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A COUNTEREXAMPLE TO A WEAK-TYPE ESTIMATE FOR POTENTIAL SPACES AND TANGENTIAL APPROACH REGIONS

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Abstract. We show that for every potential space $L^1_K(\mathbb{R}^n)$, there exists an approach region for which the associated maximal function is of weak-type, but the boundedness for the completed region is false, which is in contrast with the nontangential case.

1. Introduction

In [NS84] it was proved that Fatou's theorem holds on regions Ω larger than cones (but still nontangential), by means of the boundedness of the associated maximal function M_Ω. One of the key points in that proof is that one could replace the given region by a larger region $\widehat{\Omega}$ obtained by adding a cone at any point of Ω, and then prove that the boundedness of the two maximal functions M_Ω and $M_{\widehat{\Omega}}$ are equivalent. This seems geometrically very natural, since the difference, at any point, between $\widehat{\Omega}$ and Ω, is just the canonical approach region (i.e., a cone).

In [NRS82] Fatou's theorem was extended to some tangential approach regions, when the functions were assumed to have some a priori smoothness (they belonged to a potential space). This result was later on generalized in [RS97] to characterize all the approach regions (under a completion hypothesis similar to the one in [NS84]) for which convergence holds for the potential spaces.

The main result of this paper is to show that, contrary to the case of [NS84], the assumptions on the region assumed in [RS97], which is natural as we mentioned before, from the point of view of convergence, turn out to give different boundedness results for the corresponding maximal operators. In order to clarify this statement, let us introduce some notation.

Let $P_t(x)$ be the Poisson kernel in $\mathbb{R}^{n+1}_+$. Given a set $\Omega \subset \mathbb{R}^{n+1}_+$, we define the maximal function

$$M_\Omega f(x) = \sup_{(y,t)\in\Omega_x} |P_t * f(y)|,$$

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where $\Omega_x = x + \Omega$. If $r : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function, then we define the "cone" for the function r as

$$\Gamma_r(x,t) = \{(y,s) : |x-y| \le r(s) - r(t)\}.$$

If $r(t) = t$, then $\Gamma_t = \Gamma$ is a nontangential cone. We say that Ω satisfies the r-condition if $\Gamma_r(x,t) \subset \Omega$ for all $(x,t) \in \Omega$. For example, in the case of nontangential approach, $r(t) = t$ and the r-condition is the cone condition of [NS84]. The function r is determined, in each case, from the potential space under consideration. In our case,

$$L^1_K(\mathbb{R}^n) = \{f : f = K * F,\ F \in L^1(\mathbb{R}^n)\}.$$

The kernel K is positive and integrable, but unbounded ($K(0) = \infty$), nonnegative and radial (if $|x| = |y|$, then $K(x) = K(y)$), and decreasing (if $|x| \le |y|$, then $K(x) \ge K(y)$). We consider the following norm on the potential space $L^1_K(\mathbb{R}^n)$:

$$\|f\|_{L^1_K(\mathbb{R}^n)} = \inf_{f = K*F} \|F\|_{L^1(\mathbb{R}^n)}.$$

For the space $L^1_K(\mathbb{R}^n)$, we have that if $r_K(t) = \|P_t * K\|_\infty^{-1/n}$, then the region $\Gamma_K = \Gamma_{r_K}$ is tangential, under the above assumptions on the kernel K (see [NRS82]). This can be expressed as

$$\lim_{t\to 0} \frac{r_K(t)}{t} = \infty. \tag{1.1}$$

In the case of the Bessel potential spaces $L^1_\alpha(\mathbb{R}^n) = \{F * G_\alpha : F \in L^1(\mathbb{R}^n)\}$ (where G_α is the Bessel potential), then $r_{G_\alpha}(t) = t^{1-\alpha/n}$. As a consequence of Theorem 2.6 in [RS97], we know that if Ω satisfies the r_K-condition, then $M_\Omega : L^1_K(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ if and only if $|\Omega(t)| \le C(r_K(t))^n$, for all $t > 0$, where $\Omega(t) = \{x : (x,t) \in \Omega\}$. Given an approach region Ω, we can always define the smallest region containing Ω, satisfying the r_K-condition as follows:

$$\widehat{\Omega}_K = \{(y,t) \in \mathbb{R}^{n+1}_+ : \exists (x,s) \in \Omega, |x-y| \le r_K(t) - r_K(s)\}.$$

