THE DEGREE OF SMOOTH NON-ARITHMETICALLY COHEN-MACAULAY THREEFOLDS IN $\mathbf{P}^5$

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Abstract. In [B], Banica considers the problem of determining the integers $d$ such that there are smooth threefolds which are not arithmetically Cohen-Macaulay. Moreover, he gives a partial answer to this question. In this note, using liaison, we will complete his answer.

Introduction

In a recent work [B], Banica determines the integers $d$ such that there exist smooth surfaces of degree $d$ in $\mathbf{P}^4$ which are not arithmetically Cohen-Macaulay. Concretely, these are precisely the integers $d \geq 4$ with the exception $d = 6$. Furthermore, he considers the problem of determining the integers $d$ such that there exist smooth threefolds in $\mathbf{P}^5$ which are not arithmetically Cohen-Macaulay, and he gives a partial answer to this question. Namely, for any odd integer $d \geq 7$ or any even integer $d = 2k > 8$ with $k = 5s + 1$, $5s + 2$, $5s + 3$ or $5s + 4$, there exist smooth threefolds in $\mathbf{P}^5$ of degree $d$ which are not arithmetically Cohen-Macaulay.

On the other hand, Beltrametti-Schneider-Sommese prove that any smooth threefold of degree 10 is arithmetically Cohen-Macaulay [BBS]. So, the problem of determining the integers $d = 10n$, $n > 1$, such that there exist smooth threefolds in $\mathbf{P}^5$ which are not arithmetically Cohen-Macaulay, remains open.

The goal of this note is to prove that, for any integer $d = 10n$, $n > 1$, there exist smooth threefolds in $\mathbf{P}^5$ of degree $d$ which are not arithmetically Cohen-Macaulay. To this end, we begin with well known smooth non-arithmetically Cohen-Macaulay threefolds in $\mathbf{P}^5$ of low degree, and we use the fact that the property of being arithmetically Cohen-Macaulay is preserved under liaison.

1. Let $k$ be an algebraically closed field of characteristic zero, $S = k[x_0, \ldots, x_5]$ and $\mathbf{P}^5 = \text{Proj}(S)$. Recall that a threefold $X$ in $\mathbf{P}^5$ is arithmetically Cohen-Macaulay if and only if $\bigoplus_{t \in \mathbf{Z}} H^i(\mathbf{P}^5, I_X(t)) = 0$ for $1 \leq i \leq 3$. The notion of
liaison among closed subschemes of \( \mathbf{P}^d \) was introduced in [PS]; we will quote from that paper what we need in our proofs.

Our aim is to show

**Proposition 1.1.** For any integer \( d = 10n \), \( n > 1 \), there exist smooth threefolds in \( \mathbf{P}^5 \) of degree \( d \) which are not arithmetically Cohen-Macaulay.

**Proof.** Let \( Y \subset \mathbf{P}^5 \) be a smooth non-arithmetically Cohen-Macaulay threefold of degree 12 having a locally free resolution of the following kind (see \([B, \S 2.5]\) for the existence of \( Y \)):

\[
0 \to \mathcal{O} \oplus \mathcal{O}(1)^3 \to \Omega(3) \to \mathcal{I}_Y(6) \to 0.
\]

In particular, \( \mathcal{I}_Y(6) \) is globally generated. Let \( X \) be the threefold linked to \( Y \) by means of two general hypersurfaces of degree 6 and 7, respectively, passing through \( Y \). By [PS, Proposition 2.5], the ideal sheaf of \( X \) has resolution

\[
0 \to T(-10) \to \mathcal{O}(-8)^3 \oplus \mathcal{O}(-7)^2 \oplus \mathcal{O}(-6) \to \mathcal{I}_X \to 0.
\]

In particular, the degree of \( X \) is 30, it is not arithmetically Cohen-Macaulay and \( \mathcal{I}_X(8) \) is globally generated. Now we use \( X \) in order to construct non-arithmetically Cohen-Macaulay threefolds of degree \( d = 10n \), \( n \geq 5 \).

In fact, for all \( n \geq 5 \), write \( d + 30 = 10(n + 3) \) and take two general hypersurfaces of degree 10 and \( n + 3 \), respectively, passing through \( X \). As a residual, we get a smooth non-arithmetically Cohen-Macaulay threefold, \( Z \subset \mathbf{P}^5 \), of degree \( d = 10n \), \( n \geq 5 \).

Finally, it remains to construct smooth non-arithmetically Cohen-Macaulay threefolds of degree \( d = 20, 40 \).

**Case** \( d = 20 \). Let \( Y \subset \mathbf{P}^5 \) be a smooth non-arithmetically Cohen-Macaulay threefold of degree 9 having a locally free resolution of the following kind (see \([B, \S 2.5]\) for the existence of \( Y \)):

\[
0 \to T(-6) \to \mathcal{O}(-4)^6 \to \mathcal{I}_Y \to 0.
\]

Note that \( \mathcal{I}_Y(4) \) is globally generated. So, taking two general hypersurfaces of degree 5 passing through \( Y \), we get, as a residual, a smooth non-arithmetically Cohen-Macaulay threefold, \( X \subset \mathbf{P}^5 \), of degree 16. By [PS, Proposition 2.5], the ideal sheaf of \( X \) has resolution

\[
0 \to \mathcal{O}(-6)^6 \to \Omega(-4) \oplus \mathcal{O}(-5)^2 \to \mathcal{I}_X \to 0.
\]

Finally, taking two general hypersurfaces of degree 6 passing through \( X \) we get, as a residual, a smooth threefold of degree 20, which is not arithmetically Cohen-Macaulay.

**Case** \( d = 40 \). We take \( Y \), a smooth non-arithmetically Cohen-Macaulay threefold of degree 9 as above, and two general hypersurfaces of degree 7 passing through \( Y \). The residual threefold is smooth of degree 40 and it is not arithmetically Cohen-Macaulay. \( \Box \)
Corollary 1.2. For any integer $d \geq 7$ with exception $d = 8, 10$, there exist smooth threefolds in $\mathbb{P}^5$ which are not arithmetically Cohen-Macaulay.

Proof. It follows from [B], [BBS], and Proposition 1.1. □

Remark 1.3. Until now there is no example of smooth subvariety of codimension 2 in $\mathbb{P}^n$, $n > 5$, which is not arithmetically Cohen-Macaulay. Furthermore, Hartshorne conjectures that such an example does not exist [H].

REFERENCES


