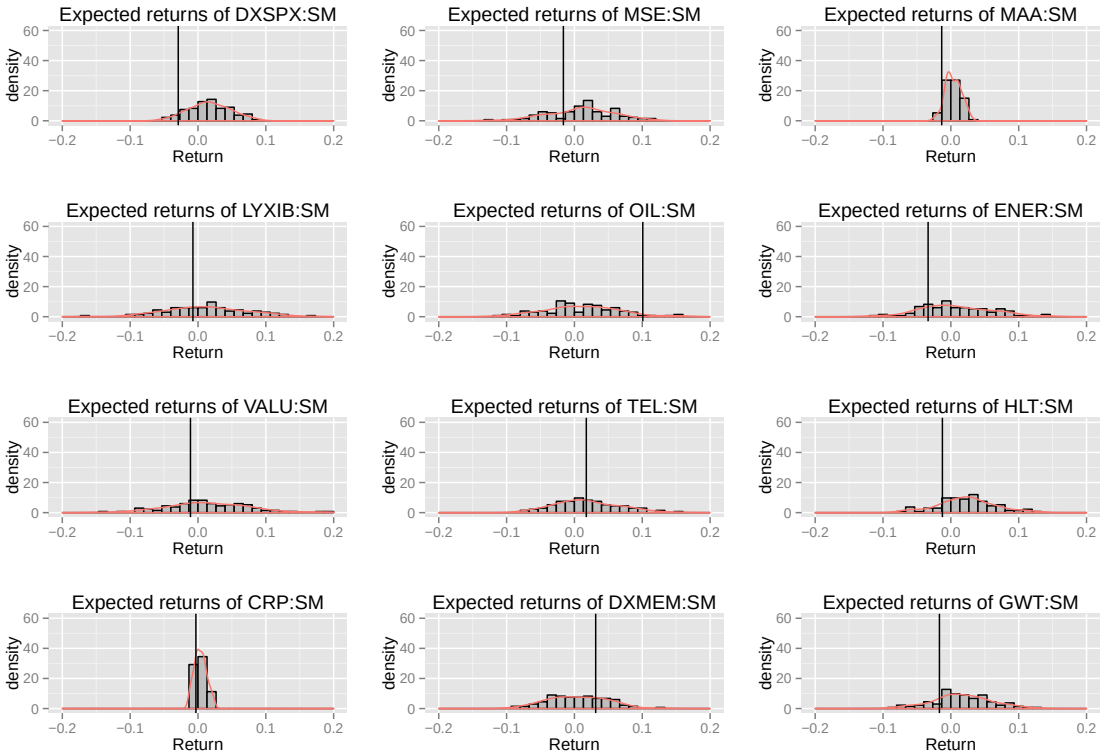

FIGURE 1. Histograms of the expected returns obtained using the Asset Correlated Paths

The previous graph makes it possible to check that all the real returns are contained in all the simulated expected returns distributions in a manner that all the real returns are been simulated.

The uncertainty of the corresponding views represented in the figure 1, are defined as a diagonal matrix with the values $p_k \sum p_k^{'} \tau$ where p_k is 1 because the matrix P is a diagonal matrix of ones because there is one view for each ETF and in consequence Ω (the matrix that represents the uncertainty in the Black-Litterman model) is defined as a matrix with the diagonal values $diag(\sum)\tau$ and the off-diagonal zeros, which can be interpreted as there are no covariance among views.

Then, it is clear that what really determines the confidence that the investor has in their views is τ and with a τ close to zero the uncertainty of the investors' views will be less than when τ is one.

The standard τ value determined in this work is 0.089, which is a number close to zero. With the purpose of to compare how the results are affected by the scalar τ, the Black-Litterman model will be executed also with the values 0.5and 1 of τ to determine the impact of τ in the model.

The investor views and their uncertainties depending on the value of τ are needed to define the views distributions of the Black-Litterman model that are use to compute the New Combined Return Distribution of the model. The prior equilibrium distribution defined by the Implied Equilibrium Return

Vector and its uncertainty is calculated on subsection 4.3 and it is the other distribution needed to compute the Black-Litterman model.

Once the New Combined Return Distribution is calculated, the expected returns and the covariance matrix of the Black-Litterman model are obtained. 100 vectors of expected returns and 100 covariance matrices (for each value of tau) will be obtained with a 100 simulations.

Each vector of expected returns of the Black-Litterman model contains an expected return for each asset and in consequence after finishing the simulations, 100 expected returns are obtained for each ETF. The expected returns histograms, are shown in the next graph and compared with the densities of the simulated returns obtained with the Asset Correlated Paths used in the views distributions. The implied returns incorporated in the prior equilibrium distributions are represented with a red vertical line and the real returns given that month with a black vertical line:

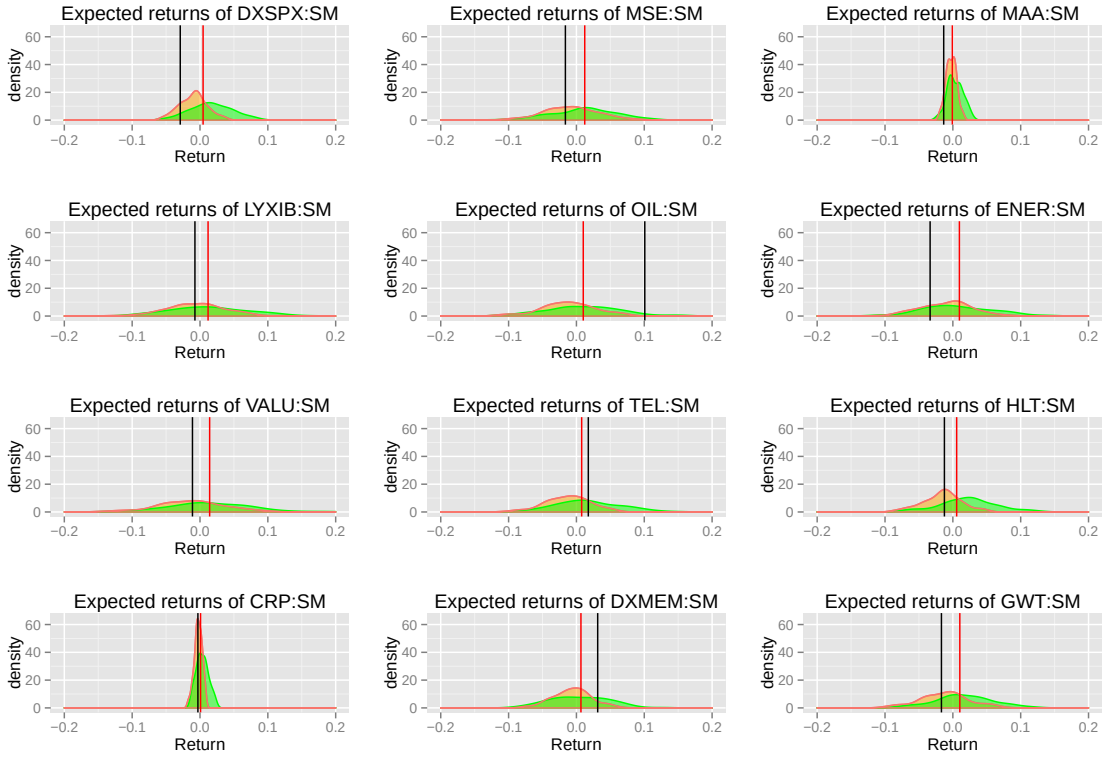

FIGURE 2. Densities of the Black-Litterman and the Asset Correlated Paths expected returns

The main characteristic that can be observed for all ETFs is that the distribution of the Black-Litterman expected returns has a greater kurtosis than the distribution of the simulated returns so that the last distribution tends to have highest values on the extremes, specially on the right tail. This fact makes that the ETF OIL:SM real return is contained in the distribution of the simulated returns but not in the Black-Litterman expected returns distribution.

The differences between the distributions is explained by Ω, the extreme cases are the most intuitive:

- When Ω takes the value 0 for the asset i, the Black-Litterman expected return i corresponds to the value Q_i, which is the simulated expected return
- When Ω takes the value ∞ for the asset i, the Black-Litterman expected return i corresponds to the value π_i, which is the implied equilibrium return

The median of the Black-Litterman expected returns tends to be near to the implied expected returns as a consequence of the fact that Ω is near to zero, and for this reason there are differences between the Black-Litterman expected returns distribution and the distribution of the simulated returns.

Having the Black-Litterman expected returns and the Black-Litterman covariance matrices, the following step is to find the portfolio weights for each simulation that maximizes the Sharpe ratio, 100 simulations means that 100 optimal portfolios weights have to be found. In order to find the optimal portfolios, the standard procedure of the Black-Litterman theory to obtain the portfolio weights is to solve the unconstrained maximization problem: max $w'\mu - \lambda w' \sum w/2$, resulting in the formula $w = (\lambda \sum)^{-1}\mu$.
This solution is feasible when the investor views simulated are relative because when the relative views are used the weights that result from the solution of the unconstrained problem sum up to one, while when absolute views are used the weights do not add to one and the solution can result in an investment several times the initial capital.

