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# DETERMINACY AND WEAKLY RAMSEY SETS IN BANACH SPACES

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ABSTRACT. We give a sufficient condition for a set of block subspaces in an infinite-dimensional Banach space to be weakly Ramsey. Using this condition we prove that in the Levy-collapse of a Mahlo cardinal, every projective set is weakly Ramsey. This, together with a construction of W. H. Woodin, is used to show that the Axiom of Projective Determinacy implies that every projective set is weakly Ramsey. In the case of $c_0$ we prove similar results for a stronger Ramsey property. And for hereditarily indecomposable spaces we show that the Axiom of Determinacy plus the Axiom of Dependent Choices imply that every set is weakly Ramsey. These results are the generalizations to the class of projective sets of some theorems from W. T. Gowers, and our paper "Weakly Ramsey sets in Banach spaces."

## INTRODUCTION

In this paper we continue the study we started in [2] of the new Ramsey-style property for Banach spaces introduced by W. T. Gowers in [6], [8], the *weakly Ramsey* property (see Definition 1 below). This new combinatorial notion is extremely powerful for the analysis of the infinite-dimensional closed subspaces of a given (infinite-dimensional and separable) Banach space. This is exemplified by Gowers' famous dichotomy for Banach spaces ([7]), which is a direct consequence of the fact that certain simple sets (intersections of open sets) are weakly Ramsey. The weakly Ramsey property is a property of sets of block subspaces, and the set of all block subspaces of a Banach space, with the natural topology, is a Polish space, which makes it suitable for a set-theoretic treatment. Indeed, in [2] we gave a proof, using ideas from set theory, of a theorem first announced in [6] (see also [8]), namely, that every analytic set is weakly Ramsey. For the proof, we used the Suslin decomposition of analytic sets and introduced a family of partial orderings, which can be thought of as sets of approximations to a particular block subspace. Dense subsets of these partial orderings are used to guarantee that any filter that meets the dense sets will produce the required block subspace. Assuming some reasonable combinatorial principles (e.g., a form of Martin's Axiom), we also showed in [2] that for a more complex class of sets of block subspaces, the class of all continuous images of co-analytic sets, all sets in the class are weakly Ramsey. For the proof we introduced again a new family of partial orderings and used the canonical decomposition of such sets into $\aleph_1$ Borel sets.

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These results parallel the situation for the classical Ramsey theory in $[\mathbb{N}]^\omega$, the space of all infinite subsets of integers. J. Silver [19] showed that every analytic set is Ramsey, and that Martin's Axiom implies that continuous images of co-analytic sets are Ramsey. As for more complex sets, A. R. D. Mathias [16] showed that in the model obtained by collapsing an inaccessible cardinal using the collapse of Levy, every projective subset of the space $[\mathbb{N}]^\omega$ is Ramsey. A similar situation also holds for weakly Ramsey sets: we show in Section 3 that in a certain inner model of the Levy-collapse of a Mahlo cardinal, every projective set of block subspaces is weakly Ramsey. However, due to the asymmetry of the weakly Ramsey property, we need an entirely different argument: we show that every projective set has a good decomposition (see Definition 5 below), a sufficient condition for a set to be weakly Ramsey, and we use the absoluteness properties of the Solovay model that results from the Levy-collapse.

As one might expect by looking at the definition of weakly Ramsey (Definition 1), the axioms of determinacy, which assert that certain games in $\mathbb{N}$ are determined, have strong consequences in the theory of weakly Ramsey sets. We show that the Axiom of Projective Determinacy implies that every projective set of block subspaces is weakly Ramsey. This answers a question of W. T. Gowers ([8]). And if the space is hereditarily indecomposable, the full Axiom of Determinacy plus a weak form of the Axiom of Choice imply that every set is weakly Ramsey.

A special situation holds in the space $c_0$ (the space of subspaces of scalars with limit 0). This is the Banach space that, in a combinatorial sense, is nearest to the space $[\mathbb{N}]^\omega$. For this space we prove similar results as before, but for a stronger Ramsey property, that of being *almost-Ramsey*, i.e., either avoids or almost-contains a cube (see Section 6).

This paper is organized as follows: in Section 1 we recall from [2] the fundamental notions and prove some basic facts that will be used in the subsequent sections. Section 2 is related to the new notion of having a good decomposition. In Section 3 we prove that in the Levy-collapse of a Mahlo cardinal over $L$ of a small set every projective set of block subspaces has a good decomposition, and hence is weakly Ramsey. We use this fact, together with a construction of Woodin ([23]) of a small sufficiently-correct model, to show in Section 4 that, under the Axiom of Projective Determinacy, every projective set of block subspaces is weakly Ramsey. We also consider two special cases, for which we prove stronger results. The first is when the Banach space is hereditarily indecomposable. We show in Section 5 that the Axiom of Determinacy implies that every set of block subspaces is weakly Ramsey. The second special case is $c_0$. We show in Section 6 that under the Axiom of Projective Determinacy, every projective set of block subspaces of $c_0$ is almost-Ramsey.

## 1. Definitions and basic facts

We are interested in infinite-dimensional and separable Banach spaces. So, in this paper, a Banach space will always be infinite-dimensional and separable. Also, subspaces of a given Banach space will always be assumed to be closed.

Let $\mathcal{X}(=(\mathcal{X},\|\cdot\|))$ be a Banach space over $\mathbb{K}\in\{\mathbb{C},\mathbb{R}\}$. A sequence $(x_n)_n\in\mathcal{X}^\omega$ is a *Schauder basis* if for every $x\in\mathcal{X}$ there exists a unique $(\lambda_n)_n\in\mathbb{K}^\omega$ such that $x=\sum_{n\geq 1}\lambda_n x_n$. It is well known that every Banach space has a basic sequence (see [14]). Note that a Banach space having a Schauder basis is always separable.

We say that $(x_n)_n$ is a *basic sequence* iff $(x_n)_n$ is a Schauder basis in the closed linear span of $(x_n)_n$, i.e., the closure of the subspace generated by $\{x_n \mid n \geq 1\}$.

Given a Schauder basis $(\mathbf{e}_n)_n$ of $\mathcal{X}$ (we may assume that $\|\mathbf{e}_n\| = 1$), and $x \in \mathcal{X}$, $x = \sum_{n=1}^{\infty} \lambda_n \mathbf{e}_n$, the support of $x$ is supp $x := \{n \in \omega \mid \lambda_n \neq 0\}$. Suppose that $x, y$ have finite support. We write $x < y$ if $\max \operatorname{supp} x < \min \operatorname{supp} y$. $(y_n)_n$ is a *block basic sequence* (with respect to $(\mathbf{e}_n)_n$) iff every $y_n$ has finite support and for every $n \geq 1$, $y_n < y_{n+1}$. For conciseness, we usually refer to block basic sequences simply as *block sequences*, or *block bases.* A *block vector* is a normalized vector with finite support. In most cases, and without ambiguity, we will confuse $\mathcal{X}$ with the basic sequence $(\mathbf{e}_n)_n$. Let $\boldsymbol{B_1} = \boldsymbol{B_1}(\mathcal{X})$ be the set of normalized block basic sequences of $\mathcal{X}$. For notational efficiency, we sometimes identify a block basic sequence with the closed subspace it generates. Thus, we use upper-case letters $X, Y, Z, \ldots$ to refer to normalized block basic sequences as well as the corresponding subspaces. We reserve the lower-case letters $s, t, u, ..$ for finite segments of normalized block basic sequences and the corresponding subspaces (we refer to those as *finite block basic sequences*). Also, for a finite sequence $s$, we will write $|s|$ for the cardinality of $s$.

Fix a Banach space $\mathcal{X}$. For $\boldsymbol{a}$ and $\boldsymbol{b}$ finite or infinite block basic sequences, we define $\boldsymbol{a} \preceq \boldsymbol{b}$ iff $\boldsymbol{a} \subseteq \boldsymbol{b}$ (as subspaces). Note that $\preceq$ is a transitive relation. $Y \preceq^* Z$ iff there is $n_0$ so that $(y_n)_{n \geq n_0} \preceq Z$, where $Y = (y_n)_n$. For a finite block sequence $s = (x_1, \ldots, x_k)$, define $Y \setminus s = (y_n)_{n \geq m}$, where $m$ is the least such that $\max \operatorname{supp} x_k < \min \operatorname{supp} y_m$. $Y \restriction n = (y_1, \ldots, y_n)$ and $Y \setminus n = (y_k)_{k>n}$. Also, for a sequence $\Delta = (\delta_k)_k > 0$, define $\Delta \setminus n = (\delta_k)_{k>n}$. Let $[Z] := \{Y | Y \preceq Z\}$. If $s = (x_1, \ldots, x_n)$, then we will write $[s]$ for $\{t \mid t \preceq (x_1, \ldots, x_n)\}$. For $s, A$ define $[s; A] = \{Y \in \boldsymbol{B_1} \mid$ there is $n$ such that $Y \restriction n = s$ and $Y \setminus s \preceq A\}$. Note that $[s; A] = [s; (A \setminus s)]$.

