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ON A GENERALIZED CORONA PROBLEM ON THE UNIT DISC

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ABSTRACT. Let $g, f_1, \dots, f_n \in H^\infty$. We give a sufficient condition on the size of a function g in order for it to be in the ideal generated by $f_1, \dots, f_n$. In particular, this improves Cegrell's result on this problem.

INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane, and let $H^\infty = H^\infty(\mathbb{D})$ be the Banach algebra of bounded analytic functions on $\mathbb{D}$. Carleson's corona theorem says that the unit disc is dense in the space M_{H^∞} of maximal ideals of H^∞ with the weak-star topology. This result is equivalent to the following fact: if we have functions $f_1, \dots, f_n \in H^\infty$ such that

$$\sum_{j=1}^{n} |f_j(z)| \geq \delta > 0, \quad \forall z \in \mathbb{D},$$

then there exist solutions $g_1, \dots, g_n \in H^\infty$ of the equation

$$\sum_{j=1}^{n} f_j g_j = 1.$$

In order to generalize this result, it is natural to ask if it is possible to replace the function 1 by an arbitrary function $g \in H^\infty$; that is, one asks if the condition

$$|g(z)| \leq C \sum_{j=1}^{n} |f_j(z)|, \quad \forall z \in \mathbb{D}, \tag{1}$$

implies that the function g belongs to the ideal I generated by $f_1, \dots, f_n$. Condition (1) is clearly a necessary condition, but an example given by Rao (see [Ra]) shows that the answer is, in general, negative. Thus the following problem arises naturally.

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Problem A. Let h be a positive continuous function on $[0,\infty)$ increasing in a neighbourhood of zero, and let $g, f_1, \dots, f_n \in H^\infty$. For which functions h does the condition

$$|g(z)| \le h(|f_1(z)| + \cdots + |f_n(z)|), \quad \forall z \in \mathbb{D}, \tag{2}$$

imply that the function g is in the ideal generated by $f_1, \dots, f_n$?

For functions of the form $h(s) = s^\alpha$, with $\alpha \ge 1$, the problem is completely solved. For $1 \le \alpha < 2$, a variation of Rao's example shows that the answer is negative, and for $\alpha > 2$, work of Wolff, Cegrell and others gives an affirmative answer (see [Ce1], [Ga]). The problem for $\alpha = 2$ was an old question of Wolff, which remained open for twenty years. However, Treil (see [Tr]) has recently shown (using a connection with the best estimates of the solutions of the corona theorem) that the answer is, in general, negative.

In [Li], Lin gave an affirmative answer for this problem for the function

$$h(s) = s^2(-\log s)^{-(3/2+\varepsilon)}$$

with $\varepsilon > 0$, and in [Ce2] Cegrell established the following strongest known positive case for this problem.

Theorem A (Cegrell)**.** *Let $f_1, \dots, f_n \in H^\infty$ with $|f(z)|^2 = \sum_{j=1}^n |f_j(z)|^2 > 0$, for all $z \in \mathbb{D}$. Then, Problem A has an affirmative answer for*

$$h(s) = \frac{s^2}{(-\log s)^{3/2}(\log(-\log s))^{3/2} \log\,\log(-\log s)}.$$

Our main result below is an improvement of Cegrell's theorem.

Theorem 1. *Let $k : (0,1) \to [0,\infty)$ be a nondecreasing bounded continuous function such that $k(x)/x$ is nonincreasing and*

$$\int_0^1 \frac{k(x)}{x} |\log x|\, dx < +\infty,$$

and let $H(x) = \sqrt{k(x)\int_0^x \frac{k(s)}{s}\, ds}$. Furthermore, let $g, f_1, \dots, f_n \in H^\infty$, where $0 < |f|^2 := \sum_{j=1}^n |f_j|^2 < 1$. Then the condition

$$|g| \le |f|^2\, H(|f|^2)$$

implies the existence of solutions $g_1, \dots, g_n \in H^\infty$ of the equation

$$g = f_1 g_1 + \cdots + f_n g_n.$$

For example, if we take $k(x) = |\log x|^{-2}(\log|\log x|)^{-3/2}$, we see that Problem A has an affirmative answer for the function

$$h(s) = s^2(-\log s)^{-3/2}(\log(-\log s))^{-1},$$

and this clearly improves Cegrell's result.

