SAMPLING SEQUENCES FOR HARDY SPACES OF THE BALL

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ABSTRACT. We show that a sequence $a := \{a_k\}_k$ in the unit ball of $\mathbb{C}^n$ is sampling for the Hardy spaces $H^p$, $0 < p < \infty$, if and only if the admissible accumulation set of $a$ in the unit sphere has full measure. For $p = \infty$ the situation is quite different. While this condition is still sufficient, when $n > 1$ (in contrast to the one dimensional situation) there exist sampling sequences for $H^\infty$ whose admissible accumulation set has measure 0. We also consider the sequence $a(\omega)$ obtained by applying to each $a_k$ a random rotation, and give a necessary and sufficient condition on $\{|a_k|\}_k$ so that, with probability one, $a(\omega)$ is of sampling for $H^p$, $p < \infty$.

§1. INTRODUCTION

Let $\mathbb{B}_n$ denote the unit ball of $\mathbb{C}^n$. Let $S^n$ denote the unit sphere and $d\sigma$ its normalized Lebesgue measure. Recall that for any $0 < p \leq \infty$ the Hardy space $H^p(\mathbb{B}_n)$ is the set of functions $f$ holomorphic in $\mathbb{B}_n$ such that

$$
\|f\|_p := \left( \sup_{r<1} \int_{S^n} |r\zeta|^p \, d\sigma(\zeta) \right)^{1/p} < \infty,
$$

where the integral is replaced by a supremum in the case $p = \infty$.

Roughly speaking, we would like to say that a sequence $a := \{a_k\}_k$ of the unit ball is sampling for the space $H^p$ when the values of any function $f \in H^p$, restricted to the sequence, determine the function uniquely, and moreover some inequalities between the $H^p$-norm and an appropriate norm on the space of functions on the sequence $a$ hold. In [Th] it was shown that a natural notion of sampling for $H^p$ is the following.

Given $\alpha > 1$, let

$$
\Gamma_\alpha(\zeta) := \{z \in \mathbb{B}_n : |1 - \zeta \cdot \bar{z}| < \frac{\alpha}{2} (1 - |z|^2)\}
$$

be the admissible approach region with vertex at $\zeta \in S^n$ and aperture $\alpha$ (see [Ru1, p. 72] for the properties of these regions).

The admissible maximal function on $S^n$ is then defined, for every $\alpha > 1$, as

$$
M^\alpha f(\zeta) := \sup_{z \in \Gamma_\alpha(\zeta)} |f(z)|.
$$

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For any $0 < p \leq \infty$ and for any $\alpha > 1$ we have $\|M^\alpha f\|_p \leq C_{p, \alpha}\|f\|_p$, where, for functions defined on the unit sphere, $\| \cdot \|_p$ stands for the usual norm in the space $L^p(\mathbb{S}^n)$ ([Ru1, 5.6.5]).

Following [Br-Ni-Oy] we also consider the corresponding maximal function associated to the sequence $a$:

$$M^\alpha_a(f)(\zeta) := \sup_{z \in a \cap \Gamma_\alpha(\zeta)} |f(z)|.$$ 

From the above it follows that $\|M^\alpha_a(f)\|_p \leq C_{p, \alpha}\|f\|_p$.

**Definition.** A sequence $a$ is called a sampling sequence for $H^p$ when there exists $\alpha > 1$ and a constant $C > 0$ such that $\|M^\alpha_a(f)\|_p \geq C\|f\|_p$ for every $f \in H^p$.

In the case where $p = \infty$ this simply says that $\sup_a |f| \geq C\|f\|_\infty$, and by taking powers of $f$ we see that $\sup_a |f| = \|f\|_\infty$.

Given a sequence $a$ let

$$AD_a(a) = \{ \zeta \in S^n : \zeta \in a \cap \Gamma_\alpha(\zeta) \}$$

and define the admissible accumulation set as $AD(a) := \bigcup_{n \geq 1} AD_a(a)$.

Brown, Shields and Zeller showed that the condition $\sigma(AD(a)) = 1$ characterizes the sampling sequences for $H^\infty$ of the disk ([Br-Sh-Ze, Th. 3, (iii)-(iv)])). It will be important to keep in mind that $\sigma(AD(a)) = 1$ if and only if $\sigma_\alpha(AD(a)) = 1$ for some $\alpha > 1$ large enough (see [Th]). Recently the second author showed that the same condition is actually necessary and sufficient for $a$ to be of sampling for any $H^p$ of the disk, $p < \infty$ [Th, Theorem 1].