$y \in \boldsymbol{a}$ always means that $y$ is not only a vector in $\boldsymbol{a}$ but that it is also normalized and has finite support.

For $\boldsymbol{a} = (x_n)_n$ and $\boldsymbol{b} = (y_n)_n$ such that $|\boldsymbol{a}| = |\boldsymbol{b}|$ (i.e., either both $\boldsymbol{a}$ and $\boldsymbol{b}$ are infinite block basic sequences or they are both finite and of the same cardinality) and $\Delta = (\delta_n)_n$, define $d(\boldsymbol{a}, \boldsymbol{b}) \leq \Delta$ iff for every $n$, $\|x_n - y_n\| \leq \delta_n$.

Let $\Delta = (\delta_n)_n > 0$, and $\sigma \subseteq \boldsymbol{B_1}$. Then we define

$$\sigma_\Delta = \{(x_n)_n \mid d((x_n)_n, (y_n)_n) \leq \Delta \text{ for some } (y_n)_n \in \sigma\}.$$

Let $Y = (y_n)_n$ and $\tilde{Y} = (\tilde{y}_n)_n$ be block sequences, and let $Z = (z_n)_n \in [Y]$. We say that $\tilde{Z} = (\tilde{z}_n)_n \in [\tilde{Y}]$ *is defined as* $Z \in [Y]$ iff for every $n$,

$$\text{if } z_n = \sum_{k=1}^{m} \lambda_k y_k, \text{ then } \tilde{z}_n = \frac{1}{\lambda} \sum_{k=1}^{m} \lambda_k \tilde{y}_k$$

where $\lambda = \| \sum_{k=1}^{m} \lambda_k \tilde{y}_k \|$, i.e., if $\tilde{Z} = T(Z)$, where $T : Y \to \tilde{Y}$ is the isomorphism defined by $T(y_n) = \tilde{y}_n$.

**Fact 1.1** (see [2]). Given $\Delta = (\delta_n)_n > 0$, there is $0 < \Gamma < \Delta/2$ decreasing which satisfies the following:

For every $Y, \tilde{Y}$ such that $d(Y, \tilde{Y}) \leq \Gamma$, if $Z \in [Y]$, and $\tilde{Z} \in [\tilde{Y}]$ is defined as $Z \in [Y]$, then $d(Z, \tilde{Z}) \leq \Delta$.

Given $Y \in \boldsymbol{B_1}$ and $\sigma \subseteq \boldsymbol{B_1}$, we define the *game* $\eth_\sigma[Y]$ as follows: There are two players, $I$, and $II$. $I$ always plays a block vector of $Y$, and $II$ can play either a block vector of $Y$ or 0, the latter denoting that $II$ does not play any vector at that moment. The game starts with $I$ playing a block vector $x_1^{(1)} \in Y$, to which $II$ responds by playing either a (block) vector $y_1 \in [x_1^{(1)}]$, or 0. If $II$ plays a vector, then the game restarts with $I$ playing a vector $x_1^{(2)} \in Y$. However, if $II$ plays 0, then $I$ must play a vector $x_2^{(1)} > x_1^{(1)}$, to which $II$ responds by playing either a block vector $y_1 \in [x_1^{(1)}, x_2^{(1)}]$ or 0, and so on. It is also required that if $y_n$ and $y_m$ are vectors played by $II$ and $n < m$, then $y_n < y_m$. Thus, the game looks like this:

$$\begin{array}{c||cccccccccc} \mathbf{I} & x_1^{(1)} & \cdots & x_{n_1-1}^{(1)} & x_{n_1}^{(1)} & x_1^{(2)} & \cdots & x_{n_2-1}^{(2)} & x_{n_2}^{(2)} & \cdots \\ \hline \mathbf{II} & 0 & \cdots & 0 & y_1 & 0 & \cdots & 0 & y_2 & \cdots \end{array}$$

where $y_1 \in [x_1^{(1)}, ..., x_{n_1}^{(1)}]$, $y_2 \in [x_1^{(2)}, ..., x_{n_2}^{(2)}]$, and $y_1 < y_2$, etc. $II$ wins the game if she produces a sequence $(y_n)_n \in \sigma$. Otherwise (i.e., if $II$ does not produce an infinite sequence, or if $(y_n)_n \notin \sigma$) $I$ wins.

A strategy for $I$ or $II$ is a function from the set of finite runs of the game to block vectors, or 0, such that the value of the function on a finite run is a legal move. A strategy $S$ for $I$ ($II$) is a *winning strategy* if whenever $I$ ($II$) plays according to $S$, then he (she) wins the game.

Given a strategy for $I$ in $X$, $S$, we say that a finite block sequence $(y_1, ..., y_n)$ is *coherent* with $S$ iff $(y_1, ..., y_n)$ is the sequence of vectors played by $II$ in a finite run of the game in which $I$ plays according to $S$. An infinite block sequence $(y_n)_n$ is coherent with $S$ iff for every $n$, $(y_1, ..., y_n)$ is. For a sequence $Y$ coherent with $S$, let $S * Y$ be the sequence of vectors played by $I$ following the strategy $S$ against $Y$. For a strategy $S$ for $II$ in $X$, the definition of being coherent with $S$ is analogous, replacing $I$ for $II$.

Given $s$, a finite block sequence, and $\boldsymbol{b}$, either a finite or infinite block sequence, $s < \boldsymbol{b}$ has the obvious meaning. For $s < t$ and $s < A$, let $s^\frown t$ and $s^\frown A$ be the concatenation of $s$ with $t$ and of $s$ with $A$, respectively. Finally we define, for $s \in [\mathcal{X}]^{<\omega}$, the game $\eth_\sigma^s[Y]$: It is a game played in $Y \setminus s$, and if $II$ produces $Z$, then she wins iff $s^\frown Z \in \sigma$.

We can consider a natural topology on $\boldsymbol{B_1}$, *the N-topology*: The topology inherited from $\mathcal{X}^\omega$, where $\mathcal{X}$ has the norm topology and $\mathcal{X}^\omega$ the product topology. Note that $\mathcal{X}^\omega$ is a Polish space. It is easy to show that $\boldsymbol{B_1}$ is an N-closed subset of $\mathcal{X}^\omega$ (see [2]), and hence it is also a Polish space.

A set $\sigma \subseteq \boldsymbol{B_1}$ is *large in* $[Y]$ iff for every $Z \preceq Y$ there exists $Z' \preceq Z$ such that $Z' \in \sigma$. $\sigma$ is large in $[s; A]$ iff for every $Z \preceq A$, there exists $Z' \preceq Z$ such that $s^\frown Z' \in \sigma$.

The main notion is that of a weakly Ramsey set ([6], see also [2]).

**Definition 1.** Let $\Delta > 0$. A set $\sigma \subseteq \boldsymbol{B_1} = \boldsymbol{B_1}(\mathcal{X})$ is $\Delta$-*weakly Ramsey* iff there exists $Y \in \boldsymbol{B_1}$ such that either $[Y] \cap \sigma = \emptyset$, or $II$ has a winning strategy for the game $\eth_{\sigma_\Delta}[Y]$. $\sigma$ is *weakly Ramsey* iff it is $\Delta$-weakly Ramsey for every $\Delta > 0$. (Notice that without loss of generality we can always assume that $\Delta$ is decreasing and $\Delta < 1$.) Note that saying that $\sigma$ is $\Delta$-weakly Ramsey is equivalent to saying that if $\sigma$ is large (in $\mathcal{X}$), then there is some $X$ such that $II$ has a winning strategy for the game $\eth_{\sigma_\Delta}[X]$.

We defined in [2] two classes of partial orderings, which will also play a key role in this paper. The first is the following: For $Y \in \boldsymbol{B_1}$, $\mathbb{P} = \mathbb{P}(Y)$ is the partial ordering whose elements are pairs $(s, A)$, where $s$ and $A$ are block sequences of $Y$, $s$ finite and $A$ infinite, and such that $s < A$.

The ordering is given by: $(s, A) \leq (t, B)$ iff $t$ is a subsequence of $s$, $A \preceq B$ and $s \setminus t \in [B]$. (Note that this implies $t$ is an initial segment of $s$.)