When the investor includes some restriction to the Black-Litterman model, the unconstrained maximization problem cannot be applied and in order to avoid the problems mentioned, the most efficient way to use the Black-Litterman model with constraints and/or absolute views is introduce the output of the Black-Litterman model (expected returns and the covariance matrix) to the Mean-Variance optimization problem seen in the section 3.1. Introducing the Black-Litterman output to the Mean-Variance optimization problem the maximum Sharper ratio portfolio, can be found for each simulation after calculating the efficient frontier.

The maximum Sharpe ratio portfolios obtained through the Black-Litterman efficient frontiers can be obtained for different values of τ and be compared with the Markowitz model portfolios and the portfolios obtained from the use of an equal weight asset allocation strategy, including and no including the restriction of short sells.

1.1. Results obtained without short sells.

This subsection includes the results of the use of the Mean-Variance optimization problem with the restriction of avoiding short sells using the expected returns and covariances matrices of the Black-Litterman simulations.

The process to obtain the portfolios that maximize the Sharpe ratio given a value of τ, consists on creating the efficient frontiers for the 100 simulations of the Black-Litterman model to obtain the maximum Sharpe ratio portfolio of each efficient frontier. The next plot shows the efficient frontiers

obtained using the standard τ value (0.0189 in this case) and the maximum Sharpe ratio portfolios are represented by blue dots:

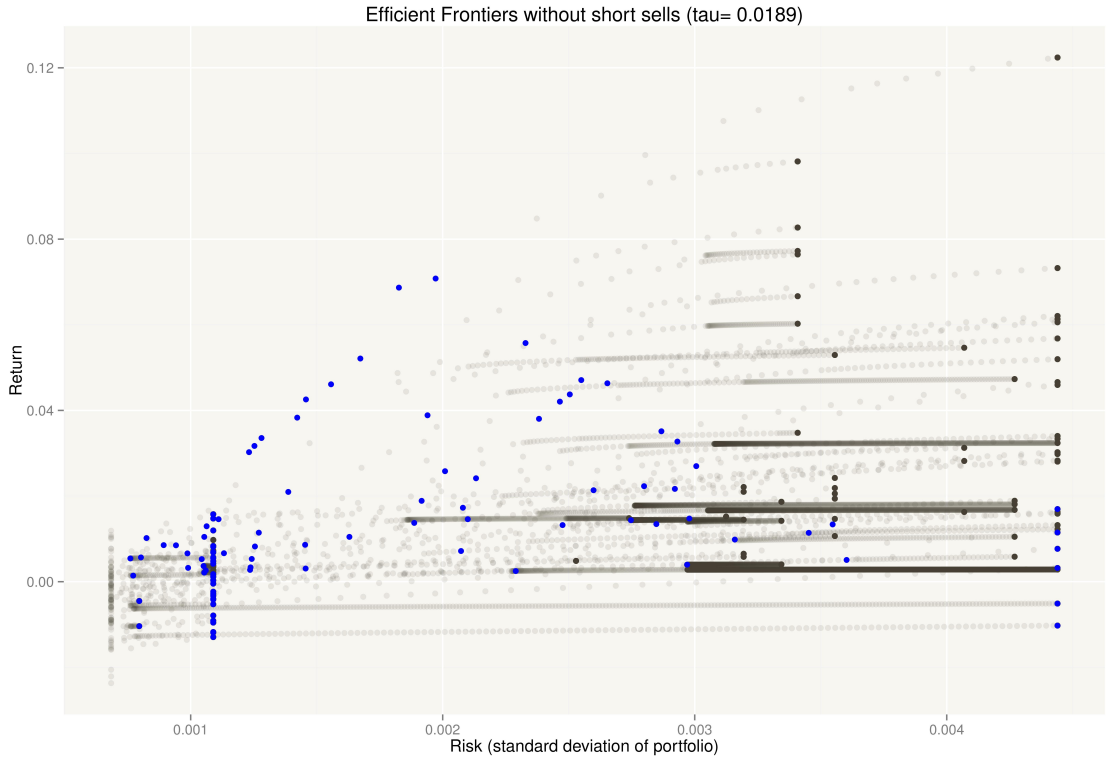

FIGURE 3. Efficient frontiers of Black-Litterman model ($\tau = 0.0189$)

The portfolio of each efficient frontier that has the maximum levels of return and risk, consists of the ETF with the highest expected return that has been simulated in the corresponding simulation of the Black-Litterman model and in consequence in this portfolio, all the capital is invested in this ETF. Therefore, the maximum returns of the efficient frontiers are the the highest ETFs returns obtained in each simulation of the Black-Litterman model, so it is possible to see the diversification of the returns obtained through the scenarios generated.

The maximum Sharpe ratio portfolios points are dispersed all over the graph with a special case being on the efficients frontiers that have a low maximum return, where the maximum Sharpe ratio increments linearly until it achieves the maximum on the extreme of the efficient frontier which is the ETF with the maximum expected return simulated.

The previous graph only represents the efficient frontiers of the Black-Litterman simulations with the value $\tau = 0.0189$ (the standard value selected for τ). In order to see the incidence of this parameter that affects the covariance matrix the efficient frontiers obtained with the τ values 0.0189, 0.05 and 1 are represented and also compared with the Markowitz efficient frontier represented in red and its maximum Sharpe ratio portfolio in purple:

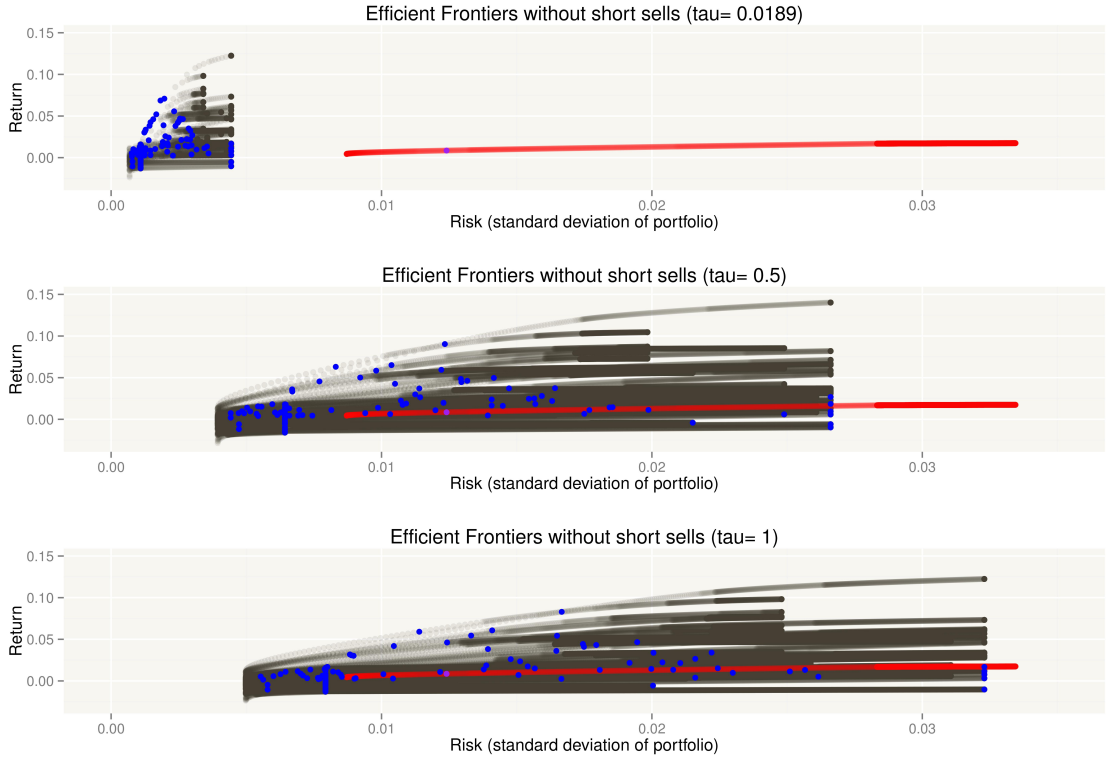

FIGURE 4. Efficient frontiers of Black-Litterman and Markowitz without short sells for different values of τ

Differences between the Black-Litterman efficient frontiers by τ can be observed. The first value of τ, 0.0189, makes the covariance matrices much smaller than the values 0.5 or 1 respectively, due to this fact the Black-Litterman efficient frontiers do not coincide at any point with the Markowitz efficient frontier. With the τ value 0.5, the covariance matrices obtained results in a major overlapping of the Black-Litterman efficient frontiers with the Markowitz efficient frontier although there are significant differences, which are further reduced using a τ value of 1.

