ON VECTORIAL POLYNOMIALS AND COVERINGS IN CHARACTERISTIC 3

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Abstract. For \( K \) a field containing the finite field \( \mathbb{F}_9 \) we give explicitly the whole family of Galois extensions of \( K \) with Galois group \( 2S_4 \times Q_8 \) or \( 2S_4 \times D_8 \) and determine the discriminant of such an extension.

1. Introduction

The motivation of this work is the problem of resolution of singularities in positive characteristic, more precisely the ideas presented by S.S. Abhyankar in \cite{Abhyankar:1994}. Following Abhyankar, loc. cit. Section 18, let \( N_{k,t}^d \) denote a neighborhood of a simple point on a \( d \)-dimensional algebraic variety over an algebraically closed field \( k \) of characteristic \( p \) from which we have deleted a divisor having a \( t \)-fold normal crossing at the simple point and let \( \pi^L_A(N_{k,t}^d) \) be the set of all Galois groups of finite unramified local Galois coverings of \( N_{k,t}^d \). In his landmark paper \cite{Abhyankar:1966}, Abhyankar, while working on local uniformization of algebraic varieties in a positive characteristic, proved the inclusion \( \pi^L_A(N_{k,t}^d) \subset P_t(p) \), where \( P_t(p) \) denotes the set of finite groups \( G \) such that the quotient \( G/p(G) \) of \( G \) by the subgroup \( p(G) \) generated by its \( p \)-Sylow subgroups is abelian, generated by \( t \) generators. Later, using so-called projective and vectorial polynomials, he proved (see \cite{Abhyankar:1967,Abhyankar:1994}) that \( \pi^L_A(N_{k,t}^d) \) contains \( \text{PGL}(m,q) \) and \( \text{GL}(m,q) \), for every integer \( m > 1 \) and every power \( q > 1 \) of \( p \). Recently D. Harbater et al. \cite{Harbater:2001} proved that for a group \( G \) to belong to \( \pi^L_A(N_{k,t}^d) \) it is necessary that \( p(G) \) admit an abelian supplement in \( G \) of rank \( \leq t \). In \cite{Abhyankar:1994}, Abhyankar exhibited some examples due to G. Stroth of groups contained in \( P_t(p) \) but not satisfying the abelian supplement condition. In characteristic 3, and for \( t = 3 \), the Stroth groups are the groups \( 2S_4 \times H \), where \( 2S_4 \) denotes a double cover of the symmetric group \( S_4 \), \( H \) is either the quaternion group \( Q_8 \) or the dihedral group \( D_8 \) of order 8 and \( \ast \) denotes central product. In this paper, for \( K \) a field containing the finite field \( \mathbb{F}_9 \) of nine elements, we give explicitly the whole family of Galois extensions of \( K \) with Galois group \( 2S_4 \times H \), and determine the discriminant of such an extension. We note that in \cite{Crespo:2004}, the first author provided an explicit construction of \( 2S_4 \times Q_8 \)-extensions of fields containing \( \mathbb{F}_9 \) using her previous results on Galois embedding problems based on Serre’s trace formula, \cite{Serre:1979}. Here we use a different method of construction combining Abhyankar’s embedding criterion \cite{Abhyankar:1994}.
and Serre’s trace formula, and reach a more explicit and simple formula both for $2S_4 \ast Q_8$- and $2S_4 \ast D_8$-extensions as well as an explicit formula for the discriminant of such extensions. The explicit determination of the discriminant of these extensions is a step towards local uniformization for three-dimensional varieties in positive characteristic.

2. Preliminaries

Let us first recall the definitions and fix the notation. We denote by $2S_n$ one of the two double covers of the symmetric group $S_n$ reducing to the nontrivial double cover $2A_n$ of the alternating group $A_n$, and by $H$ either the quaternion group $Q_8$ or the dihedral group $D_8$, double covers of the Klein group $V_4$. Let $K$ be a field of characteristic different from 2 and let $\tilde{\mathcal{L}}|K$ be a Galois extension with Galois group the group $2S_4 \ast H$. Then if $L$ is the field fixed by the center of $2S_4 \ast H$, we have $\text{Gal}(L|K) \simeq S_4 \times V_4$, and for $L_1, L_2$ the fixed subfields of $L$ by $V_4$ and $S_4$, respectively, we have $\text{Gal}(L_1|K) \simeq S_4$ and $\text{Gal}(L_2|K) \simeq V_4$. Therefore we obtain the whole family of Galois extensions with Galois group $2S_4 \ast H$ of a field $K$ by constructing the whole family of $2S_4 \ast H$-extensions containing a given arbitrary $S_4$-extension of the field $K$. Let us now be given a polynomial $f(X) \in K[X]$ of degree 4 with Galois group $S_4$ and splitting field $L_1$ over $K$. We want to determine when $L_1$ is embeddable in a Galois extension of $K$ with Galois group $2S_4 \ast H$. This fact is equivalent to the existence of a Galois extension $L_2|K$ with Galois group $V_4$, disjoint from $L_1$, and such that, if $L$ is the compositum of $L_1$ and $L_2$, the Galois embedding problem

