Gen Relativ Gravit (2015) 47:112
DOI 10.1007/s10714-015-1947-8

RESEARCH ARTICLE

Perturbation theory and harmonic gauge propagation in general relativity, a particular example

Jesús Martín[1] · **Alfred Molina**[2] · **Eduardo Ruiz**[3]

Received: 22 May 2015 / Accepted: 6 August 2015

Abstract We study how the changes of coordinates between the class of harmonic coordinates affect the analitycal solutions of Einstein's equations and we apply it to an analytical approach for stationary and axisymmetric solutions of Einstein equation used by Cabezas et al. (Gen. Relativ. Gravit. 39:707–736, 2007) and Cuchí et al. (Gen. Relativ. Gravit. 45:1433–1456, 2013) to solve the problem of a self-gravitating rigidly rotating perfect fluid compact source.

Keywords Harmonic coordinates · Analitic solutions · Einstein equations · Stationary and axisymmetric metrics

1 Introduction

In several previous articles [1,2] to solve Einstein's equations we used a new method to obtain approximate stationary and axisymmetric solutions to describe the gravitational field (interior and exterior) of a compact stellar object in rigid rotation. This

✉ Alfred Molina
alfred.molina@ub.edu

Jesús Martín
chmm@usal.es

Eduardo Ruiz
eruiz@usal.es

1 Instituto Universitario de Física Fundamental y Matemáticas (IUFFyM), Universidad de Salamanca, Salamanca, Spain

2 Dept. Física Fonamental, Institut de Ciències del Cosmos, Universitat de Barcelona, Barcelona, Spain

3 Depto. de Física Fundamental, Facultad de Ciencias, Universidad de Salamanca, Salamanca, Spain

Published online: 05 September 2015

method is a combination of a postminkowskian approach and another one that could be qualified as post spherical. At every postminkowskian order in a dimensionless parameter, λ that we set as usual to be proportional to the mass of the object, a series expansion is performed in powers of a parameter, Ω, also dimensionless, which takes into account the deformation of the object relative to spherical symmetry. This parameter classically represents the ratio in the equatorial plane between the centrifugal force and the gravitational one, or in terms of the Maclaurin ellipsoids [3,4], some function of the eccentricity. In this sense this second subordinate series expansion is similar to the slow rotation approach used by Hartle [5], but with the advantage that we do not start from an spherical exact solution in λ, so that we can go easily to higher Ω terms and we do not need any numerical computation to obtain algebraic results in the slow rotation parameter.

In this article we are going to study only some aspects related to the postminkowskian approach without any care with respect to the post spherical approach.

To explain our current goal let's recall first that the approximate general solution of the Einstein equations to the postminkowskian linear order in harmonic coordinates contains four groups of constants. As regards the exterior solution two of these groups of constants are the static and dynamic multipole moments of Geroch–Hansen, Thorne et al. [6–8] (and therefore they are invariant), while the other two groups are removable by a coordinate change, so that they can be described by *gauge constants*. In the interior a similar situation occurs, there are two sets of non removable constants, but they do not have a specific meaning (as those from the exterior solution have), and two other groups of gauge constants. A priori it seems natural to choose a coordinate system in which all gauge constants become equal to zero. This is the choice of Thorne [9] and that is known as *canonical gauge*. However we are interested in making a match on the surface of the stellar object by using the Lichnerowicz's prescription [10], i.e. in this matching the metric is continuous and has continuous derivatives on the surface. On this basis the gauge constants (both external and internal) are needed and also are uniquely fixed when the matching is performed [2].

As a consequence of this we need to maintain the gauge constants at the first order and drag them to the second order (and so on) in an iterative process inferred from the postminkowskian approach. It is easy to see that this results in quite long and cumbersome calculations, even with the use of the algebraic software, which must be implemented with elaborate routines when we go over the second order. The following questions can be naturally raised:

- What is the aspect of the solution at the second order (for instance) that comes from the first order gauge constants?
- You may obtain this part of the second order solution just analyzing infinitesimal coordinate changes up to the second order?

The main objective of this paper is to answer these questions, showing that the global solution to the second order contains the first order gauge constants transferred to the second order solution that matches exactly what we expect from the analysis of infinitesimal coordinate changes. One might object that the result is trivial because of the covariance of the theory; however this argument presents some difficulties which are specified in the text, and in any case our contribution is a explicit verification of how gauge constants are transferred, which can simplify the calculations significantly.

The paper is organized as follows. In Sect. 2 the expressions for the harmonicity conditions and the Ricci tensor are written in an appropriate way to develop the post-minkowskian approximation. In Sect. 3, these expressions are used to write Einstein's equations to different orders in the postminkowskian expansion. In Sect. 4 the infinitesimal coordinate changes are studied up to the second order including the peculiarities of the *harmonic changes*, i.e., those that go from harmonic coordinates to harmonic coordinates, specially the results are written for our problem i.e. for a stationary and axisymmetric spacetime with a Papapetrou structure and asymptotically flat. In Sect. 5 the properties of the stationary and axisymmetric metrics are specified. In Sect. 6 we address the problem of the external vacuum solution, solving Einstein's equations to first and second order of the postminkowskian approximation in harmonic coordinates. Besides we prove the fundamental thesis of this work, i.e., the part of the second order solution transferred by the first order gauge constants can be obtained by a simple analysis of the infinitesimal changes. It is also argued that the result must be true to third order. In Sect. 7 the same process is repeated for the case of an interior perfect fluid solution in rigid rotation and with a linear equation of state. Finally in Sect. 8 the conclusions of the work are presented.

As usual the space–time metric is written as follows

$$ds^2 = g_{\alpha\beta}(x^\rho)dx^\alpha dx^\beta$$

The Greek and Latin indexes take the values

$$\alpha, \beta, \lambda, \ldots = 0, 1, 2, 3; \qquad i, j, k, \ldots = 1, 2, 3$$

We use the Einstein summation convention, the following definitions $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$, $\triangle = \delta^{ij}\partial_i\partial_j$, and the signature of the space–time is $(-, +, +, +)$.

2 Harmonicity conditions and the Ricci tensor

2.1 Harmonicity conditions

It is well known that the harmonic coordinates condition is

$$\Gamma^\alpha := g^{\lambda\mu}\Gamma^\alpha_{\lambda\mu} = -\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\alpha}\right) = 0 \tag{1}$$

Since

$$\Gamma^\rho_{\mu\rho} = \partial_\mu \log\sqrt{-g} \tag{2}$$

we have

$$\Gamma^\alpha = -\partial_\mu g^{\mu\alpha} - g^{\mu\alpha}\Gamma^\rho_{\mu\rho} \tag{3}$$

so that we can also define

$$\Gamma_\alpha := g_{\alpha\lambda}\Gamma^\lambda = -g_{\alpha\lambda}\partial_\mu g^{\mu\lambda} - \Gamma^\mu_{\alpha\mu} \tag{4}$$

If we define the metric deviation $h_{\alpha\beta}$ from the Minkowski metric as

$$g_{\alpha\beta} := \eta_{\alpha\beta} + h_{\alpha\beta} \tag{5}$$

and the inverse metric

$$g^{\alpha\beta} := \eta^{\alpha\beta} + k^{\alpha\beta} \quad \left(\Rightarrow\; k^{\alpha\rho}\eta_{\rho\beta} + \eta^{\alpha\rho}h_{\rho\beta} + k^{\alpha\rho}h_{\rho\beta} = 0\right) \tag{6}$$

the Eq. (4) can be written in terms of the metric deviation as follows

$$\Gamma_\alpha = \eta^{\lambda\mu}\Gamma_{\alpha,\lambda\mu} + k^{\lambda\mu}\Gamma_{\alpha,\lambda\mu} = l_\alpha + P_\alpha = 0 \tag{7}$$

where

$$\begin{aligned} l_\alpha &:= \partial^\mu h_{\mu\alpha} - \tfrac{1}{2}\partial_\alpha h \\ P_\alpha &:= k^{\lambda\mu}\Gamma_{\alpha,\lambda\mu} \end{aligned} \qquad \left(\partial^\mu := \eta^{\mu\rho}\partial_\rho, \quad h := \eta^{\lambda\mu}h_{\lambda\mu}\right) \tag{8}$$

by splitting the linear terms in the deviation from the nonlinear terms.