For $1 \le p < \infty$, let H^p be the Hardy space of analytic functions in the unit disc such that

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \,\frac{d\theta}{2\pi} < \infty.$$

It is well known that an analytic function f belongs to H^p if and only if the nontangential maximal function $Mf(e^{i\theta}) = \sup\{|f(z)| : z \in \Gamma(\theta)\}$ belongs to the usual Lebesgue space $L^p(\mathbb{T})$, where

$$\Gamma(\theta) := \big\{z \in \mathbb{D} : |e^{i\theta} - z| < (1+\alpha)(1-|z|)\big\}$$

is the Stolz angle with vertex at $e^{i\theta}$ and fixed aperture $\alpha > 0$ (the choice of α is irrelevant here), and $\mathbb{T}$ denotes the unit circle. Several H^p versions of the corona theorem have been considered. Concretely, one is interested in conditions on functions $f_1, \dots, f_n \in H^\infty$ such that the equation

$$1 = f_1 g_1 + \cdots + f_n g_n \tag{3}$$

has solutions $g_1, \dots, g_n$ in H^p. If $|f|^2 = \sum_{j=1}^n |f_j|^2$ and $|g|^2 = \sum_{j=1}^n |g_j|^2$, it follows from (3) that $1 \le |f||g|$, and hence $M(|f|^{-1}) \in L^p(\mathbb{T})$ is a necessary condition. We note that when $p = \infty$, this is the usual corona condition. However, for $1 \le p < \infty$, this condition is far from being sufficient. In [ABN], it is shown that, for any $\varepsilon > 0$, the stronger condition $M(|f|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ is not sufficient. Our next result is the H^p version of Theorem 1.

Theorem 2. *Let k be as in Theorem 1, let $H(x) = \Big(k(x)\displaystyle\int_0^x k(s)/s\,ds\Big)^{1/2}$ and let $1 \le p < \infty$. Given functions $g, f_1, \dots, f_n \in H^\infty$, the condition*

$$M\Big(\frac{g}{|f|^2 H(|f|^2)}\Big) \in L^p(\mathbb{T})$$

implies the existence of solutions $g_1, \dots, g_n \in H^p$ of the equation

$$g = f_1 g_1 + \cdots + f_n g_n.$$

Finally, we want to remark that in both theorems, only the behavior of the function k, and hence H, near zero is essential.

1. Carleson measures and the $\overline{\partial}$-equation

Solutions of the $\overline{\partial}$-equation with boundary control will be of vital importance in the proofs of the main theorems, and Carleson measures play an important role in obtaining these solutions. We recall that a positive Borel measure μ on $\mathbb{D}$ is called a *Carleson measure* if there exists a constant C such that

$$\int_{\mathbb{D}} |h|^2\, d\mu \le C\|h\|_2^2, \tag{4}$$

for every function h in the Hardy space H^2. It is well known that Carleson measures are those positive measures μ for which there exists a constant A such that

$$\mu(Q) \le A\, l(Q)$$

for every *Carleson square* Q defined by

$$Q = \{re^{i\theta} \in \mathbb{D} : 1 - r < l(Q),\, |\theta - \theta_0| < l(Q)\}.$$

Denote by $N(\mu) = \sup\{\mu(Q)/\, l(Q)\}$ the Carleson norm of μ, where the supremum is taken over all Carleson squares Q. The operators ∂ and $\overline{\partial}$ are defined by

$$\partial f = \frac{\partial f}{\partial z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big), \qquad \overline{\partial} f = \frac{\partial f}{\partial \bar z} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big).$$

By the Cauchy-Riemann equations, a function f is analytic if and only if $\overline{\partial} f = 0$. Recall that we can rewrite the Laplacian operator as $\Delta = 4\partial\overline{\partial}$. We need the following result of T. Wolff on the existence of bounded solutions of the $\overline{\partial}$-equation (see, for example, [Ga] p. 322).