\begin{equation}
2S_4 \ast H \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K)
\end{equation}

is solvable. We recall that a solution to this embedding problem is a quadratic extension $\tilde{L}$ of the field $L$, which is a Galois extension of $K$ with Galois group $2S_4 \ast H$ and such that the restriction epimorphism between the Galois groups $\text{Gal}(\tilde{L}|K) \rightarrow \text{Gal}(L|K)$ agrees with the given epimorphism $2S_4 \ast H \rightarrow S_4 \times V_4$.

If $\tilde{L} = L(\sqrt{r})$ is a solution, then the general solution is $L(\sqrt[r]{r})$, $r \in K^*$. Given a Galois extension $L_1|K$ with Galois group $S_4$, in order to obtain all $2S_4 \ast H$-extensions of $K$ containing $L_1$, we have to determine all $V_4$-extensions $L_2$ of $K$, disjoint from $L_1$, and such that the embedding problem (1) is solvable.

Let us consider the double covers $2S_4 \rightarrow S_4$ and $H \rightarrow V_4$ and let $\varepsilon_1 \in H^2(S_4, \pm 1)$, $\varepsilon_2 \in H^2(V_4, \pm 1)$ denote the corresponding cohomology elements. Let $\pi_1 : S_4 \times V_4 \rightarrow S_4$ and $\pi_2 : S_4 \times V_4 \rightarrow V_4$ be the two projections and let $\pi_1^*, \pi_2^*$ be the induced morphisms between the 2-cohomology groups. Then the element $\varepsilon = \pi_1^*(\varepsilon_1) \cdot \pi_2^*(\varepsilon_2) \in H^2(S_4 \times V_4, \{\pm 1\})$ corresponds to the double cover $2S_4 \ast H$ of $S_4 \times V_4$. This implies that the element in $H^2(G_K, \{\pm 1\})$ giving the obstruction to the solvability of the embedding problem (1) is equal to the product of the elements giving the obstructions to the solvability of the embedding problems $2S_4 \rightarrow S_4 \simeq \text{Gal}(L_1|K)$ and $H \rightarrow V_4 \simeq \text{Gal}(L_2|K)$.

Let us now specify notation by writing $2^+ S_n$ or $2^- S_n$ depending on whether transpositions in $S_n$ lift in the double cover to involutions or to elements of order 4. Let $E = K[X]/(f(X))$, for $f(X)$ the polynomial of degree 4 realizing $L_1$, let $Q_E$ denote the trace form of the extension $E|K$, i.e. $Q_E(x) = \text{Tr}_{E|K}(x^2)$, and let $d$ be the discriminant of the polynomial $f(X)$. Let $L_2 = K(\sqrt{a}, \sqrt{b})$. We denote by $w$ the Hasse-Witt invariant of a quadratic form and by $(\cdot, \cdot)$ a Hilbert symbol.
By [3] the obstruction to the solvability of the embedding problem $2^+ S_4 \rightarrow S_4 \simeq \text{Gal}(L_4|K)$ is equal to $w(Q_E)(\pm 2, d) \in H^2(G_K, \{\pm 1\})$. By [10], the obstruction to the solvability of $Q_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$ is equal to $(a, b).(-1, ab) \in H^2(G_K, \{\pm 1\})$ and by e.g. [2], the obstruction to the solvability of $D_8 \rightarrow V_4 \simeq \text{Gal}(L_2|K)$ is equal to $(a, b) \in H^2(G_K, \{\pm 1\})$ (here we assume that the order 4 elements of $D_8$ are mapped on the nontrivial element in $\text{Gal}(L_2|K)$ fixing $\sqrt{ab}$).

