PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 102, Number 3, March 1988

A NOTE ON QUADRICS THROUGH AN ALGEBRAIC CURVE

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(Communicated by William C. Waterhouse)

ABSTRACT. In this note we describe the intersection of all quadric hypersurfaces containing a given linearly normal smooth projective curve of genus n and degree $2n + 1$.

Let C be an irreducible nonsingular curve of genus g, defined over an algebraically closed field of any characteristic. Let C be embedded in $\mathbf{P}^r$ by a complete linear system $|L|$. Saint-Donat [5] has proved that if $\deg L \geq 2g+2$ then the homogeneous ideal I_C of $C \subseteq \mathbf{P}^r$ is generated by quadrics, and if $\deg L = 2g + 1$ then I_C is generated by quadrics and cubics (see also Fujita [1]). In [2], Green and Lazarsfeld have announced the following result: In case $\deg L = 2g+1$, I_C fails to be generated by quadrics if and only if C is hyperelliptic or L embeds C with a trisecant line, i.e., $H^0\mathcal{O}_C(L - K_C) \neq 0$, where K_C denotes the canonical divisor on C. In this note we describe the intersection of all quadric hypersurfaces passing through $C \subseteq \mathbf{P}^r$ in the borderline situation $\deg L = 2g + 1$. The main ingredient of the proof is a theorem of Castelnuovo on the postulation of points.

A g_d^1 on a curve is, by definition, a base-point free linear system of degree d and dimension 1. For the definition and properties of rational normal scrolls see [3].

Our result is the following.

THEOREM. *Let $C \subseteq \mathbf{P}^{n+1}$ be a linearly normal smooth irreducible curve of genus $n \geq 4$ and degree $2n + 1$. If $W(C)$ denotes the intersection of all quadric hypersurfaces of $\mathbf{P}^{n+1}$ which contain C, then either $W(C)$ consists of C plus (possibly) a line and finitely many isolated points, or $W(C)$ is a rational normal scroll of dimension 2. In case $W(C)$ is a scroll, one of the following situations occurs:*

(i) *$W(C)$ is smooth and C meets every fiber of $W(C)$ at three points. C is trigonal and embedded by the linear system $|K_C + g_3^1|$.*

(ii) *$W(C)$ is a cone with vertex P, and C passes through P and meets every fiber of $W(C)$ at P plus two other points. C is hyperelliptic and embedded by $|P + ng_2^1|$.*

(iii) *$W(C)$ is smooth and C is a divisor in $W(C)$ of class $2H + R$, where H denotes a hyperplane and R a fiber of the ruling. In particular C is hyperelliptic, the g_2^1 being given by restriction of the ruling of $W(C)$.*

PROOF. Throughout this proof we will assume that $W(C)$ is not the union of C and (possibly) a line plus finitely many points. Consequently, there exists a curve $G \subseteq W(C)$, $G \neq C$, with degree of $G \geq 2$. G is allowed to be a pair of distinct lines. Pick two distinct general points Q_1 and Q_2 on G, none of them on C. If G is a

Received by the editors October 20, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 14E25, 14H45, 14N05.
Key words and phrases. Rational scroll, hyperelliptic, trigonal.

union of two lines, then by a general pair we mean that Q_1 is a general point on one of the lines and Q_2 is general on the other line. Choose now a general hyperplane $\mathbf{P}^n$ in $\mathbf{P}^{n+1}$ passing through Q_1 and Q_2, and set $\Gamma = C \cap \mathbf{P}^n$. Let $W(\Gamma)$ be the intersection of all quadric hypersurfaces of $\mathbf{P}^n$ which contain Γ. If $\mathcal{I}_C, \mathcal{I}_\Gamma$ denote the ideal sheaves of C in $\mathbf{P}^{n+1}$ and Γ in $\mathbf{P}^n$ respectively, then the exact sequence

$$0 = H^0(\mathbf{P}^{n+1}, \mathcal{I}_C(1)) \to H^0(\mathbf{P}^{n+1}, \mathcal{I}_C(2)) \\ \to H^0(\mathbf{P}^n, \mathcal{I}_\Gamma(2)) \to H^1(\mathbf{P}^{n+1}, \mathcal{I}_C(1)) = 0$$

yields $W(\Gamma) = W(C) \cap \mathbf{P}^n$.

CLAIM 1. Γ consists of $2n+1$ points in general linear position (i.e., any subset of $n+1$ points of Γ spans $\mathbf{P}^n$).