2.2 The Ricci tensor

Now we are going to write the Ricci tensor in terms of the left hand side of the harmonicity condition and the metric deviation. First of all from the definition of the Ricci tensor we have

$$R_{\alpha\beta} = \partial_\lambda\Gamma^\lambda_{\alpha\beta} - \partial_\beta\Gamma^\lambda_{\alpha\lambda} + \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\alpha\beta} - \Gamma^\lambda_{\rho\beta}\Gamma^\rho_{\alpha\lambda} \tag{9}$$

but using (3) the first term of the right hand side can be written

$$\begin{aligned} \partial_\lambda\Gamma^\lambda_{\alpha\beta} &= g^{\lambda\mu}\partial_\lambda\Gamma_{\mu,\alpha\beta} + \partial_\lambda g^{\lambda\mu}\Gamma_{\mu,\alpha\beta} \\ &= g^{\lambda\mu}\partial_\lambda\Gamma_{\mu,\alpha\beta} - \Gamma^\mu\Gamma_{\mu,\alpha\beta} - \Gamma^\lambda_{\rho\lambda}\Gamma^\rho_{\alpha\beta} \end{aligned} \tag{10}$$

and using this result in (9)

$$R_{\alpha\beta} = g^{\lambda\mu}\partial_\lambda\Gamma_{\mu,\alpha\beta} - \partial_\beta\Gamma^\lambda_{\alpha\lambda} - \Gamma_\mu\Gamma^\mu_{\alpha\beta} - \Gamma^\lambda_{\rho\beta}\Gamma^\rho_{\alpha\lambda}\,. \tag{11}$$

In this expression only the first and the second terms on the right hand side contain linear terms in the deviation. We are going to separate them from the non linear ones; since

$$g^{\lambda\mu}\partial_\lambda\Gamma_{\mu,\alpha\beta} - \partial_\beta\Gamma^\lambda_{\alpha\lambda} = \frac{1}{2}\eta^{\lambda\mu}\left(\partial_{\lambda\alpha}h_{\mu\beta} + \partial_{\lambda\beta}h_{\mu\alpha} - \partial_{\lambda\mu}h_{\alpha\beta}\right) + k^{\lambda\mu}\partial_\lambda\Gamma_{\mu,\alpha\beta} - \frac{1}{2}\eta^{\lambda\mu}\partial_{\beta\alpha}h_{\lambda\mu} - \partial_\beta(k^{\lambda\mu}\Gamma_{\mu,\alpha\lambda}) \tag{12}$$

then the Ricci tensor can be expressed as

$$R_{\alpha\beta} = L_{\alpha\beta} - \Gamma_\mu\Gamma^\mu_{\alpha\beta} + N_{\alpha\beta} \tag{13}$$

where we have defined

$$L_{\alpha\beta} := -\frac{1}{2}\Box h_{\alpha\beta} + \frac{1}{2}\partial_\alpha l_\beta + \frac{1}{2}\partial_\beta l_\alpha \tag{14}$$

$$N_{\alpha\beta} := k^{\lambda\mu}\partial_\lambda\Gamma_{\mu,\alpha\beta} - \frac{1}{2}\partial_\beta(k^{\lambda\mu}\partial_\alpha h_{\lambda\mu}) - \Gamma^\lambda_{\rho\beta}\Gamma^\rho_{\alpha\lambda} \tag{15}$$

and where all the linear terms in the deviation are in $L_{\alpha\beta}$. If we use the harmonicity condition $\Gamma_\alpha = 0$ we can substitute $l_\alpha = -P_\alpha$ and we finally get

$$R_{\alpha\beta} = -\frac{1}{2}\Box h_{\alpha\beta} - \frac{1}{2}\partial_\alpha P_\beta - \frac{1}{2}\partial_\beta P_\alpha + N_{\alpha\beta} \tag{16}$$

where only the first term on the right hand side contains linear terms in the deviation.

3 Einstein equations and the series expansion

The Einstein equations read

$$R_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta} := \mathcal{T}_{\alpha\beta} \quad (8\pi G = c = 1) \tag{17}$$

Let us assume that the deviation $h_{\alpha\beta}$ of the metric and $\mathcal{T}_{\alpha\beta}$ can be expanded in a series of a parameter λ, i.e.

$$\begin{cases} h_{\alpha\beta} = \lambda h^{(1)}_{\alpha\beta} + \lambda^2 h^{(2)}_{\alpha\beta} + \lambda^3 h^{(3)}_{\alpha\beta} + \cdots = \sum_{n=1}\lambda^n h^{(n)}_{\alpha\beta} \\ \mathcal{T}_{\alpha\beta} = \lambda\mathcal{T}^{(1)}_{\alpha\beta} + \lambda^2\mathcal{T}^{(2)}_{\alpha\beta} + \lambda^3\mathcal{T}^{(3)}_{\alpha\beta} + \cdots = \sum_{n=1}\lambda^n\mathcal{T}^{(n)}_{\alpha\beta} \end{cases} \tag{18}$$

Then the linear term, as in the harmonic coordinates condition (7), can be written as

$$l_\alpha = \sum_{n=1}\lambda^n\left[\partial^\beta h^{(n)}_{\alpha\beta} - \frac{1}{2}\partial_\alpha h^{(n)}\right] := \sum_{n=1}\lambda^n l^{(n)}_\alpha, \quad h^{(n)} := \eta^{\mu\nu}h^{(n)}_{\mu\nu} \tag{19}$$

For the nonlinear term we have

$$P_\alpha = \lambda^2 P^{(2)}_\alpha + \lambda^3 P^{(3)}_\alpha + \cdots = \sum_{n=2}\lambda^n P^{(n)}_\alpha := \sum_{n=2}\lambda^n \sum_{r+s=n} k^{(r)\mu\nu}\,\Gamma^{(s)}_{\alpha,\mu\nu}$$

$(r, s = 1, 2, \ldots)$ where in $\Gamma^{(s)}_{\alpha,\lambda\mu}$ only the terms $\partial_\nu h^{(s)}_{\beta\rho}$ are needed. The terms $k^{(r)\mu\nu}$ of the inverse metric are obtained as follows; from the definition

$$k^{\alpha\beta} = \sum_{n=1} \lambda^n k^{(n)\alpha\beta} \quad \text{and the relation} \quad g^{\alpha\mu} g_{\mu\beta} = \delta^\alpha_\beta$$

we obtain

$$\sum_{n=1} \lambda^n \left[k^{(n)\alpha\rho} \eta_{\rho\beta} + \eta^{\alpha\rho} h^{(n)}_{\rho\beta} + \sum_{r+s=n} k^{(r)\alpha\rho} h^{(s)}_{\rho\beta} \right] = 0$$

If we know $h^{(1)}_{\mu\nu}$ this gives us $k^{(1)\mu\nu}$ and if we know

$$\left\{ h^{(1)}_{\mu\nu}, \ldots, h^{(n)}_{\mu\nu} \right\} \quad \text{and} \quad \left\{ k^{(1)\mu\nu}, \ldots, k^{(n-1)\mu\nu} \right\}$$

the term with λ^n gives us the term $k^{(n)\mu\nu}$. Then, to obtain the term $P^{(n)}_\alpha$ we need the deviation of the metric to order $n - 1$.

The same happens with the Ricci tensor (16). The remaining terms to analyze are in $N_{\alpha\beta}$, in which the two first terms have the same structure we have already analyzed, and the third one contains products of the type

$$(\eta^{\lambda\mu} + k^{\lambda\mu}) \Gamma_{\mu,\alpha\rho} (\eta^{\rho\nu} + k^{\rho\nu}) \Gamma_{\nu,\beta\lambda}$$

The terms of order n are obtained with the deviation of the metric to order less than order n, and like P_α in $N_{\alpha\beta}$ the lower term is of order λ^2, i.e.

$$N_{\alpha\beta} = \lambda^2 N^{(2)}_{\alpha\beta} + \lambda^3 N^{(3)}_{\alpha\beta} + \cdots = \sum_{n=2} \lambda^i N^{(n)}_{\alpha\beta} \tag{20}$$