In this note we first prove that when $n > 1$, the condition $\sigma(AD(a)) = 1$ also characterizes the sampling sequences for $H^p$, $p < \infty$.

**Theorem 1.** A sequence $a$ is sampling for $H^p$, $p < \infty$, if and only if $\sigma(AD(a)) = 1$.

In particular, sampling sequences for $H^p$ are the same for all values $p < \infty$.

For $p = \infty$ the situation is more complicated.

On the one hand it is clear that if $a$ is sampling for $H^\infty$, then necessarily $\overline{a} \cap S^n = S^n$ (if $a$ avoids an open set $\{ \zeta \in S : |1 - \zeta \cdot \eta| < \delta \}$, any peak function for $\eta$, for instance $f(z) = z \cdot \eta$, violates the sampling condition). Although $\overline{a} \cap S^n = S^n$ is also sufficient for sampling in the ball algebra $A(B_n)$, for general $H^\infty$ functions this is far from being sufficient: there are sequences which are contained in an $H^\infty$ zero set such that $\overline{a} \cap S^n = S^n$ (for example any sequence $a$ with $\sum_k (1 - |a_k|) < \infty$ having $S^n$ as cluster set).

The proof of Theorem 1 shows that $\sigma(AD(a)) = 1$ is as well sufficient for $a$ to be sampling for $H^\infty$. This condition is far from being necessary.

**Definition.** A set $E$ in $S^n$ is a max-set when $\text{esssup}_E |f| = \|f\|_\infty$ for all $f \in H^\infty$.

It is clear from the definition that if $AD(a)$ contains a max-set, then $a$ must be sampling for $H^\infty$. Since there exist max-sets of arbitrarily small measure [Ru2, 13.4] it is possible to construct, for every $\varepsilon > 0$, a sampling sequence $a$ with $\sigma(AD(a)) < \varepsilon$. This can be pushed a little further:

**Theorem 2.** If $n > 1$, there exist sampling sequences for $H^\infty$ with $\sigma(AD(a)) = 0$.

Several conjectures can be made regarding necessary or sufficient conditions for sampling in $H^\infty$, although we have not been able to prove any of them. All the
attempts to prove any of these conjectures have led us to the well-known Fatou problem on radial behaviour of holomorphic bounded functions in higher dimension (see [Ru1, Chapter 11]).

We also prove a probabilistic result on random sampling sequences for $H^p$ with prescribed radii, along the lines of the results in [Bo], [Co] and [Ma].

Consider the probability space $\Omega = \prod_{k=1}^{\infty} \Omega_k$, where $\Omega_k$ is the unit sphere $S^n$ for all $k$. $\mathcal{A}_k$ denotes the $\sigma$-algebra of Lebesgue measurable sets on $S^n$, and $P_k$ denotes the normalized Lebesgue measure $\sigma$ on the sphere. An element of $\Omega$ is denoted by $\omega = (\zeta_1, \zeta_2, \ldots)$, where $\zeta_k \in S^n$. Each $\zeta_k : \Omega \rightarrow S^n$ can be viewed as a random variable defined on $S^n$, with values on $S^n$ as well. To construct the space of probability, one can alternatively take $\Omega_k = O(2^n)$, the group of rotations of $\mathbb{C}^n$, $P_k$ the Haar measure on $O(2n)$ and $\mathcal{A}_k$ the $\sigma$-algebra of measurable sets with respect to the Haar measure in $O(2n)$. Then the elements of $\Omega$ are denoted by $\omega = (\mathcal{R}_1, \mathcal{R}_2, \ldots)$.

Given a sequence $a$ we consider a sequence of independent and uniformly distributed random variables $\zeta_k(\omega)$ in $S^n$ (resp. $\mathcal{R}_k^\omega$ in $O(2n)$) and define the associated random sequence as $a(\omega) := \{a_k(\omega)\}_k$, where $a_k(\omega) = |a_k|\zeta_k(\omega)$ (resp. $a_k(\omega) = \mathcal{R}_k^\omega(a_k)$). Notice that $|a_k| = |a_k(\omega)|$ for all $\omega$ and for all $k$.

**Theorem 3.** Let $a$ be a sequence in $\mathbb{B}_n$.

(a) If $\sum_{k=1}^{\infty} (1 - |a_k|)^n = \infty$, then $P(\{\omega : \sigma(AD(a(\omega))) = 1\}) = 1$.
(b) If $\sum_{k=1}^{\infty} (1 - |a_k|)^n < \infty$, then $\sigma(AD(a(\omega))) = 0$ for all $\omega$.