To define the second class of partial orderings we need the notion of $\Delta$-cover: Given $s$ and $\Delta = (\delta_n)_n > 0$, we say that a set $\{t_1, ..., t_k\} \subseteq [s]$ is a $\Delta$-*cover* of $[s]$ iff for every $t \in [s]$ there exists $1 \leq i \leq k$ such that supp $t_i =$ supp $t$ and $d(t, t_i) \leq \Delta$ (this means that if $t = (y_1, ..., y_m)$ and $t_i = (z_1, ..., z_m)$, then for every $1 \leq j \leq m$, $d(y_j, z_j) \leq \delta_j$).

Also, for $\delta > 0$, a set $\{y_1, ..., y_n\} \subseteq s$ is a $\delta$-*cover* of $s$ iff for every $y \in s$ there is $1 \leq i \leq k$ such that supp $y =$ supp $y_i$ and $d(y, y_i) \leq \delta$. Note that for our convenience, the notation $x \in s$ implies that $x$ is normalized. Clearly, if $\delta > 0$ and $[s]$ has a $\Delta = (\delta)_n$-cover, then $s$ has a $\delta$-cover.

For every $s = (x_1, ..., x_n)$ and $\Delta > 0$, there is a $\Delta$-cover of $[s]$.

**Example 1.1.** Given $s = (x_1, ..., x_n)$ and $\delta > 0$, we give an example of a $\delta$-cover of $s$ in the case of real Banach spaces (the complex case is similar): For $1 \leq k \leq n$, let $M_k$ be the smallest positive integer such that $\frac{1}{(1+\delta)^{M_k}} < \frac{\delta}{k}$. Note that $1 > \frac{1}{1+\delta} > \frac{1}{(1+\delta)^2} > \cdots > \frac{1}{(1+\delta)^{M_k}}$. Let

$$L(\delta, k) = \{\pm 1, \pm\frac{1}{1+\delta}, \pm\frac{1}{(1+\delta)^2}, ..., \pm\frac{1}{(1+\delta)^{M_k}}, 0\}, \text{ and}$$
$$F(\delta, k) = \{f : \{1, ..., k\} \to L(\delta, k) \mid \text{for some } l,\ f(l) = \pm 1\}.$$

Let $V$ be the set of vectors $\sum_{k=1}^n f(k)x_k$ such that $f \in F(\delta, |\text{supp } f|)$. Then $C = \{x/\|x\| \mid x \in V\}$ is a $\delta$-cover of $s$: For suppose that $x \in s$, $x = \sum_{k=1}^n \lambda_k x_k$, $a =$ supp $x$. Let $f \in F(\delta, |a|)$ such that for every $k \in a$, $|\lambda_k - f(k)| \leq \frac{\delta}{2|a|}$. Let $y = \sum_{k=1}^n f(k)x_k$. Since $f \in F(\delta, |\text{supp } f|)$, $x \in V$ and $z = x/\|x\| \in C$. But

$$\|x - y\| = \|\sum_{k=1}^n (\lambda_k - f(k))x_k\| \leq \sum_{k \in a} \frac{\delta}{2|a|} = \frac{\delta}{2}.$$

Now, it is easy to show that if $v_1, v_2 \in \mathcal{X}$ are such that $\|v_2\| = 1$, and $\|v_1 - v_2\| \leq \gamma$, then $\|v_1 - v_1/\|v_1\|\| \leq \delta$. So,

$$\|z - y\| \leq \|z - x\| + \|x - y\| \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Note that our definition guarantees the following fact: Suppose that $s = (x_1, ..., x_n)$, $t = (x_1, ..., x_m)$ and $m \geq n$. Then the $\delta$-cover constructed above is exactly the set of vectors $x \in s$ that are in the $\delta$-cover of $t$.

Given $\Delta > 0$ and $Y = (y_n)_n \in \boldsymbol{B_1}$, we define the partial order $\mathbb{P}(\Delta, Y)$ as follows: First, choose for every finite subset of positive integers $a$ and every $m$, a finite $\delta_m$-cover of $[(y_k)_{k\in a}]$, $\mathcal{C}(a, m)$. We also require that for every $m$, if $a \subseteq b$ then $\mathcal{C}(a, m) = \{x \in \mathcal{C}(b, m) \mid \text{supp } x \subseteq a\}$ (see the Example 1.1).

Elements of $\mathbb{P}(\Delta, Y)$ are $(s, A) \in \mathbb{P}(Y)$ such that if $s = (x_1, ..., x_n)$, then for every $i \leq n$, $x_i \in \mathcal{C}(\text{supp } x_i, i)$. We call $s$ a *finite* $(\Delta, Y)$-*block basis*.

$(s, A) \leq (t, B)$ iff:

1. $t \subseteq s$,
2. $A \preceq B$, and
3. for every $|t| < i \leq |s|$, there exists $u \in B$ such that supp $u =$ supp $x_i$ and $d(u, x_i) \leq \delta_i$.

Given $A$ and $\Delta > 0$, we define

$$\wr A\wr_\Delta = \{B \in \boldsymbol{B_1} \mid \text{ there is some } \tilde{B} \in [A] \text{ such that } d(B, \tilde{B}) \leq \Delta \text{ and } \text{supp } B = \text{supp } \tilde{B}\},$$

where if $B = (b_n)_n$ and $\tilde{B} = (\tilde{b}_n)_n$, then supp $B =$ supp $\tilde{B}$ means that for every $n$ supp $b_n =$ supp $\tilde{b}_n$. We can also define $\wr A\wr_\Delta^{<\omega}$ and $\wr s\wr_\Delta$ in the obvious way. Then, we can re-state condition 3 as:

3′. $s \setminus t \in \wr B\wr^{<\omega}_{(\delta_i)_{i>|t|}}$.

Subsets of Polish spaces can be classified according to their topological complexity, which yields the projective (or Lusin) hierarchy of classes (see [11]). We use the following (standard) notation: $\utilde{\Sigma}^1_1$ is the class of *analytic* sets, i.e., the continuous images of Borel sets. $\utilde{\Pi}^1_1$ is the class of *co-analytic* sets, i.e., the complements of analytic sets. $\utilde{\Sigma}^1_{n+1}$ is the class of the continuous images of $\utilde{\Pi}^1_n$ sets, and $\utilde{\Pi}^1_{n+1}$ is the class of complements of $\utilde{\Sigma}^1_{n+1}$ sets. The projective sets are the sets that belong to one of the projective classes.

Using the notions defined above, in particular, the partial orderings $\mathbb{P}(Y)$, we gave in [2] a proof of the following (see also [8]):

**Theorem 1.1.** *Every analytic set of block sequences is weakly Ramsey.*

To prove that every $\utilde{\Sigma}^1_2$ set of block sequences is weakly Ramsey, we used in [2] the partial orderings $\mathbb{P}(\Delta, Y)$. We will call the partial orderings of the form $\mathbb{P}(\Delta, Y)$ relevant.

A fundamental fact from [2] is that every $\mathbb{P}(\Delta, Y)$ satisfies Baumgartner's Axiom A. Let $\mathrm{MA}_{\omega_1}(\mathcal{P})$ be the Martin's Axiom for the class $\mathcal{P}$ of the partial orderings $\mathbb{P}(\Delta, Y)$.

We quote the following result from [2]:

**Theorem 1.2** ($\mathrm{MA}_{\omega_1}(\mathcal{P})$). *Every $\utilde{\Sigma}^1_2$ subset of $\boldsymbol{B_1}$ is weakly Ramsey.*

We also showed in [2] that some additional axiom of set theory is needed to prove that every $\utilde{\Sigma}^1_2$ subset of $\boldsymbol{B_1}$ is weakly Ramsey.

The aim of this paper is to extend these results to all projective sets. For this, we give in the next section a sufficient condition for a set to be weakly Ramsey, namely, to have a good decomposition.

## 2. Good decompositions

We assume the reader is familiar with the basic notions of the forcing technique (see, for example, [9] or [12]).

We start with a bit of set-theoretic study of $\boldsymbol{B_1}$.