Lemma 1. *Let $G(z)$ be bounded and C^1 on the disc $\mathbb{D}$, and assume that the measures $d\mu(z) = |G(z)|^2 \log \frac{1}{|z|}\, dx\, dy$ and $d\sigma(z) = |\partial G(z)| \log \frac{1}{|z|}\, dx\, dy$ are Carleson measures. Then there exists a function $u \in C(\overline{\mathbb{D}}) \cap C^1(\mathbb{D})$ such that $\overline{\partial} u = G$ and*

$$\|u\|_{L^\infty(\mathbb{T})} \le C_1 N(\mu)^{1/2} + C_2 N(\sigma).$$

We will also need an L^p-version of the Wolff criteria. The next lemma is a refinement of the version given in [ABN].

Lemma 2. *Let $1 \le p < \infty$, and let G be a C^1 function in $\overline{\mathbb{D}}$ such that:*

(a) *$G = \varphi\psi$, where $M(\varphi) \in L^p(\mathbb{T})$, and $|\psi(z)|^2 \log \frac{1}{|z|}\, dx\, dy$ is a Carleson measure;*

(b) *for every function $k \in H^q$, where $1/p + 1/q = 1$, we have*

$$\int_{\mathbb{D}} |k(z)||\partial G(z)| \log \frac{1}{|z|}\, dx\, dy \le B\|k\|_q.$$

Then there exists a function $u \in C(\overline{\mathbb{D}}) \cap C^1(\mathbb{D})$ such that $\overline{\partial} u = G$ and

$$\int_0^{2\pi} |u(e^{i\theta})|^p\, d\theta \le C,$$

where C depends only on the L^p-norm of $M(\varphi)$, the constant B and the Carleson norm of the measure of (a).

Proof. Let q be the conjugate exponent of p, $1 < q \le \infty$. By duality,

$$\inf\left\{\|b\|_p : \overline{\partial} b = G\right\} = \sup\left\{\Big|\frac{1}{2\pi}\int_0^{2\pi} Fk\, d\theta\Big| : k \in H^q, k(0) = 0, \|k\|_q \le 1\right\}$$

where F is a priori solution, say the one given by the Cauchy kernel, which is continuous on $\overline{\mathbb{D}}$. By Green's formula,

$$\frac{1}{2\pi}\int_0^{2\pi} Fk\, d\theta = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta(Fk) \log\frac{1}{|z|}\, dx\, dy$$
$$= \frac{2}{\pi}\int_{\mathbb{D}} k'(z)G(z) \log\frac{1}{|z|}\, dx\, dy + \frac{2}{\pi}\int_{\mathbb{D}} k(z)\partial G(z) \log\frac{1}{|z|}\, dx\, dy = I_1 + I_2.$$

It is proved in [ABN] that if $|\psi|^2 \log \frac{1}{|z|}$ is a Carleson measure with Carleson norm K, then

$$\int_{\mathbb{D}} |k'(z)|\, |\varphi(z)|\, |\psi(z)| \log\frac{1}{|z|}\, dx\, dy \le\ C\, \|k\|_{H^q} \|M\varphi\|_p\, K$$

where C is an absolute constant. This implies the required bound for I_1, and the boundness of I_2 follows from condition (b). □

The following lemma can be found in [Ni]. For completeness we will give a proof here.

Lemma 3. *Let $u \in C^2(\overline{\mathbb{D}})$ be a bounded subharmonic function. Then*

$$d\lambda(z) = \Delta u(z) \log\frac{1}{|z|}\, dx\, dy$$

is a Carleson measure with Carleson norm bounded by $2\pi e\|u\|_\infty$.