As a consequence of Theorem 1 we have the following:

**Corollary.** Let $a$ be a sequence in $\mathbb{B}_n$.

(a) If $\sum_{k=1}^{\infty} (1 - |a_k|)^n = \infty$, then $P(\{\omega : a(\omega) \text{ is sampling for } H^p\}) = 1$ for any $p \leq \infty$.
(b) If $\sum_{k=1}^{\infty} (1 - |a_k|)^n < \infty$, then $P(\{\omega : a(\omega) \text{ is sampling for } H^p\}) = 0$ for any $p < \infty$.

Some remarks are in order.

When the generalized Blaschke condition $\sum_k (1 - |a_k|)^n < \infty$ holds, the sequence is actually almost surely contained in an $H^p$ zero set, for all $p < \infty$ [Ma, Theorem 1.2]. Thus the generalized Blaschke condition distinguishes two sharply contrasting situations: either $a(\omega)$ is almost surely sampling for $H^p$ or it is almost surely contained in an $H^p$ zero set.

In the unit disk the Blaschke condition on $a$ implies that every $a(\omega)$ is an $H^p$ zero sequence, for all $p \leq \infty$. In particular, $a(\omega)$ is never a sequence of sampling for $H^p$, even for $p = \infty$. On the other hand, when $n > 1$ and $\sum_k (1 - |a_k|)^n < \infty$ we have $\sigma(AD\alpha(a(\omega))) = 0$ for all $\omega$, but as seen in Theorem 2, this is not enough to deduce that $a(\omega)$ is not sampling for $H^\infty(\mathbb{B}_n)$.

As in [Co, Corollary 1] one can also randomize the moduli $|a_k|$ independently of $\{\zeta_k(\omega)\}_k$, and show that Theorem 3 also holds for $a_k(\omega) = r_k(\omega)\zeta_k(\omega)$, where the $\zeta_k(\omega)$ are as before and $\{r_k(\omega)\}_k$ satisfy:

(i) $r_k(\omega) \in (0, 1)$ for all $k$ and: in case (a) almost surely $\sum_k (1 - r_k(\omega))^n = \infty$; in case (b) almost surely $\sum_k (1 - r_k(\omega))^n < \infty$.
(ii) each $r_k(\omega)$ is independent of $\{\zeta_k(\omega)\}_k$.

In the following three sections we prove respectively Theorems 1, 2 and 3.
§2. Proof of Theorem 1

A function $f$ defined on $\mathbb{B}_n$ is said to have admissible limit at $\zeta \in S^n$ when the limit $\lim_{z \to \zeta} f(z)$ exists, is finite and is the same for all $\alpha > 1$. The limit is denoted by $f^*(\zeta)$.

The proof that $\sigma(AD(a)) = 1$ implies $a$ sampling for $H^p$, $p \leq \infty$, is essentially due to Brown, Shields and Zeller, and we include it for the sake of completeness.

For $\alpha > 1$, every $f \in H^p$ has admissible limit at almost every $\zeta$ and $\|f^*\|_p = \|f\|_p$ [Ru1, 5.6.8]. Thus for almost every $\zeta \in AD\alpha(a)$,

$$M_\alpha^p(f) \geq \lim_{z \to \zeta} |f(z)| = |f^*(\zeta)|.$$

Hence if $\sigma(AD\alpha(a)) = 1$, then necessarily $\|M_\alpha^p(f)\|_p \geq \|f^*\|_p$.

Let us see now that $\sigma(AD\alpha(a)) = 1$ is also necessary, if $p < \infty$.

Assume $\sigma(AD\alpha(a)) < 1$. By the same argument as in [Br-Sh-Ze], we may assume that there exist a compact $A \subset S^n$ and $N \in \mathbb{N}$ such that $\sigma(A) > 0$ and $\Gamma_\alpha(\zeta) \cap a \subset B(0, 1 - 1/N)$ for all $\zeta \in A$. We will use the following technical result.

**Lemma.** For any $m \in \mathbb{N} \setminus \{0\}$ and $p > 0$, there exists a positive real function $\psi_m \in C(\overline{\mathbb{B}_n})$ such that:

(i) $\psi_m(z) \leq m$ for all $z \in \overline{\mathbb{B}_n}$, and $\psi_m(z) = m$ for $z \in A$;

(ii) $\psi_m(z) \leq 1$ for $z \in B(0, 1 - 1/N)$;

(iii) $\sigma(\{\zeta \notin A : M_\alpha^p\psi_m(\zeta) \geq 1\}) \leq m^{-p}$.