Then to first order the Einstein's equation and the harmonicity conditions are

$$\begin{cases} \Box h^{(1)}_{\alpha\beta} = -2 \mathcal{T}^{(1)}_{\alpha\beta} \\ \partial^\rho h^{(1)}_{\rho\alpha} - \dfrac{1}{2} \partial^\alpha h^{(1)} = 0 \end{cases} \tag{21}$$

and to order $\boldsymbol{n \geq 2}$

$$\begin{cases} \Box h^{(n)}_{\alpha\beta} = -2 \mathcal{T}^{(n)}_{\alpha\beta} + 2 \left[N^{(n)}_{\alpha\beta} - \partial_{(\alpha} P^{(n)}_{\beta)} \right] \\ \partial^\rho h^{(n)}_{\rho\alpha} - \dfrac{1}{2} \partial^\alpha h^{(n)} = -P^{(n)}_\alpha \end{cases} \tag{22}$$

where for instance for $n = 2\, \mathcal{T}^{(2)}_{\alpha\beta}$, $N^{(2)}_{\alpha\beta}$ and $P^{(2)}_\alpha$ are built with the linear deviation $h^{(1)}_{\alpha\beta}$ i.e.

$$
\begin{aligned}
P_{\alpha}^{(2)} &= k^{(1)\lambda\mu}\Gamma^{(1)}_{\alpha,\lambda\mu} \\
N_{\alpha\beta}^{(2)} &= k^{(1)\lambda\mu}\partial_{\lambda}\Gamma^{(1)}_{\mu,\alpha\beta} - \frac{1}{2}k^{(1)\lambda\mu}\partial_{\alpha\beta}h(1)_{\lambda\mu} - \frac{1}{2}\partial_{\alpha}h^{(1)}_{\lambda\mu}\partial_{\beta}k^{(1)\lambda\mu} - \Gamma^{(1)\rho}_{\beta\lambda}\Gamma^{(1)\lambda}_{\alpha\rho}
\end{aligned} \tag{23}
$$

and $\mathcal{T}^{(2)}_{\alpha\beta}$ is built from $h^{(1)}_{\lambda\mu}$. The same happens with the higher order equations, the second member of the equations are built with lower orders that are already known.

4 Infinitesimal coordinates change to second order

Now we are going to study how the infinitesimal changes of coordinates affect the deviation of the metric of the previous section and how this gauge propagates to higher order. Let us assume that we take the deviation to second order in the parameter, i.e.

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + \lambda h_{\alpha\beta}(x) + \lambda^2 q_{\alpha\beta}(x) \tag{24}$$

and the inverse metric

$$g^{\alpha\beta}(x) = \eta^{\alpha\beta} + \lambda k^{\alpha\beta}(x) + \lambda^2 p^{\alpha\beta}(x) \tag{25}$$

Let us point out that we have changed the notation, the terms $h_{\alpha\beta}$ and $k^{\alpha\beta}$ represent from now on the first order in the parameter λ for the deviation and not all the deviation as in equations (5) and (6). i.e.

$$\begin{cases} h^{(1)}_{\alpha\beta} \to h_{\alpha\beta}, & h^{(2)}_{\alpha\beta} \to q_{\alpha\beta} \\ k^{(1)\alpha\beta} \to k^{\alpha\beta}, & k^{(2)\alpha\beta} \to p^{\alpha\beta} \end{cases} \tag{26}$$

Then from (6)

$$k^{\alpha\rho}\eta_{\rho\beta} + \eta^{\alpha\rho}h_{\rho\beta} = 0; \quad p^{\alpha\rho}\eta_{\rho\beta} + \eta^{\alpha\rho}q_{\rho\beta} + k^{\alpha\rho}h_{\rho\beta} = 0 \tag{27}$$

If we perform now the infinitesimal coordinates change to second order

$$\tilde{x}^{\alpha} = x^{\alpha} + \lambda\,\xi^{\alpha}(x) + \lambda^2\zeta^{\alpha}(x) \tag{28}$$

the metric in the new coordinates can be written as

$$\tilde{g}_{\alpha\beta}(\tilde{x}) = \eta_{\alpha\beta} + \lambda\tilde{h}_{\alpha\beta}(\tilde{x}) + \lambda^2\tilde{q}_{\alpha\beta}(\tilde{x}) \tag{29}$$

$$= \eta_{\alpha\beta} + \lambda\tilde{h}_{\alpha\beta}(x) + \lambda^2\Big[\xi^{\rho}(x)\partial_{\rho}\tilde{h}_{\alpha\beta}(x) + \tilde{q}_{\alpha\beta}(x)\Big] \tag{30}$$

but also

$$g_{\alpha\beta}(x) = \frac{\partial\tilde{x}^{\lambda}}{\partial x^{\alpha}}(x)\frac{\partial\tilde{x}^{\mu}}{\partial x^{\beta}}(x)\tilde{g}_{\lambda\mu}\big[\tilde{x}(x)\big] \tag{31}$$

and then

$$
\begin{aligned}
\eta_{\alpha\beta} + \lambda h_{\alpha\beta}(x) + \lambda^2 q_{\alpha\beta}(x) = {} & (\delta_\alpha^\lambda + \lambda \partial_\alpha \xi^\lambda + \lambda^2 \partial_\alpha \zeta^\lambda)(\delta_\beta^\mu + \lambda \partial_\beta \xi^\mu + \lambda^2 \partial_\beta \zeta^\mu) \\
& \times \Big\{ \eta_{\lambda\mu} + \lambda \tilde{h}_{\lambda\mu}(x) + \lambda^2 \Big[\xi^\rho(x) \partial_\rho \tilde{h}_{\lambda\mu}(x) \\
& + \tilde{q}_{\lambda\mu}(x) \Big] \Big\}
\end{aligned} \tag{32}
$$

Now we can identify the two first orders in λ in Eqs. (30) and (32)

$$
\tilde{h}_{\alpha\beta}(x) = h_{\alpha\beta}(x) - \partial_\alpha \xi_\beta(x) - \partial_\beta \xi_\alpha(x) \tag{33}
$$

$$
\begin{aligned}
\tilde{q}_{\alpha\beta}(x) = {} & q_{\alpha\beta}(x) - \partial_\alpha \zeta_\beta(x) - \partial_\beta \zeta_\alpha(x) - \partial_\alpha \xi^\mu(x) \partial_\beta \xi_\mu(x) \\
& - \xi^\rho(x) \partial_\rho \Big[h_{\alpha\beta}(x) - \partial_\alpha \xi_\beta(x) - \partial_\beta \xi_\alpha(x) \Big] \\
& - \Big[h_{\alpha\mu}(x) - \partial_\alpha \xi_\mu(x) - \partial_\mu \xi_\alpha(x) \Big] \partial_\beta \xi^\mu(x) \\
& - \Big[h_{\mu\beta}(x) - \partial_\beta \xi_\mu(x) - \partial_\mu \xi_\beta(x) \Big] \partial_\alpha \xi^\mu(x)
\end{aligned} \tag{34}
$$

which can be written as

$$
\tilde{q}_{\alpha\beta}(x) = q_{\alpha\beta}(x) - \partial_\alpha \zeta_\beta(x) - \partial_\beta \zeta_\alpha(x) - \pounds(\boldsymbol{\xi}) \tilde{h}_{\alpha\beta}(x) - \partial_\alpha \xi^\mu(x) \partial_\beta \xi_\mu(x) \tag{35}
$$

where the two last terms are the first order gauge propagation to second order.

For later use it is important to specify the equations that the vectors $\xi^\alpha(x)$ and $\zeta^\alpha(x)$ should verify to ensure that the change (28) transforms harmonic coordinates to harmonic coordinates ("harmonic changes"). If x^α are harmonic coordinates for the metric $g_{\lambda\mu}(x^\alpha)$ then $\Gamma^\alpha = 0$ (1), and the necessary and suficient condition for $\tilde{x}^\alpha(x)$ to be harmonic is

$$
g^{\lambda\mu} \partial_{\lambda\mu} \tilde{x}^\alpha(x) - \Gamma^\beta \partial_\beta \tilde{x}^\alpha = 0
$$

but since $\Gamma^\beta = 0$, then

$$
g^{\lambda\mu} \partial_{\lambda\mu} \tilde{x}^\alpha(x) = 0 \tag{36}
$$

which results in the following equations

$$
\begin{aligned}
& \eta^{\lambda\mu} \partial_{\lambda\mu} \xi^\alpha(x) = 0 \\
& \eta^{\lambda\mu} \partial_{\lambda\mu} \zeta^\alpha(x) = h^{\lambda\mu}(x) \partial_{\lambda\mu} \xi^\alpha(x)
\end{aligned} \tag{37}
$$

having taken into account (25), (26), (27) and (28), and where $h^{\lambda\mu} = \eta^{\lambda\alpha} \eta^{\mu\beta} h_{\alpha\beta}$.

