Backreaction from non-zero mass conformally coupled scalar fields and the quantum creation of the Universe

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Abstract: The purpose of this paper is to study the homogeneous backreaction of non-zero mass conformally coupled scalar fields on a cosmological background, including flat spacetime, at one-loop order. For this study, the semiclassical Friedmann equation is solved using a method called order reduction. For a nearly flat background we get a scale factor that evolves to a stationary scale factor depending on the initial quantum state. We also prove that a de Sitter scale factor is also a mathematical solution of the semiclassical Friedmann equation, and we argue this is a runaway type solution.

I. INTRODUCTION

In this project we want to consider the coupling of non-zero mass matter fields interacting with the metric perturbations over a cosmological background spacetime with a conformal coupling to the curvature. We will study the homogeneous backreaction of these quantum fields on the spacetime background at one-loop order. One of the reasons that makes this calculation interesting, even when the background is flat, is that Brout et al suggested already in 1978 (ref. [1], [2]), i.e. before the inflationary model was introduced by Guth in 1981, that a quantum fluctuation in flat spacetime of a non-zero mass conformally coupled scalar field could lead to the creation of a universe with accelerated expansion.

We reanalyze such scenario of the universe creation. It is likely that such a creation, even if it could be energetically possible as discussed in ref. [3], should be discarded in the framework of an effective field theory (EFT) approach. We argue that the accelerated universe expansion that one finds is a runaway type of solution. For this analysis we need to solve the so called semiclassical Einstein equations where the source is the expectation value of the stress tensor of conformally coupled scalar fields of arbitrary mass. These semiclassical equations include terms which are quadratic in the curvature due to the ultraviolet renomalization of the stress tensor of the quantum matter and, thus, have higher derivatives than the Einstein equations. It is known that these backreaction equations have, consequently, more degrees of freedom than the classical equations, and thus may include some spurious solutions which are not physical. To extract the physical solutions is a subtle issue that we need to confront.

II. GENERAL MODEL

The model we will follow is based in references [3] and [4]. We will consider an isotropic and spatially homogeneous Friedmann, Lemaitre, Robertson, Walker (FLRW) metric, which can be written as:

$$g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu} \tag{1}$$

where $a(\eta)$ is the scale factor as a function of the conformal time, η, and $\eta_{\mu\nu} = diag(-1,1,1,1)$ is the 4-dimensional Minkowski metric. In such a spacetime we will study a free quantum scalar field of mass m, conformally coupled to the curvature scalar. The action of this field is:

$$S[a,\Phi] = -\frac{1}{2}\int d^4x \left[\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi + \left(m^2 + \frac{1}{6}R\right)\Phi^2\right], \tag{2}$$

where R is the Ricci curvature scalar associated with the metric. Now we will introduce the rescaled field $\phi(x) = a(\eta)\Phi(x)$, in terms of which we can rewrite the action in a simpler way:

$$S[a,\phi] = -\frac{1}{2}\int d^4x[\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + M^2(\eta)\phi^2], \tag{3}$$

where $M = a(\eta)m$. Setting the variation of this action with respect to ϕ equal to zero gives the usual Klein-Gordon equation in Minkowski spacetime with a time-dependent mass:

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - M^2(\eta))\phi = 0. \tag{4}$$

The solutions of this equation depend on the initial conditions at some initial time η_i. Moreover, the quantum state $|\psi\rangle$ is the vacuum associated with this initial time. The way we choose this state is the following: we define an auxiliary scale factor $a_\psi(\eta)$ for the domain $(-\infty, \eta_i]$ such that

$$\lim_{\eta\to-\infty} M_\psi^2 = 0 \tag{5}$$

$a_\psi(\eta)$ is completely arbitrary except for this condition and the condition of continuity at η_i, i.e. $a_\psi(\eta_i) = a(\eta_i)$. The states defined this way correspond to the in-vacuum state when the scale factor is $a_\psi(\eta)$ for $\eta < \eta_i$. The computations regarding the scalar field can be done then as if it were a FLRW spacetime with a scale factor which is $a(\eta)$ for $\eta > \eta_i$ and $a_\psi(\eta)$ for $\eta < \eta_i$ and the state were the in-vacuum evolving from $\eta = -\infty$. The evolution of this in-vacuum from $\eta \to -\infty$ to η_i defines the initial quantum state $|\psi\rangle$ at the initial time η_i.