PROOF OF CLAIM 1. Let $(\mathbf{P}^{n+1})^*$ be the space of hyperplanes of $\mathbf{P}^{n+1}$. It is a well-known fact that the set

$$\mathcal{U} = \{H \in (\mathbf{P}^{n+1})^* \mid H \cap C \text{ is in general linear position}\}$$

is dense in $(\mathbf{P}^{n+1})^*$. For $i = 1, 2$, the set $M(Q_i) = \{H \in (\mathbf{P}^{n+1})^* \mid Q_i \in H\}$ is a hyperplane of $(\mathbf{P}^{n+1})^*$. Since degree of $G \geq 2$ we have

$$\bigcup_{Q_1, Q_2 \in G} (M(Q_1) \cap M(Q_2)) = (\mathbf{P}^{n+1})^*$$

and thus $M(Q_1) \cap M(Q_2) \cap \mathcal{U} \neq \varnothing$ for a generic choice of Q_1 and Q_2. This proves Claim 1.

Choose linear subvarieties $\hat{\mathbf{P}}^{n-1}$, $\hat{\mathbf{P}}^n$ of $\mathbf{P}^{n+1}$ of dimensions $n-1$ and n respectively. Let $\pi\colon C \to \hat{\mathbf{P}}^{n-1}$ be the projection of C from the line $\overline{Q_1Q_2}$ spanned by Q_1 and Q_2, and let $\pi_1\colon C \to \hat{\mathbf{P}}^n$ be the projection of C from Q_1.

CLAIM 2. π and π_1 are generically one-to-one.

PROOF OF CLAIM 2. It suffices to prove the statement for π. Since $n+1 \geq 5$, any hyperplane passing through Q_1 and Q_2 contains at least three fibers of π. If π has degree $k \geq 2$ then those three fibers consist of $3k \geq 6$ points which span a $\mathbf{P}^3$ or a $\mathbf{P}^4$, so that they are not in general linear position. But this contradicts Claim 1.

CLAIM 3. A general hyperplane of $\hat{\mathbf{P}}^n$ passing through the point $\overline{Q_1Q_2} \cap \hat{\mathbf{P}}^n$ cuts $\pi_1(C)$ at a set of points in general linear position.

PROOF OF CLAIM 3. We argue as in Claim 1. The set

$$\mathcal{U}' = \{H \in (\hat{\mathbf{P}}^n)^* \mid H \cap \pi_1(C) \text{ is in general linear position}\}$$

is dense in $(\hat{\mathbf{P}}^n)^*$, and $N(Q_2) = \{H \in (\hat{\mathbf{P}}^n)^* \mid \overline{Q_1Q_2} \cap \hat{\mathbf{P}}^n \in H\}$ is a hyperplane of $(\hat{\mathbf{P}}^n)^*$. Fix Q_1. Since $\deg G \geq 2$, the points $\overline{Q_1Q_2} \cap \hat{\mathbf{P}}^n$ describe a curve in $\hat{\mathbf{P}}^n$ as Q_2 varies along G. Therefore

$$(\hat{\mathbf{P}}^n)^* = \bigcup_{Q_2 \in G} N(Q_2),$$

and thus $N(Q_2) \cap \mathcal{U}' = \varnothing$ for at most finitely many Q_2's.

CLAIM 4. $\Gamma \cup \{Q_1, Q_2\}$ is in general linear position in $\mathbf{P}^n$.

PROOF OF CLAIM 4. Choose any subset Ω of $n+1$ points in $\Gamma \cup \{Q_1, Q_2\}$. We have to show that Ω spans $\mathbf{P}^n$.

Case 1. $\Omega \subseteq \Gamma$. The claim is obvious because Γ is in general linear position.

Case 2. $\Omega = \{Q_1, T_1, \ldots, T_n\}$ with $\{T_1, \ldots, T_n\} \subseteq \Gamma$. By Claim 3, a general hyperplane $\mathbf{P}^n \subseteq \mathbf{P}^{n+1}$ containing $\overline{Q_1Q_2}$ cuts $\hat{\mathbf{P}}^n$ along an $(n-1)$-plane $\tilde{\mathbf{P}}^{n-1}$ such that $\tilde{\mathbf{P}}^{n-1} \cap \pi_1(C)$ is in general linear position. By Claim 2, $\pi_1(T_1), \ldots, \pi_1(T_n)$ are all distinct and belong to $\tilde{\mathbf{P}}^{n-1} \cap \pi_1(C)$. Since $\{\pi_1(T_1), \ldots, \pi_1(T_n)\}$ spans $\tilde{\mathbf{P}}^{n-1}$, it follows that $\{Q_1, \pi_1(T_1), \ldots, \pi_1(T_n)\}$ spans $\mathbf{P}^n$, and so does Ω.