Proof. By considering the function $b(z) = u(z) + \|u\|_\infty$ we can assume that our function u is positive. Let $h \in H^2$. Then, for $t > 0$,

$$\begin{aligned}\Delta(|h|^2 e^{tu}) &= 4|h'|^2 e^{tu} + |h|^2 \Delta e^{tu} + 8\,\mathrm{Re}\,(\overline{\partial}(|h|^2)\,\partial e^{tu})\\ &= 4|h'|^2 e^{tu} + |h|^2(4t^2|\partial u|^2 + t\,\Delta u)e^{tu} + 8te^{tu}\mathrm{Re}\,(h\,\overline{\partial} h\,\partial u)\\ &= t|h|^2 e^{tu}\Delta u + e^{tu}|2\partial h + 2th\,\partial u|^2\\ &\geq t|h|^2 e^{tu}\Delta u \,\geq\, t|h|^2\Delta u.\end{aligned}$$

Thus we have

$$\int_{\mathbb{D}} |h(z)|^2 \Delta u(z)\,\log\frac{1}{|z|}\,dx\,dy \,\leq\, \inf_{t>0}\frac{1}{t}\int_{\mathbb{D}} \Delta(|h(z)|^2 e^{tu(z)})\,\log\frac{1}{|z|}\,dx\,dy,$$

which, by Green's formula, is bounded by

$$\inf_{t>0}\frac{1}{t}\int_{\partial\mathbb{D}} |h|^2 e^{tu} \leq \inf_{t>0}\frac{2\pi}{t}\|e^{tu}\|_\infty\|h\|_2^2 = 2\pi e\|u\|_\infty\|h\|_2^2\,.$$

(The last identity is obtained by computation of the minimum of $t^{-1}\|e^{tu}\|_\infty$, which is attained at the point $t_0 = 1/\|u\|_\infty$.) Hence the measure λ is a Carleson measure with Carleson norm bounded by $2\pi e\|u\|_\infty$. □

Given functions $f_1,\ldots,f_n \in H^\infty$, we write $|f|^2 = \sum_{i=1}^n |f_i|^2$, and $|f'|^2 = \sum_{i=1}^n |f_i'|^2$. The next result is the key for the proof of Theorems 1 and 2.

Lemma 4. *Let $k : (0,1) \to [0,\infty)$ be a bounded continuous function such that $\int_0^1 \frac{k(x)}{x}|\log x\,|\,dx < \infty$. Let $f_1,\ldots,f_n \in H^\infty$ with $0 < |f|^2 < 1$. Then the measures*

(*a*) $\dfrac{|\partial(|f|^2)|^2}{|f|^4}\,k(|f|^2)\,\log\dfrac{1}{|z|}\,dx\,dy,$

(*b*) $\dfrac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^4}\,\Big(\displaystyle\int_0^{|f|^2}\frac{k(s)}{s}\,ds\Big)\,\log\frac{1}{|z|}\,dx\,dy$

are Carleson measures with Carleson norm bounded by $C\int_0^1 \frac{k(s)}{s}|\log s|\,ds$.

We note that when $n = 1$, part (*b*) is vacuous and part (*a*) is a known result that is also true for bounded analytic functions vanishing in $\mathbb{D}$ (see [ABN]).

Proof. Consider the function

$$U(z) = \log|f(z)|\int_0^{|f(z)|^2}\frac{k(s)}{s}\,ds + \int_0^{|f(z)|^2}\frac{k(s)}{s}|\log s|\,ds.$$

It clearly satisfies

$$0 \leq U(z) \leq 2\int_0^{|f(z)|^2}\frac{k(s)}{s}|\log s|\,ds,$$

and a computation gives

$$\frac{1}{4}\Delta\,U(z) = S(z),$$

where

$$S = \frac{|\partial(|f|^2)|^2}{|f|^4}\,k(|f|^2) + \frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^4}\Big(\int_0^{|f|^2}\frac{k(s)}{s}\,ds\Big).$$