To study the effect that this quantum field has in the metric we need to introduce the Feynman-Vernon influence action $S_{IF}[g^+, g^-]$ which is an action that can be computed integrating the degrees of freedom of the scalar field ϕ. In this integral there appears the action of the quantum field, $S[g, \phi]$, for two different configurations of the field ϕ^+ and ϕ^-. These configurations are supposed to coincide at some final time η_f and each one of them has associated a metric g^+ and g^- and, thus, a scale factor a^+ and a^-, respectively.

The dynamics of the metric is derived from the closed-time-path (CTP) effective action, which takes into account not only the Feynman-Vernon influence action, but also the classical gravitational action $S_g[g]$. A more extense explanation of the derivation and renormalization process of this effective action can be found in ref. [5].The CTP effective action is:

$$\Gamma_{CTP}[g^+, g^-] = S_g[g^+] - S_g[g^-] + S_{IF}[g^+, g^-], \quad (6)$$

where $S_g[g]$ is the classical gravitational action:

$$S_g[g] = S_g^{ren}[g] - S_{div}[g] = S_{EH}[g] + S_C[g] - S_{div}[g], \quad (7)$$

where $S_{div}[g]$ include the divergences that cancel the divergent behavior of the influence action $S_{IF}[g^+, g^-]$ and $S_g^{ren}[g]$ is the renormalized gravitational action, which is given by the sum of the Einstein-Hilbert action S_{EH} and the additional counterterms S_C. The first term is:

$$S_{EH}[g] = \frac{1}{16\pi l_p^2} \int d^4x \sqrt{-g}(R - 2\Lambda), \quad (8)$$

where Λ is the cosmological constant, that we will suppose equal to 0. And the second:

$$S_C[g] = \int d^4x \sqrt{-g} \left[\alpha \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - R_{\mu\nu} R^{\mu\nu}\right) + \beta R^2\right], \quad (9)$$

where α and β are dimensionless parameters. Then we can rewrite the total effective action as:

$$\Gamma_{CTP}[g^+, g^-] = S_{EH}[g^+] - S_{EH}[g^-] + S_{IF}^{ren}[g^+, g^-], \quad (10)$$

where $S_{IF}^{ren}[a^+, a^-]$ is the renormalized influence action, which includes all the counterterms:

$$\begin{aligned} S_{IF}^{ren}[g^+, g^-] = S_{IF}[g^+, g^-] &- S_{div}[g^+] + S_{div}[g^-] \\ &+ S_C[g^+] - S_C[g^-]. \end{aligned} \quad (11)$$

A. Semiclassical Einstein equations

Once we have the CTP effective action (10) we can derive the semiclassical Einstein equations (as in ref [3]) by first functionally differentiating with respect to g^+ and then taking $g^+ = g^- = g$. The Einstein-Hilbert action gives the Einstein tensor, $G_{\mu\nu}$, and the renormalized influence action gives the renormalized expectation value of the stress tensor operator:

$$G_{\mu\nu} = 8\pi l_p^2 \langle T_{\mu\nu}\rangle_\Psi. \quad (12)$$

Here l_p is the Planck length. As we assume spatial homogeneity and isotropy only the equations for two components are independent. These two equations can be chosen to be the 00 component and the stress tensor conservation law. The first one is the semiclassical analog of the Friedmann equation:

$$\dot{a}^2 = \frac{8\pi l_p^2}{3} a^2 \langle T_{00}\rangle_\Psi \quad (13)$$

In this equation and in the following the overdots mean derivatives with respect to the conformal time, $\dot{} \equiv \frac{d}{d\eta}$. The second equation we have is the stress-tensor conservation law

$$\nabla_\mu \langle T^{\mu\nu}\rangle = 0, \quad (14)$$

which will be needed later to compute $\langle T_{00}\rangle_\Psi$, which is necessary for studying this semiclassical Friedman equation (13)

B. Expectation value of the stress-energy tensor

The classical stress tensor of the matter field is defined as:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \quad (15)$$

As we are supposing an homogeneous metric of the form $g_{\mu\nu} = a^2(\eta)\eta_{\mu\nu}$ the only degree of freedom with respect to which we can differentiate the classical action is $a(\eta)$; consequently we are only able to obtain the trace of the stress tensor T^μ_μ, since $\delta g_{\mu\nu} = 2a\eta_{\mu\nu}\delta a = 2a^{-1} g_{\mu\nu}\delta a$ we can write:

$$\begin{aligned} \delta S = \int d^4x \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} &= \int d^4x \sqrt{-g} T^{\mu\nu} g_{\mu\nu} \frac{\delta a}{a} \\ &= \int d\eta \int d^3x a^3 T^\mu_\mu \delta a, \end{aligned} \quad (16)$$

then,

$$T^\mu_\mu = \frac{1}{Va^3} \frac{\delta S}{\delta a}, \quad (17)$$

where V is the spatial comoving volume. A similar procedure can be done with the renormalized influence action S_{IF}^{ren} to obtain the expectation value of the trace of the stress-tensor operator:

$$\langle T^\mu_\mu\rangle = \frac{1}{Va^3} \frac{\delta S_{IF}^{ren}[a^+, a^-]}{\delta a^+}\bigg|_{a^+ = a^- = a}. \quad (18)$$