The maximum Sharpe ratio portfolios, defined as the weights combination that maximize the Sharpe ratio, have been obtained for each frontier simulated (blue points on the previous graph). The graph below shows the density weights distributions for the three values of τ: $\tau = 0.0189$ is represented in red, $\tau = 0.5$ in green and $\tau = 1$ in blue. The market weight capitalization is represented with green vertical lines, the maximum Sharpe ratio portfolio weights of the Markowitz model is represented with red vertical lines and the weights $\frac{1}{12}$ corresponding to an equal weight asset allocation strategy are represented with black vertical lines:

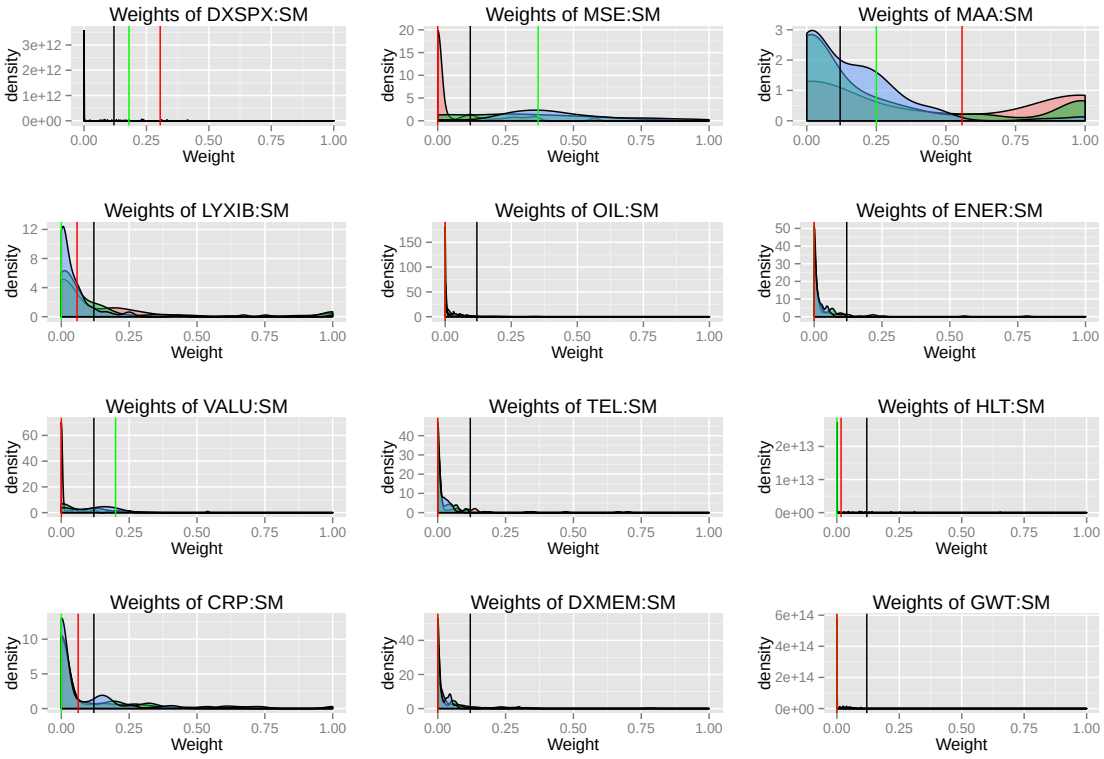

FIGURE 5. Black-Litterman weights of the maximum Sharpe ratio portfolios without short sells for different values of τ

There are several things that can be appreciated in the graph above. The main point, is that between the graphs there are big differences, for example for the ETFs DXSPX:SM, HLT:SM and GWT:SM the densities, at least for one value of τ are fully concentrated on the value zero. However, for the first and the second ETFs the market weights capitalization and the Markowitz weights differ from zero. The rest of the ETFs have their weight values more diversified and, with the exception of MSE:SM, MAA:SM and LYXIB:SM, this diversification is not made on the full range of the weight values.

Another thing that is important to consider is that there does not seem to be a relation between the Black-Litterman weight distributions, the market weight capitalization weights and the Markowitz weights. The weights distributions depending on the τ value seems to follow an order whereby the weight distribution with $\tau = 1$ takes values closer to zero than the weights distribution with τ values of 0.5 and 0.0189 respectively for the ETFs MAA:SM, LYXIB:SM, ENER:SM and TEL:SM, but that order is inverse for the ETFs MSE:SM and OIL:SM.

To appreciate the details among the distributions, the next table shows the first, second and third quartiles of the weights distributions according to their τ values:

	0.25			0.5			0.75		
	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$
DXSPX:SM	0.0000	0.0000	0.0000	0.0000	0.0310	0.0137	0.0062	0.1170	0.1605
MSE:SM	0.0000	0.0959	0.3153	0.0000	0.2799	0.4068	0.0538	0.4663	0.5590
MAA:SM	0.0000	0.0000	0.0000	0.0974	0.0490	0.1009	1.0000	0.2960	0.2406
LYXIB:SM	0.0000	0.0000	0.0000	0.0000	0.0286	0.0180	0.1826	0.1334	0.0643
OIL:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0554	0.0140	0.0056
ENER:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0389	0.0231	0.0172
VALU:SM	0.0000	0.0000	0.0256	0.0000	0.0317	0.1307	0.0153	0.1296	0.1780
TEL:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0318	0.0275	0.0205
HLT:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0048
CRP:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1041	0.0979	0.0786
DXMEM:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0633	0.0258	0.0176
GWT:SM	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

TABLE 1. Quartiles of the weights distributions

The table above, shows that in the first quartile of the weights distribution all the ETFs take the value 0 for the standard value of τ 0.0189, while one value takes a value different from zero with $\tau = 0.5$ and two have values that differ from zero with $\tau = 1$.
In the second quartile, all the ETFs have the value 0 for $\tau = 0.0189$, except the MAA:SM and when $\tau = 0.5$ and $\tau = 1$ are used five ETFs have a value greater than zero.
In the third quartile only the ETF GWT:SM continues to have the value zero for all values of τ , and the relation of the τ value between the weights distributions can be seen clearly. For the ETFs MAA:SM, LYXIB:SM, OIL:SM, ENER:SM, TEL:SM, CRP:SM and DXMEM:SM, lowest τ values results in higher values of weights on these ETFs. On the other hand, this relation is inverse for the ETFs DXSPX:SM, MSE:SM and VALU:SM, which obtain higher weights values with higher values of τ.

The weight values of each maximum Sharpe ratio portfolio resulted from the simulations allow us to obtain the portfolio returns of the Black-Litterman simulations by τ for the period $T+1$, which are the main interest results for a portfolio manager that applies whatever technique of portfolio management. To determine the success of the Black-Litterman portfolio returns (the median of the portfolio returns is represented with a green vertical line) in comparison with the Markowitz model portfolio returns and the portfolio returns of an equal weight asset allocation strategy, the three methods are represented in the following graph, the returns from the Markowitz model in red vertical lines and the returns from the equal weight strategy in black vertical lines:

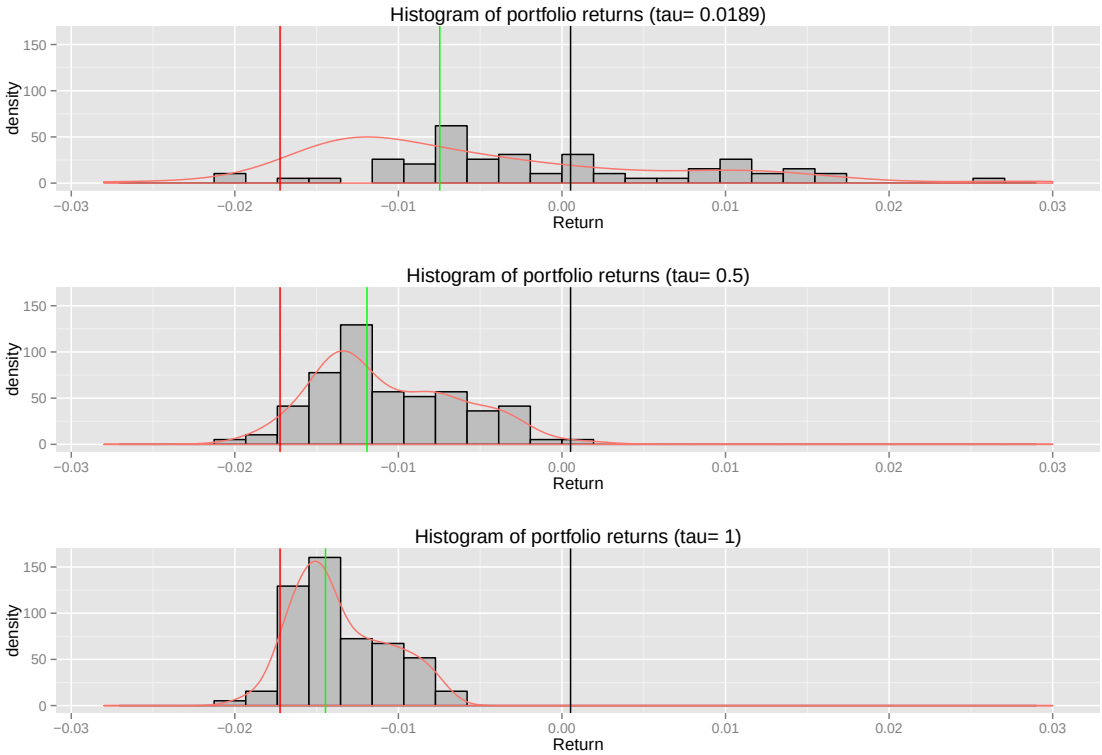

FIGURE 6. Histograms of the real portfolio returns obtained with short sells forbidden on April 2015 using different values of τ

The first effect of the parameter τ over the Black-Litterman portfolio returns that can be appreciated is the skewness obtained over the three distributions, using a τ value of 0.0189 the positive skewness obtained is higher than using τ values of 0.5 and 0.1 respectively.
Due to this fact, the comparisons of the Black-Litterman portfolio returns bt τ with the portfolio return of the Markowitz model seem to be similar for all the τ values because the Markowitz portfolio return is situated to the left of the distributions, whereas if the Black-Litterman portfolio returns are compared with the portfolio return of the equal weight asset allocation strategy the results obtained by τ are completely different.