Then it is easy to show that $\widehat{\Omega}_K$ satisfies the r_K-condition, and $\Omega \subset \widehat{\Omega}_K$.

In the nontangential case it was proved in [NS84] that the operator $M_\Omega : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ if and only if $M_{\widehat{\Omega}} : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$. However, we will show in Theorem 2.1 that under the above conditions on K, and hence (1.1) holds, then this equivalence fails in general. This is somehow surprising, since $M_{\Gamma_K} : L^1_K(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ (see [NRS82]). Therefore, even though the boundary convergence holds within both Ω and the "cone" Γ_K, it fails for the completed region $\widehat{\Omega}_K$.

2. Main theorem

We now prove our main result, namely that the characterization in [NS84] does not hold for tangential regions: a maximal operator M_Ω can be of weak-type (1,1) while the maximal operator for the completed region, $M_{\widehat{\Omega}_K}$ fails to be of weak-type (1,1).

Theorem 2.1. *For each of the potential spaces $L^1_K(\mathbb{R}^n)$, there exists a region Ω with the following properties:*

(i) *Ω satisfies the cone condition.*
(ii) $|\Omega(t)| \le C(r_K(t))^n$.

(iii) $|\{M_\Omega f > \lambda\}| \le C \dfrac{\|f\|_{L^1_K}}{\lambda}$.

(iv) $\dfrac{|\widehat{\Omega}_K(t)|}{(r_K(t))^n}$ *is unbounded.*

(v) $M_{\widehat{\Omega}_K}$ *is not of weak type* $(1,1)$.

The proof uses the following lemma from [Sjö83].

Lemma 2.2. *Assume the operators T_k, $k = 1, 2, \ldots,$ are defined in $\mathbb{R}^n$ by*

$$T_k f(x) = \sup_{v \in I_k} (K_v * |f|(x)), \tag{2.1}$$

where the K_v are integrable and nonnegative in $\mathbb{R}^n$, and the index sets I_k are such that $T_k f$ are measurable for any measurable f. For each $i = 1, \ldots, n$, let a sequence $\{\gamma_{ki}\}_{k=1}^\infty$ be given with $\gamma_{ki} \ge \gamma_{k+1,i} > 0$, and assume the T_k are uniformly of weak-type $(1,1)$, with

$$\operatorname{supp} K_v \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon |x_i| \le \gamma_{ki},\ i = 1, \ldots, n\},\ v \in I_k,$$

and

$$\int K_v^* \le C_0, \quad v \in \bigcup_k I_k,$$

where for $v \in I_k$,

$$K_v^*(x) = \sup\{K_v(x+y) \colon |y_i| \le \gamma_{k+\mathcal{N},i},\ i = 1, \ldots, n\} \tag{2.2}$$

for some fixed natural number $\mathcal{N}$. Then the operator

$$Tf(x) = \sup_k T_k f(x)$$

is of weak-type $(1,1)$.

Proof of Theorem 2.1. For simplicity, we will usually drop the subscript K, and we will write $r(t) = r_K(t)$, although for the regions Γ_K we will keep it. Also, we only consider the case $n = 1$ (higher dimensions require minor modifications).