**Proposition 2.1.** *The relation $\preceq$ over block sequences of $\mathcal{X}$ is a closed subset of $\boldsymbol{B_1} \times \boldsymbol{B_1}$.*

*Proof.* For a given $X = (x_n)_n, Y = (y_n)_n \in \boldsymbol{B_1}$, $X \preceq Y$ iff $X \subseteq Y$ as subspaces, i.e., $X \preceq Y$ iff for every $n$, $x_n \in Y$. Define for each $n$,

$$A_n = \{(Z, W) \in \boldsymbol{B_1}^2 \mid z_n \in W\}.$$

If we prove that every $A_n$ is a closed subset of $\boldsymbol{B_1}^2$, then we are done since $\preceq = \bigcap_n A_n$. So, suppose that $(X_m, Y_m)_m$ is any sequence of pairs of block sequences, each one in $A_n$, and suppose that $(X_m, Y_m)_m$ has limit $(X, Y)$. Suppose that for every $m$, $X_m = (x_k^{(m)})_k$, $Y_m = (y_k^{(m)})_k$, $X = (x_k)_k$, and $Y = (y_k)_k$. By definition of $A_n$, for every $m$, $x_n^{(m)} \in Y_m$ and the sequence $(x_n^{(m)})_m$ has limit $x_n$. There are $m'$ and $l$ such that for every $m \geq m'$, supp $x_n^{(m)} =$ supp $x_n$ and $x_n^{(m)} = \lambda_1^{(m)} y_1^{(m)} + \cdots + \lambda_l^{(m)} y_l^{(m)}$, for some $|\lambda_i^{(m)}| \leq 2C$, where $C$ is the basic constant associated to $(\mathbf{e}_n)_n$. For $j = 1, ..., l$ (and passing to a subsequence if necessary), the sequences $(\lambda_j^{(m)})_m$ converge. Let $\lambda_1$,...,$\lambda_l$ be their limits. It is easy to prove that $x_n = \lambda_1 y_1 + \cdots + \lambda_l y_l$. □

We need to code elements of $\boldsymbol{B_1}$. So, let $g : \mathbb{N} \to Fin$ be any primitive recursive coding of $Fin$, the set of all finite subsets of $\mathbb{N}$. For every block vector $x = \sum_{n=1}^m \lambda_n \mathbf{e}_n$, let $c(x) = (\lambda_n)_{n=1}^m$. Let $C \subseteq \mathbb{K}^\omega \times \mathcal{N}$ be the set of sequences $((\lambda_n)_n, (k_n)_n)$ such that

1. $g(k_n) < g(k_{n+1})$ (i.e., $g(k_n) \cap g(k_{n+1}) = \emptyset$, and $\max g(k_n) < \min g(k_{n+1})$).
2. If $M_n = \sum_{m=1}^n |g(k_m)|$, then $x_{n+1} = \sum_{m \in g(k_{n+1})} \lambda_{m+M_n} \mathbf{e}_m$ is a normalized block vector. Note that $(x_n)_n \in \boldsymbol{B_1}$.

It is not difficult to show that $C$ is a perfect subset of $\mathbb{K}^\omega \times \mathcal{N}$ (i.e., closed and without isolated points), and that the map $\boldsymbol{c} : C \to \boldsymbol{B_1}$, defined by $\boldsymbol{c}(((\lambda_n)_n, (k_n)_n)) = (x_n)_n$ is continuous. The map $\boldsymbol{d} : \boldsymbol{B_1} \to \mathbb{K}^\omega \times \mathcal{N}$ defined by

$$\boldsymbol{d}((x_n)_n) = (c(x_1)^\frown c(x_2)^\frown \cdots, (g^{-1}(\text{supp } x_n)_n))$$

is the inverse of $\boldsymbol{c}$. So, $\boldsymbol{c}$ is a homeomorphism.

It is well-known that there is a canonical Borel isomorphism between any non-empty perfect subset of $\mathbb{K}^\omega \times \mathcal{N}$ and the Baire space $\mathcal{N}$ (see [11]). For now, fix some (canonical) Borel isomophism $\boldsymbol{b} : \mathcal{X} \to \mathcal{N}$.

We recall the following definitions from model theory and descriptive set theory (see, for example, [9]):

**Definition 2.** Let $M \subseteq N$ be transitive classes and let $\varphi(x_1, ..., x_n)$ be a formula of the language of set theory. We say that $\varphi$ is *absolute* for $M, N$ if for every $a_1, ..., a_n \in M$,

$$M \models \varphi(a_1, ..., a_n) \text{ iff } N \models \varphi(a_1, ..., a_n).$$

**Definition 3.** A formula is $\Sigma_n^1$ if it is of the form:

$$\exists x_1 \subseteq \omega \forall x_2 \subseteq \omega \exists x_3 \subseteq \omega \cdots \forall\!\!\!\exists x_n \subseteq \omega \psi(x_1, ..., x_n, y_1, ..., y_m)$$

where $y_1, ..., y_m$ are variables ranging over subsets of $\omega$ and all the quantifiers in $\psi$ range over $\omega$.

A formula is $\Pi_n^1$ if it is the negation of a $\Sigma_n^1$ formula.

A formula is *projective* if it is $\Sigma_n^1$ or $\Pi_n^1$, for some $n$.

**Definition 4.** We will say that a model of set theory $V$ is $\boldsymbol{\Sigma_n^1}$*-absolute* iff all the $\Sigma_n^1$ formulas are absolute for $V$ and $V[G]$, for every $G$ $V$-generic for a relevant partial ordering, i.e., of the form $\mathbb{P}(\Delta, Y)$. Note that a model is $\Sigma_n^1$-absolute iff it is $\Pi_n^1$-absolute. A model of set theory $V$ is *projective absolute* if it is $\Sigma_n^1$-absolute for every $n$.

It is well-known that every well-founded model of set theory is $\Sigma^1_1$-absolute (see, [9]).

The following well-known fact relates projective formulas and projective subsets of $\mathcal{N}$:

**Fact 2.1.** A subset $A \subseteq \mathcal{N}$ is a $\underset{\sim}{\Sigma}^1_n$ subset of $\mathcal{N}$ iff there exists a $\Sigma^1_n$ formula $\varphi(y, y_1, ..., y_k)$ and $\alpha_1, ..., \alpha_k \in \mathcal{N}$ such that $A = \{r \in \mathcal{N} \mid \varphi(r, \alpha_1, ..., \alpha_k)\}$.

*Proof.* See, for example, [11] or [12]. □

**Proposition 2.2.** *Suppose that $V$ is a $\Sigma^1_n$-absolute model of set theory, and $G$ is $V$-generic for $\mathbb{P}(\Delta, Y)$. Then for every $\underset{\sim}{\Sigma}^1_n$ set of block sequences $\sigma \in V$,*

$$V[G] \models (\sigma \neq \emptyset) \textit{ iff } V \models (\sigma \neq \emptyset).$$

*In particular, for analytic sets $\sigma$, $V \models \sigma \neq \emptyset$ iff $V[G] \models \sigma \neq \emptyset$.*

*Proof.* $\sigma \neq \emptyset$ iff $\emptyset \neq \boldsymbol{c}(\sigma) \subseteq \mathbb{K}^\omega \times \mathcal{N}$, and use Fact 2.1. □

**Proposition 2.3.** *Let $\sigma$ be any subset of $\boldsymbol{B_1}$, and let $s$ be any finite block sequence. Define*

$$\begin{aligned} \sigma^s &= \{X \in \boldsymbol{B_1} \mid s^\frown X \in \sigma\}, \\ c(\sigma) &= \{X \in \boldsymbol{B_1} \mid [X] \subseteq \sigma\}, \text{ and} \\ l(\sigma) &= \{X \in \boldsymbol{B_1} \mid \sigma \text{ is large in } [X]\}. \end{aligned}$$

*Then,*

1. *the operator $\sigma \mapsto \sigma^s$ does not change the complexity.*
2. *If $\sigma$ is $\underset{\sim}{\Pi}^1_n$ then $c(\sigma)$ is also $\underset{\sim}{\Pi}^1_n$, $l(\sigma)$ is $\underset{\sim}{\Pi}^1_{n+2}$ and $\sigma_\Delta$ is $\underset{\sim}{\Sigma}^1_{n+1}$.*
3. *If $\sigma$ is $\underset{\sim}{\Sigma}^1_n$ then $c(\sigma)$ is $\underset{\sim}{\Pi}^1_{n+1}$, $l(\sigma)$ is $\underset{\sim}{\Pi}^1_{n+1}$ and $\sigma_\Delta$ is $\underset{\sim}{\Sigma}^1_n$.*

*Proof.* 1 holds because the map $T_s : \boldsymbol{B_1} \to \boldsymbol{B_1}$ defined by $T_s(X) = s^\frown(X \setminus s)$ is continuous. For 2 and 3, suppose that $\sigma$ is a $\underset{\sim}{\Pi}^1_n$ subset of $\boldsymbol{B_1}$. Fix any $\underset{\sim}{\Sigma}^1_{n-1}$ subset $B \subseteq \boldsymbol{B_1} \times \mathcal{N}$ such that $\sigma = \{X \mid \text{for every } \alpha \in \mathcal{N},\ (X, \alpha) \in B\}$. Then define

$$\begin{aligned} B' &= \{(W_1, W_2, \alpha) \in \boldsymbol{B_1}^2 \times \mathcal{N} \mid (W_1, \alpha) \in B\}, \text{ and} \\ A &= (\boldsymbol{B_1}^2 \setminus \preceq) \times \mathcal{N}. \end{aligned}$$

$B'$ is a $\underset{\sim}{\Sigma}^1_{n-1}$ subset of $\boldsymbol{B_1}^2 \times \mathcal{N}$, and $A$ is an open subset of $\boldsymbol{B_1}^2 \times \mathcal{N}$. It is easy to prove that $\sigma = \{X \in \boldsymbol{B_1} \mid \forall (Y, \alpha) \in \boldsymbol{B_1} \times \mathcal{N},\ (Y, X, \alpha) \in (A \cup B')\}$, which is a $\underset{\sim}{\Pi}^1_n$ set, because $A \cup B'$ is $\underset{\sim}{\Sigma}^1_{n-1}$.