From now on, we assume that $K$ is a field of characteristic 3. We write $f(X) = X^4 + s_2 X^2 - s_3 X + s_4$. By computation of the trace form $Q_E$, we obtain

$$w(Q_E) = (ds_2, (s_4^2 - s_4)s_2) \cdot (-1, s_2^2 - s_4).$$

If we further assume that $K$ contains $\mathbb{F}_9$, i.e. that $-1 \in K^2$, the solvability of the embedding problem [1] is equivalent to

$$(2) \quad (ds_2, (s_2^2 - s_4)s_2) = (a, b),$$

that is, the equality of two Hilbert symbols.

We now recall the isomorphisms $S_4 \simeq \text{PGL}(2, 3)$ and $2^+ S_4 \simeq \text{GL}(2, 3)$ and state Abhyankar’s Embedding Criterion [3] (1.1), and Polynomial Theorem [3] (2.1), (3.7), in our particular case.

**Proposition 1.** Let $K$ be a field of characteristic 3, and let $M|K$ be a Galois extension with Galois group $\text{PGL}(2, 3)$. The embedding problem

$$(3) \quad \text{GL}(2, 3) \rightarrow \text{PGL}(2, 3) \simeq \text{Gal}(M|K)$$

is solvable $\iff M|K$ is the splitting field of a projective polynomial $Y^4 + c_3 Y + c_4 \in K[Y]$. Moreover, if $|K| \geq 9$, the splitting field of the vectorial polynomial $Y(Y^8 + c_3 Y^2 + c_4)$ is a solution to the embedding problem [3].

3. **Main results**

Under the hypothesis $\text{char} K = 3$, the two equivalent conditions to the solvability of the Galois embedding problem $2^+ S_4 \rightarrow S_4 \simeq \text{Gal}(M|K)$ obtained by applying Serre’s trace formula and Abhyankar’s Embedding Criterion can directly be seen to be equivalent. Indeed, let $M|K$ be a Galois extension with the Galois group $S_4$ given as the splitting field of a polynomial $f(X) = X^4 + s_2 X^2 - s_3 X + s_4 \in K[X]$, let $d$ be the discriminant of $f(X)$, let $x$ be a root of $f(X)$ in $M$ and let $E = K(x)$. Then $M$ is the splitting field of a polynomial of the form $Y^4 + c_3 Y + c_4 \in K[Y]$ if and only if there exists elements $a_0, a_1, a_2, a_3 \in K$ such that the irreducible polynomial over $K$ of the element $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ has such a form. By computation, this is equivalent to the conditions $a_0 = -a_2 s_2$ and $Q(a_1, a_2, a_3) := s_2 a_1^2 + (s_2^2 - s_4) a_2^2 + s_3 a_3^2 + (s_4^2 + s_4) a_1 a_3 + 2s_2 s_3 a_2 a_3 = 0$. Now the quadratic trace form $Q_E$, for $E = K(x)$, is equivalent to $1 + Q$, for $Q$ the quadratic form in $a_1, a_2, a_3$ in the second condition. If we assume $w(Q_E) = (2, d)$, then we have $Q_E \sim \langle 1, 1, 2, 2d \rangle$ (see [3] 3.2) which implies $Q \sim \langle 1, 2, 2d \rangle$ and this last quadratic form represents 0 over any field $K$ of characteristic 3. Reciprocally, assume that $Q$ represents 0 over $K$. Diagonalizing $Q$, we obtain $(s_2, m, s_2 md)$, with $m = s_2^2 - s_4$, and so we have $s_2 b_1^2 + m b_2^2 + s_2 m b_3^2 = 0$, for some $b_1, b_2, b_3 \in K$, which implies $(-ds_2, -ms_2) = 1$, and so $ds_2, ms_2 = (-1, md) \cdot (-1, -1)$. Hence we get $w(Q_E) = (ds_2, ms_2) \cdot (-1, m) = (-1, d) = (2, d)$.