Now we can compute $\langle T^\mu_\mu \rangle$ but for the semiclassical Friedman equation we need the expectation value of the energy density $\langle T_{00}\rangle$. We can calculate it from the expectation value of the trace making use of some useful relationships between them. We know the stress-tensor conservation law eq. (14), and that $\vec{\xi} = \partial/\partial\eta$ is a conformal Killing field, i.e., $2\,\nabla_{(a}\,\xi_{b)} = \lambda g_{\mu\nu}$ with $\lambda = 2\dot{a}/a$. These two equations lead to:

$$\nabla_\mu(\langle T^{\mu\nu}\rangle \xi_\nu) = \frac{\dot{a}}{a}\langle T^\mu_\mu\rangle \tag{19}$$

which can be integrated over the spacetime volume bounded by the spacelike hypersurfaces corresponding to $\eta' = -\infty$ and $\eta' = \eta$. Using Gauss theorem we get:

$$\begin{aligned}\langle T_{00}(\eta)\rangle a^2(\eta) - \langle T_{00}(-\infty)\rangle a^2(-\infty) \\ = -\int_{-\infty}^{\eta} d\eta' \dot{a}(\eta') a^3(\eta')\langle T^\mu_\mu(\eta')\rangle\end{aligned} \tag{20}$$

where we have divided by the spatial volume V, which appears due to the spatial homogeneity of the expected value of the stress-tensor. As $\langle T_{00}(-\infty)\rangle$ is the expectation value of the energy density of the Minkowski vacuum, the term proportional to it should vanish.

This calculation is done in ref [3] for the case that interests us, that is a weakly non-conformal field ($M^2/a^2 \ll L^{-2}$) conformally coupled and with non-zero mass. Where L^{-2} characterizes the curvature of the spacetime. This is needed in order to be able to treat the terms $M^2\phi^2$ in eq. (3) pertubatively. An example for a nearly flat background is when one considers a scale factor $a(\eta) = 1+\lambda(\eta)\sin\Omega\eta$ where $\Omega \gg m$ and $\lambda(\eta)$ is a suitable smooth function such that $|\lambda(\eta)| < 1$; another example is when the background is de Sitter. The result of this calculation is:

$$\begin{aligned}a^2\langle T_{00}\rangle = \beta\left[\dot{a}\frac{d}{d\eta}\left(\frac{\ddot{a}}{a^2}\right) - \frac{1}{2}\left(\frac{\ddot{a}}{a}^2\right)\right] + 3\alpha\left(\frac{\dot{a}}{a}\right)^4 \\ -90\alpha m^4[a^4\ln a + F[a^2]]\end{aligned} \tag{21}$$

where the renormalization parameter β is arbitrary and α is fixed and depends on the type and number of fields, in our case $\alpha = \frac{N}{2880\pi^2}$, where N is the number of scalar fields. The functional F is defined by:

$$F[a^2;\eta) = -2\int_{-\infty}^{\eta} d\eta' \frac{da^2}{d\eta'}\kappa[a^2,\eta') \tag{22}$$

where κ is another functional defined by:

$$\kappa[f;\eta) = -\lim_{\epsilon\to 0^+}\left[\int_{-\infty}^{\eta-\epsilon}\frac{d\eta'}{\eta-\eta'}f(\eta') + (\ln\epsilon + \ln\mu + \gamma)f(\eta)\right] \tag{23}$$

where μ is an arbitrary mass scale which plays the role of the renormalization scale, and γ is the Euler-Mascheroni constant.

With all these expressions now we are able to work with the semiclassical Friedmann equation (13).

III. ORDER REDUCTION

In this section the order reduction method is explained and used to solve equation (13).