Similarly for the case $\underset{\sim}{\Sigma}^1_n$. □

The following definition provides a sufficient condition for a set to be weakly Ramsey. We will use it to prove the main results of this paper.

**Definition 5.** $\sigma = \bigcup_{i \in J} \sigma_i$ is a *good decomposition* iff for every $\Delta > 0$ the following hold:

1. For every $(s, A)$, if $\sigma$ is large in $[s; A]$, then there is some $i \in J$ and $(t, B) \leq (s, A)$ such that $(\sigma_i)_\Delta$ is large in $[t; B]$, and
2. For every $i \in J$, $s$ and $X$, $((\sigma_i)_\Delta)^s$ is weakly Ramsey in $\boldsymbol{B_1}(X)$.

The interest in good decompositions is due to the following:

**Theorem 2.1.** *Every subset of $\boldsymbol{B_1}$ with a good decomposition is weakly Ramsey.*

We need some lemmas.

**Lemma 2.1.** *There is $Y$ such that for every $(s,A) \in \mathbb{P}(Y)$, if $II$ has a winning strategy for the game $\Game_{(\sigma_\Delta)^s}[A]$, then also for the game $\Game_{((\sigma_\Delta)^s)_\Delta}[Y]$.*

*Proof.* Consider the following subsets of $\mathbb{P}(\mathcal{X})$:

$$D_m := \{(s,A) \in \mathbb{P}(\mathcal{X}) \mid |s| \geq m \text{ and for every } t \in [s] \text{ if for some } B \preceq A,\ II \text{ has a winning strategy for } \Game_{(\sigma_\Delta)^t}[B] \text{ then } II \text{ has a winning strategy for } \Game_{((\sigma_\Delta)^t)_\Delta}[A]\}.$$

Every $D_m$ is a dense subset of $\mathbb{P}(\mathcal{X})$: Fix any $(s,A)$, and we can suppose that $|s| \geq m$. Let $\{t_1, ..., t_k\}$ be any $\frac{\Delta}{2}$-cover of $s$. Define $A = A_0 \succeq \cdots \succeq A_k$ as follows: Suppose $A_{i-1}$ is defined. If there is some $B \preceq A_{i-1}$ such that $II$ has a winning strategy for the game $\Game_{((\sigma_\Delta)^{t_i})_{\Delta/2}}[B]$, then put $A_i = B$; otherwise $A_i = A_{i-1}$. Then $(s,A_k) \leq (s,A)$, and $(s,A_k) \in D_m$: Suppose that $B \preceq A_k$ is such that $II$ has a winning strategy for the game $\Game_{(\sigma_\Delta)^t}[B]$, for some $t \in [s]$. Let $i$ be such that $d(t,t_i) \leq \Delta/2$. Then $II$ also has a winning strategy for the game $\Game_{((\sigma_\Delta)^{t_i})_{\Delta/2}}[B]$, and by definition of $A_i$, $II$ has also a winning strategy for the game $\Game_{((\sigma_\Delta)^{t_i})_{\Delta/2}}[A_i]$, and hence, also for $\Game_{((\sigma_\Delta)^t)_\Delta}[A_i]$.

Then let $G$ be a $\{D_m\}_m$-generic filter (it always exists because the set of dense subsets is countable), and $Y = Y_G$ satisfies what we wanted: Let $(t,B) \in \mathbb{P}(Y)$ and suppose that $II$ has a winning strategy for the game $\Game_{(\sigma_\Delta)^t}[B]$. Let $m$ be large enough so that $t \in [s]$, for some $(s,A) \in G \cap D_m$. Choose any $k$ such that $B^* = B \setminus k \preceq A$, and $II$ has a winning strategy for the game $\Game_{(\sigma_\Delta)^t}[B^*]$, and, by definition of $D_m$, also for the game $\Game_{((\sigma_\Delta)^t)_\Delta}[A]$. But $Y \preceq^* A$ and hence $II$ also has a winning strategy for the game $\Game_{((\sigma_\Delta)^t)_\Delta}[Y]$. □

**Lemma 2.2.** *Let $\sigma = \bigcup_{i \in J} \sigma_i$ be a good decomposition. If $\sigma$ is large in $Y$, then there is $Z \preceq Y$ such that $II$ has a winning strategy, in $Z$, for producing a sequence $t$ for which there is some $i \in J$ such that $(\sigma_i)_\Delta$ is large in $[t;A]$, for some $A \preceq Y$.*

*Proof.* Consider

$$\tilde{\sigma}(\Delta, Y) = \{(y_m)_m \preceq Y \mid \text{ there exist } i \in J \,\&\, k \geq 1 \text{ and } A \preceq Y \text{ such that } (\sigma_i)_\Delta \text{ is large in } [s^\frown(y_1, ..., y_k); A]\}.$$

Note that $\tilde{\sigma}(\frac{\Delta}{2}, Y)$ is open below $Y$ (i.e., is an open subset of $[Y]$). And also large: For suppose that $Z \preceq Y$. Since $\sigma$ is large in $[Z]$ and $\bigcup_{i \in J} \sigma_i$ is a good decomposition of $\sigma$, there is some $i$ and $(s,A) \in \mathbb{P}(Z)$ such that $(\sigma_i)_{\Delta/8}$ is large in $[s;A]$. But then $s^\frown A \in \tilde{\sigma}(\frac{\Delta}{2}, Y) \cap [Z]$, and we are done.

By Theorem 1.1 (working with $\mathcal{X} = Y$), there is some $Z \preceq Y$ such that $II$ has a winning strategy for the game $\Game_{(\tilde{\sigma}(\Delta/2,Y))_{\Delta/2}}[Z]$, hence also for the game $\Game_{\tilde{\sigma}(\Delta,Y)}[Z]$.

In other words, there is $Z \preceq Y$ such that $II$ has a winning strategy in $Z$ for producing a sequence $s \preceq Z$ such that for some $i$ and some $A \preceq Y$, $(\sigma_i)_\Delta$ is large in $[s;A]$. □

*Proof of Theorem 2.1.* Let $Y$ satisfy Lemma 2.1 and let $Z \preceq Y$ satisfy Lemma 2.2. We give a winning strategy for $II$ in the game $\Game_{\sigma_\Delta}[Z]$: First, $II$ plays in $Z$ for producing a sequence $s$ such that for some $i \in J$, $(\sigma_i)_{\Delta/3}$ is large in $[s;A]$, for some

$A \preceq Y$ (i.e., $((\sigma_i)_{\Delta/3})^s$ is large in $[A]$). $\bigcup_{i\in J}\sigma_i$ is a good decomposition of $\sigma$, and hence, $((\sigma_i)_{\Delta/3})^s$ is weakly Ramsey in $A$. Choose any $B \preceq A$ such that $II$ has a winning strategy for $\Game_{(((\sigma_i)_{\Delta/3})^s)_{\Delta/3}}[B]$. But

$$(((\sigma_i)_{\frac{\Delta}{3}})^s)_{\frac{\Delta}{3}} \subseteq ((\sigma_{\frac{\Delta}{3}})^s)_{\frac{\Delta}{3}} \subseteq ((\sigma_{\frac{\Delta}{3}})_{\frac{\Delta}{3}})^s \subseteq (\sigma_{\frac{2\Delta}{3}})^s,$$

hence $II$ also has a winning strategy for the game $\Game_{(\sigma_{2\Delta/3})^s}[B]$. By our assumption over $Y$ (Lemma 2.1 for $B \preceq A \preceq Y$), $II$ also has a winning strategy for the game $\Game_{((\sigma_{2\Delta/3})^s)_{\Delta/3}}[Y]$. But $Z \preceq Y$, and hence $II$ has a winning strategy for the game $\Game_{(\sigma_\Delta)^s}[Z]$.