**Theorem 1.** Let $K$ be a field of characteristic 3 containing $\mathbb{F}_9$, and let $f(X) = X^4 + s_2 X^2 - s_3 X + s_4 \in K[X]$, with Galois group $S_4$ and $L_4$ the splitting field of $f(X)$
over $K$. Let $d = s_3^2 + s_2^2 s_1^2 + s_2^2 s_4 - s_2^2 s_3^2$ be the discriminant of the polynomial $f(X)$. The family of elements $a, b$ in $K$ such that $(a, b) = (ds_2, ms_3)$, where $m = s_2^2 - s_4$, can be given in terms of an arbitrary invertible matrix $P = (p_{ij})_{1 \leq i, j \leq 3} \in GL(3, K)$ as $a = dA, b = s_2 mF$, where

$$A = s_2 p_{11}^2 + mp_{21}^2 + ds_2 p_{31}^2,$$

$$F = dm p_{13}^2 + ds_2 p_{23}^2 + p_{33}^2, \quad \text{with} \quad P_{ij} = \begin{vmatrix} p_{ii} & p_{ij} \\ p_{ji} & p_{jj} \end{vmatrix}.$$ 

Let $L_2 = K(\sqrt{a}, \sqrt{b})$ and assume that $L_2 | K$ has Galois group $V_4$ and $L_1 \cap L_2 = K$ (i.e. that the elements $a, b, ab, da, db, dab$ are not squares in $K$). Let $L = L_1 \cdot L_2$. For $x$ a root of the polynomial $f(X)$, take $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$, with $a_0 = -s_2 a_2$, $a_1 = dm \sqrt{-1} (ns_2 p_{11} P_{23} - p_{21} P_{33} + m n p_{21} P_{23}) + m \sqrt{a} (d P_{13} + n P_{33})$, $a_2 = ds_2 \sqrt{-1} (p_{11} P_{33} - s_2^2 s_3 p_{11} P_{23} - ms_2 s_3 p_{21} P_{13} - d m p_{31} P_{13}) - s_2 \sqrt{a} (s_2 s_3 P_{33} + d P_{23})$, $a_3 = ds_2 \sqrt{-1} (s_{211} P_{23} + m p_{21} P_{13}) + ms_2 \sqrt{a} P_{33}$, 

where $n = s_2^2 + s_4$. Then $L(\sqrt{r})$, $r \in K^*$, is the general solution to the embedding problem

$$2^+ S_4 \ast D_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L/K).$$

\textbf{Proof.} By [8, 3.2], the equality of Hilbert symbols [2] is equivalent to the $K$-equivalence of quadratic forms

$$(4) \quad \langle ds_2, ms_3, dm \rangle \sim \langle a, b, ab \rangle.$$ 

The family of quadratic forms $K$-equivalent to $R := \langle ds_2, ms_3, dm \rangle$ is given by $P^3 R P$, for $P$ running over $GL(3, K)$. By diagonalizing $P^3 R P$, we obtain $\langle dA, s_2 mF, dA s_2 mF \rangle$, with $A$ and $F$ as in the statement. Let $a = dA, b = s_2 mF$. Now, we have $(a, b) = 1 \in H^2(G_{K(\sqrt{a}), \pm 1})$ and, as $a \notin K^2$ and $L_1 \cap K(\sqrt{a}) = K$, the extension $L_1(\sqrt{a}) | K(\sqrt{a})$ has Galois group $S_4$, and the Galois embedding problem $2^+ S_4 \rightarrow S_4 \simeq \text{Gal}(L_1(\sqrt{a}) | K(\sqrt{a}))$ is solvable. By the argument preceding Theorem 1, there exist $a_1, a_2, a_3 \in K(\sqrt{a})$ such that $Q(a_1, a_2, a_3) = 0$, and for the element $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ we have that $\text{Irr}(y, K(\sqrt{a}))$ is a projective polynomial. Also, by Abhyankar’s Polynomial Theorem (see Proposition 1), the splitting field of the vectorial polynomial $Y \text{Irr}(\sqrt{a}, K(\sqrt{a}))$, that is, the field $L_1(\sqrt{\alpha})(\sqrt{\beta})$, is a solution to the Galois embedding problem $2^+ S_4 \rightarrow S_4 \simeq \text{Gal}(L_1(\sqrt{\alpha}) | K(\sqrt{\alpha}))$. Now our aim is to compute explicitly such elements $a_i$. Diagonalizing $Q$, we obtain $\langle s_2, m, s_{2m} \rangle$ and from [2] we get that $\langle s_2, m, s_{2m} \rangle \sim \langle A, s_2 mF, s_2 mF d \rangle$ and the basis change matrix can be written down explicitly in terms of the matrix $P$. Now the vector $(0, d \sqrt{-1}, \sqrt{a}) \in K(\sqrt{a})^3$ annihilates the quadratic form $\langle A, s_2 mF, s_2 mF d \rangle$, and from it we obtain the values for $a_1, a_2, a_3 \in K(\sqrt{a})$ such that $Q(a_1, a_2, a_3) = 0$.

