



---

Embedding Galois Problems and Reduced Norms

Author(s): Teresa Crespo

Source: *Proceedings of the American Mathematical Society*, Vol. 112, No. 3 (Jul., 1991), pp. 637-639

Published by: American Mathematical Society

Stable URL: <http://www.jstor.org/stable/2048683>

Accessed: 06/02/2009 06:46

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=ams>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We work with the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).



American Mathematical Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the American Mathematical Society*.

<http://www.jstor.org>

## EMBEDDING GALOIS PROBLEMS AND REDUCED NORMS

TERESA CRESPO

(Communicated by William Adams)

**ABSTRACT.** For certain embedding problems  $\tilde{G} \rightarrow G \simeq \text{Gal}(L|K)$  associated to a representation  $t: G \rightarrow \text{Aut } A$  of the group  $G$  by automorphisms of a central simple  $K$ -algebra  $A$  of dimension  $n^2$ , we prove that the solutions are the fields  $L((rN(z))^{1/n})$ , with  $r$  running over  $K^*/K^{*n}$  and  $N(z)$  the reduced norm of an invertible element  $z$  in the algebra  $B \otimes L$ , for  $B$  the twisted algebra of  $A$  by  $t$ .

In a previous paper [1], we explicitly solved embedding problems associated with orthogonal Galois representations. Our method exploited the relationship between the solutions of such embedding problems with Clifford algebras and spin norms. In the present work, our aim is to generalize this relationship to the case of embedding problems given by a representation of a Galois group by automorphisms of a central simple algebra (cf. [2, §10]).

For  $K$  a field of characteristic not dividing a given integer  $n$  and containing the group  $\mu_n$  of  $n$ -roots of unity, we consider a finite Galois extension  $L|K$  with Galois group  $G$ . Unless here noted, we use the notations in [2, §10].

We are interested in embedding problems  $\tilde{G} \rightarrow G \simeq \text{Gal}(L|K)$ , where  $\tilde{G}$  is an  $n$ -covering of the group  $G$  obtained in the following way. For  $t: G \rightarrow \text{PGL}(A) = \text{Aut } A$  a representation of the group  $G$  by  $K$ -automorphisms of a central simple  $K$ -algebra  $A$  of dimension  $n^2$  over  $K$ , we define  $\tilde{G}$  as the pullback of the diagram:

$$\begin{array}{ccc} & & G \\ & & \downarrow t \\ & & \text{PGL}_n(K^s) \\ \text{SL}_n(K^s) & \longrightarrow & \end{array}$$

(cf. [2, 10.2]). The obstruction to the solvability of such an embedding problem is given by the element  $j^*(s_i)$  in  $H^2(\Omega_K, \mu_n)$  (cf. [2, 10.11]). The two invariants related to this element by [2, 10.14], can be computed as a sum of

---

Received by the editors March 20, 1990.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 11R32.

Partially supported by a grant of the CIRIT (Generalitat de Catalunya) and by PRE 8802 (Universitat Politècnica de Catalunya).

Galois symbols. For  $\delta[j \circ t]$ , it follows from [4, Corollaire 1]; for  $PN_2[j \circ t]$ , we give the explicit expression in the next proposition.

**Proposition 1.** *Let  $t : \Omega_K \rightarrow \text{Aut } A$  be a representation of  $\Omega_K$  by  $K$ -automorphisms of a central simple  $K$ -algebra  $A$ . Let  $d_1, d_2, \dots, d_r$  be elements in  $K^*$ , independent modulo  $K^{*n}$ , such that the fixed field of  $\text{Ker}(PN \circ t)$  is contained in  $K((d_1)^{1/n}, (d_2)^{1/n}, \dots, (d_r)^{1/n})$ . Let  $\omega_1, \omega_2, \dots, \omega_r$  be elements in  $\Omega_K$  satisfying*

$$((d_j)^{1/n})^{\omega_i - 1} = \zeta^{\delta_{ij}}$$

for  $\zeta$  a primitive  $n$ -root of 1, and let  $PN(t(\omega_i)) = a_i \text{ mod } K^{*n}$ ,  $1 \leq i \leq r$ . We then have

$$PN_2[t] = \sum_{i=1}^r (d_i, a_i),$$

where  $(, )$  denotes the Galois symbol (cf. [4, I.2]).

*Proof.* We can follow the calculations in [2, 7.8], for the 2-cocycle  $c : \Omega_K \times \Omega_K \rightarrow \mu_n$  representing  $PN_2[t]$  and, taking into account [3, Chapter IV, §2, Lemma 1], we obtain

$$PN_2[t] = \sum_{i=1}^r (d_i) \cup (a_i),$$

where  $(d_i), (a_i)$  are the elements in  $H^1(\Omega_K, \mu_n) \simeq \text{Hom}(\Omega_K, \mu_n)$  corresponding to  $d_i, a_i$  and  $\cup$  denotes the cup product. The proposition follows from the definition of the Galois symbol.