A. The method

The semiclassical Friedmann equation (13) contains terms with up to third-order derivatives of the scale factor, as seen from eq. (21). Such kind of higher-order time derivatives are common in backreaction problems. In fact, they are a generic feature of effective fields theories (EFTs), where the effects of the UV sector on the dynamics of the the low-energy degrees of freedom are encoded at the level of the action through an expansion of local terms with an increasing number of derivatives. The validity of the EFT expansion relies on the fact that for length scales much larger than the inverse cut-off scale of the UV sector the higher-order terms in the expansion become increasingly smaller. In this regime their contribution amounts to a small correction to the equation of motion which results, when treated pertubatively, into locally small perturbations of the classical solutions. In contrast, solving the corresponding higher-order equations exactly gives rise to additional solutions exhibiting exponential instabilities with characteristic time scales comparable to the inverse cutoff scale of the EFT, often referred to as "runaway" solutions. These are spurious solutions which should not be taken seriously since they involve characteristic scales for which the EFT expansion breaks down and the contributions from the higher-order terms to the equation of motion no longer correspond to small corrections but to dominant terms.

One way to avoid such spurious solutions is the order reduction method, which is explained in references [6]-[8] (see also ref. [9]). It consists in taking the equation with corrections up to a finite order and write an alternative equation which is equivalent to up to that order but contains no higher derivative terms. This method can be illustrated with the following example for a first order differential equation in time for a function $f(\eta)$ with a perturbative correction of order κ^2. Given

$$\dot{f} + bf = \kappa^2 P(f, \dot{f}, \ddot{f}, ...), \tag{24}$$

where b is a constant and P is a function. Order reduction uses that $\dot{f} = -bf + O(\kappa^2)$ and by deriving one more time $\ddot{f} = -b\dot{f} + O(\kappa^2) = b^2 f + O(\kappa^2)$. Substituting this into the equation we get:

$$\dot{f} + bf = \kappa^2 P(f, -bf, b^2 f, ...) + O(\kappa^4), \tag{25}$$

which is a first order equation valid to the same order in κ^2 as the the original eq. (24) but does not have unphysical solutions.

B. Application of the method

In the semiclassical Friedmann equation (13) the perturbative correction is the right hand side term, which is of order l_p^2. Applying the method we have: $\dot{a} = O(l_p^2)$, $\ddot{a} = O(l_p^2)$,... Substituting this we get:

$$\dot{a}^2 = -240\pi l_p^2 m^4 \alpha [a^4 \ln a + F[a_\Psi^2]] + O(l_p^4) \tag{26}$$

where the term $F[a_\Psi^2]$ only depends on a_Ψ and μ. Therefore it is a constant independent of a and η. This equation can be solved numerically for a given a_Ψ, i.e., for given initial conditions. The solution is a scale factor that rapidly evolves to a stationary solution with a scale factor that satisfies:

$$a^4 \ln a = -F[a_\Psi^2] \tag{27}$$

Depending on the initial vacuum we choose, $F[a_\Psi^2]$ can be zero: $F[a_\Psi^2] = 0$, and $a = 1$ in this case. The result is that the scale factor becomes a constant and we find that the spacetime evolves to the Minkowski spacetime, which is thus asymptotically stable with respect to fluctuations of this scale factor produced by the quantum massive fields.

However, a word of caution should be made about this conclusion. As already explained before equation (21) the equation we have for $\langle T_{00} \rangle$ was derived under some pertubative assumptions that are not generally valid when the background is exactly Minkowski. This is strictly valid when the background has large frecuency oscillations. The correct $\langle T_{\mu\nu} \rangle$ that one needs to use for the Minkowski background was derived in ref. [10]. The problem, however does not arise for the use of $\langle T_{00} \rangle$ in the following section where the perturbative assumptions are perfectly realized.

IV. DE SITTER SOLUTION

As we have explained in the introduction Brout et. al. proposed that quantum fluctuations could be the cause of inflation with no need of a cosmological constant (ref. [1]). That means that a scale factor of de Sitter would be a solution of the semiclassical Friedmann equation. Now we will prove this by substituting a de Sitter scale factor in our equations. We take the Bunch-Davies vacuum for our initial state. That means that for $\eta < \eta_i$ we will choose $a_\psi = a_{dS}$,

The de Sitter scale factor is:

$$a_{dS}(\eta) = -\frac{1}{H\eta} \tag{28}$$

where we have introduced an arbitrary H, which plays the role of the Hubble constant that needs to be determined. With a scale factor of this form we can calculate the expectation value of the energy density using the expressions we have detailed before (21)-(23):

$$a^4 \ln a = \frac{1}{H^4 \eta^4} \ln\left(\frac{-1}{H\eta}\right), \tag{29}$$

$$\kappa[a_{dS}^2, \eta) = \frac{1}{H^2\eta^2}(1 - \gamma - \ln \mu\eta), \tag{30}$$

$$F[a_{dS}^2, \eta) = \frac{1}{H^4\eta^4}(\ln \mu\eta + \gamma - 3/4), \tag{31}$$

$$\langle T_{00} \rangle^{dS} = \frac{\alpha}{(-H\eta)^2}\left[3H^4 - 90m^4\left(\ln\frac{\mu}{H} + \gamma - \frac{3}{4}\right)\right]. \tag{32}$$