The next table shows the median of the Black-Litterman portfolio returns by τ and the number of times that the Black-Litterman portfolio returns beats the portfolio return of the Markowitz model and the portfolio return obtained by the use of an equal weight asset allocation strategy:

	BL median			BL>Markowitz			BL>E-W		
	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$
04/2014	-0.0075	-0.01192	-0.1446	97	96	96	26	1	0

TABLE 2. Summary of the portfolio returns without short sells for the period $T + 1$

The median of the Black-Litterman portfolios returns decreases with high values of τ, with a greater decrease between values $\tau = 0.0189$ and $\tau = 0.5$ than values $\tau = 0.5$ to $\tau = 1$. More over, the portfolio return of the Markowitz model is beaten 97 times using $\tau = 0.0189$ and 96 times using $\tau = 0.5$ or $\tau = 1$, but the portfolio return of the equal weight strategy is beaten 26, 1 and 0 times for the τ values 0.0189, 0.5 and 1 respectively.

Consequently, the standard value of the scalar τ defined in the subsection 4.2.3 is the best value to obtain Black-Litterman portfolio returns that can beat the equal weight asset allocation strategy a significant number of times.

The distribution of the ETFs weights obtained plays an important role in relation to the results obtained. The Markowitz weights are zero for seven weights of ETFs and in consequence are very different to the ETFs equal weights. The Black-Litterman weights distributions obtained from the simulations are not so extreme as the Markowitz weights but are clearly not equal for all the ETFs in the majority of cases.

1.2. Results obtained with short sells.

This subsection includes the results of the use of the Mean-Variance optimization problem without the restriction of avoiding short sells, using the expected returns and covariances of the Black-Litterman simulations. A restriction to limit the possible maximum debt or investment on a ETF has been added to the Mean-Variance optimization problem that allows short sells due to the problem that for the optimization of some Black-Litterman simulations, the debt achieved on one asset could be more than 300 times the initial capital and obviously any portfolio manager would not be willing to accept the portfolios obtained (see figures 1 and 2 in the subsection 4 of the appendix).

As in the previous subsection, the process to obtain the portfolios that maximize the Sharpe ratios given a value of τ, consists on creating the efficient frontiers for the 100 simulations of the Black-Litterman model that have been done to obtain the maximum Sharpe ratio portfolio of each efficient frontier.

To see the effect of the parameter τ and in consequence how it affects the covariance matrix to the efficient frontiers of the Black-Litterman simulations, the Black-Litterman efficient frontiers are represented below with the τ values 0.0189, 0.05 and 1, as well as the maximum Sharpe portfolios and are compared with the Markowitz efficient frontier (that allow short sells) represented in red and its maximum Sharpe ratio portfolio in purple:

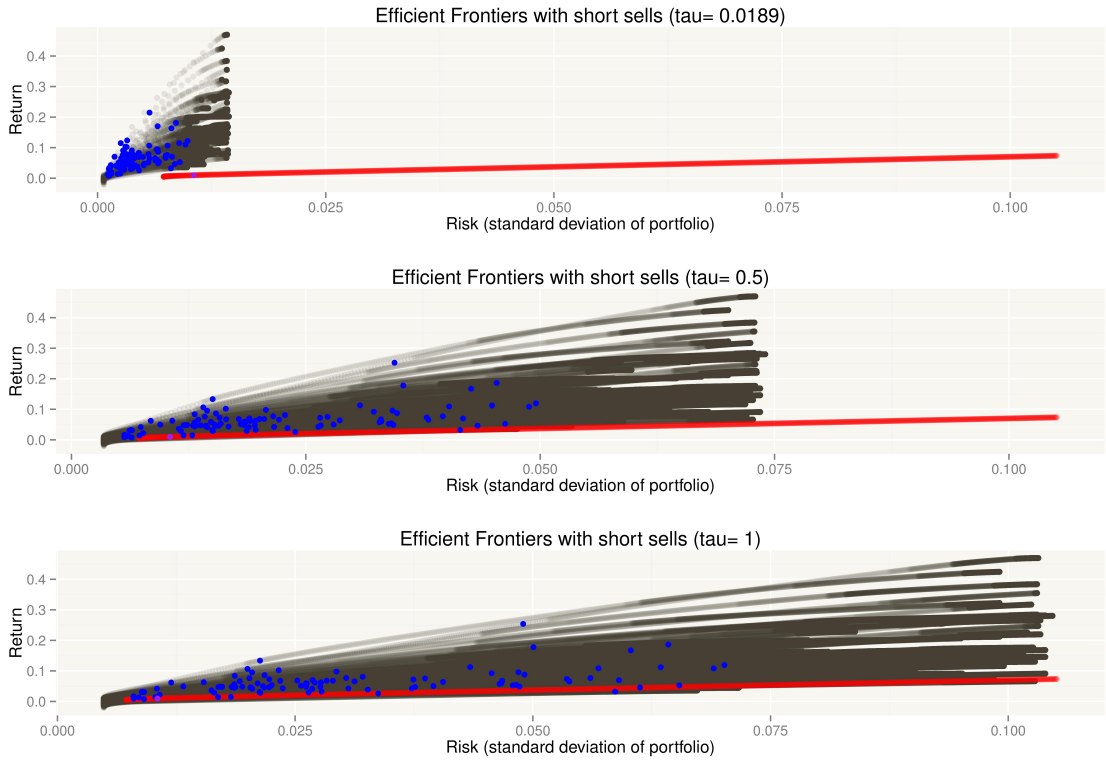

FIGURE 7. Efficient frontiers of Black-Litterman and Markowitz with short sells for different values of τ

In comparison to the Markowitz efficient frontier, the Black-Litterman efficient frontiers are above it with few exceptions and have greater slopes, which are interpreted as the obtainment of higher levels of return with lowest risk increments.

As in the case of the prohibition of short sells, the value of the scalar τ and in consequence the covariance matrix, have the same effect on the efficient frontiers, resulting on longer efficient frontiers with higher values of τ and also cause the dispersion of the maximum Sharpe ratio portfolios.

The ETFs weights contained in the maximum Sharpe ratio portfolios of each Black-Litterman frontier simulated are represented on the next graph, where the density weight distribution of $\tau = 0.0189$ is represented in red, $\tau = 0.5$ in green and $\tau = 1$ in blue. The market weights capitalization is represented with green vertical lines, the maximum Sharpe ratio portfolio weights of the Markowitz model is represented with red vertical lines and the weights $\frac{1}{12}$ corresponding to a equal weight asset allocation strategy is represented with black vertical lines:

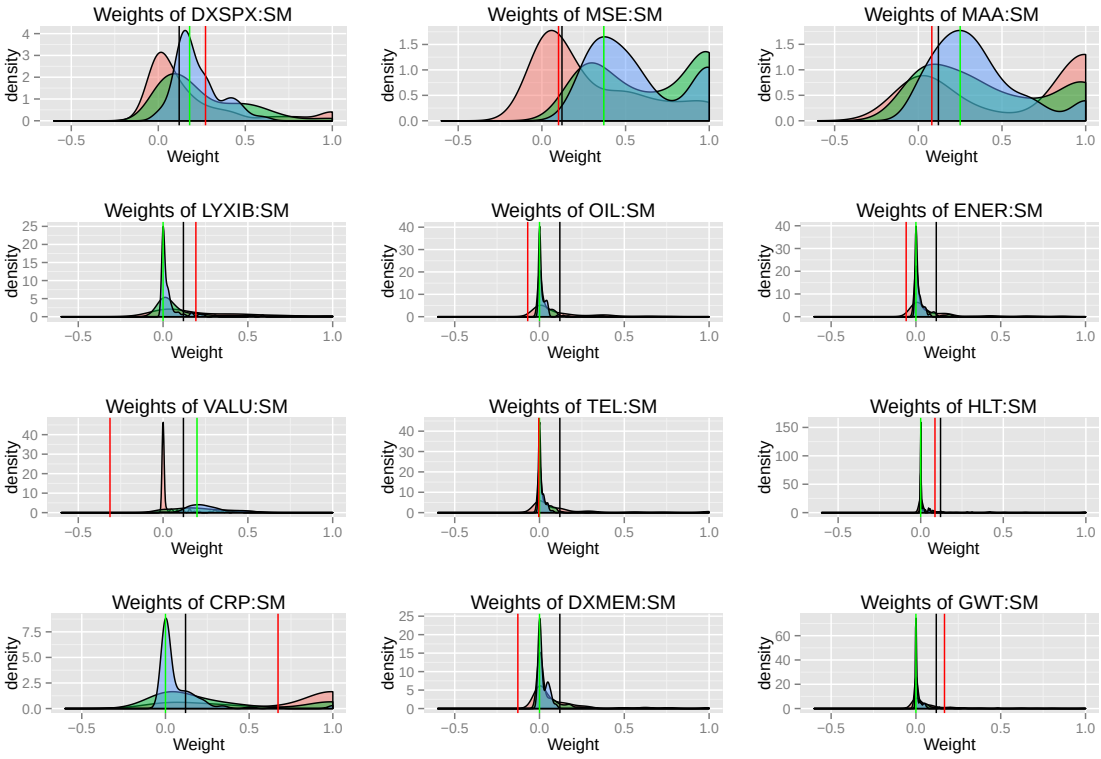

FIGURE 8. Black-Litterman weights of the maximum Sharpe ratio portfolios with short sells for different values of τ

Comparing the graph above with the graph obtained when short sells are forbidden (figure 5), it can be seen that a greater diversification is achieved when the possibility of debt exists because the weights are not so focused on the value zero. Despite improvement of the diversification , the weights of the ETFs DXSPX:SM, MSE:SM, MAA:SM and CRP:SM are diversified for all values of τ while the other ETFs are less diversified or the diversification depends on the scalar τ.

The table below shows the first, second and third quartiles of the weights distributions according to their τ values:

	0.25			0.5			0.75		
	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$
DXSPX:SM	-0.1966	0.0665	0.2594	0.0684	0.1632	0.1849	0.2594	0.3962	0.2806
MSE:SM	0.0113	0.3320	0.3490	0.1383	0.6719	0.5008	0.4984	1.0000	0.9066
MAA:SM	-0.1931	0.0685	0.1684	0.9594	0.3240	0.2914	1.0000	0.8440	0.4761
LYXIB:SM	0.0014	-0.0354	-0.0344	0.1079	0.0235	0.0033	0.4190	0.1676	0.0326
OIL:SM	-0.1238	-0.1293	-0.0624	0.0092	-0.0192	-0.0084	0.1675	0.0417	0.0214
ENER:SM	-0.1135	-0.1766	-0.0699	0.0061	-0.0006	-0.0003	0.1441	0.0399	0.0210
VALU:SM	-0.0517	0.1031	0.1775	-0.2278	0.1896	0.2372	0.0214	0.3168	0.3118
TEL:SM	-0.1565	-0.1790	-0.1054	-0.0126	-0.0347	-0.0216	0.1429	0.0385	0.0206
HLT:SM	-0.4534	-0.3249	-0.1595	-0.1679	-0.1088	-0.0423	0.0421	0.0128	0.0044
CRP:SM	0.2658	-0.1544	-0.2821	1.0000	0.1321	-0.0260	1.0000	0.4914	0.1049
DXMEM:SM	-0.2719	-0.2898	-0.0923	-0.0335	-0.0095	-0.0043	0.1424	0.0597	0.0440
GWT:SM	-0.1138	-0.1260	-0.0738	0.0119	-0.0121	-0.0249	0.0952	0.0349	0.0101

TABLE 3. Quartiles of the weights distributions

The first quartile includes short sells for all the ETFs on some τ values with the exception of the MSE:SM while in the second quartile the number of ETFs that do not include short sells decrease to four and in the third quartile no ETFs have short sells.

The previous table shows interesting relations between the τ values in that a different behavior can be observed between $\tau = 0.0189$ and $\tau = 0.5$ or $\tau = 1$. The lasts two values of the scalar τ in general have similar values. For example, the ETFs with short sells for $\tau = 0.5$ and $\tau = 1$ are the same with one exception on the second quartile.

The weight values of the portfolios with the maximum Sharpe ratio obtained through the Black-Litterman simulations allow us to obtain the Black-Litterman returns when short sells are allowed for the period $T + 1$. The following graph represents the histograms of the Black-Litterman portfolio returns by τ, the median of the portfolio returns is represented with a green vertical line, the portfolio returns from the Markowitz model with red vertical lines and the portfolio returns from the equal weight strategy with black vertical lines:

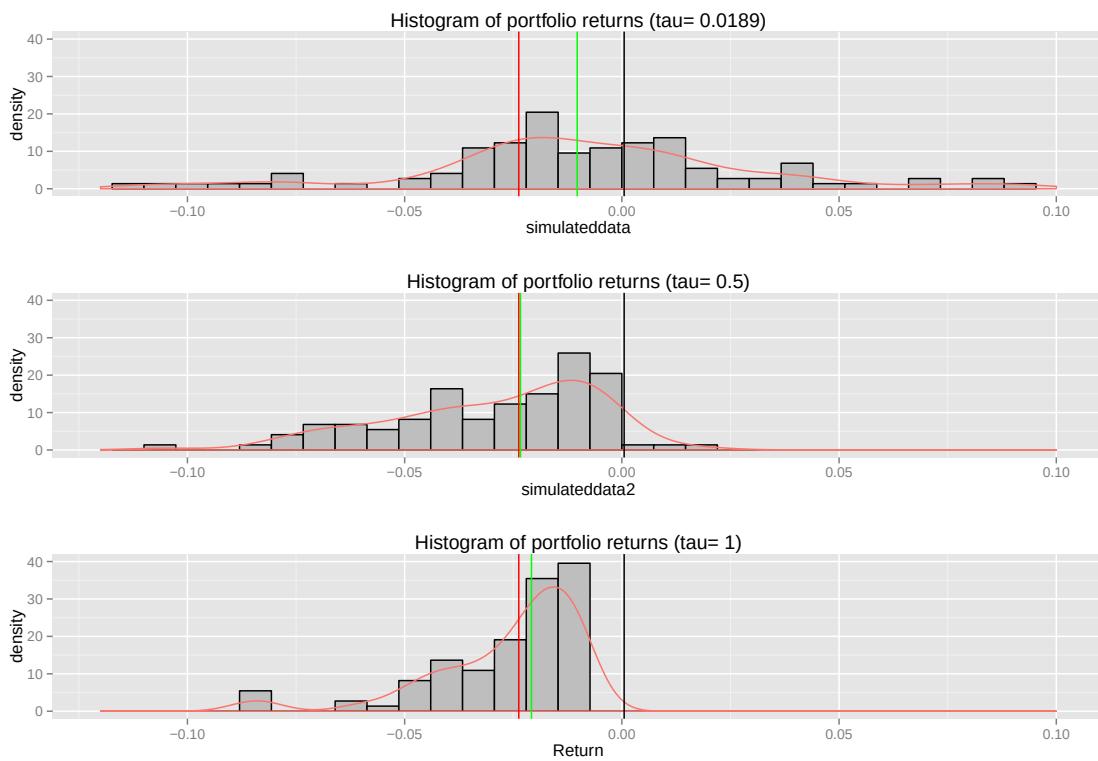

FIGURE 9. Histograms of the real portfolio returns obtained with short sells allowed on April 2015 using different values of τ

The difference on the ETFs weights by τ are reflected on the distributions obtained for the Black-Litterman portfolio returns. The Black-Litterman portfolio return distribution with $\tau = 0.0189$ does not have significant skewness, while the distributions using $\tau = 0.05$ and $\tau = 1$ have a clear negative skewness. The number of Black-Litterman portfolio returns that beat the Markowitz portfolio return

and the equal weight asset allocation portfolio return is clearly higher using $\tau = 1$, but also a major number of portfolio returns are located at the most negative part of the x axis.

	BL median			BL>Markowitz			BL>E-W		
	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$	$\tau = 0.0189$	$\tau = 0.5$	$\tau = 1$
04/2014	-0.0103	-0.0233	-0.0208	71	50	61	39	2	0

TABLE 4. Summary of the portfolio returns with short sells for the period $T + 1$

The median of the Black-Litterman portfolios returns does not follow the same behavior that has been seen when short sells are forbidden. The best results are also obtained using $\tau = 0.0189$, but now using $\tau = 1$ the number of times that the Markowitz portfolio return is beaten in comparison to using $\tau = 0.05$ is 11 times higher although the equal weight portfolio return is beaten two times using $\tau = 0.5$ and no one using $\tau = 1$.

Therefore, with the results of the previous graph and table, $\tau = 0.0189$ is also the best choice when short sells are included but it is more difficult to decide whether $\tau = 0.5$ is better than $\tau = 1$ or not.