5 Stationary and axisymmetric metrics

We require our space-time to be a stationary and axisymmetric semi–Riemannian asymptotically flat manifold admitting a global system of spherical-like coordinates $\{t, r, \theta, \varphi\}$ which verifies the following properties:

A. Coordinates are adapted to the space-time symmetry, that is to say, $\boldsymbol{\xi} = \partial_t$ and $\boldsymbol{\zeta} = \partial_\varphi$ are respectively the timelike and spacelike Killing vectors, so that the metric components do not depend on the coordinate t nor φ.
B. Coordinates $\{r, \theta\}$ parametrize two dimensional surfaces orthogonal to the orbits of the symmetry group, that is, the metric tensor has Papapetrou structure,

$$\begin{aligned} \boldsymbol{g} = \gamma_{tt}\boldsymbol{\omega}^t\otimes\boldsymbol{\omega}^t + \gamma_{t\varphi}(\boldsymbol{\omega}^t\otimes\boldsymbol{\omega}^\varphi + \boldsymbol{\omega}^\varphi\otimes\boldsymbol{\omega}^t) + \gamma_{\varphi\varphi}\boldsymbol{\omega}^\varphi\otimes\boldsymbol{\omega}^\varphi \\ + \gamma_{rr}\boldsymbol{\omega}^r\otimes\boldsymbol{\omega}^r + \gamma_{r\theta}(\boldsymbol{\omega}^r\otimes\boldsymbol{\omega}^\theta + \boldsymbol{\omega}^\theta\otimes\boldsymbol{\omega}^r) + \gamma_{\theta\theta}\boldsymbol{\omega}^\theta\otimes\boldsymbol{\omega}^\theta, \end{aligned} \tag{38}$$

where $\boldsymbol{\omega}^t = dt,\ \boldsymbol{\omega}^r = dr,\ \boldsymbol{\omega}^\theta = r\,d\theta,\ \boldsymbol{\omega}^\varphi = r\sin\theta\,d\varphi$ is the Euclidean orthonormal co basis associated to these coordinates.
C. Coordinates $\{t, x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = \cos\theta\}$ associated with the spherical-like coordinates are Cartesian coordinates at spacelike infinity, that is the metric in these coordinates tends to the Minkowsky metric in standard Cartesian coordinates for large values of the coordinate r.

All these properties are compatible with the harmonic coordinates we use in this paper. Time coordinate t is always harmonic under these assumptions.

The coordinate change to another system of adapted coordinates $\{t', \varphi'\}$ that preserve the regularity at the axis of symmetry, with closed compact orbits of periodicity 2π for the axial Killing vector, are $t = a\,t'$ and $\varphi = \varphi' + b\,t'$, where a and b are constants. If at spacelike infinity the metric tends to the Minkowski metric then $a = 1$ and this implies for the infinitesimal change of Sect. 4 that $\xi^0(x) = 0, \zeta^0(x) = 0$. Taking also into account the independence of the metric on time coordinate, we have $\xi^i(x^j)$ and $\zeta^i(x^k)$, i.e. do not depend on the time coordinate. Furthermore the change $\varphi = \varphi' + bt'$ does not maintain the harmonicity condition, i.e. if the Cartesian coordinates x, y, z associated to the spherical r, θ, φ are harmonic the Cartesian ones associated to r, θ, φ' are not.

Consequently, splitting time and space components in the Eqs. (33) and (35) we have

First order

$$\tilde{h}_{00}(x^k) = h_{00}(x^k) \tag{39}$$

$$\tilde{h}_{0j}(x^k) = h_{0j}(x^k) \tag{40}$$

$$\tilde{h}_{ij}(x^k) = h_{ij}(x^k) - \partial_i\xi_j(x^k) - \partial_j\xi_i(x^k) \tag{41}$$

the metric components h_{00} and h_{0j} of the metric deviation are invariants to first order.

Second order

$$\tilde{q}_{00}(x^k) = q_{00}(x^k) - \xi^l(x^k)\partial_l h_{00}(x^k) \tag{42}$$

$$\tilde{q}_{0j}(x^k) = q_{0j}(x^k) - \xi^l(x^k)\partial_l h_{0j}(x^k) - h_{0l}\partial_j \xi^l(x^k) \tag{43}$$

$$\tilde{q}_{ij}(x^k) = q_{ij}(x^k) - \partial_i \zeta_j(x^k) - \partial_j \zeta_i(x^k) - \pounds(\boldsymbol{\xi})\tilde{h}_{ij}(x^k) - \partial_i \xi^l(x^k)\partial_j \xi_l(x^k) \tag{44}$$

With respect to the "harmonic changes" (37), the stationarity and axial symmetry with a Papapetrou structure leads to the following equations

$$\Delta \xi^i(x) = 0 \tag{45}$$

$$\Delta \zeta^i(x) = h^{jk}(x)\partial_{jk}\xi^i(x) \tag{46}$$

i.e. the functions ξ^i must be three independent harmonic functions. On the other hand, the functions ζ^i must be the sum of a harmonic function and a particular solution of equation (46).

Finally, the harmonic condition and Einstein's equations of Sect. 3 with the splitting of time and space components become:

First order

$$\Delta h_{00}^{(1)} = -2\mathcal{T}_{00}^{(1)} \tag{47}$$

$$\Delta h_{0j}^{(1)} = -2\mathcal{T}_{0j}^{(1)}; \quad \partial^j h_{0j}^{(1)} = 0 \tag{48}$$

$$\Delta h_{ij}^{(1)} = -2\mathcal{T}_{ij}^{(1)}; \quad \partial^j h_{ij}^{(1)} - \frac{1}{2}\partial_j \hat{h}^{(1)} = -\frac{1}{2}\partial_j h_{00}^{(1)} \tag{49}$$

where $\hat{h} := \delta^{ij} h_{ij}$.

Order $n \geq 2$

$$\Delta h_{00}^{(n)} = -2\mathcal{T}_{00}^{(n)} + 2N_{00}^{(n)} \tag{50}$$

$$\Delta h_{0j}^{(n)} = -2\mathcal{T}_{0j}^{(n)} + 2N_{0j}^{(n)} - \partial_j P_0^{(n)}; \quad \partial^j h_{0j}^{(n)} = -P_0^{(n)} \tag{51}$$

$$\Delta h_{ij}^{(n)} = -2\mathcal{T}_{ij}^{(n)} + 2\left[N_{ij}^{(n)} - \partial_{(i} P_{j)}^{(n)}\right]; \quad \partial^j h_{ij}^{(n)} - \frac{1}{2}\partial_i \hat{h}^{(n)} = -P_i^{(n)} - \frac{1}{2}\partial_i h_{00}^{(n)} \tag{52}$$

6 Vacuum exterior solution in harmonic coordinates

Let's assume that we have a compact stationary gravitational source with axial symmetry and we would like to study the exterior 2–postminkowskian solution [1]. First of all we need to solve Eqs. (47), (48) and (49) with $\mathcal{T}_{\mu\nu}^{(1)} = 0$, and then to solve the Eqs. (53), (54) and (55) for $n = 2$ and $\mathcal{T}_{\mu\nu}^{(2)} = 0$. Henceforth we will use the notations of (26).

6.1 First order in λ

• For the **scalar term h_{00}** the solution of (47) is the solution of the Laplace equation in spherical coordinates, introduced in Sect. 5. The solution is [1,6]

$$h_{00}(r,\theta) = 2\sum_{n=0}^{\infty} \frac{M_n}{r^{n+1}} P_n(\cos\theta) \tag{53}$$

where M_n are arbitrary constants, representing the static multipole moments of Thorne [6,9], and $P_n(x)$ is the Legendre polynomial of order n.