Now we want to see that $L(\sqrt{y}) | K$ is a Galois extension with Galois group $2^+ S_4 \ast D_8$. By the assumption $L_1 \cap L_2 = K$, we have $\text{Gal}(L(\sqrt{y}) | L_2) \simeq 2^+ S_4$. We now consider the behaviour of $y$ under the action of $\text{Gal}(L_2 / K)$. Let $r, s, t$ be the nontrivial elements of $\text{Gal}(L_2 / K)$ fixing respectively $\sqrt{ab}, \sqrt{b}, \sqrt{a}$. By computation we obtain $y^r y = d^2 h^2 b$, where $h = ms_2 p_{11} x^3 + (p_{21} - s_2^2 s_3 p_{11}) x^2 + (m n p_{31} + p_{11}) x + s_3 s_3 p_{11} - s_2 p_{21}$. Now $y \in K(\sqrt{a})(x)$, so $y^r = y$ and $y^r = y^s$, so $L(\sqrt{y})$ is Galois over
Let the fields $K$ and $L$ and the elements $d$, $a$, $b$, and $y$ be as in Theorem 1, let $\mu = d \pm (d+1)\sqrt{a}$ and let $\rho = ab \pm (ab+1)\sqrt{ab}$. Then

1. $L(\sqrt{\mu y}), r \in K^*$, is the general solution to the embedding problem
   \[ 2^{-1} S_4 \ast D_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K). \]

2. $L(\sqrt{\mu y}), r \in K^*$, is the general solution to the embedding problem
   \[ 2^+ S_4 \ast Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K). \]

3. $L(\sqrt{\mu y}), r \in K^*$, is the general solution to the embedding problem
   \[ 2^- S_4 \ast Q_8 \rightarrow S_4 \times V_4 \simeq \text{Gal}(L|K). \]

Proof. For $\sigma \in S_4 \setminus A_4$, we have $\mu^\sigma \mu = -(d-1)^2d$ and so, $L(\sqrt{\mu y})|K$ Galois. Now $(d-1)^2d$ changes sign under the action of $\sigma$, so $\text{Gal}(L(\sqrt{\mu y})|L) \simeq 2^{-1} S_4$, hence $\text{Gal}(L(\sqrt{\mu y})|K) \simeq 2^{-1} S_4 \ast D_8$.

For $r, s, t \in \text{Gal}(L_2|K)$ fixing $\sqrt{ab}, \sqrt{b}, \sqrt{a}$, resp., we have $\rho^s \rho = \rho^t \rho = -(ab-1)^2ab$ and $\rho^s \rho = \rho^t$, so $L(\sqrt{\mu y})|K$ Galois. Now $(ab-1)^2\sqrt{ab}$ changes sign under the action of $s$ and under the action of $t$, so $\text{Gal}(L(\sqrt{\mu y})|L) \simeq Q_8$, hence $\text{Gal}(L(\sqrt{\mu y})|K) \simeq 2^+ S_4 \ast Q_8$.

Combining both arguments, we obtain the third statement in the theorem. □

Proposition 2. Let the fields $K$ and $L$ and the elements $s_2, s_3, s_4, d, a, b, m, p_{ij}$ and $y$ be as in Theorem 1; $\mu, \rho$ as in Theorem 2. We have

\[
\begin{align*}
\text{disc}(L(\sqrt{\mu y})|K) &= d^{104}a^{96}b^{100}D^2, \\
\text{disc}(L(\sqrt{\mu y})|K) &= d^{102}a^{96}b^{100}D^2(d-1)^{48}, \\
\text{disc}(L(\sqrt{\mu y})|K) &= d^{104}a^{144}b^{148}D^2(ab-1)^{48}, \\
\text{disc}(L(\sqrt{\mu y})|K) &= d^{152}a^{144}b^{148}D^2(d-1)^{48}(ab-1)^{48},
\end{align*}
\]

where

\[
\begin{align*}
D &= s_4p_{11}^4 - s_2s_4p_{11}^4p_{21} + ms_2p_{11}^3p_{21}^2 - ms_3p_{11}p_{21}^3 + (m^2 - s_2s_4^2)p_{21}^4 \\
&\quad + dp_{31}(p_{11}^3 + ms_2p_{11}p_{21} - s_3p_{21}^3 + m^2s_2p_{11}^2) + d^2p_{31}^3(s_2s_3p_{21} + mp_{11}) + d^3p_{31}^3.
\end{align*}
\]