**Theorem 2.** *Let  $\tilde{G} \rightarrow G \simeq \text{Gal}(L|K)$  be the embedding problem associated to a representation  $t : G \rightarrow \text{Aut } A$  of the group  $G$  by automorphisms of a central simple  $K$ -algebra  $A$  of dimension  $n^2$ . Let  $B$  denote the twisted algebra of  $A$  by  $t$  (cf. [2, §10]) and assume that  $PN \circ t = 1$ . If the embedding problem is solvable, all its solutions are the fields  $L((rN(z))^{1/n})$ , with  $r$  running over  $K^*/K^{*n}$ ,  $z$  an invertible element in the  $L$ -algebra  $B_L = B \otimes_K L$  and  $N$  denoting the reduced norm in  $B_L$ .*

*Proof.* Applying [2, 10.14], the solvability of the embedding problem implies that the classes of similarity of the algebras  $A$  and  $B$  are equal, and, as  $\dim_K A = \dim_K B$ , we have an isomorphism  $g : A \rightarrow B$ . Let  $f : A_L = A \otimes_K L \rightarrow B \otimes_K L = B_L$  be an isomorphism such that  $f^{-1} \circ f^\sigma = t(\sigma)$ , for  $\sigma$  in  $\tilde{G}$ . Now, as  $PN \circ t = 1$ , we can choose a system of representatives  $x_\sigma$  of  $G$  in  $\tilde{G}$  so that the  $x_\sigma$  are in  $\text{SL}(A)$ . For a  $K$ -basis  $\{e_i\}_{1 \leq i \leq n^2}$  of  $A$ , we define  $v_i = f(e_i)$ ,  $w_i = g(e_i)$ . Applying the Skolem-Noether theorem, we obtain an invertible element  $z$  in  $B_L$  such that

$$(1) \quad v_i z = z w_i, \quad 1 \leq i \leq n^2.$$

Now, we have  $x_\sigma e_i x_\sigma^{-1} = f^{-1} \circ f^\sigma(e_i)$  and so:

$$(2) \quad f(x_\sigma) v_i f(x_\sigma)^{-1} = f^\sigma(e_i).$$

From (1) and (2), we obtain:

$$f(x_\sigma)^{-1} z^\sigma w_i = v_i f(x_\sigma)^{-1} z^\sigma$$

and this relation, together with (1), implies that

$$b_\sigma := f(x_\sigma)^{-1} z^\sigma z^{-1}$$

is an element in  $L^*$ . From the identity

$$(zz^{-\sigma})(zz^{-\tau})^\sigma(zz^{-\sigma\tau})^{-1} = 1,$$

we get that the elements  $b_\sigma$  satisfy

$$(3) \quad b_\sigma b_\tau^\sigma b_{\sigma\tau}^{-1} = x_\sigma x_\tau x_{\sigma\tau}^{-1}.$$

Now, applying the reduced norm of  $B_L$  to  $z^\sigma = b_\sigma f(x_\sigma)^{-1} z$ , we obtain

$$(4) \quad N(z)^\sigma = b_\sigma^n N(z).$$

From relations (3) and (4), it follows that  $L((N(z))^{1/n})$  is a solution to the embedding problem.

*Remark.* In the case that the number  $n$  is prime and the extension  $1 \rightarrow \mu_n \rightarrow \tilde{G} \rightarrow G \rightarrow 1$  does not split, all solutions to the considered embedding problem are proper.

### REFERENCES

1. T. Crespo, *Explicit solutions to embedding problems associated to orthogonal Galois representations*, J. Reine Angew. Math. **409** (1990), 180–189.
2. A. Fröhlich, *Orthogonal representations of Galois groups, Stiefel-Whitney classes and Hasse-Witt invariants*, J. Reine Angew. Math. **360** (1985), 84–123.
3. S. Lang, *Rapport sur la cohomologie des groupes*, Benjamin, New York and Amsterdam, 1966.
4. C. Soulé,  $K_2$  et le groupe de Brauer, Séminaire Bourbaki, vol. 601, 1982/83.

UNIVERSITÄT REGENSBURG, FACHBEREICH MATHEMATIK, UNIVERSITÄTSSTRASSE 31, 8400 REGENSBURG, GERMANY

*Current address:* Departament de Matemàtiques I, E. U. Politècnica de Barcelona, Universitat Politècnica de Catalunya, Avinguda Dr. G. Marañón s/n, 08028 Barcelona, Spain