- For the **vectorial h_{0j} terms** the solution of (48) is

$$h_{0j}(r,\theta) = 2\sum_{n=1}^{\infty} \frac{J_n}{r^{n+1}} P_n^1(\cos\theta) e_{\varphi j} \tag{54}$$

  where J_n are arbitrary constants, representing the dynamic multipole moments of Thorne, $e_\varphi := (-\sin\varphi, \cos\varphi, 0)$ is the tangent vector associated to the φ coordinate and $P_n^1(x)$ is the Legendre function of the first kind.
- For the **tensorial h_{ij} terms** the solution of (49) is

$$h_{ij} = h_{00}\delta_{ij} + \partial_i w_j + \partial_j w_i \tag{55}$$

  where

$$w_j := \sum_{n=0}^{\infty} \frac{Q_n}{r^{n+1}} P_n(\cos\theta) e_{zj} + \sum_{n=1}^{\infty} \frac{R_n}{r^{n+1}} P_n^1(\cos\theta) e_{\rho j} \tag{56}$$

  and Q_n and R_n are arbitrary constants, $e_\rho := (\cos\varphi, \sin\varphi, 0)$ is the tangent vector associated to the $\rho = r\sin\theta$ cylindrical coordinate and $e_z := (0, 0, 1)$ is the tangent vector associated to the z cylindrical coordinate. The constants Q_n and R_n are related to the A_n and B_n constants used in [1] by the following expressions

$$A_{n+1} = Q_n + nR_n,\; B_{n+1} = -2R_n$$

It is clear that the solution of the homogeneous part of the tensor Eq. (55) is *pure gauge* [see (41)]. That is, it could be interpreted as a first order infinitesimal change of coordinates with $\xi^i(x) := -w^i(x)$; furthermore it would be a "harmonic change" because it is easily verified that $\Delta w^i = 0$. On the other hand, the complete equation, that contains the derivative of the scalar equation solution, has the simple particular solution $h_{00}\delta_{ij}$.

Now, assuming that the first order solution contains this gauge dependence in the tensor term we would like to know how these gauge terms, w_i, propagate to the second order term, of the deviation, $q_{\alpha\beta}$. To simplify the resulting equations we are going to write down only the terms containing the first order gauge.

In particular, we would like to verify that the resulting terms coincide with those that come from an infinitesimal change of coordinates up to the second order.

6.2 Second order in λ

The second order equations to solve are (50), (51) and (52) with $\mathcal{T}^{(n)} = 0$ and $n = 2$, then we need to use equation (23) where only the deviation to first order in λ appears.

It has been already assumed that the gauge dependence in the first order deviation is given by Eqs. (53), (54) and (55), i.e. only the terms h_{ij} depend on the first order gauge. From (27) we have $k^{\mu\nu} = -\eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$ thus in $k^{\mu\nu}$ only the terms k^{ij} depend on the first order gauge. Then the Christoffel symbols to first order are

$$\Gamma^{(1)}_{0,00} = 0, \quad \Gamma^{(1)}_{0,0j} = \frac{1}{2}\partial_j h_{00}, \quad \Gamma^{(1)}_{0,ij} = \frac{1}{2}\left(\partial_i h_{0j} + \partial_j h_{0i}\right) \tag{57}$$

$$\Gamma^{(1)}_{k,00} = -\frac{1}{2}\partial_k h_{00}, \quad \Gamma^{(1)}_{k,0j} = \frac{1}{2}\left(\partial_j h_{0k} - \partial_k h_{0j}\right) \tag{58}$$

$$\Gamma^{(1)}_{k,ij} = \frac{1}{2}\left(\delta_{kj}\partial_i h_{00} + \delta_{ki}\partial_j h_{00} - \delta_{ij}\partial_k h_{00}\right) + \partial_{ij} w_k \tag{59}$$

Only the terms $\Gamma_{i,jk}$, h_{ij} and k^{ij} depend on the gauge. Now we can write the gauge dependent terms splitting the space and time components of $P^{(2)}$ and $N^{(2)}$ in (23) using the sign $\simeq$ to denote that we omit the terms with no gauge dependence.

$$P^{(2)}_0 = k^{\lambda\mu}\Gamma^{(1)}_{0,\lambda\mu} \simeq -\frac{1}{2}\left(\partial^k w^l + \partial^l w^k\right)\left(\partial_k h_{0l} + \partial_l h_{0k}\right) \tag{60}$$

$$P^{(2)}_k = k^{\lambda\mu}\Gamma^{(1)}_{k,\lambda\mu} \simeq -\left(\partial_i w_k + \partial_k w_i\right)\partial^i h_{00} + \partial_i w^i \partial_k h_{00} - 2\partial^i w^j \partial_{ij} w_k \tag{61}$$

where we have taken into account that $\Delta w^i = 0$, and

$$N^{(2)}_{00} = k^{\lambda\mu}\partial_\lambda \Gamma^{(1)}_{\mu,00} - \Gamma^{(1)\lambda}_{0\mu}\Gamma^{(1)\mu}_{0\lambda} \simeq \partial^i w^j \partial_{ij} h_{00} \tag{62}$$

$$\begin{aligned} N^{(2)}_{0j} &= k^{\lambda\mu}\partial_\lambda \Gamma^{(1)}_{\mu,0j} - \Gamma^{(1)\lambda}_{0\mu}\Gamma^{(1)\mu}_{j\lambda} \\ &\simeq -\frac{1}{2}\left(\partial^k w^l + \partial^l w^k\right)\left(\partial_{kj} h_{0l} - \partial_{kl} h_{0j}\right) - \frac{1}{2}\left(\partial^l h^k_0 - \partial^k h^l_0\right)\partial_{jk} w_l \end{aligned} \tag{63}$$

$$\begin{aligned} N^{(2)}_{ij} &= k^{\lambda\mu}\partial_\lambda \Gamma^{(1)}_{\mu,ij} - \Gamma^{(1)\lambda}_{i\mu}\Gamma^{(1)\mu}_{j\lambda} - \frac{1}{2}\left(k^{\lambda\mu}\partial_{ij} h_{\lambda\mu} + \partial_i h_{\lambda\mu}\partial_j k^{\lambda\mu}\right) \\ &\simeq \delta_{ij}\partial_{kl} h_{00}\partial^k w^l + \partial_{ij} h_{00}\partial_k w^k - \partial_{(ik} h_{00}\left(\partial_{j)} w^k + \partial^k w_{j)}\right) \\ &\quad + \partial^k h_{00}\left(\partial_{k(i} w_{j)} - \partial_{ij} w_k\right) + \partial_{(i} h_{00}\partial_{j)k} w^k + \partial_{ik} w_l \partial^k_j w^l \end{aligned} \tag{64}$$

where as usual $(i\ j)$ means simmetryzation on the indexes i, j.

Once we have determined the gauge dependent terms on the right hand side of the equations we are going to find the second order solution. As we did before, let us begin solving the scalar equation

$$\Delta q_{00} = 2N^{(2)}_{00} \simeq 2\partial^i w^j \partial_{ij} h_{00} \tag{65}$$

The solution would be equal to the homogeneous solution, formally identical to the first order plus a particular solution of the inhomogeneous equation. The complete right hand side of the above equation has also terms which do not depend on w^k but they had been omitted. It turns out that the particular solution can be written as the sum of a canonical particular solution plus a pure gauge particular solution. Therefore, for our purposes, we need to find out a particular solution of the differential equation:

$$\Delta q_{00} = 2\partial^i w^j \partial_{ij} h_{00} \tag{66}$$

Taking into account that h_{00} and w^j solve the flat Laplace equation, then

$$\Delta\left(w^i \partial_i h_{00}\right) = 2\partial^j w^i \partial_{ji} h_{00} \tag{67}$$

Thus, the particular solution $w^i \partial_i h_{00}$ has exactly the structure required by the propagation of the first order gauge to second order (42), remember that $w_j := -\xi_j$.

Let us now solve the vector equations

$$\begin{aligned}\Delta q_{0j} &= 2N^{(2)}_{0j} - \partial_j P^{(2)}_0 \simeq 2\partial^k w^l \partial_{kl} h_{0j} + 2\partial_{jk} w^l \partial^k h_{0l}\\ \partial^j q_{0j} &= -P^{(2)}_0 \simeq \partial^k w^l \left(\partial_k h_{0l} + \partial_l h_{0k}\right)\end{aligned} \tag{68}$$

As in the scalar equation, the homogeneous solution is formally the same as the first order one, and taking into account that $\Delta w_k = 0$ and $\partial^j h_{0j} = 0$ we obtain that a pure gauge particular solution is

$$w^k \partial_k h_{0j} + h_{0k} \partial_j w^k \tag{69}$$

i.e., the solution required by the propagation of the first order gauge to second order as given in Eq. (43).