Proof. We have disc($L(\sqrt{\mu y})|K) = \text{disc}(L(K)|K) \cdot N_{L/K}(y)$ and disc($L(K) = (ab)^{48}$. Now $N_{L/K}(y) = (N_{L_{\nu_1}((\sqrt{\nu_1})|K)}/(c_4))^{2}$, for $c_4$ the degree 0 coefficient in the irreducible polynomial of $y$ over $K(\sqrt{\nu_1})$. By computation we obtain $c_4 = \frac{4}{d^2}(a_1a_2 - a_1^2 - s_2a_2^2)^2$ and, by substituting the values of $a_1, a_2, a_3$ and computing the norm, $N_{K(\sqrt{\nu_1})|K}(c_4) = d^4b^4D^2$ for $D$ as in the statement.

To obtain the other three discriminants, it is now enough to compute $N_{L/K}(\mu) = N_{K(\sqrt{\nu_1})|K}(\mu)^{48} = (d-1)^{48}d^{48}$ and $N_{L/K}(\rho) = N_{K(\sqrt{\nu_1})|K}(\rho)^{48} = (ab-1)^{48}(ab)^{48}$. □
4. Examples

Let \( K = k((Z_1, Z_2, Z_3)) \) be the quotient field of the formal power series ring in 3 variables over a field \( k \) containing \( \mathbb{F}_9 \). We consider the polynomial

\[
   f(X) = X^4 + Z_1X^2 + Z_2X + Z_3 \in K[X],
\]

i.e. we are taking \( s_2 = Z_1, s_3 = -Z_2, s_4 = Z_3 \). We can check that the polynomial \( f \) has Galois group \( S_4 \) over \( K \) and let \( L_1 \) be the splitting field of \( f \) over \( K \). We consider the extension \( L_2|K \) generated by the elements \( \sqrt{ds_2}, \sqrt{ms_2}, \sqrt{dm} \), which corresponds to taking the matrix \( P \) in Theorem 1 to be one of the matrices

\[
   \begin{pmatrix}
   1 & 0 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1
   \end{pmatrix}, \quad
   \begin{pmatrix}
   0 & 0 & 1 \\
   1 & 0 & 0 \\
   0 & 1 & 0
   \end{pmatrix}, \quad
   \begin{pmatrix}
   0 & 1 & 0 \\
   0 & 0 & 1 \\
   1 & 0 & 0
   \end{pmatrix}.
\]

We can check that the elements \( ds_2, ms_2, dm, s_2, dms_2, m \) are not squares in \( K \), and so \( L_2|K \) has Galois group \( V_4 \) and is disjoint with \( L_1|K \). Let \( L = L_1 \cdot L_2 \).

We denote by \( y_i \) the element \( y \) given by Theorem 1 for each of the matrices \( P_i \), \( i = 1, 2, 3 \). Then we have

\[
   \text{Gal}(L(\sqrt{y_i}|K)) \cong 2^+S_4 * D_8, \quad i = 1, 2, 3,
\]

with \( L(\sqrt{y_1})|L_1(\sqrt{dm}), L(\sqrt{y_2})|L_1(\sqrt{ms_2}), L(\sqrt{y_3})|L_1(\sqrt{ds_2}) \) cyclic. The factors appearing in \( \text{disc}(L(\sqrt{y_i}|K)) \) are \( d, s_2, m \) and \( s_4 \), the factors appearing in \( \text{disc}(L(\sqrt{y_2}|K)) \) are \( d, s_2, m \) and \( m^2 - s_2s_3^2 \), and the factors appearing in \( \text{disc}(L(\sqrt{y_3}|K)) \) are \( d, s_2, m \). In particular the discriminant locus remains unchanged when going from \( L \) to \( L(\sqrt{y_3}) \).

We observe that the elements \( d - 1 \) and \( ab - 1 \), for each choice of \( a, b \) among \( ds_2, ms_2, dm \), are invertible elements in the ring \( k[[Z_1, Z_2, Z_3]] \), and so the discriminant locus will not change if we realize any other of the groups \( 2S_4 * H \) over the same field \( K \) by means of Theorem 2.

References


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