Now, applying Lemma 3, we obtain the desired conclusion. □

2. Proof of Theorem 1

The proof follows standard arguments with the use of Lemma 4 as a new ingredient. By a standard normal families argument, we can assume that the functions $f_1, \dots, f_n$ are analytic in a neighborhood of the closed unit disc. For $j = 1, \dots, n$ we define

$$\varphi_j(z) = \frac{\overline{f_j(z)}}{|f(z)|^2}.$$

We see that the functions φ_j belong to $\mathcal{C}^\infty(\mathbb{D})$ and satisfy the equation $\sum_{j=1}^n f_j\varphi_j = 1$. For $j, k = 1, \dots, n$, let

$$G_{jk} = \varphi_j \overline{\partial} \varphi_k.$$

Assume that for $j, k = 1, \dots, n$ we can solve the $\overline{\partial}$-equations

$$\overline{\partial} u_{jk} = g\, G_{jk}, \tag{5}$$

with $\|u_{jk}\|_{L^\infty(\mathbb{T})} \le M$. Then, for $j = 1, \dots, n$, the functions

$$g_j = g\varphi_j + \sum_{k=1}^n (u_{jk} - u_{kj}) f_k$$

are bounded, satisfy $\overline{\partial} g_j = 0$ and so are analytic, and satisfy the equation

$$g = \sum_{j=1}^n f_j g_j.$$

It only remains to show that (5) has bounded solutions. To see this, we will use Lemma 1. Fix j, k, and denote G_{jk} by G. A computation gives

$$G_{jk} = \frac{\bar{f}_j}{|f|^6} \sum_{l \ne k} f_l\,(\overline{f_l f_k' - f_k f_l'}) \tag{6}$$

and

$$\sum_{j,k=1}^n |f_k' f_j - f_k f_j'|^2 = |f|^2\,|f'|^2 - |\partial(|f|^2)|^2\,. \tag{7}$$

By (6) and (7),

$$|G| \le 2\frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)^{1/2}}{|f|^4}. \tag{8}$$

Using (8) and our condition on the size of $|g|$ we see that

$$|gG|^2 \le \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4}\Big(\int_0^{|f|^2} \frac{k(s)}{s}\,ds\Big)\,k(|f|^2),$$

and because k is bounded,

$$|g(z)G(z)|^2 \,\log\frac{1}{|z|}\,dx\,dy$$

is a Carleson measure by Lemma 4.

We have $\partial(gG) = g'G + g\partial G$, and since $|g| \le |f|^2\, H(|f|^2)$, we have that the measure $|g(z)\partial G(z)|\,\log\frac{1}{|z|}\,dx\,dy$ is a Carleson measure by the following result.

Lemma 5. *The measure $|f(z)|^2\, H(|f(z)|^2)\,|\partial G(z)|\,\log\frac{1}{|z|}\,dx\,dy$ is a Carleson measure.*

To prove this, we note that $k(|f|^2) \leq \int_0^{|f|^2} \frac{k(s)}{s}\, ds$, since the function $k(x)/x$ is nonincreasing. Also, a computation gives

$$|\partial G| \leq 2|G|\frac{|\partial(|f|^2)|}{|f|^2} + \frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^6}. \tag{9}$$

So, by (9) and then (8),

$$\begin{aligned} |f|^2\, H(|f|^2)\, |\partial G| &\leq 2H(|f|^2)\, |G|\, |\partial(|f|^2)| + \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4} \int_0^{|f|^2} \frac{k(s)}{s}\, ds \\ &\leq 2\frac{|\partial(|f|^2)|^2}{|f|^4}\, k(|f|^2) + 2\frac{|f|^2|f'|^2 - |\partial(|f|^2)|^2}{|f|^4}\, \Big(\int_0^{|f|^2} \frac{k(s)}{s}\, ds\Big), \end{aligned}$$

and the result follows by Lemma 4.

It remains to check that

$$|g'(z)G(z)|\, \log\frac{1}{|z|}\, dx\, dy$$

is a Carleson measure. Let $h \in H^2$. Then

$$\int_{\mathbb{D}} |h(z)|^2 |(g'G)(z)|\, \log\frac{1}{|z|}\, dx\, dy = \int_A + \int_{\mathbb{D}\setminus A} = I_1 + I_2,$$