The tensor equations are

$$\Delta q_{ij} = 2N^{(2)}_{ij} - \partial_i P^{(2)}_j - \partial_j P^{(2)}_i, \quad \partial^j q_{ij} - \frac{1}{2}\partial_i \hat{q} = -\frac{1}{2}\partial_i q_{00} - P^{(2)}_i \tag{70}$$

where

$$\begin{aligned}2N^{(2)}_{ij} - \partial_i P^{(2)}_j - \partial_j P^{(2)}_i &\simeq 2\partial^k w^l \partial_{kl} h_{00}\delta_{ij} + 2\partial^k h_{00}\left(\partial_{ki} w_j + \partial_{kj} w_i\right)\\ &\quad + 2\partial_{ik} w_l \partial^k_j w^l + 2\left(\partial^{kl} w_i \partial_{jk} w_l + \partial^{kl} w_j \partial_{ik} w_l\right) + 2\partial^k w^l \left(\partial_{kli} w_j + \partial_{klj} w_i\right)\end{aligned} \tag{71}$$

and using the result from the scalar equation we have

$$\begin{aligned}-\frac{1}{2}\partial_i q_{00} - P^{(2)}_i &\simeq -\frac{1}{2}\partial_i w^k \partial_k h_{00} - \frac{1}{2} w^k \partial_{ki} h_{00} + 2\partial^k w^l \partial_{kl} w_i\\ &\quad + \partial^k h_{00}\left(\partial_k w_i + \partial_i w_k\right) - \partial_k w^k \partial_i h_{00}\end{aligned} \tag{72}$$

Following the same reasoning we have done above, the part of the solution of the differential system (70) which depends on the first order gauge term is

$$q^*_{ij} \simeq w^k \partial_k \tilde{h}_{ij} + \tilde{h}_{kj} \partial_i w^k + \tilde{h}_{ik} \partial_j w^k - \partial_i w^k \partial_j w_k \tag{73}$$

where

$$\tilde{h}_{ij} = h_{00} \delta_{ij} + \partial_i w_j + \partial_j w_i \tag{74}$$

i.e. the gauge term transferred from the first to the second order (43). We must bear in mind that this is only a part of the particular solution which depends on the first order gauge, the terms depending on the rest of the first order solution that appear in P and N and the terms depending on the gauge of second order are not computed.

6.3 The possible validity of the result to third order

If we replace (74) in (73), after a short calculation we obtain

$$\begin{aligned} q^*_{ij} &= w^k \partial_k h_{00}\, \delta_{ij} + h_{00} \left(\partial_i w_j + \partial_j w_i \right) \\ &\quad + \partial_i w_k \partial_j w^k \partial_i \left(w^k \partial_k w_j \right) + \partial_j \left(w^k \partial_k w_i \right) \end{aligned} \tag{75}$$

Therefore the exact solution of (70) can be written as follows

$$\begin{aligned} q_{ij} &= q^c_{ij} + w^k \partial_k h_{00}\, \delta_{ij} + h_{00} \left(\partial_i w_j + \partial_j w_i \right) + \partial_i w_k \partial_j w^k \\ &\quad + \partial_i \left(w^{(2)}_j + w^k \partial_k w_j \right) + \partial_j \left(w^{(2)}_i + w^k \partial_k w_i \right) \end{aligned} \tag{76}$$

where q^c_{ij} represents a particular solution of (70) with $w^i = 0$ and where the sum $\partial_i w^{(2)}_j + \partial_j w^{(2)}_i$ is the general solution of the homogeneous system, therefore $\Delta w^{(2)}_i = 0$, i.e. with $w^{(2)}_i$ formally identical to (56).

The second line of (72) could be interpreted as the main part of an infinitesimal change of coordinates to second order. It is also easy to see that $\zeta_i = \xi^{(2)}_i + w^k \partial_k \xi_i$ is the general solution of (46) with $\xi^i = -w^i$ and $\xi^{(2)}_i = w^{(2)}_i$, that is, it would be an harmonic coordinate change. This suggests that the results concerning the gauge transfer would also be valid to the third order.

7 Interior solution in harmonic coordinates for a rigidly rotating perfect fluid with a linear equation of state

We are going to use the solution obtained in [2].

From the metric Eq. (38) and

$$e_{ri}\, dx^i = dr, \quad e_{\theta i}\, dx^i = r d\theta, \quad e_{\varphi i}\, dx^i = r \sin\theta\, d\varphi. \tag{77}$$

we have

$$ds^2 = \gamma_{tt} dt^2 + 2\gamma_{t\varphi} e_{\varphi j} dt dx^j + (\gamma_{rr} e_{ri} e_{rj} + 2\gamma_{r\theta} e_{ri} e_{\theta j} + \gamma_{\theta\theta} e_{\theta i} e_{\theta j} + \gamma_{\varphi\varphi} e_{\varphi i} e_{\varphi j}) dx^i dx^j \tag{78}$$

that is

$$g_{ij} := \gamma_{rr} e_{ri} e_{rj} + 2\gamma_{r\theta} e_{ri} e_{\theta j} + \gamma_{\theta\theta} e_{\theta i} e_{\theta j} + \gamma_{\varphi\varphi} e_{\varphi i} e_{\varphi j}, \quad g_{0j} := \gamma_{t\varphi} e_{\varphi j}$$

We assume that the source of the gravitational field is a perfect fluid,

$$\mathcal{T}_{\alpha\beta} := T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} = (\mu + p) u_\alpha u_\beta + \frac{1}{2}(\mu - p) g_{\alpha\beta} \tag{79}$$

whose density μ and pressure p depend only on r and θ. We also assume that the fluid has no convective motion, so its velocity lies on the plane spanned by the two Killing vectors

$$\boldsymbol{u} = \psi(\partial_t + \omega \partial_\varphi) = \psi\left(\partial_t + \omega r \sin\theta \, e^i_\varphi \partial_i\right) \tag{80}$$

where

$$\psi \equiv \left[-\left(\gamma_{tt} + 2\omega\, \gamma_{t\varphi}\, r \sin\theta + \omega^2\, \gamma_{\varphi\varphi}\, r^2 \sin^2\theta\right)\right]^{-\frac{1}{2}} \tag{81}$$

is a normalization factor. From now on we are going to assume that the fluid rotates rigidly; so that $\omega =$ constant.

In order to obtain a postminkowskian expansion for the metric we need to have the density μ of order λ i.e. $\mu = \lambda\tilde{\mu}$.

In the postminkowskian approximation we have

$$g_{00} = -1 + \lambda h_{00} + \lambda^2 q_{00}, \quad g_{0j} = \omega\left(\lambda h_{0j} + \lambda^2 q_{0j}\right), \quad g_{ij} = \delta_{ij} + \lambda h_{ij} + \lambda^2 q_{ij} \tag{82}$$

This implies that

$$\begin{aligned} \gamma_{tt} &= -1 + \lambda h_{tt} + \lambda^2 q_{tt}, \quad \gamma_{t\varphi} = \omega(\lambda h_{t\varphi} + \lambda^2 q_{t\varphi}), \quad \gamma_{\varphi\varphi} = 1 + \lambda h_{\varphi\varphi} + \lambda^2 q_{\varphi\varphi} \\ \gamma_{rr} &= 1 + \lambda h_{rr} + \lambda^2 q_{rr}, \quad \gamma_{r\theta} = \lambda h_{r\theta} + \lambda^2 q_{r\theta}, \quad \gamma_{\theta\theta} = 1 + \lambda h_{\theta\theta} + \lambda^2 q_{\theta\theta} \end{aligned} \tag{83}$$

Let us consider the Euler equations for the fluid (or, equivalently, the energy-momentum tensor conservation law) [11]

$$\partial_a p = (\mu + p)\partial_a \ln\psi \qquad (a, b, \ldots = r\,, \theta)\,. \tag{84}$$

We are going to use the following linear equation of state (EOS)

$$\mu + (1-n)p = \mu_0. \tag{85}$$

With this EOS the Euler equations are integrated easily, to give

$$p = \frac{\mu}{n}\left\{\left(\frac{\psi}{\psi_\Sigma}\right)^n - 1\right\}, \quad \mu = \frac{\mu}{n}\left\{(n-1)\left(\frac{\psi}{\psi_\Sigma}\right)^n + 1\right\} \tag{86}$$

where ψ_Σ is the value of the normalization factor ψ on the surface of zero pressure, which in turn leads to the following implicit equation for the matching surface

$$p = 0 \quad \Longleftrightarrow \quad \psi = \psi_\Sigma\,. \tag{87}$$

With this dependence on the parameter, the lowest order in λ of this equation gives us cylindrical surfaces instead of spherical deformed ones unless the rotation parameter depends on λ. For that we need $\omega^2 \propto \lambda$ at least. Then, to the lowest order in λ

$$\psi \approx 1 + \frac{1}{2}\lambda\big(h_{tt} + \tilde{\omega}^2 r^2 \sin^2\theta\big) \tag{88}$$

where we have defined $\omega^2 := \lambda\tilde{\omega}^2$ ($\tilde{\omega} \propto \Omega$ slow rotation parameter) and for the constant ψ_Σ

$$\psi_\Sigma := 1 + \lambda S \tag{89}$$

This approach is consistent with the post spherical approach carried out in previous articles (see details in references [1] and [2]). This implies that some quantities, including g_{0j}, have a series expansion in half powers of λ. However, this situation does not affect the present work, in which we only deal with the postminkowskian approach, so we are just to consider the integer part of the orders of the series in the computation.