The distribution of the ETFs weights obtained plays an important role to the results obtained. The Markowitz weights are zero for seven values and in consequence are very different to the equal weights. The Black-Litterman weights distributions obtained from the simulations are not so extreme as the Markowitz weights but clearly not equal for all the ETFs in the majority of cases.

Comparing the portfolio returns obtained with short sells and without short sells, the main difference is that when short sells are included a major risk is assumed but the returns obtained can also be higher. The worst portfolio return with short sells is five time worst than the worst portfolio return without short sells, while the difference between the best portfolios returns is about 3.3 times better with short sells.

2. Black-Litterman simulations using absolute views in the sample-period

The section above, has shown the results of applying the Black-Litterman model to the out-of-sample period $T + 1$ using the historical data from November 2010 until March 2015. Due to the fact that it is impossible to determine which asset allocation model is better than other with one case, this section provides the results of applying the Black-Litterman model (forbidding and allowing short sells) to obtain the real Black-Litterman portfolio returns for every month from January 2014 to March 2015 in order to be able to compare the Black-Litterman model, the Markowitz model and an equal weight asset allocation strategy.

In order to compare the results obtained between the different months, an annual window of data is going to be used to calculate the statistics needed to compute the Black-Litterman model for the considered month, as well as the Markowitz model and the equal weight strategy. For example, to calculate the Black-Litterman models to obtain the real Black-Litterman portfolio returns on January 2014, the historical data used in the model will be from January 2013 until December 2013.

Remembering the determination of the standard value for τ that has been commented in the paragraph 4.2.3, it is obvious that the standard value of τ has to be $\tau = \frac{1}{12}$ because the number of observations that are considered is 12.

The value of the risk-free rate is determined as in the previous sections because the interest rates have remained flat over the sample-period selected and the change would be minimum.

The steps followed to calculate the portfolio returns of the models mentioned above are the same as those that have been done in the previous subsection:

- Select the period of the ETFs returns data to use. This will be the previous year of the month that has been selected to obtain the portfolios returns
- Calculate the prior equilibrium distribution with the data defined
- Generate 100 absolute views using the Asset Correlated Asset Paths algorithm
- Calculate the views distributions through the absolute views generated on the previous step
- Calculate the expected returns and the covariance matrices of the Black-Litterman model using the 100 views distributions for each value of τ
- Maximize the Sharpe ratio of the efficient frontiers obtained using the Mean-Variance optimization with the expected returns and the covariance matrices from the Black-Litterman model as inputs
- Calculate the Black-Litterman portfolio returns that would be obtained the month selected using the known real returns

- Compare the results with the Markowitz portfolio return that maximize the Sharpe ratio and the portfolio return obtained from an equal weight asset allocation strategy calculated with the real returns obtained as was done previously for the Black-Litterman portfolio returns

2.1. Results obtained without short sells.

This subsection describes the results obtained for the portfolio returns obtained with the Black-Litterman model, the Markowitz model and the equal weight strategy forbidding short sells over the 15 months selected from the sample data.

The following figure represents the Black-Litterman portfolio returns of each month. The Markowitz model portfolios returns are represented with vertical red lines and the equal weight asset allocation portfolios returns are represented with black vertical lines. Three color histograms represent the Black-Litterman portfolio returns: the color yellow corresponds to the returns obtained using $\tau = \frac{1}{12}$, the Black-Litterman portfolio returns calculated with $\tau = 0.5$ can be seen in red and the green histograms represent the Black-Litterman portfolio returns calculated using $\tau = 1$. The median of all the Black-Litterman portfolios returns is represented with a green vertical line:

FIGURE 10. Black-Litterman portfolio returns over the 15 months without short sells

The initial thing that is striking about this picture is that all the histograms are painted in green. However,upon closer inspection there are some small variations. The first result that can be obtained is that the parameter τ does not cause a significant impact on the Black-Litterman portfolio returns using yearly windows of data, despite the fact that the returns obtained over the different months are not equal.

With this graph it is not easy to determine whether one strategy is better than another. For this reason the following table summarizes the median of the Blak-Litterman portfolio returns by the value of τ used, and the number of times that the Black-Litterman portfolio returns beats the Markowitz portfolio returns and the portfolio return resulted of an equal weight asset allocation strategy for every month:

	BL median			BL>Markowitz			BL>E-W		
	$\tau = \frac{1}{12}$	$\tau = 0.5$	$\tau = 1$	$\tau = \frac{1}{12}$	$\tau = 0.5$	$\tau = 1$	$\tau = \frac{1}{12}$	$\tau = 0.5$	$\tau = 1$
01/2014	0.0047	0.0045	0.0045	50	49	49	72	72	72
02/2014	0.0130	0.0131	0.0131	9	10	10	23	23	23
03/2014	0.0050	0.0050	0.0050	97	97	97	29	29	28
04/2014	0.0075	0.0075	0.0075	11	10	9	20	20	20
05/2014	0.0186	0.0192	0.0191	36	35	34	34	33	32
06/2014	0.0075	0.0075	0.0075	36	32	30	28	25	25
07/2014	0.0052	0.0052	0.0052	63	63	63	85	85	86
08/2014	0.0186	0.0186	0.0186	70	71	71	32	31	31
09/2014	0.0001	0.0002	0.0002	21	20	21	29	29	28
10/2014	0.0022	0.0022	0.0022	56	56	56	73	74	73
11/2014	0.0174	0.0174	0.0174	52	51	51	39	40	39
12/2014	-0.0191	-0.0200	-0.0191	1	1	1	49	48	49
01/2015	0.0520	0.0523	0.05234	79	80	80	47	47	47
02/2015	0.0624	0.0624	0.0625	67	67	66	53	54	54
03/2015	-0.0031	-0.0031	-0.0031	70	70	70	25	25	25
Mean				47.87	47.47	47.2	42.53	42.33	42.13

TABLE 5. Summary of the portfolios obtained by month

The table shows that the τ effect on the portfolio returns is minimal for the three measures that have been observed, so that the results of the τ sensibility seen on the previous section are quite different even though on average the results based on the three values of τ are slightly better with a higher values.

There are several differences between the results obtained on the different months, for example on March 2014 the Markowitz portfolio return is beaten 96 times by the Black-Litterman simulated portfolios returns (the three value of τ gives the same result) while on December 2014 the return of the Markowitz portfolio is beaten only once by the Black-Litterman portfolios returns (the three value of τ gives the same result).

On average, over the period considered, the Black-Litterman portfolio returns beat the Markowitz portfolio returns 47.87 times considering the standard $\tau = \frac{1}{12}$ and the equal weight portfolios returns 42.53 times. The results show that the the Black-Litterman portfolio returns beat both portfolio management techniques a significant number of times with the views incorporated to the Black-Litterman through simulations.

2.2. Results obtained with short sells.

This subsection describes the results obtained for the portfolio returns with the Black-Litterman model, the Markowitz model and the equal weight strategy allowing short sells over the 15 months selected

from the sample data.

The following figure represents the Black-Litterman portfolio returns of each month. The Markowitz model portfolios returns are represented with vertical red lines and the equal weight asset allocation portfolios returns are represented with black vertical lines. Three color histograms represent the Black-Litterman portfolio returns: the color yellow corresponds to the returns obtained using $\tau = \frac{1}{12}$, the Black-Litterman portfolio returns calculated with $\tau = 0.5$ can be seen in red and the green histograms represent the Black-Litterman portfolio returns calculated using $\tau = 1$. The median of all the Black-Litterman portfolios returns is represented with a green vertical line:

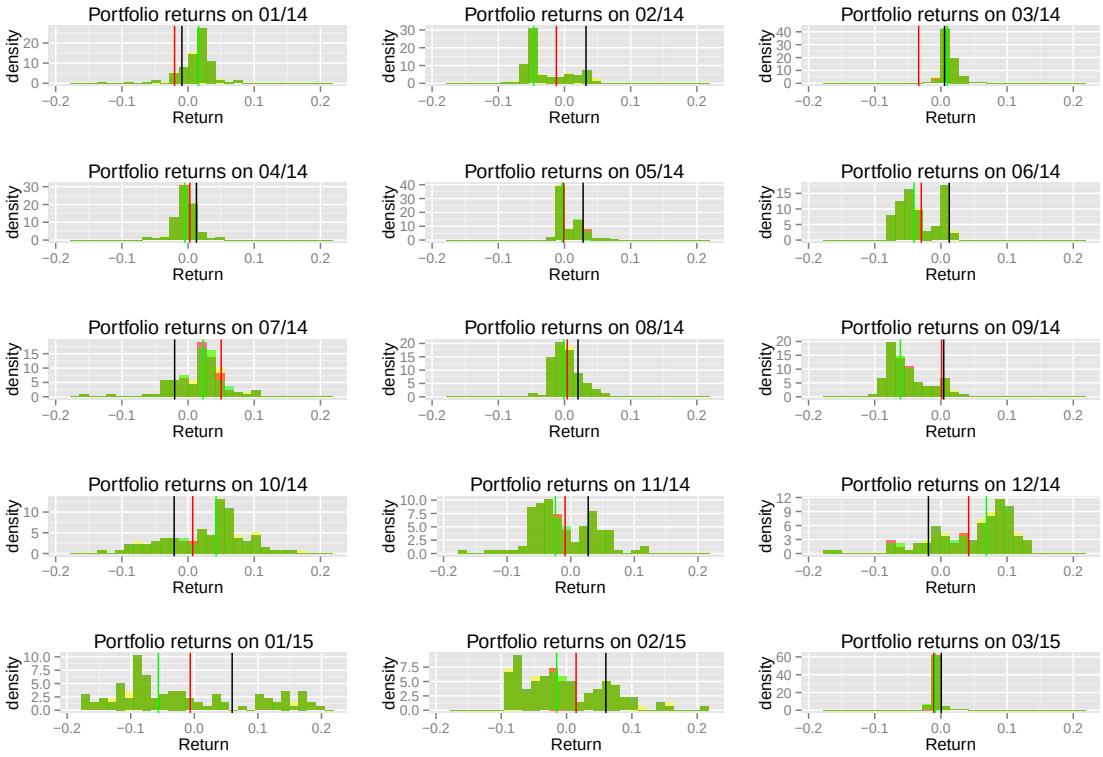

FIGURE 11. Black-Litterman portfolio returns over the 15 months with short sells

As has been seen in the previous section, where the short sells restriction was allowed, due to the effect of the short sells the range of the portfolio returns obtained are higher and as a result the volatility increases on the portfolio returns distributions of the Black-Litterman model but the returns that can be achieved are also higher.

The τ value, as is the case when short sells are forbidden, does not have a significant impact on the Black-Litterman portfolio returns distributions even though some small differences can be appreciated.

The next table shows the median of the Black-Litterman portfolio returns by τ and the number of times that the Black-Litterman portfolio returns beat the Markowitz portfolio returns and the portfolio

returns that resulted from an equal weight asset allocation strategy for the 15 months:

	BL median			BL>Markowitz			BL>E-W		
	$\tau = \frac{1}{12}$	$\tau = 0.5$	$\tau = 1$	$\tau = \frac{1}{12}$	$\tau = 0.5$	$\tau = 1$	$\tau = \frac{1}{12}$	$\tau = 0.5$	$\tau = 1$
01/2014	0.0159	0.0158	0.0159	93	92	92	86	86	86
02/2014	-0.0463	-0.0462	-0.0463	30	30	30	10	7	8
03/2014	0.0096	0.0096	0.0097	100	100	100	74	75	76
04/2014	-0.0050	-0.0050	-0.0050	30	30	30	11	11	11
05/2014	-0.0035	-0.0037	-0.0036	45	45	45	14	13	13
06/2014	-0.0403	-0.0408	-0.0408	38	37	37	7	4	4
07/2014	0.0244	0.0216	0.0222	13	12	12	86	85	85
08/2014	-0.0009	0.0010	0.0010	37	37	37	18	19	19
09/2014	-0.0613	-0.0611	-0.0611	12	12	12	9	9	9
10/2014	0.0424	0.0422	0.0422	70	70	70	78	76	76
11/2014	-0.0247	-0.0240	-0.0237	39	39	39	27	27	27
12/2014	0.0685	0.0687	0.0687	63	63	63	85	85	85
01/2015	-0.0564	-0.0566	-0.0565	36	36	36	24	24	24
02/2015	-0.0155	-0.0150	-0.0149	34	34	34	19	19	19
03/2015	-0.0084	-0.0085	-0.0084	79	79	79	7	8	8
Mean				47.93	47.73	47.73	37	36.53	36.67

TABLE 6. Summary of the portfolios obtained by month

The results observed on the table allows us to see that on average the best τ election is the standard $\tau = \frac{1}{12}$ again, while the second best τ is 1 with a minimum difference on the number of times that the Black-Litterman portfolio returns beat the equal weight portfolio returns in comparison with $\tau = 0.5$. A major difference between the τ values 0.5 and 1 has been seen in the previous subsection. It is difficult to decide which τ between the values 0.5 and 1 is better, but what is also reflected here is that these values of τ do not follow the same pattern when the short sells restriction is included or not.

The number of times that the Markowitz portfolio return is beaten on average by the Black-Litterman portfolio returns when short sells are included is slightly higher than when short sells are prohibited due to the fact of that the inclusion of short sells generates Markowitz portfolios with lower returns than the ones obtained without short sells. In consequence, the same thing does not happen when the equal weight portfolio return is compared with the Black-Litterman portfolio returns. The result is that the difference of the number of times that the equal weight portfolio return is beaten by the Black-Litterman portfolio returns when short sells are included is about 5.5 times less than when short sells are not included and this leads to the conclusion that in general the inclusion of short sells does not provide enough returns to exceed the equal weight portfolio return.

Chapter 4
Conclusions

In this project several scenarios have been simulated using Monte Carlo simulations that allow the inclusion of correlations in order to generate vectors of absolute views corresponding to the expected returns of 12 ETFs. Those absolute views have been used as inputs to generate simulations of the Black-Litterman asset allocation model for three values of the scalar τ, which is an input of the model that affects the Black-Litterman covariance matrix.

From the simulations of the Black-Litterman model, the expected returns and the covariance matrices obtained have been introduced into a Mean-Variance optimizer in the presence of constraints such as the total investment of the capital or the forbidding of short sells.

The Black-Litterman portfolios returns have been calculated with and without short sells for the out-of-sample period $T + 1$ using all the historical data analyzed in this work and over 15 months of the sample-period using yearly windows of historical data. The results obtained have been compared with the portfolio returns obtained from a Markowitz model and an equal weight asset allocation strategy.

After analyzing the ETFs characteristics and implementing portfolio management techniques, it has been seen that the results obtained can be implemented with ETFs, as well as with common stocks.

In the majority of cases the inclusion of short sells in the models involves more risk but can also result in higher portfolio returns.

In the out-of-sample period $T + 1$ has been concluded that the effect of the scalar τ over the Black-Litterman portfolio returns is significant, obtaining higher portfolio returns with the the τ assumed as standard following the references of He and Litterman (1999) and Blamont and Firoozye (2003) which give a τ value close to zero, than the τ values 0.5 and 1. On the other hand, in the sample-period the differences on the Black-Litterman portfolio returns by τ, on average over the 15 months considered, have been not so significant due to the fact that, although the best value of τ has been the standard selected, the differences obtained have been minimal. Divergences in the conclusions have resulted from the election of different windows of historic data in the application of the Black-Litterman model.

The number of times that the Black-Litterman portfolios returns beat the Markowitz portfolios returns (including short sells and not) can be considered high due to the fact that on average in the sample period more than 47% of the times the Black-Litterman model beats the Markowitz model and the

results obtained in the out-of-sample period are even higher, especially if the standard τ considered is used.

In the comparison of the Black-Litterman portfolio returns with the equal weight portfolios returns in the out-of-sample period, the results obtained using the standard value of the scalar τ are much better than when $\tau = 0.5$ and $\tau = 1$ are used, beating 26 times the equal weight asset allocation model when short sells are forbidden and 39 times when short sells are allowed. In the sample period on average, for the Black-Litterman model all the τ values give similar results and the results obtained are worse when short sells are included in comparison with the equal weight asset allocation strategy, beating the equal weight asset allocation model about 42 times with short sells forbidden and approximately 37 times when short sells are allowed.

The next step of this project would be to investigate the optimal windows of data that have to be determined to obtain the best estimations for the inputs that use historical data of the Black-Litterman asset allocation model and whether the scenarios selected by professional investors really beat the Markowitz model and the equal weight asset allocation strategy since it has been seen that it is possible to beat both models a significant number of times with the scenarios simulated corresponding to possible investor's views.

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Appendix

R code

Efficient frontiers.

```
efficient.frontier <- function (returns, short="no",risk.premium.up=0.5,
                       risk.increment=0.00005,riskfree,maxweights=NULL){

  covariance <- cov(returns)
  n <- ncol(covariance)

  Amat <- matrix (1, nrow=n)
  bvec <- 1
  meq <- 1

  #If short sells are forbidden:
  if(short=="no"){
    Amat <- cbind(1, diag(n))
    bvec <- c(bvec, rep(0, n))
  }

  #If maximum weights are defined:
  if(!is.null(maxweights)){
    Amat <- cbind(cbind(Amat, +diag(n)),-diag(n))
    bvec <- c(bvec, rep(-maxweights, n),rep(-maxweights, n))
  }


  # Number of loops to do:
  loops <- risk.premium.up / risk.increment + 1
  loop <- 1

  # Matrix that contains the results:
  results <- matrix(nrow=loops, ncol=n+3)
  # Now I need to give the matrix column names
  colnames(results) <- c(colnames(returns), "Std.Dev", "Exp.Return", "sharpe")

  # Calculate the optimal portfolios:
  for (i in seq(from=0, to=risk.premium.up, by=risk.increment)){
    dvec <- colMeans(returns) * i
```

```
    sol <- solve.QP(covariance, dvec=dvec, Amat=Amat, bvec=bvec, meq=meq)
    results[loop,"Std.Dev"] <- sqrt(sum(sol$solution*colSums((covariance*sol$solution))))
    results[loop,"Exp.Return"] <- as.numeric(sol$solution %*% colMeans(returns))
    results[loop,"sharpe"] <- (results[loop,"Exp.Return"]-riskfree) / results[loop,"Std.Dev"]
    eff[loop,1:n] <- sol$solution
    loop <- loop+1
  }
  return(as.data.frame(results)
}
```