We start with the construction of the region Ω: for this we choose a set of points ω from which we obtain the region Ω by completing ω with nontangential cones. To construct ω, we define a curve $\gamma(t)$,

$$\gamma(t) = N(t) r(t),$$

where $N(t)$ is a function that tends to infinity as $t \to 0$. The curve $(\gamma(t), t)$ stays well outside Γ_K ($\gamma(t)/r(t) = N(t) \to \infty$ as $t \to 0$). There are some restrictions on how fast $N(t)$ may increase. The first condition on $N(t)$ is that the curve $\gamma(t)$ approaches the origin as $t \to 0$, i.e.,

$$\lim_{t \to 0} \gamma(t) = \lim_{t \to 0} N(t) r(t) = 0. \tag{2.3}$$

Now choose a starting level t_1 satisfying

$$\gamma(t_1) - ([N(t_1)] - 1) r(t_1) \ge r(t_1) > 3t_1.$$

(If t_1 is small enough, the last inequality holds, due to the tangentiality of Γ_K (see (1.1).) The first $[N(t_1)]$ points in the set ω are:

$$\{(x_i^1, t_1) :\ x_i^1 = \gamma(t_1) - i r(t_1),\ 0 \le i \le [N(t_1)] - 1\}.$$

Observe that the points (x_i^1, t_1) are well outside the cone Γ.

We proceed inductively, assuming that we have chosen t_{k-1} and added the $[N(t_{k-1})]$ points at this level to ω. Now choose any $t_k < t_{k-1}$ satisfying

$$\gamma(t_k) + (t_{k-1} - t_k) < 2t_{k-1}, \tag{2.4}$$

which implies that after adding Γ to $(\gamma(t_k), t_k)$, the region thus obtained is contained in the nontangential cone Γ_{2t}, at height t_{k-1}. It is obvious that this cone does not intersect the previously chosen points in ω. Now add the following $[N(t_k)]$ points to the set ω:

$$\{(x_i^k, t_k) : \; x_i^k = \gamma(t_k) - ir(t_k), \; 0 \le i \le [N(t_k)] - 1\}.$$

This finishes the construction on the level t_k. If we continue this way, the set of points ω is obtained. It is clear that ω contains points arbitrarily close to the boundary, whose number at height t_k increases to infinity as $t_k \to 0$. The region Ω is then defined by completing ω with the nontangential cone Γ. We now check condition (ii).

We start at any level t_k and move upwards to t_{k-1}. At the level t_k the region Ω consists of one part that is contained in a fixed nontangential cone with vertex at the origin, which comes from the lower levels (see (2.4)), and here (ii) is obvious.

The other part consists of $[N(t_k)]$ intervals: as we move upwards, first each interval (which is of the form $\{x : |x - x_i^k| < t - t_k\} \times \{t\}$, for $t > t_k$ and $0 \le i \le [N(t_k)] - 1$) will have a size at height t which is bounded from above by t (this is the case if t is below $t_k + \frac{1}{2}r(t_k)$), and the size of the union of these intervals is then bounded from above by $tN(t_k)$. For $t > t_k + \frac{1}{2}r(t_k)$ the intervals will have met, and the size estimate follows, if it holds while they are disjoint. Thus, if we impose on $N(t)$ that

$$tN(t) < r(t), \tag{2.5}$$

then the size of the disjoint intervals will have the correct upper bound. Since the region Γ_K is tangential (see (1.1)), this can be achieved, while $N(t)$ tends to infinity as $t \to 0$.

If we instead complete the region Ω with the tangential region associated with the potential space L_K^1, that is Γ_K, then $|\widehat{\Omega}_K(t)|/r(t)$ will not be bounded, since otherwise we could find a constant C such that

$$|\widehat{\Omega}_K(t)| \le Cr(t). \tag{2.6}$$

Let $T > 0$ be the level where the $[N(t_k)]$ regions added at the level t_k have met: this T satisfies that $r(T) - r(t_k) = r(t_k)/2$. A lower bound on $|\widehat{\Omega}_K(T)|$ is then $[N(t_k)]r(t_k)$. For (2.6) to hold we must have

$$[N(t_k)]r(t_k) \le Cr(T) = \frac{3}{2}Cr(t_k), \tag{2.7}$$

and this is only possible if $N(t)$ is bounded. Therefore, $\widehat{\Omega}_K$ cannot satisfy (ii), and hence, $M_{\widehat{\Omega}_K}$ cannot be of weak-type (1,1) (by Theorem 2.6 in [RS97]).

