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THE DEGREE OF SMOOTH NON-ARITHMETICALLY COHEN-MACAULAY THREEFOLDS IN $\mathbf{P}^5$

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ABSTRACT. In [B], Banica considers the problem of determining the integers d such that there are smooth threefolds which are not arithmetically Cohen-Macaulay. Moreover, he gives a partial answer to this question. In this note, using liaison, we will complete his answer.

INTRODUCTION

In a recent work [B], Banica determines the integers d such that there exist smooth surfaces of degree d in $\mathbf{P}^4$ which are not arithmetically Cohen-Macaulay. Concretely, these are precisely the integers $d \geq 4$ with the exception $d = 6$. Furthermore, he considers the problem of determining the integers d such that there exist smooth threefolds in $\mathbf{P}^5$ which are not arithmetically Cohen-Macaulay, and he gives a partial answer to this question. Namely, for any odd integer $d \geq 7$ or any even integer $d = 2k > 8$ with $k = 5s+1$, $5s+2$, $5s+3$ or $5s+4$, there exist smooth threefolds in $\mathbf{P}^5$ of degree d which are not arithmetically Cohen-Macaulay.

On the other hand, Beltrametti-Schneider-Sommese prove that any smooth threefold of degree 10 is arithmetically Cohen-Macaulay [BBS]. So, the problem of determining the integers $d = 10n$, $n > 1$, such that there exist smooth threefolds in $\mathbf{P}^5$ which are not arithmetically Cohen-Macaulay, remains open.

The goal of this note is to prove that, for any integer $d = 10n$, $n > 1$, there exist smooth threefolds in $\mathbf{P}^5$ of degree d which are not arithmetically Cohen-Macaulay. To this end, we begin with well known smooth non-arithmetically Cohen-Macaulay threefolds in $\mathbf{P}^5$ of low degree, and we use the fact that the property of being arithmetically Cohen-Macaulay is preserved under liaison.

1. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, $S = \mathbf{k}[x_0, \dots, x_5]$ and $\mathbf{P}^5 = \mathrm{Proj}(S)$. Recall that a threefold X in $\mathbf{P}^5$ is arithmetically Cohen-Macaulay if and only if $\oplus_{t\in\mathbf{Z}} H^i(\mathbf{P}^5, I_X(t)) = 0$ for $1 \leq i \leq 3$. The notion of

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liaison among closed subschemes of $\mathbf{P}^n$ was introduced in [PS]; we will quote from that paper what we need in our proofs.

Our aim is to show

Proposition 1.1. *For any integer $d = 10n$, $n > 1$, there exist smooth threefolds in $\mathbf{P}^5$ of degree d which are not arithmetically Cohen-Macaulay.*

Proof. Let $Y \subset \mathbf{P}^5$ be a smooth non-arithmetically Cohen-Macaulay threefold of degree 12 having a locally free resolution of the following kind (see [B,§2.5] for the existence of Y):

$$0 \to \mathscr{O} \oplus \mathscr{O}(1)^3 \to \Omega(3) \to I_Y(6) \to 0.$$

In particular, $I_Y(6)$ is globally generated. Let X be the threefold linked to Y by means of two general hypersurfaces of degree 6 and 7, respectively, passing through Y. By [PS, Proposition 2.5], the ideal sheaf of X has resolution

$$0 \to T(-10) \to \mathscr{O}(-8)^3 \oplus \mathscr{O}(-7)^2 \oplus \mathscr{O}(-6) \to I_X \to 0.$$

In particular, the degree of X is 30, it is not arithmetically Cohen-Macaulay and $I_X(8)$ is globally generated. Now we use X in order to construct non-arithmetically Cohen-Macaulay threefolds of degree $d = 10n$, $n \geq 5$.

In fact, for all $n \geq 5$, write $d + 30 = 10(n + 3)$ and take two general hypersurfaces of degree 10 and $n + 3$, respectively, passing through X. As a residual, we get a smooth non-arithmetically Cohen-Macaulay threefold, $Z \subset \mathbf{P}^5$, of degree $d = 10n$, $n \geq 5$.

Finally, it remains to construct smooth non-arithmetically Cohen-Macaulay threefolds of degree $d = 20, 40$.

Case $d = 20$. Let $Y \subset \mathbf{P}^5$ be a smooth non-arithmetically Cohen-Macaulay threefold of degree 9 having a locally free resolution of the following kind (see [B, §2.5] for the existence of Y):

$$0 \to T(-6) \to \mathscr{O}(-4)^6 \to I_Y \to 0.$$

Note that $I_Y(4)$ is globally generated. So, taking two general hypersurfaces of degree 5 passing through Y, we get, as a residual, a smooth non-arithmetically Cohen-Macaulay threefold, $X \subset \mathbf{P}^5$, of degree 16. By [PS, Proposition 2.5], the ideal sheaf of X has resolution

$$0 \to \mathscr{O}(-6)^6 \to \Omega(-4) \oplus \mathscr{O}(-5)^2 \to I_X \to 0.$$

Finally, taking two general hypersurfaces of degree 6 passing through X we get, as a residual, a smooth threefold of degree 20, which is not arithmetically Cohen-Macaulay.

Case $d = 40$. We take Y, a smooth non-arithmetically Cohen-Macaulay threefold of degree 9 as above, and two general hypersurfaces of degree 7 passing through Y. The residual threefold is smooth of degree 40 and it is not arithmetically Cohen-Macaulay. $\square$

Corollary 1.2. *For any integer $d \geq 7$ with exception $d = 8, 10$, there exist smooth threefolds in $\mathbf{P}^5$ which are not arithmetically Cohen-Macaulay.*

Proof. It follows from [B], [BBS], and Proposition 1.1. □

Remark 1.3. Until now there is no example of smooth subvariety of codimension 2 in $\mathbf{P}^n$, $n > 5$, which is not arithmetically Cohen-Macaulay. Furthermore, Hartshorne conjectures that such an example does not exist [H].

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