**Proof.** Let $\varrho(\zeta, \eta) = |1 - \zeta \cdot \eta|$ denote the non-isotropic pseudodistance on $\overline{\mathbb{B}_n}$ and let $\varrho(z, A) := \inf_{\zeta \in A} \varrho(z, \zeta)$. Define

$$\psi_m(z) := |z|^{\mu_m} \max\left(m(1 - \lambda_m \varrho(z, A)), 1/m\right),$$

where $\lambda_m$ and $\mu_m$ are sequences of positive numbers increasing to infinity whose growth will be determined later on.

The property (i) is then clear, and we ensure (ii) by choosing $\mu_m$ large enough so that $m(1 - 1/N)^{\mu_m} \leq 1$.

Let us now prove (iii). Take $\zeta \notin A$ and suppose that there exists $z \in \Gamma_\alpha(\zeta)$ such that $\psi_m \geq 1$. This implies that $m(1 - \lambda_m \varrho(z, A)) \geq 1$; thus $\varrho(z, A) \leq 1/\lambda_m$.

Then, by the triangle inequality for $\varrho^{1/2}$,

$$\varrho(\zeta, A)^{1/2} \leq \left(\frac{\alpha}{2} (1 - |z|^2)\right)^{1/2} + \varrho(z, A)^{1/2} \leq (\sqrt{\alpha} + 1)\varrho(z, A)^{1/2} \leq \frac{(\sqrt{\alpha} + 1)}{\sqrt{\lambda_m}}.$$

Since $S^n \setminus A$ is an open set of finite measure, we may end the proof by choosing $\lambda_m$ large enough so that

$$\sigma(\{\zeta \notin A : \varrho(\zeta, A) \leq (\sqrt{\alpha} + 1)^2 / \lambda_m\}) \leq m^{-p}.$$

□

This Lemma and [Ru2, Theorem 3.5] give us, for any $\varepsilon > 0$, a polynomial $P_m$ such that $|P_m| \leq \psi_m$ on the closed ball, and

$$\sigma(\{\zeta \in S^n : |P_m(\zeta)| < \psi_m(\zeta) - \varepsilon\}) < \varepsilon.$$
For a given \( p > 0 \), and taking \( \varepsilon \) small enough, we use Lemma (i) to obtain the following lower bound:

\[
\int_{S^n} |P_m|^p d\sigma \geq \int_A |P_m|^p d\sigma \geq \frac{1}{2} \int_A |\psi_m|^p d\sigma = \frac{m^p}{2} \sigma(A).
\]

On the other hand, \( M^\alpha P_m(\zeta) \leq M^\alpha \psi_m(\zeta) \) for \( \zeta \notin A \), and by Lemma (ii) also \( M^\alpha P_m(\zeta) \leq \sup_{z \in B(0,1-1/N)} \psi_m(z) \leq 1 \) for \( \zeta \in A \). This and Lemma (iii) yield:

\[
\int_{S^n} (M^\alpha P_m)^p d\sigma \leq \sigma(A) + \int_{S^n \setminus A} (M^\alpha \psi_m)^p d\sigma
\]

\[
\leq \sigma(A) + \int_{\{\zeta \notin A : M^\alpha \psi_m > 1\}} (M^\alpha \psi_m)^p d\sigma + \int_{\{\zeta \notin A : M^\alpha \psi_m \leq 1\}} (M^\alpha \psi_m)^p d\sigma
\]

\[
\leq \sigma(A) + m^{-p}(\sup_{B_n} \psi_m)^p + \sigma(S^n \setminus A) \leq 2.
\]

Since this is bounded independently of \( m \), \( a \) cannot be sampling for \( H^p \). This finishes the proof of Theorem 1.

§3. PROOF OF THEOREM 2

Let \( \{\eta_k\}_k \) be a dense sequence on the sphere, and consider for each \( k \) the big circle \( C_{\eta_k} = \{e^{i\theta} \eta_k : \theta \in [0, 2\pi)\} \). Denote \( E = \bigcup_k C_{\eta_k} \).

Take next a dyadic decomposition of each big circle \( C_{\eta_k} \): for any \( m \in \mathbb{N} \) consider the intervals

\( f_{m,j}^{(k)} = \{ e^{i\theta} \eta_k \in S^n : (j-1)2^{-m} \leq \frac{\theta}{2\pi} < j2^{-m} \}, \quad j = 1, \ldots, 2^m. \)

Let \( c_{m,j}^{(k)} = e^{i2\pi 2^{-m}(j-1/2)} \eta_k \) denote the center of the subinterval \( f_{m,j}^{(k)} \).