Case 3. $\Omega = \{Q_1, Q_2, T_1, \ldots, T_{n-1}\}$ with $\{T_1, \ldots, T_{n-1}\} \subseteq \Gamma$. If $\tilde{\mathbf{P}}^{n-2} = \hat{\mathbf{P}}^{n-1} \cap \mathbf{P}^n$ then $\tilde{\mathbf{P}}^{n-2} \cap \pi(C)$ is in general linear position. The points $\pi(T_1), \ldots, \pi(T_{n-1})$ are all distinct because of Claim 2, and they belong to $\tilde{\mathbf{P}}^{n-2} \cap \pi(C)$. Inasmuch as $\{\pi(T_1), \ldots, \pi(T_{n-1})\}$ spans $\tilde{\mathbf{P}}^{n-2}$ we get that $\{Q_1, Q_2, \pi(T_1), \ldots, \pi(T_{n-1})\}$ spans $\mathbf{P}^n$, and so does Ω.

Let us summarize the results obtained so far. For a general hyperplane section $\Gamma = C \cap \mathbf{P}^n$ of C we can find two points $Q_1, Q_2 \in W(\Gamma)$ such that $\Gamma \cup \{Q_1, Q_2\}$ is in general linear position. Since Γ imposes exactly $2n+1$ conditions on quadrics [**3**, p. 36], so does $\Gamma \cup \{Q_1, Q_2\}$. Hence $\Gamma \cup \{Q_1, Q_2\}$ is a set of $2n+3$ points in general linear position in $\mathbf{P}^n$ which imposes $2n+1$ conditions on quadrics. Here we use the main ingredient of the proof: a lemma of Castelnuovo states that $\Gamma \cup \{Q_1, Q_2\}$ must lie on a rational normal curve $B \subseteq \mathbf{P}^n$ [**3**, p. 36].

Pick a quadric R in $\mathbf{P}^n$ which contains Γ. If B is not contained in R then $2n+1 =$ cardinal of $\Gamma \leq$ cardinal of $(R \cap B) = 2n$, absurd. Hence $B \subseteq R$. Since the ideal of B is generated by quadrics we get $W(\Gamma) = B$. Now recall that $W(\Gamma) = W(C) \cap \mathbf{P}^n$. Notice that the above considerations hold for a general hyperplane $\mathbf{P}^n$ of $\mathbf{P}^{n+1}$. It follows that $W(C)$ is a surface of minimal degree. $W(C)$ cannot be the Veronese surface in $\mathbf{P}^5$ because C has odd degree and is contained in $W(C)$. Therefore $W(C)$ is a rational normal scroll of dimension 2 [**3**, p. 51]. The homogeneous ideal of C in $\mathbf{P}^{n+1}$ is generated by quadrics and cubics [**5**] and thus C meets every fiber of $W(C)$ at no more than three points. Next we are going to classify the possible configurations $(W(C), C)$.

Assume first that $W(C)$ is a cone. The vertex P of $W(C)$ must belong to C (otherwise C would have degree $2n$ or $3n$), and C meets every fiber of $W(C)$ at two other points. Now it is obvious that C is hyperelliptic, and that any hyperplane section of C passing through P belongs to the system $|P + ng_2^1|$.

Suppose that $W(C)$ is nonsingular, and denote by F a general fiber of $W(C)$. If C meets F at three points then C is trigonal, and an easy application of the Riemann-Roch formula shows that the divisors of the g_3^1 span lines only when the hyperplane divisor belongs to the system $|K_C + g_3^1|$. In case C meets F at two points and H denotes a hyperplane divisor of $W(C)$ we have $H^2 = n$, $CH = 2n+1$ and C is linearly equivalent to $2H + bF$. One concludes that $b = 1$.

REMARK. By Green-Lazarsfeld's claim, quoted in the Introduction, it follows that in case $W(C)$ is not a scroll and $W(C)$ contains a line, then this line is a trisecant of C.

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