We check that this is a winning strategy in $Z$ for $II$. For suppose that $X$ is a coherent sequence with this strategy. Then, let $s$ be an initial segment of $X$ such that $((\sigma_i)_{\Delta/3})^s$ is large in $A \preceq Z$ for some $i \in J$. Then, $X \setminus s$ is played so that $X \setminus s \in (\sigma_\Delta)^s$, i.e., $X = s^\frown(X \setminus s) \in \sigma_\Delta$, and we are done. $\square$

We will now prove that in a $\Sigma^1_3$-absolute model every $\underset{\sim}{\Sigma}^1_2$ set is weakly Ramsey.

**Lemma 2.3.** *Suppose that for every $t$ there is some $C \preceq A$ such that $[s^\frown t; C] \cap \sigma_\Delta = \emptyset$. Then, for every $(t, B)$, there is some $C \preceq B$ such that for every $u \in [t]$, $[s^\frown u; t \setminus u^\frown C] \cap \sigma_{\Delta/2} = \emptyset$.*

*Proof.*

**Claim 2.3.1.** *For every $(t, B)$ there is $\check{B} \preceq B$ such that for every $u \in [t]$, $[s^\frown u; \check{B}] \cap \sigma_{\Delta/2} = \emptyset$.*

*Proof of Claim.* Fix $(t, B)$, and let $\{t_1, ..., t_k\}$ be a $\Delta/2$-cover of $t$. Then, find $B = A_0 \succeq A_1 \succeq \cdots \succeq A_k$ so that $[s^\frown t_i; A_i] \cap \sigma_\Delta = \emptyset$. Take $\tilde{B} = A_k$, and we are done. $\square$

Find $B = C_0 \succeq C_1 \succeq \cdots \succeq C_k \succeq \cdots$ so that for every $k$:

1. $C_{k+1} \preceq C_k \setminus c_k$, where $c_k$ is the first element of the block sequence $C_k$.
2. For every $u \in [t]$ and every $w \in [(t\setminus u)^\frown(c_0, ..., c_k)]$, $[s^\frown u^\frown w; C_{k+1}] \cap \sigma_{\Delta/2} = \emptyset$.

We check that this is possible: For suppose we have defined $C_k$. By Claim 2.3.1, there is $\tilde{C} \preceq C_k \setminus c_k$ such that for every $v \in [t^\frown(c_0, ..., c_k)]$, $[s^\frown v; \tilde{C}] \cap \sigma_{\Delta/2} = \emptyset$. But if $u \in [t]$ and $w \in [(t \setminus u)^\frown(c_0, ..., c_k)]$, then $u^\frown w \in [t^\frown(c_0, ..., c_k)]$. So take $C_{k+1} = \tilde{C}$.

Let $C = (c_k)_{k\geq 0}$. We check that $(t, C)$ satisfies what we want: Let $u \in [t]$, and let $Z = (z_m)_m \preceq (t \setminus u)^\frown C$. There is $m_0$ and $k_0$ such that $(z_1, ..., z_{m_0}) \in [(t\setminus u)^\frown(c_1, ..., c_{k_0})]$ and $(z_m)_{m>m_0} \preceq (c_k)_{k>k_0}$. By construction, $[s^\frown u^\frown(z_1, ..., z_{m_0}); C_{k_0+1}] \cap \sigma_{\Delta/2} = \emptyset$. But $(z_m)_{m>m_0} \preceq C_{k_0+1}$ and then $s^\frown u^\frown(z_1, ..., z_{m_0})^\frown(z_m)_{m>m_0} \notin \sigma_{\Delta/2}$, and so we are done. $\square$

**Lemma 2.4.** *Suppose that $G$ is $\mathbb{P}(\Delta, Y)$-generic over $\mathbb{V}$, and let*

$$Y = Y_G = \bigcup_{(s,A)\in G} s.$$

*Then, in $V[G]$, for every $(s, A) \in G$, there is some $k \geq |s|$ such that $Y \setminus k \in \wr A\wr_{\Delta\setminus k}$.* $\square$

**Theorem 2.2.** *If $V$ is $\Sigma^1_3$-absolute for $\mathbb{P}$, then every $\underset{\sim}{\Sigma}^1_2$ set has a good decomposition.*

*Proof.* Using the Shoenfield tree for $\underset{\sim}{\Sigma}^1_2$ sets (see [9]), we can show that every $\underset{\sim}{\Sigma}^1_2$ set of block sequences is the union of $\omega_1$ Borel sets, $\sigma = \bigcup_{\alpha<\omega_1} \sigma_\alpha$. We check that this is a good decomposition: By Proposition 2.3, every $((\sigma_\alpha)_\Delta)^s$ is analytic, and also for every $X$, $((\sigma_\alpha)_\Delta)^s \cap [X]$ is analytic. Hence, $((\sigma_\alpha)_\Delta)^s$ is weakly Ramsey in $X$.

Now suppose that $\sigma$ is large in $\mathcal{X}$. We need to check that there is some $\alpha < \omega_1$ and some $(t, B)$ such that $(\sigma_\alpha)_\Delta$ is large in $[t; B]$. Otherwise, for every $\alpha < \omega_1$, and every $(t, B)$, there is $C \preceq B$, such that $(\sigma_\alpha)_\Delta \cap [t; C] = \emptyset$. By Lemma 2.3, for every $\alpha < \omega_1$, and every $(t, B)$, there is $C \preceq B$, such that for every $u \in [t]$, $(\sigma_\alpha)_{\Delta/2} \cap [u; (t \setminus u)^\frown C] = \emptyset$. For every $\mathbb{P}$-name for an ordinal $< \omega_1$, $\dot{\alpha}$, and for every $m$, let

$$D_{\dot{\alpha},m} = \{(t, B) \in \mathbb{P}(\frac{\Delta}{2}, \mathcal{X}) \mid , |t| \geq m \,\&\, \forall u \in [t],\ (t, B) \Vdash'' [u; t \setminus u^\frown B] \cap (\sigma_{\dot{\alpha}})_{\frac{\Delta}{2}} = \emptyset''\}.$$

**Claim 2.2.1.** *Every $D_{\dot{\alpha},m}$ is dense.*

*Proof of Claim.* $\mathbb{P}$ does not collapse $\omega_1$, so, we can suppose that $\dot{\alpha} = \check{\alpha}$, where $\alpha < \omega_1$. Fix $(t, B)$. We may assume that $|t| \geq m$. Let $C$ be such that for every $u \in [t]$, $(\sigma_\alpha)_{\Delta/2} \cap [u; (t \setminus u)^\frown C] = \emptyset$, i.e., $(t \setminus u)^\frown C \in c((\boldsymbol{B_1} \setminus (\sigma_\alpha)_{\Delta/2})^u)$. By Proposition 2.3, $c((\boldsymbol{B_1} \setminus (\sigma_\alpha)_{\Delta/2})^u)$ is a $\underset{\sim}{\Pi}^1_1$ set, and by Proposition 2.2,

$$(t, C) \Vdash'' (\sigma_{\check{\alpha}})_{\frac{\Delta}{2}} \cap [u; (t \setminus u)^\frown C] = \emptyset''.$$

□

Let $\mathcal{D} = \{D_{\check{\alpha},m} \mid \alpha < \omega_1,\ m \in \mathbb{N}\}$.

**Claim 2.2.2.** *Suppose that $G$ is a filter generic over $V$. In $V[G]$, there is some $Y$ such that for every $\alpha < \omega_1$, $\sigma_\alpha \cap [Y] = \emptyset$.*

*Proof of Claim.* Fix $G$. Let $Y = Y_G$, and work in $V[G]$. Fix $\alpha < \omega_1$, and let $u^\frown C \preceq Y$. Let $(t, B) \in D_{\check{\alpha},m} \cap G$ be such that $u \in [t]$. Then $(t, B) \Vdash'' [u; t \setminus u^\frown B] \cap (\sigma_{\check{\alpha}})_{\Delta/2} = \emptyset''$, and hence, in $V[G]$, $[u; t \setminus u^\frown B] \cap (\sigma_\alpha)_{\Delta/2} = \emptyset$. But $Y \preceq \wr t \setminus u^\frown B \wr_{\Delta/2 \setminus u}$, and we are done. □

Let $G$ be a filter generic over $V$. By the previous Claim, in $V[G]$ there is some $Y$ such that for every $\alpha < \omega_1$, $\sigma_\alpha \cap [Y] = \emptyset$. The decomposition $\sigma = \bigcup_{\alpha<\omega_1} \sigma_\alpha$ remains true in $V[G]$ (the decomposition is absolute). So, in $V[G]$, there is some $Y$ such that $\sigma \cap [Y] = \emptyset$. We know that in $V$, $\sigma$ is large, i.e., $\mathcal{X} \in l(\sigma)$, and by Proposition 2.3, $l(\sigma)$ is a $\underset{\sim}{\Pi}^1_3$ set. Hence (by our assumption on $V$), in $V[G]$ $\sigma$ is also large in $\mathcal{X}$. A contradiction. □

**Corollary 2.2.1.** *Suppose that $V$ is $\Sigma^1_3$-absolute for $\mathbb{P}$. Then, every $\underset{\sim}{\Sigma}^1_2$ set is weakly Ramsey.* □

Let us remark that this is a stronger result than Theorem 1.2. Indeed, in [1] it is shown that for any class of partial orderings $\mathcal{P}$, MA($\mathcal{P}$) implies $\Sigma^1_3$-absoluteness for partial orderings in $\mathcal{P}$, but not the converse.