Asset Correlated Paths.

```
###Asset Correlated Paths ###
assetcorrelatedpaths<-function(S0,mu,volatility,correlationsmatrix,dt,numbersteps,
nsimulations){

  assetsnumber<-length(S0)    #number of assets
  drift<- mu - ((volatility^2)/2) #Calculating rhe drift

  #Cholesky factorization on the correlation matrix:
  R<-chol(correlationsmatrix)
  S<-list(NA)

  #Generating correlated random sequences and paths:
  for(i in 1:nsimulations){

    #Generate uncorrelated random sequences
    x=matrix(data = NA,numbersteps,dim(correlationsmatrix)[2])

    for(z in 1:dim(correlationsmatrix)[2]){
      x[,z]<-rnorm(n=numbersteps,mean=mu[z],sd =volatility[z])
    }

    #Correlate the sequences:
    ep=x%*%R

    #Generate potential paths:
    S[[i]]<-
      rbind( matrix(1,1,assetsnumber),
             matrix(  exp(

               repmat(matrix(drift*dt,1,assetsnumber),numbersteps,1) +ep%*%diag(volatility)
               *sqrt(dt)  ),
               numbersteps,assetsnumber))
```

```
    for(j in 2:(numbersteps+1)) S[[i]][j,] = S[[i]][j,] * S[[i]][(j-1),]
    S[[i]]<- S[[i]]%*%diag(S0)
  }

  return(S)
}
```

Reverse optimization.

```
reverse_optimization <- function (returnsr,Wmkt,riskfree){

  covariance <- cov(returnsr)
  marketvariance<- sum(Wmkt*colSums((covariance*Wmkt)))
  marketreturn <- as.numeric(Wmkt %*% colMeans(returnsr))

  #Calculation of lambda:
  lambda<- ((marketreturn-riskfree)/marketvariance)

  #Reverse optimization:
  er <- t(lambda * covariance %*% Wmkt)

  return(er)
}
```

Black-Litterman asset allocation model.

```
black_litterman <- function(impliedreturns,covariancematrix, tau,lambda, P, Q, Omega) {

  ts <- tau * covariancematrix
  n <- nrow(Q)
  m <- ncol(Q)


  # Posterior estimate of the mean:
  er<- solve(solve(ts)+  t(P)%*%solve(Omega)%*%P)  %*% (solve(ts)%*%impliedreturns+
                                                t(P)%*%solve(Omega)%*%as.numeric(Q))

  # Posterior estimate of the uncertainty in the mean:
  posteriorsigma<- solve(solve(ts)+(t(P)%*%solve(Omega)%*%P))

  # Posterior weights based on uncertainty in mean:
  w <- solve(lambda * posteriorsigma)%*%er

  #Results:
```

```
  colnames(er) <- "Return"
  results <- list()
  results$er <- er
  results$w <- w
  results$posteriorsigma <- posteriorsigma
  return(results)
}
```

Figures

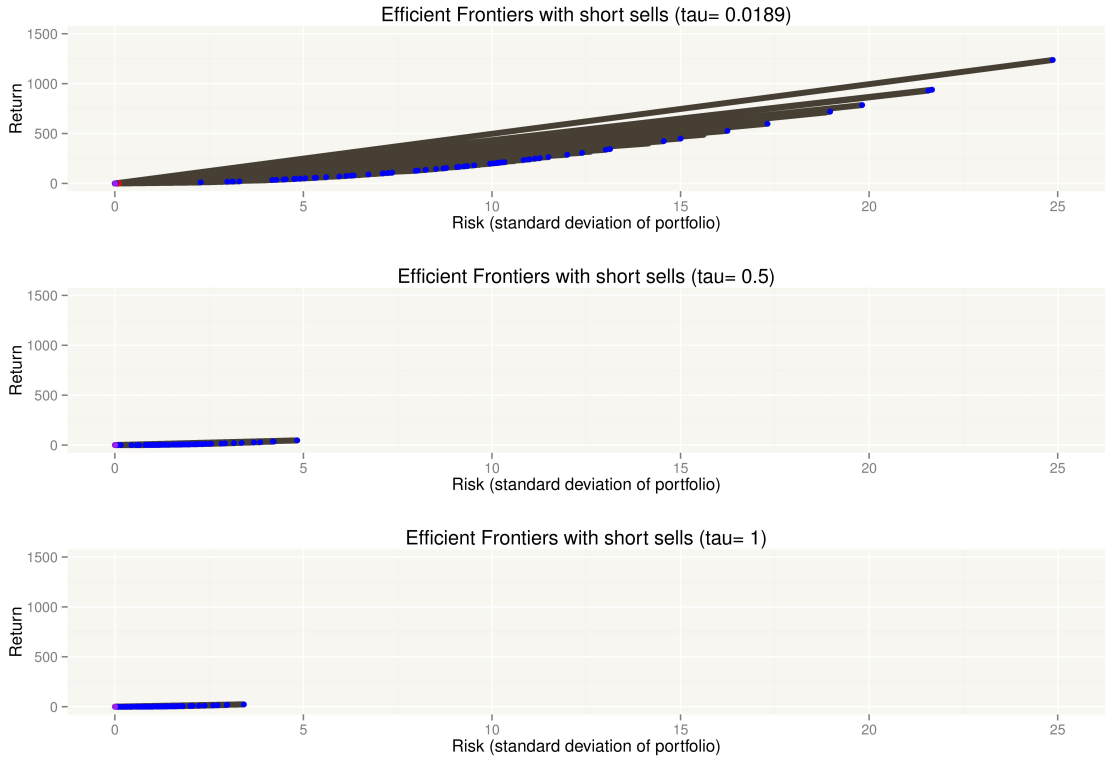

FIGURE 1. Efficient frontiers of Black-Litterman and Markowitz with short sells and without the maximum weight constriction for different values of τ

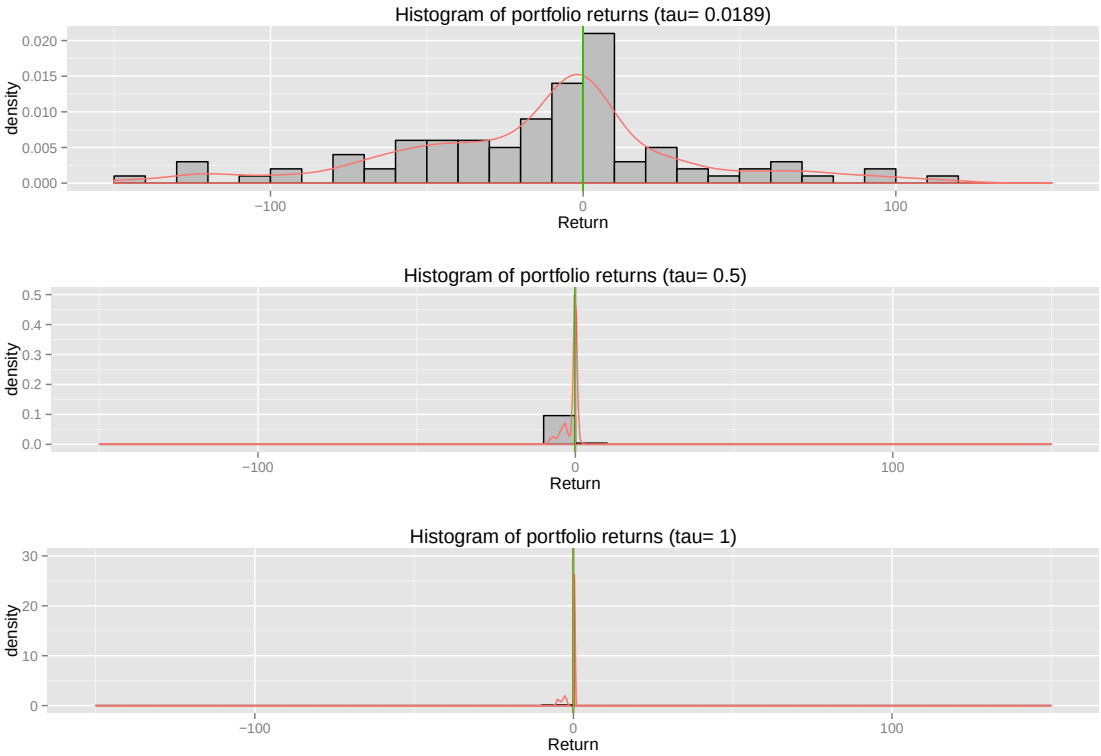

FIGURE 2. Histogram of the real portfolio returns obtained with short sells and without the maximum weight constriction allowed on April of 2015 using different values of τ