Our sequence is defined as \( a = \{a_{m,j}^{(k)}\}_{k,m,j} \), where \( a_{m,j}^{(k)} = (1 - \frac{\alpha_k}{2}) c_{m,j}^{(k)} \) and \( \{\alpha_k\}_k \) is such that \( \sum_k (k \alpha_k)^{-1} < \infty \).

Let us see first that \( a \) is sampling for \( H^\infty \). By construction, and according to the theorem of Brown, Shields and Zeller, on each slice \( D_{\eta_k} = \eta_k \mathbb{D} \) the sequence \( \{a_{m,j}^{(k)}\}_{m,j} \subset D_{\eta_k} \) is sampling for \( H^\infty(D_{\eta_k}) \), since the non-tangential accumulation set is all \( C_{\eta_k} \). Thus, given \( f \in H^\infty \), every slice function \( f_{\eta_k}(\lambda) = f(\lambda \eta_k), \lambda \in \mathbb{D} \), has radial limits \( f_{\eta_k}^{*} \in L^\infty(\mathbb{T}) \) satisfying \( \|f_{\eta_k}^{*}\|_{L^\infty(\mathbb{T})} \leq S_f \), where \( S_f = \sup_a |f| \).

The maximum principle then yields \( |f| \leq S_f \) in \( \bigcup_k D_{\eta_k} \), which by the density of \( \{\eta_k\}_k \) in \( S^n \) already implies \( \|f\|_{\infty} \leq S_f \).

It remains to prove that \( \sigma(AD(a)) = 0 \). Define

\( F = \{ \zeta \in S^n : \varrho(\zeta, C_{\eta_k}) < k\alpha_k \text{ for infinitely many } k \} \),

where \( \varrho \) is the non-isotropic pseudodistance defined at the beginning of the proof of the Lemma. Since \( \sigma(F) \leq \sum_{k \geq p} (k \alpha_k)^{n-1} \) for all \( p \in \mathbb{N} \), we deduce that \( \sigma(F) = 0 \).

On the other hand, for every \( \zeta \notin E \cup F \), the quotient \( \alpha_k/\varrho(\zeta, C_{\eta_k}) \) (which is bounded by \( 1/k \) for \( k \) big enough) tends to 0, so \( \zeta \) is not approachable within an admissible region by points of \( a \). Hence \( AD(a) \) is contained in the zero measure set \( E \cup F \).
§4. Proof of Theorem 3

Proof of (a). It will be enough to show that for some \( \alpha > 1 \)

\[
\int_\Omega \sigma(AD_\alpha(a(\omega))) \, dP(\omega) = 1.
\]

Notice that

\[
(1) \quad AD_\alpha(a(\omega)) = \bigcap_{p \in \mathbb{N}} \bigcup_{k \geq p} I_\alpha(a_k(\omega))
\]

\[= \{ \zeta \in S : \zeta \in I_\alpha(a_k(\omega)) \text{ for infinitely many } k \} \]

where \( I_\alpha(a_k(\omega)) = \{ \zeta \in S^n : a_k(\omega) \in \Gamma_\alpha(\zeta) \} \).

Since the random variables \( \zeta_k(\omega) \) are uniformly distributed one has:

\[
P(\{ \omega : \zeta \in I_\alpha(a_k(\omega)) \}) = \sigma(I_\alpha(a_k)) = C(1 - |a_k|^2)^n
\]

for some constant \( C > 0 \) depending only on \( \alpha \) and the dimension.

Now \( \sum_k P(\{ \omega : \zeta \in I_\alpha(a_k(\omega)) \}) = \infty \), so the Borel-Cantelli lemma yields:

\[
P(\{ \omega : \zeta \in I_\alpha(a_k(\omega)) \text{ for infinitely many } k \}) = 1.
\]

In particular

\[
P(\{ \omega : \zeta \in \bigcup_{k \geq p} I_\alpha(a_k(\omega)) \}) = 1
\]

for all \( \zeta \in S^n \) and all \( p \in \mathbb{N} \). Thus

\[
\int_\Omega \sigma(\bigcup_{k \geq p} I_\alpha(a_k(\omega))) \, dP(\omega) = \int_{S^n} \int_\Omega \bigcup_{k \geq p} I_\alpha(a_k(\omega))(\zeta) \, dP(\omega) \, d\sigma(\zeta) = 1
\]

for all \( p \in \mathbb{N} \). This together with (1) shows that the required equality holds. \( \square \)

Proof of (b). This is immediate from (1) and the fact that \( \sigma(I_\alpha(a_k(\omega))) = C(1 - |a_k|^2)^n \) for all \( \omega \). \( \square \)

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