Then, the approximate pressure is

$$p \approx \frac{1}{2}\lambda^2\tilde{\mu}\big(-2S + h_{tt} + \tilde{\omega}^2 r^2 \sin^2\theta\big) \tag{90}$$

and for the velocity field we have

$$\begin{cases} u_0 \approx -1 + \frac{1}{2}\lambda\big(h_{tt} - \tilde{\omega}^2 r^2 \sin^2\theta\big) \\ u_j \approx \tilde{\omega}\lambda^{1/2}\left\{r\sin\theta + \lambda\left[h_{t\varphi} + \frac{1}{2}r\sin\theta\big(h_{tt} + 2h_{\varphi\varphi} + \tilde{\omega}^2 r^2 \sin^2\theta\big)\right]\right\} e_{\varphi j} \end{cases} \tag{91}$$

Now we can compute the energy momentum tensor to second order in λ, and we obtain

$$\mathcal{T}_{00} \approx \frac{1}{2}\lambda\tilde{\mu} + \frac{1}{4}\lambda^2\tilde{\mu}\big(-2(n+2)S + nh_{00} + (n+6)\tilde{\omega}^2 r^2 \sin^2\theta\big)$$

$$\begin{aligned}\mathcal{T}_{0j} &\approx -\lambda^{3/2}\tilde{\mu}\tilde{\omega} r\sin\theta e_{\varphi j} - \frac{1}{2}\lambda^{5/2}\tilde{\mu}\tilde{\omega}\\ &\quad\times\Big[h_{0k}e^k_\varphi + r\sin\theta\big(-2nS + nh_{00} + 2h_{kl}e^k_\varphi e^l_\varphi + (n+2)\tilde{\omega}^2 r^2\sin^2\theta\big)\Big]e_{\varphi j}\\ \mathcal{T}_{ij} &\approx \frac{1}{2}\lambda\tilde{\mu}\delta_{ij} + \frac{1}{4}\lambda^2\tilde{\mu}\Big[(2-n)\big(2S - h_{00} - \tilde{\omega}^2 r^2\sin^2\theta\big)\delta_{ij} + 2h_{ij}\\ &\quad + 4\tilde{\omega}^2 r^2\sin^2\theta e_{\varphi i}e_{\varphi j}\Big] \end{aligned} \tag{92}$$

7.1 First order in λ

The Einstein's equations to first order are (47), (48) and (49). The scalar equation

$$\Delta h_{00} = -2\mathcal{T}^{(1)}_{00} = -\tilde{\mu} \tag{93}$$

has as solution well behaved in $r = 0$ given by

$$h_{00}(r,\theta) = -\frac{1}{6}\tilde{\mu}\, r^2 + 2\sum_{n=0}^{\infty} m_n r^n P_n(\cos\theta) \tag{94}$$

The vector equations are

$$\begin{aligned}\Delta h_{0k} &= -2\mathcal{T}^{(1)}_{0k} = 2\tilde{\mu}\,\omega\, r\sin\theta\, e_{\varphi k}\\ \partial^k h_{0k} &= 0\end{aligned} \tag{95}$$

and their solution is

$$h_{0k}(r,\theta) = \left[\frac{1}{5}\tilde{\mu}\,\omega\, r^3\sin\theta + 2\sum_{n=1}^{\infty} j_n r^n P^1_n(\cos\theta)\right] e_{\varphi k} \tag{96}$$

As in the exterior solution, there are not gauge dependent terms in the first order solution for h_{00} and h_{0j}.

The tensor equations are

$$\begin{aligned}\Delta h_{ij} &= -2\mathcal{T}^{(1)}_{ij} = -\tilde{\mu}\,\delta_{ij}\\ \partial^k h_{kj} - \frac{1}{2}\partial_j\hat{h} &= -\frac{1}{2}\partial_j h_{00}\end{aligned} \tag{97}$$

and their solution is

$$h_{ij} = h_{00}\,\delta_{ij} + \partial_i\bar{w}_j + \partial_j\bar{w}_i \tag{98}$$

where now

$$\bar{w}_j := \sum_{n=1}^{\infty} \bar{Q}_n r^n P_n(\cos\theta)\, e_{zj} + \sum_{n=1}^{\infty} \bar{R}_n r^n P_n^1(\cos\theta)\, e_{\rho j} \tag{99}$$

The constants $\bar{Q}_n$ and $\bar{R}_n$ are related to the a_n and b_n constants used in [1] by the following expressions

$$a_n = \bar{Q}_{n+1} + (n+2)\bar{R}_{n+1}, \quad b_n = 2\bar{R}_{n+1}.$$

As in the exterior problem, the solution of the homogeneous part of this tensor equation is *pure gauge*, see (41) with $\xi_j := -\bar{w}_j$. Since the complete equation that contains the derivative of the scalar equation solution has the simple particular solution $h_{00}\delta_{ij}$ too, we can use the same expressions (60), (61), (62), (63) and (64) we have used above for the exterior solution.

Now we also want to determine how the *"pure gauge"* terms of the first order solution are transmitted to the second order solution and, in particular, verify that the resulting terms coincide with those coming from an infinitesimal second order change of coordinates.

7.2 Second order in λ

We can now go to second order i.e. solve (50), (51) and (52) with $\mathcal{T}$ given by Eqs. (92) when $n = 2$. Do not confuse this n, related to the order of the approximation we perform, with the n appearing in the EOS. As in the exterior problem we are going to look only for the terms in the solution corresponding to the propagation of the first order gauge.

The scalar equation is

$$\Delta q_{00} = 2N_{00}^{(2)} - 2\mathcal{T}_{00}^{(2)} \simeq 2\partial^i \bar{w}^j \partial_{ij} h_{00} \tag{100}$$

The term $\mathcal{T}_{00}^{(2)}$ does not contain the gauge vectors $\bar{w}^i$. The terms to solve have the same aspect as in the exterior case. And we have

$$\Delta\left(\bar{w}^i \partial_i h_{00}\right) = \Delta\bar{w}^i\, \partial_i h_{00} + 2\partial^j \bar{w}^i\, \partial_j(\partial_i h_{00}) + \bar{w}^i \Delta(\partial_i h_{00}) = 2\partial^j \bar{w}^i\, \partial_{ji} h_{00} \tag{101}$$

where we have taken into account that

$$\Delta\bar{w}^i = 0, \quad \Delta(\partial_i h_{00}) = \partial_i\left(\Delta h_{00}\right) = 0 \tag{102}$$

Then, the particular solution for the terms depending on the first order gauge has the aspect required by a gauge change to second order (35).