where $A = \{z : |g(z) \leq |f(z)|^5\}$. For $z \in A$ we have

$$|(g'G)(z)|\ \leq\ \frac{|g'(z)|^2}{|g(z)|} + \frac{|f'(z)|^2}{|f(z)|}.$$

Since for $F \in H^\infty$, the measure $\frac{|F'(z)|^2}{|F(z)|}\, \log\frac{1}{|z|}\, dx\, dy$ is Carleson (see [Ga], p. 327, or apply Lemma 4 with $k(x) = x^{1/2}$), we see that $I_1 \leq C_1\|h\|^2_{H^2}$, by (4). To estimate I_2, let

$$B(|f|^2) = \frac{(|f|^2|f'|^2 - |\partial(|f|^2)|^2)}{|f|^4}\Big(\int_0^{|f|^2} \frac{k(s)}{s}\, ds\Big).$$

Since k is nondecreasing, we see that

$$\begin{aligned} |g'G| &\leq |g'|^2\, |g|^{-2}k(|f|^2) + B(|f|^2) \\ &\leq |g'|^2\, |g|^{-2}s(|g|^2) + B(|f|^2) \end{aligned}$$

in $\mathbb{D} \setminus A$, where $s(x) = k(x^{1/5})$. One easily verifies that s satisfies the condition $\int_0^1 \frac{s(x)}{x}\, |\log x|\, dx < \infty$. Then $I_2 \leq C_2\, \|h\|^2_{H^2}$ by Lemma 4 and (4). Hence the measure

$$|(g'G)(z)|\, \log\frac{1}{|z|}\, dx\, dy$$

is a Carleson measure. By Lemma 1, the proof is complete.

3. Proof of Theorem 2

For $j, k = 1, \ldots, n$, let $G_{jk} = \varphi_j\overline{\partial}\varphi_k$, where $\varphi_j = \bar{f}_j|f|^{-2}$. As in the proof of Theorem 1, it is sufficient to solve the $\overline{\partial}$-equations $\overline{\partial}u_{jk} = g\, G_{jk}$, with $\|u_{jk}\|_{L^p(\mathbb{T})} \leq M$. For this, we will make use of Lemma 2. Fix j, k, and for ease of notation, denote G_{jk} by G. We can write gG in the form $gG = \phi\psi_1$, where $\phi = g\,|f|^{-2}(H(|f|^2))^{-1}$ and $\psi_1 = |f|^2H(|f|^2)\, G$. By hypothesis, $M(\phi) \in L^p(\mathbb{T})$, and the proof of Theorem 1 shows that

$$|\psi_1(z)|^2\, \log\frac{1}{|z|}\, dx\, dy$$

is a Carleson measure. So condition (a) of Lemma 2 is satisfied. To check condition (b), let $k \in H^q$, where $1/p + 1/q = 1$. We have $\partial(gG) = g'G + g\partial G$, and we can write $|g\partial G|$ as $|\phi|\psi_2$, where $\psi_2 = |f|^2 H(|f|^2)\,|\partial G|$. By Lemma 5, the measure

$$d\mu(z) = \psi_2(z)\,\log\frac{1}{|z|}\,dx\,dy$$

is a Carleson measure. Then

$$\begin{aligned}\int_{\mathbb{D}} |k(z)|\,|(g\partial G)(z)|\,\log\frac{1}{|z|}\,dx\,dy &\le \Big(\int_{\mathbb{D}} |\phi|^p\,d\mu\Big)^{1/p}\Big(\int_{\mathbb{D}} |k|^q\,d\mu\Big)^{1/q}\\ &\le C\|M(\phi)\|_{L^p(\mathbb{T})}\,\|k\|_{H^q},\end{aligned}$$

since, if μ is a Carleson measure and $M\psi \in L^p(\mathbb{T})$, then $\int_{\mathbb{D}} |\psi|^p\,d\mu \;\le\; \|M\psi\|^p_{L^p(\mathbb{T})}$ (see [Ga], p. 32). An argument similar to that in the proof of Theorem 1 shows that

$$\int_{\mathbb{D}} |k(z)|\,|(g'G)(z)|\,\log\frac{1}{|z|}\,dx\,dy \;\le\; C\,\|k\|_{H^q},$$

and condition (b) of Lemma 2 is satisfied. This completes the proof.

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