Now that the region Ω is defined, and we have dealt with (i), (ii), (iv), and (v) as long as $N(t)$ satisfies (2.3) and (2.5) (e.g., for the case of the Bessel potentials G_α, one can take $N(t) = [t^{-\alpha(1-\alpha)^3}/\log(1/t)]$), we need to prove the weak-type of the maximal operator (i.e., (iii)). For a set $\Omega \subset \mathbb{R}_+^{n+1}$ and a function u defined in $\mathbb{R}_+^{n+1}$ we define the maximal operator $\mathcal{M}_\Omega u(x) = \sup_{\Omega_x} |u|$. Hence, $\mathcal{M}_\Omega(P_t * f)(x) = M_\Omega f(x)$. We can, without loss of generality, assume that the function F is positive.

We split the kernel $K_t(x) = P_t * K(x)$ into two parts, the local part of the kernel and the tail:

$$K_t(x) = \left(\chi_{|x|<3\gamma(t)} + \chi_{|x|>3\gamma(t)}\right) K_t(x) = K_{1,t} + K_{2,t}.$$

First we consider the tail, $K_{2,t}$. We need to estimate the following:

$$(K_{2,t} * F)(x + x'), \text{ where } (x', t) \in \Omega \subset \{(y, t) : \ |y| \leq \gamma(t)\}.$$

Assuming $|x'| \leq \gamma(t)$, we have

$$\begin{aligned}
(K_{2,t} * F)(x + x') &= \int_{\{|y|>3\gamma(t)\}} K_t(y)F(x + x' - y)dy \\
&= \int_{\{|y+x'|>3\gamma(t)\}} K_t(y + x')F(x - y)dy \\
&\leq \int_{\mathbb{R}} K_t(y/2)F(x - y)dy.
\end{aligned}$$

Since K is radially decreasing, the same is true for K_t, and the boundedness of $\mathcal{M}_\Omega(K_{2,t} * F)$ then follows from Lemma 2.2 in [NRS82].

We now turn to the local part of the kernel, i.e., $K_{1,t}$. Let ω_k be the part of ω whose points have the second coordinate equal to t_k: $\omega_k = \{x\colon (x, t_k) \in \omega\}$. Let

$$\Omega_k = (\omega_k + \Gamma) \cap \{(x, t)\colon x \in \mathbb{R},\ t_k \leq t \leq t_{k-1}\}$$

for $k > 1$, and for $k = 1$, let $\Omega_1 = \omega_1 + \Gamma$. Then $\Omega \subset \Gamma_{3t} \cup (\bigcup \Omega_k)$. We split the operator as

$$\begin{aligned}
\mathcal{M}_\Omega(K_{1,t} * F)(x) &\leq \sup_k \mathcal{M}_{\Omega_k}(K_{1,t} * F)(x) + \mathcal{M}_{\Gamma_{3t}}(K_{1,t} * F)(x) \\
&= \sup_k T_k F(x) + \mathcal{M}_{\Gamma_{3t}}(K_{1,t} * F)(x),
\end{aligned}$$

where $T_k F(x) = \mathcal{M}_{\Omega_k}(K_{1,t} * F)(x)$. To use Lemma 2.2 we need uniform weak-type (1,1) estimates for the operators T_k, and they also have to fit the terminology of Lemma 2.2, which we will do below. The main advantage of the lemma is that we can assume t is in a fixed interval, away from 0.

