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## THE DEGREE OF SMOOTH NON-ARITHMETICALLY COHEN-MACAULAY THREEFOLDS IN $\mathbf{P}^5$

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**ABSTRACT.** In [B], Banica considers the problem of determining the integers  $d$  such that there are smooth threefolds which are not arithmetically Cohen-Macaulay. Moreover, he gives a partial answer to this question. In this note, using liaison, we will complete his answer.

### INTRODUCTION

In a recent work [B], Banica determines the integers  $d$  such that there exist smooth surfaces of degree  $d$  in  $\mathbf{P}^4$  which are not arithmetically Cohen-Macaulay. Concretely, these are precisely the integers  $d \geq 4$  with the exception  $d = 6$ . Furthermore, he considers the problem of determining the integers  $d$  such that there exist smooth threefolds in  $\mathbf{P}^5$  which are not arithmetically Cohen-Macaulay, and he gives a partial answer to this question. Namely, for any odd integer  $d \geq 7$  or any even integer  $d = 2k > 8$  with  $k = 5s+1, 5s+2, 5s+3$  or  $5s+4$ , there exist smooth threefolds in  $\mathbf{P}^5$  of degree  $d$  which are not arithmetically Cohen-Macaulay.

On the other hand, Beltrametti-Schneider-Sommese prove that any smooth threefold of degree 10 is arithmetically Cohen-Macaulay [BBS]. So, the problem of determining the integers  $d = 10n, n > 1$ , such that there exist smooth threefolds in  $\mathbf{P}^5$  which are not arithmetically Cohen-Macaulay, remains open.

The goal of this note is to prove that, for any integer  $d = 10n, n > 1$ , there exist smooth threefolds in  $\mathbf{P}^5$  of degree  $d$  which are not arithmetically Cohen-Macaulay. To this end, we begin with well known smooth non-arithmetically Cohen-Macaulay threefolds in  $\mathbf{P}^5$  of low degree, and we use the fact that the property of being arithmetically Cohen-Macaulay is preserved under liaison.

1. Let  $\mathbf{k}$  be an algebraically closed field of characteristic zero,  $S = \mathbf{k}[x_0, \dots, x_5]$  and  $\mathbf{P}^5 = \text{Proj}(S)$ . Recall that a threefold  $X$  in  $\mathbf{P}^5$  is arithmetically Cohen-Macaulay if and only if  $\bigoplus_{i \in \mathbf{Z}} H^i(\mathbf{P}^5, I_X(t)) = 0$  for  $1 \leq i \leq 3$ . The notion of

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liaison among closed subschemes of  $\mathbf{P}^n$  was introduced in [PS]; we will quote from that paper what we need in our proofs.

Our aim is to show

**Proposition 1.1.** *For any integer  $d = 10n$ ,  $n > 1$ , there exist smooth threefolds in  $\mathbf{P}^5$  of degree  $d$  which are not arithmetically Cohen-Macaulay.*

*Proof.* Let  $Y \subset \mathbf{P}^5$  be a smooth non-arithmetically Cohen-Macaulay threefold of degree 12 having a locally free resolution of the following kind (see [B, §2.5] for the existence of  $Y$ ):

$$0 \rightarrow \mathcal{O} \oplus \mathcal{O}(1)^3 \rightarrow \Omega(3) \rightarrow I_Y(6) \rightarrow 0.$$

In particular,  $I_Y(6)$  is globally generated. Let  $X$  be the threefold linked to  $Y$  by means of two general hypersurfaces of degree 6 and 7, respectively, passing through  $Y$ . By [PS, Proposition 2.5], the ideal sheaf of  $X$  has resolution

$$0 \rightarrow T(-10) \rightarrow \mathcal{O}(-8)^3 \oplus \mathcal{O}(-7)^2 \oplus \mathcal{O}(-6) \rightarrow I_X \rightarrow 0.$$

In particular, the degree of  $X$  is 30, it is not arithmetically Cohen-Macaulay and  $I_X(8)$  is globally generated. Now we use  $X$  in order to construct non-arithmetically Cohen-Macaulay threefolds of degree  $d = 10n$ ,  $n \geq 5$ .

In fact, for all  $n \geq 5$ , write  $d + 30 = 10(n + 3)$  and take two general hypersurfaces of degree 10 and  $n + 3$ , respectively, passing through  $X$ . As a residual, we get a smooth non-arithmetically Cohen-Macaulay threefold,  $Z \subset \mathbf{P}^5$ , of degree  $d = 10n$ ,  $n \geq 5$ .

Finally, it remains to construct smooth non-arithmetically Cohen-Macaulay threefolds of degree  $d = 20, 40$ .

*Case  $d = 20$ .* Let  $Y \subset \mathbf{P}^5$  be a smooth non-arithmetically Cohen-Macaulay threefold of degree 9 having a locally free resolution of the following kind (see [B, §2.5] for the existence of  $Y$ ):

$$0 \rightarrow T(-6) \rightarrow \mathcal{O}(-4)^6 \rightarrow I_Y \rightarrow 0.$$

Note that  $I_Y(4)$  is globally generated. So, taking two general hypersurfaces of degree 5 passing through  $Y$ , we get, as a residual, a smooth non-arithmetically Cohen-Macaulay threefold,  $X \subset \mathbf{P}^5$ , of degree 16. By [PS, Proposition 2.5], the ideal sheaf of  $X$  has resolution

$$0 \rightarrow \mathcal{O}(-6)^6 \rightarrow \Omega(-4) \oplus \mathcal{O}(-5)^2 \rightarrow I_X \rightarrow 0.$$

Finally, taking two general hypersurfaces of degree 6 passing through  $X$  we get, as a residual, a smooth threefold of degree 20, which is not arithmetically Cohen-Macaulay.

*Case  $d = 40$ .* We take  $Y$ , a smooth non-arithmetically Cohen-Macaulay threefold of degree 9 as above, and two general hypersurfaces of degree 7 passing through  $Y$ . The residual threefold is smooth of degree 40 and it is not arithmetically Cohen-Macaulay.  $\square$

**Corollary 1.2.** *For any integer  $d \geq 7$  with exception  $d = 8, 10$ , there exist smooth threefolds in  $\mathbf{P}^5$  which are not arithmetically Cohen-Macaulay.*

*Proof.* It follows from [B], [BBS], and Proposition 1.1.  $\square$

*Remark 1.3.* Until now there is no example of smooth subvariety of codimension 2 in  $\mathbf{P}^n$ ,  $n > 5$ , which is not arithmetically Cohen-Macaulay. Furthermore, Hartshorne conjectures that such an example does not exist [H].

#### REFERENCES

- [B] C. Banica, *Smooth reflexive sheaves*, Preprint Series in Math. INCREST, Bucharest, Romania **17** (1989), 1–43.
- [H] R. Hartshorne, *Varieties of small codimension in projective spaces*, Bull. Amer. Math. Soc. **80** (1974), 1017–1032.
- [BSS] M. Beltrametti, M. Schneider, and A. Sommese, *Threefolds of degree 9 and 10 in  $\mathbf{P}^5$* , preprint, 1989.
- [PS] C. Peskine and L. Szpiro, *Liaison des varieties algebriques. I*, Invent. Math. **26** (1974), 271–302.

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