## 3. A model of set theory where every projective set is weakly Ramsey

An *inner model* is a transitive class that contains all ordinals and is a model of (a fragment of) Zermelo-Fraenkel (ZF) set theory. For a given set $A$, the *constructive closure* of $A$, $L(A)$, is the smallest inner model $M$ such that $A \in M$. So, $L(\mathbb{R})$ is the smallest inner model $M$ such that $\mathbb{R} \in M$. The *constructive closure relative* to $A$, $L[A]$, is the smallest inner model $M$ such that for every $x \in M$, $x \cap A \in M$.

$\kappa$ is a *Mahlo* cardinal if the set $\{\alpha < \kappa \mid \alpha \text{ is an inaccessible cardinal}\}$ is stationary in $\kappa$. A Mahlo cardinal is always an inaccessible cardinal.

For $\kappa$ an inaccessible cardinal, the *Levy-collapse*, $Coll(\omega, < \kappa)$, is the following partial ordering: Elements are maps $p : S \to \kappa$, where $S$ is a finite subset of $\kappa \times \omega$, and for every $(\alpha, m) \in S$ with $\alpha \neq 0$, $p(\alpha, m) < \alpha$. $p \leq q$ iff $q \subseteq p$. $Coll(\omega, < \kappa)$ collapses $\kappa$ to $\omega_1$.

For $\alpha < \kappa$, let $Coll(\omega, < \alpha)$ be the sub-partial ordering of $Coll(\omega, < \kappa)$ consisting of all $p : S \to \alpha$, where $S$ is a finite subset of $\alpha \times \omega$.

$M$ is a *Solovay model over* $V$ iff $M = L(\mathbb{R})^W$ where $W$ is a model obtained by Levy-collapsing an inaccessible cardinal $\kappa$ in $V$.

For two models $M \subseteq N$, $M \preccurlyeq N$ (do not confuse with $\preceq$) means that $M$ is an *elementary submodel* of $N$, i.e., given any sentence $\varphi$ with parameters in $M$, $\varphi$ holds in $M$ iff it holds in $N$. $M \preccurlyeq_n N$ means that for every $\Sigma^1_n$ formula $\varphi$ with parameters reals and ordinals in $M$, $\varphi$ holds in $M$ iff it holds in $N$.

We shall prove the following:

**Theorem 3.1.** *Suppose $\kappa$ is a Mahlo cardinal, and $A \in V_\kappa$. Then in $L[A]^{Coll(\omega,<\kappa)}$, for every infinite-dimensional separable Banach space $\mathcal{X}$, every projective set of normalized block bases of $\mathcal{X}$ is weakly Ramsey.*

We recall some well-known properties of the Levy-collapse:

**Proposition 3.1.**

1. *$Coll(\omega, < \kappa)$ is $\kappa$-cc.*
2. *$\omega_1^{V[G]} = \kappa$.*
3. *For every $x \in \mathbb{R}$, $\kappa$ is innacessible in $V[x]$.*
4. *For every $x \in \mathbb{R}$, there is some $\alpha < \kappa$ such that $x \in V[G_\alpha]$, where $G_\alpha = Coll(\omega, < \alpha) \cap G$.*
5. *(Factor Lemma) For every countable set of ordinals $X$ of $V[G]$ there is a $V[X]$-generic filter $H$ of $Coll(\omega, < \kappa)$ such that $V[X][H] = V[G]$.*
6. *The Levy-collapse is homogeneous (see [9]). Hence, every formula with parameters in the ground model has Boolean value* **0** *or* **1**.

*Proof.* See [9]. □

First, we give a characterization of Solovay models due to Woodin (see [3]):

**Lemma 3.1** (Woodin). *$L(\mathbb{R})$ is a Solovay model over $V$ iff*

1. *$\omega_1$ is inaccessible to reals, i.e, for every $x \in \mathbb{R}$, $\omega_1$ is an inaccessible cardinal in $V[x]$.*
2. *For every $x \in \mathbb{R}$, $V[x]$ is a generic extension of $V$ by some countable partial ordering.*

*Proof.* ($\Rightarrow$) Clear, by Proposition 3.1.

($\Leftarrow$) We shall force over $L(\mathbb{R})$ to create a generic filter $G$ over $V$ for the Levy-collapse $Coll(\omega, <\omega_1)$ with the property that $\mathbb{R} = \mathbb{R} \cap V[G]$. This will be enough, since then $L(\mathbb{R}) = L(\mathbb{R})^{V[G]}$. We define the forcing $\mathbb{G}$ as follows:

- $g \in \mathbb{G}$ iff there exists $\alpha < \omega_1$ such that $g \subseteq Coll(\omega, \leq \alpha)$ is a generic filter over $V$.
- $g \leq h$ iff $h \subseteq g$.

By (1), for every $g \in \mathbb{G}$, $\omega_1$ is an inaccessible cardinal in $V[g]$ and, hence, for every $\alpha < \omega_1$ there are only countably-many antichains of $Coll(\omega, \leq \alpha)$ in $V[g]$. Therefore, for every $\alpha < \omega_1$, $D_\alpha = \{g \in \mathbb{G} : g \cap Coll(\omega, \leq \alpha) \text{ is generic over } V\}$ is a dense subset of $\mathbb{G}$. Since every $g \in \mathbb{G}$ is a countable set in $L(\mathbb{R})$, given any real $x \in \mathbb{R}$, we can code $x$ and $g$ into a single real $y$. By (2), $V[y]$ is a generic extension by some countable partial ordering in $V$ (by Proposition 3.1). Hence, we can find $\alpha < \omega_1$ and a generic filter $h \subseteq Coll(\omega, \leq \alpha)$ such that $y \in V[h]$. But then, $h \leq g$ and $x \in V[h]$. Therefore, for every real $x$, $E_x = \{g \in \mathbb{G} : x \in V[g]\}$ is a dense subset of $\mathbb{G}$. Let $H$ be a $\mathbb{G}$-generic filter over $V$ and let $G = \bigcup H$. Then, by density of $D_\alpha$ $(\alpha < \omega_1)$, $G \subseteq Coll(\omega, <\omega_1)$ is a generic filter over $V$. By density of $E_x$ (for $x \in \mathbb{R}$), $\mathbb{R} \subseteq V[H]$. $\square$

The following lemma gives the property of the Levy-collapse of a Mahlo cardinal that we will need:

**Lemma 3.2.** *If $\kappa$ is a Mahlo cardinal, then there exists a stationary set $S \subseteq \kappa$ of inaccessible cardinals such that for every $\alpha \in S$,*

$$V^{Coll(\omega,<\alpha)} \preccurlyeq_\omega V^{Coll(\omega,<\kappa)}$$

*(i.e., if $G$ is $Coll(\omega, <\kappa)$-generic over $V$ and in $V[G]$, $g$ is $Coll(\omega, <\alpha)$-generic over $V$, then $V[g] \preccurlyeq_\omega V[G]$, that is, for every $n \in \omega$ and every $\Sigma^1_n$ formula $\varphi$ with parameters in $V[g]$, $V[g] \models \varphi$ iff $V[G] \models \varphi$).*

*Proof.* Let $I$ denote the stationary set of inaccessible cardinals below $\kappa$. First we prove that for every $\Sigma^1_n$ formula $\varphi(x_1, ..., x_k)$, and all $Coll(\omega, <\kappa)$-terms $\dot{b}_1, ..., \dot{b}_k$ for reals, if

$$V^{Coll(\omega,<\kappa)} \models \varphi(\dot{b}_1, ..., \dot{b}_k),$$

then there exists a club $C \subseteq \kappa$ such that for every $\lambda \in C \cap I$,

$$V^{Coll(\omega,<\lambda)} \models \varphi(\dot{b}_1, ..., \dot{b}_k).$$

The proof is by induction on the complexity of the formula:

For restricted formulas this is clear, since every real in $V^{Coll(\omega,<\kappa)}$ belongs to some $V^{Coll(\omega,<\alpha)}$, $\alpha < \kappa$, and restricted formulas are absolute for transitive models. For notational convenience, let us denote the restricted formulas by $\Sigma^1_0$, and also $\Pi^1_0$.