To obtain the weak-type (1,1) estimate, we first consider the part of Ω_k that lies between the levels t_k and $t_k + \frac{1}{2}r(t_k)$, namely $\Omega_k^1 = \{(x, t) \in \Omega_k\colon t_k < t < t_k + \frac{1}{2}r(t_k)\}$. Ω_k^1 consists of $[N(t_k)]$ truncated nontangential cones with vertices at the points (x_i^k, t_k), $i = 0, \ldots, [N(t_k)] - 1$. Let $\Omega_{k,i}^1 = ((x_i^k, t_k) + \Gamma) \cap \Omega_k^1$, for $i = 0, \ldots, [N(t_k)]$, where we define $x_{[N(t_k)]}^k = 0$. Then,

$$\begin{aligned}
\|\mathcal{M}_{\Omega_k^1}(K_{1,t} * F)\|_{1,\infty} &\leq \|\sup_{0\leq i\leq [N(t_k)]-1} \mathcal{M}_{\Omega_{k,i}^1}(K_{1,t} * F)\|_{1,\infty} \\
&\leq \sum_{i=0}^{[N(t_k)]-1} \|\mathcal{M}_{\Omega_{k,i}^1}(K_{1,t} * F)\|_{1,\infty} \qquad (2.8) \\
&\leq N(t_k)\|\mathcal{M}_{\Omega_{k,[N(t_k)]}^1}(K_{1,t} * F)\|_{1,\infty}.
\end{aligned}$$

The last inequality follows from translation invariance. The operator needs to be bounded uniformly in k; so we need to see that the factor $N(t_k)$ does not cause any problem. To proceed, we make a dyadic decomposition of the kernel $K_{1,t}$, and we get (F is positive)

$$\begin{aligned}(K_{1,t}(x)\chi_{|x|<3\gamma(t)}) * F &\leq \sum_{k=1}^{[C\log\gamma(t)/t]} (K_{1,t}(2^{k-1}t)\chi_{|x|<2^k t}) * F \\ &\leq C \sum_{k=1}^{[C\log\gamma(t)/t]} (2^{k-1}t)\left(\frac{K_{1,t}(2^{k-1}t)}{2^k t}\chi_{|x|<2^k t}\right) * F \\ &\leq C \sum_{k=1}^{[C\log\gamma(t)/t]} (2^{k-1}t)K_{1,t}(2^{k-1}t)MF(x) \\ &\leq CMF(x)\int_t^{3\gamma(t)} K_{1,t}(x)dx,\end{aligned}$$

where $MF(x)$ is the usual Hardy-Littlewood maximal function. In order to bound (2.8) uniformly in k, we must find a bound on the integral times $N(t_k)$. We replace the limits of integration with the smallest (respectively the largest) t allowed; i.e.,

$$N(t_k)\int_{t_k}^{\gamma(t_k+\frac{1}{2}r(t_k))} K_{1,t}(x)dx \leq N(t_k)\|P_t\|_{L^1}\int_{t_k}^{\gamma(t_k+\frac{1}{2}r(t_k))} K(x)dx. \tag{2.9}$$

The remaining integral in the right-hand side can easily be seen to decrease to 0 as $k\to\infty$. It may happen that $N(t)$ (which so far only needs to satisfy (2.3) and (2.5)) increases too fast for the product above to be uniformly bounded. If this is the case, we describe how to overcome this obstacle, by slightly modifying ω (and hence Ω). We start with a function $N(t)$ that satisfies conditions (2.3) and (2.5). To get uniform boundedness for (2.9) we fix a constant C, and define a new function $\widetilde{N}(t)$ (on $\{t_k\}$):

$$\widetilde{N}(t_k) = \min\left\{N(t_k), C\left(\int_{t_k}^{\gamma(t_k+\frac{1}{2}r(t_k))} K(x)dx\right)^{-1}\right\}.$$

Then, $\widetilde{N}(t_k)$ tends to infinity, as $k\to\infty$. We modify ω as follows: the curve $\gamma(t)=N(t)r(t)$ will remain the same, and the sequence $\{t_k\}_{k=1}^{\infty}$ will not be altered. But instead of adding $[N(t_k)]$ points at levels t_k, we add $[\widetilde{N}(t_k)]$ points:

$$\{(x_i^k,t_k):\ x_i^k=\gamma(t_k)-ir(t_k),\ 0\leq i\leq[\widetilde{N}(t_k)]-1\}.$$

This way we obtain a set of points $\widetilde{\omega}$, for which all previous estimates still hold, and (2.9) is uniformly bounded. By slight abuse of notation, the region obtained by completing $\widetilde{\omega}$ with nontangential cones will also be denoted Ω below.