For the vector equations, the term $\mathcal{T}_{0j}^{(2)}$ is only gauge dependent in the h_{ij} term, that is

$$\mathcal{T}^{(2)}_{0j} \simeq -2\tilde{\mu}\omega r \sin\theta \left(\partial_k \bar{w}_l e^k_\varphi e^l_\varphi\right) e_{\varphi j} \,. \tag{103}$$

now we need to find a particular solution for the gauge dependent part of the following equations

$$\begin{aligned} \Delta q_{0j} &= 2N^{(2)}_{0j} - \partial_j P^{(2)}_0 - 2\mathcal{T}^{(2)}_{0j} \\ &\simeq 2\partial^k \bar{w}^l \partial_{kl} h_{0j} + 2\partial_{jk} \bar{w}^l \partial^k h_{0l} + 4\tilde{\mu}\omega r \sin\theta \left(\partial_k \bar{w}_l e^k_\varphi e^l_\varphi\right) e_{\varphi j} \end{aligned} \tag{104}$$

$$\partial^j q_{0j} = -P^{(2)}_0 \;\simeq \partial^k \bar{w}^l \left(\partial_k h_{0l} + \partial_l h_{0k}\right) \tag{105}$$

We know from the second order gauge infinitesimal transformation (43) that the expression $\bar{w}^k \partial_k h_{0j} + h_{0k} \partial_j \bar{w}^k$ should be the right answer. Let's try this particular solution

$$\begin{aligned} \Delta\left(\bar{w}^k \partial_k h_{0j} + h_{0k} \partial_j \bar{w}^k\right) &= 2\partial^l \bar{w}^k \,\partial_{lk} h_{0j} + 2\partial^l h_{0k} \,\partial_{lj} \bar{w}^k \\ &\quad + \bar{w}^k \partial_k \Delta h_{0j} + \Delta h_{0k} \partial_j \bar{w}^k \,. \end{aligned} \tag{106}$$

Using the first order Eq. (95) and the expression for $\bar{\omega}_k$ (99) we get

$$\bar{w}^k \partial_k \Delta h_{0j} + \Delta h_{0k} \partial_j \bar{w}^k = 4\tilde{\mu}\omega \sum_{n=1}^{\infty} \bar{R}_n r^n P^1_n(\cos\theta)\, e_{\varphi j}$$

and the last term in the expression for Δq_{0j}, using (99),

$$r \sin\theta \left(\partial_k \bar{w}_l e^k_\varphi e^l_\varphi\right) = \sum_{n=1}^{\infty} \bar{R}_n r^n P^1_n(\cos\theta) \tag{107}$$

Now if we put this particular solution in the harmonicity condition (105)

$$\partial^j \left(\bar{w}^k \partial_k h_{0j} + h_{0k} \partial_j \bar{w}^k\right) = \partial^j \bar{w}^k \left(\partial_j h_{0k} + \partial_k h_{0j}\right) \tag{108}$$

then the gauge propagation gives the particular solution to second order for the vector equations too.

Let us finally solve the tensor equations

$$\Delta q_{ij} = 2N^{(2)}_{ij} - \partial_i P^{(2)}_j - \partial_j P^{(2)}_i - 2\mathcal{T}^{(2)}_{ij}, \quad \partial^j q_{ij} - \frac{1}{2}\partial_i \hat{q} = -\frac{1}{2}\partial_i q_{00} - P^{(2)}_i \tag{109}$$

The only difference with the equations we have solved for the exterior problem is the term

$$\mathcal{T}^{(2)}_{ij} \simeq \frac{1}{2}\tilde{\mu}\left(\partial_i \bar{w}_j + \partial_j \bar{w}_i\right) \tag{110}$$

The remaining terms are determined from (64) and (61) and the solution of the scalar Eq. (94).

Then, as we did above, we try the particular solution

$$q^*_{ij} = \bar{w}^k \partial_k h_{ij} + h_{kj} \partial_i \bar{w}^k + h_{ik} \partial_j \bar{w}^k - \partial_i \bar{w}^k \, \partial_j \bar{w}_k \tag{111}$$

where

$$h_{ij} = h_{00} \delta_{ij} + \partial_i \bar{w}_j + \partial_j \bar{w}_i \tag{112}$$

As in the exterior vacuum problem a direct calculation shows that this is indeed a solution of (109).

8 Conclusions

We have checked that second order gauge transformations map second order solutions of the stationary axisymmetric Einstein equations onto solutions of the same kind. This fact provides a simple way to introduce such gauge constants into an approximate metric avoiding the harder calculations and messy expressions involved in the derivation of a second order solution from a first order metric when gauge constants are included.

However, the second order metric obtained by iteration from the first order metric with gauge constants and the metric one gets by setting the gauge constants equal to zero in that metric followed by a second order gauge transformation can look very different. It does not mean they are different solutions of the same problem, they are indeed the same, only their expressions are different. The reason for this discrepancy is that the constants which appear in the metrics generated by the two procedures, even though they have the same labels in both metrics, can actually be not the same. Any time a homogeneous solution of the first order equations is added to get a higher order solution, constants are redefined in an uncontrolled way. This is not made explicit since we do not change the constant labels, but they have actually changed in the sense that an indeterminate term of the same order as the approximation we are calculating has been added to them. Obviously these corrections to the constants depend on the rule used to select the inhomogeneous solution and other choices we make.

As an example, consider two second order solutions from reference [2]: the exterior metric of Eq. (20) and the interior metric of Eq. (17). They both have been obtained from a first order solution with gauge constants. Setting $A_0 = A_2 = B_2 = 0$ in (20) we get an exterior metric without gauge constants, which can then be reintroduced by means of the following gauge transformation

$$w_j = \left[\frac{\left(A_0 - 2\Omega^2 B_2\right) r_0^3}{2r^3} P_1 + \frac{3\Omega^2 A_2 r_0^5}{2r^4} P_3 \right] e_{zj} + \left[\frac{\left(A_0 + \Omega^2 B_2\right) r_0^3}{2r^2} P_1^1 + \frac{\Omega^2 A_2 r_0^5}{2r^4} P_3^1 \right] e_{\rho j} \tag{113}$$

(there are not explicit second order terms because they have the same structure of first order terms, so we can include them in the first order by redefining gauge constants). In this case nothing strange happens and we recover metric (20). However, this is not the case when we apply the same scheme to the interior metric (17). The result of getting rid of gauge constants, $a_0 = a_2 = b_2 = 0$, and bringing them back by using the gauge transformation

$$
\begin{aligned}
w_j = & \left[\left(\frac{1}{2}a_0 - \Omega^2 b_2\right) r P_1 + \frac{3\Omega^2 a_2}{2r_0^2} r^3 P_3\right] e_{zj} \\
& + \left[\frac{1}{2}\left(a_0 + \Omega^2 b_2\right) r P_1^1 + \frac{\Omega^2 a_2}{2r_0^2} r^3 P_3^1\right] e_{\rho j}
\end{aligned} \tag{114}
$$

(there are no second order terms for the same reason as before) is a metric which differs widely from (17). However, the following change of constants in the metric (17),

$$
\begin{aligned}
& m_0 \to m_0, \quad m_2 \to m_2 + \lambda\left(b_2 + a_0 m_2\right), \\
& j_1 \to j_1 + \lambda j_1 \left(a_0 + \Omega^2 b_2\right), \quad j_3 \to j_3 + \lambda\left(2 j_3 a_0 + j_1 a_2 + \frac{12}{25} b_2\right), \\
& a_0 \to a_0 + \lambda\left(a_0 m_0 + \frac{3}{4} a_0^2\right), \quad a_2 \to a_2 + \lambda\left(a_2 m_0 + \frac{1}{5} a_0 m_2 + \frac{5}{2} a_0 a_2\right), \\
& b_2 \to b_2 + \lambda\left(b_2 m_0 + \frac{3}{2} b_2 a_0\right)
\end{aligned} \tag{115}
$$

brings it into the metric generated by the gauge transformation. The difference between both expressions of the same metric is just a solution of the homogeneous Einstein equations.

Acknowledgments This work has been supported by Projects FIS2012-30926 from Ministerio de Economía y Competividad. One of us A.M. also thanks the Universidad de Salamanca for hospitality. The authors thank the anonymous reviewer for its criticism and helpful comments that let us to improve the paper.

References

1. Cabezas, J.A., Martín, J., Molina, A., Ruiz, E.: Gen. Relativ. Gravit. **39**, 707–736 (2007)
2. Cuchí, J.E., Gil-Rivero, A., Molina, A., Ruiz, E.: Gen. Relativ. Gravit. **45**, 1433–1456 (2013)
3. Chandrasekhar, S.: ApJ **142**, 1488 (1965)
4. Chandrasekhar, S., Nutku, Y.: ApJ **158**, 55 (1969)
5. Hartle, J.B.: ApJ **150**, 1005 (1967)
6. Thorne, K.S.: Rev. Mod. Phys. **52**, 299 (1980)
7. Geroch, R.: J. Math. Phys. **11**, 1955 (1970)
8. Hansen, R.O.: J. Math. Phys. **15**, 46 (1974)
9. Thorne, K. S. In: Sachs, R. K. (eds.) International School of Physics Enrico Fermi, Course 47 (1969), General Relativity and Cosmology, p. 237. Academic Press, New York, (1971)
10. Lichnerowicz, A.: Théories relativistes de la Gravitation et de l'Electromagnetisme. Masson, Paris (1955)
11. Boyer, R.H.: Proc. Camb. Philos. Soc. **61**, 527 (1965)