We can see that the logarithmic term $a^4 \ln a$ has compensated with the logarithm in (31) that comes from the non-local term $F[a^2]$ and the term with the parameter β is identically zero. Substituting this expectation value in (13) we get the following expression:

$$1 = 8\pi l_p^2 \alpha \left[H^2 - 30\frac{m^4}{H^2}\left(\ln\frac{\mu}{H} + \gamma - \frac{3}{4}\right)\right] \tag{33}$$

which is independent of η, that means that we have an equation for H as a function of m and μ and that if this equation has a real solution for H, the de Sitter scale factor is a mathematical solution of the semiclassical Friedmann equation.

We can define now $H_0 = (\sqrt{8\pi\alpha} l_p)^{-1}$ in order to introduce adimensional variables ($\tilde{H} = H/H_0$, $\tilde{m} = m/H_0$ and $\tilde{\mu} = \mu/H_0$). We can rewrite the equation in a dimensionless form:

$$1 = \tilde{H}^2 - 30\frac{\tilde{m}^4}{\tilde{H}^2}\left(\ln\frac{\tilde{\mu}}{\tilde{H}} + \gamma - \frac{3}{4}\right) \tag{34}$$

This equation can be solved numerically for a given pair of parameters $\tilde{m}$, $\tilde{\mu}$. We can also solve it approximately writing $\tilde{H}^2$ as a Taylor series of $\tilde{m}$:

$$\tilde{H}^2 = f_0 + \tilde{m}^4 f_1 + O(\tilde{m}^8) \tag{35}$$

Substituting this series in (34) and expanding $\frac{1}{\tilde{H}^2}$ and $\ln\frac{\tilde{\mu}}{\tilde{H}}$ we have an equation for each order of the series. For the first terms we have:

$$f_0 = 1 \tag{36}$$

$$f_1 = 30\left(\ln\tilde{\mu} + \gamma - \frac{3}{4}\right) \tag{37}$$

Substituiting this terms in our expansion and returning to our initial variables we can write:

$$H = H_0\left(1 + 15\left(\frac{m}{H_0}\right)^4\left(\ln\frac{\mu}{H_0} + \gamma - \frac{3}{4}\right) + O\left(\frac{m}{H_0}\right)^8\right) \tag{38}$$

This series converges for values of m smaller than H_0 and for values of μ of the same order. In this conditions we can see that the value of the Hubble constant is approximately H_0 and the mass term only affects as a small correction. More terms can be found using this technique, but each new term is much smaller than the former as there is a factor $(\frac{m}{H_0})^4$ between them.

Note that for $m^2 \ll H_0^2$ we are in the region of validity of perturbative calculation that we have performed where the term $M^2\phi^2$ in eq. (3) is treated perturbatively and thus the expression (21) fully applies.

V. DISCUSSION

We have solved the Friedmann equation for the homogeneous backreaction of a conformally coupled scalar field with non-zero mass on a cosmological background.

First of all, we want to comment the solution we have obtained by using the order reduction method. We find that the solution evolves in time to the Minkowski spacetime when the background is nearly flatspacetime. But this solution has to be treated with caution because it is not generally valid for all nearly flat backgrounds. The suitable $\langle T^{\mu\nu}\rangle$ in that case is given in ref. [10].

We have also proved that a de Sitter spacetime is a mathematical solution of the semiclassical Friedmann equation. However, this solution is a runaway type of solution because it corresponds to a rate of expansion $H_0 = (\sqrt{8\pi\alpha}l_p)^{-1}$ which is of the order of the Planck scale which is the cut off scale of our EFT. This solution is, in fact, closely related to Starobinsky anomaly driven inflation that was obtained in 1980 (ref. [11]), which corresponds to the particular case $m = 0$, i.e. massless conformally coupled fields. It was discussed in references [6] and [9] that this solution is spurious because its characteristic scale lies beyond the domain of validity of semiclassical gravity. We can therefore extend this argument for our $m \neq 0$ case since it still admits the de Sitter solution with the same scale factor except for a small correction.

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