Thus, we can estimate the maximal operator by the usual Hardy-Littlewood maximal function, which gives the weak-type (1,1) for the operator $F\mapsto \mathcal{M}_{\Omega_k^1}(K_{1,t}* F)$ uniformly in k, if $k>1$.

For the rest of Ω_k, i.e., if $t_k+\frac{1}{2}r(t_k)\leq t\leq t_{k-1}$, if we complete this part with respect to Γ_K it will still satisfy the size condition, since the level sets consist of one interval. Hence, the weak-type (1,1) of the operator

$$F\mapsto \sup_{\substack{t>t_k+\frac{1}{2}r(t_k)\\ (x,t)\in\Omega_k}} (K_{1,t}*F)(x),$$

follows. This completes the proof of the uniform weak-type (1,1) of $T_k,\ k>1$.

The weak-type (1,1) for $\mathcal{M}_{\Omega_1}(K_{1,t} * F)$ follows by the same methods. First take that part of Ω_1 that lies between the levels t_1 and $t_1 + \frac{1}{2}r(t_1)$. Again, we will get a similar expression as above, and this can be dealt with the same way. When $t > t_1 + \frac{1}{2}r(t_1)$, the region Ω_1 is contained in the tangential region Γ_K, and the weak-type (1,1) is proved.

Finally, we must check that our operators can be defined as in Lemma 2.2, and that they satisfy the assumptions of the lemma. Let the index set I_k be equal to Ω_k, and set for $v = (x', t) \in I_k$,

$$K_v(x) = K_{1,t}(x + x').$$

Then $T_k F(x) = \sup_{v \in I_k}(K_v * F)(x)$. To estimate the support of $K_v = K_{(x,t)}$, we see that the support is largest when $t = t_{k-1}$, which is the largest t in the index set I_k. The support of $K_{1,t_{k-1}}$ is contained in the set $\{x\colon |x| \leq \gamma(t_{k-1})\}$; hence, we can bound the support of K_v, $v \in I_k$, taking $\gamma_k = 3\gamma(t_{k-1})$. If we take $\mathcal{N} = 2$, then we can bound the integral of K_v^* uniformly in $v \in \bigcup I_k$. With an x outside the support of K_v, we need only increase the support of the kernel $K_{1,t}$. If $v \in I_k$, using (2.4) we obtain:

$$\begin{aligned} \int_0^\infty K_v^*(x)dx &\leq \int_0^{\gamma_{k+2}} K_v^*(x)dx + \int_{\gamma_{k+2}}^\infty K_v^*(x)dx \\ &\leq \int_0^{3\gamma(t_{k+1})} K_t(0)dx + \int_{3\gamma(t_{k+1})}^\infty K_t(x - 3\gamma(t_{k+1}))dx \\ &\leq 3\frac{\gamma(t_{k+1})}{r(t_k)} + \int_0^\infty K_t(x)dx \leq 3\frac{t_k}{r(t_k)} + \|K_t\|_{L^1}, \end{aligned}$$

and from (1.1) it follows that this expression is uniformly bounded in k for all $v \in \bigcup I_k$. Lemma 2.2 now gives the weak-type (1,1) for $\sup_k T_k$, and hence for $\mathcal{M}_\Omega K_{1,t}$. Finally, we have proved a weak-type estimate for both $\mathcal{M}_\Omega K_{1,t}$ and $\mathcal{M}_\Omega K_{2,t}$, and we have finished the proof of the theorem. $\square$

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