Suppose $\varphi(x_1, ..., x_k)$ is $\Sigma^1_{n+1}$, $n \geq 0$. Then it follows by inductive hypothesis.

So, let $\forall y \psi(x_1, ..., x_k, y)$ be a $\Pi^1_{n+1}$, $n \geq 0$, formula, and let $\dot{b}_1, ..., \dot{b}_k$ be simple $Coll(\omega, <\kappa)$-terms for reals (i.e., each $\dot{b}_i$ is essentially an $\omega$-sequence of maximal antichains) such that $\forall y \psi(\dot{b}_1, ..., \dot{b}_k, y)$ holds in $V^{Coll(\omega,<\kappa)}$.

Fix an enumeration $\langle \dot{r}_\alpha : \alpha < \kappa \rangle$ of all simple $Coll(\omega, <\kappa)$-terms for reals so that for every $\lambda \in I$, $\langle \dot{r}_\alpha : \alpha < \lambda \rangle$ enumerates all simple $Coll(\omega, <\lambda)$-terms for reals.

Thus, for every $\alpha < \kappa$, $\psi(\dot{b}_1, ..., \dot{b}_k, \dot{r}_\alpha)$ holds in $V^{Coll(\omega,<\kappa)}$. By inductive hypothesis, let $C_\alpha \subseteq \kappa$ be a club such that for every $\lambda \in C_\alpha \cap I$,

$$V^{Coll(\omega,<\lambda)} \models \psi(\dot{b}_1, ..., \dot{b}_k, \dot{r}_\alpha).$$

Let $C = \triangle_{\alpha<\kappa} C_\alpha$. Then, for every $\lambda \in C \cap I$,

$$V^{Coll(\omega,<\lambda)} \models \forall y \psi(\dot{b}_1, ..., \dot{b}_k, y).$$

To prove the lemma, let $A = \langle a_\alpha : \alpha < \kappa \rangle$ be an enumeration of all the pairs $(\varphi(x_1, ..., x_k),\ \dot{b}_1, ..., \dot{b}_k)$, where $\varphi(x_1, ..., x_k)$ is a $\Sigma^1_n$ formula, some $n \in \omega$, and $\dot{b}_1, ..., \dot{b}_k$ is a sequence from $\langle \dot{r}_\alpha : \alpha < \kappa \rangle$. We may require that if $\lambda < \kappa$ is inaccessible, then $\langle a_\alpha : \alpha < \lambda \rangle$ enumerates all pairs $(\phi(x_1, ..., x_k), \dot{b}_1, ..., \dot{b}_k)$ such that $\dot{b}_1, ..., \dot{b}_k$ are from $\langle \dot{r}_\alpha : \alpha < \lambda \rangle$. For each $\alpha < \kappa$, let $C_\alpha$ be a club such that if $a_\alpha = (\varphi(x_1, ..., x_k), \dot{b}_1, ..., \dot{b}_k)$ and

$$V^{Coll(\omega,<\kappa)} \models \varphi(\dot{b}_1, ..., \dot{b}_k),$$

then

$$V^{Coll(\omega,<\lambda)} \models \varphi(\dot{b}_1, ..., \dot{b}_k)$$

for every inaccessible $\lambda \in C_\alpha$.

Let

$$C = \triangle_{\alpha<\kappa} C_\alpha.$$

Then, $S := I \cap C$ is as required. □

Recall that a partial ordering $\mathbb{P}$ is *proper* ([18]) if for some large-enough regular cardinal $\lambda$ (e.g., $\lambda > 2^{2^{|\mathbb{P}|}}$), for every countable elementary substructure $N$ of $H(\lambda)$ with $\mathbb{P} \in N$, and for every $p \in \mathbb{P} \cap N$, there is $q \leq p$, $(N, \mathbb{P})$-generic, i.e., whenever $A \subseteq \mathbb{P}$ is a maximal antichain, $A \in N$, then $\{a \in A \,|\, a \text{ is compatible with } q\} \subseteq N$.

$\mathbb{P} = (\mathbb{P}, <_\mathbb{P})$ is *absolute* iff the relations $p \in \mathbb{P}$, $p <_\mathbb{P} q$ and $p \perp_\mathbb{P} q$ (the incompatibility relation) are absolute for transitive models of ZF.

**Lemma 3.3.** *If $\mathbb{P}$ is a proper, definable and absolute partial ordering whose elements are reals, then every $\mathbb{P}$-extension of a Solovay model over $V$ is a Solovay model over $V$.*

*Proof.* Fix $M = L(\mathbb{R})^{V[G]}$ a Solovay model over $V$, where $G$ is $Coll(\omega, <\kappa)$-generic over $V$, some $\kappa$ inaccessible in $V$.

We need to show that in $V[G]^\mathbb{P}$:

1. $\omega_1$ is inaccessible to reals.
2. Every real is generic over $V$ for a countable partial ordering.

Working in $V[G]$, suppose $\dot{r}$ is a simple $\mathbb{P}$-term, and suppose $p \in \mathbb{P}$ forces that $\dot{r} : \omega \to \omega$. Assume, towards a contradiction, that $p$ also forces that $\omega_1^{V[\dot{r}]} = \omega_1$.

Let $N \preccurlyeq H(\lambda)$, where $\lambda$ is a big-enough regular cardinal, $N$ countable, and $\dot{r}, p, \mathbb{P} \in N$. Since $\mathbb{P}$ is proper, we can find $q \leq p$, $(N, \mathbb{P})$-generic. Let $\dot{s} = \dot{r} \cap N$. Then,

1. $TC(\dot{s})$ is countable,
2. $q \Vdash_\mathbb{P} \dot{s} = \dot{r}$.

Let $\alpha < \kappa$ be such that $\dot{s}$ is a $\mathbb{P}_\alpha$-term in $V[G_\alpha]$, where $G_\alpha = Coll(\omega, <\alpha) \cap G$ and $\mathbb{P}_\alpha = \mathbb{P}^{V[G_\alpha]} = \mathbb{P} \cap V[G_\alpha]$ (here we are using that $\mathbb{P}$ is absolute.) We may also require that $p$ and $q$ are in $V[G_\alpha]$.

Let $\langle D_n : n < \omega \rangle$ be an enumeration, in $V[G]$, of all $D \in V[G_\alpha]$ such that

$$V[G_\alpha] \models D \texttt{ is a maximal antichain below } q.$$

Let $N' \preccurlyeq H(\lambda)$, $\lambda$ a large-enough regular cardinal, $N'$ countable, with $q, \dot{s}, \mathbb{P}$, $\langle D_n : n < \omega \rangle \in N'$. Let $q' \leq q$ be $(N', \mathbb{P})$-generic. Then, for every $\mathbb{P}$-generic $F$ over $V[G]$ with $q' \in F$, $F_\alpha = F \cap V[G_\alpha]$ is $\mathbb{P}_\alpha$-generic over $V[G_\alpha]$ and, further,

$$V[G][F] \models \dot{r}[G * F] = \dot{s}[G_\alpha * F_\alpha].$$

So,

$$V[G][F] \models \omega_1^{V[\dot{s}[F]]} = \omega_1.$$

And thus,

$$V[G_\alpha][F_\alpha] \models \omega_1^{V[\dot{s}[F_\alpha]]} = \kappa$$

which is impossible, since $\kappa$ is still inaccessible in $V[G_\alpha][F_\alpha]$.

This shows 1 But it also shows 2, since we have found, given any real $\dot{r}$ in $V[G]^{\mathbb{P}}$, a partial ordering in $V$, namely $Coll(\omega, <\alpha) * \mathbb{P}_\alpha$, which is countable in $V[G]$, such that

$$V[G]^{\mathbb{P}} \models \dot{r} \in V^{Coll(\omega,<\alpha)*\mathbb{P}_\alpha}.$$

$\square$

**Lemma 3.4.** *If $L(\mathbb{R})$, $L(\mathbb{R}^*)$ are Solovay models over $V$ with $\mathbb{R} \subseteq \mathbb{R}^*$, and $\omega_1^{L(\mathbb{R})} = \omega_1^{L(\mathbb{R}^*)}$, then there is an elementary embedding $j : L(\mathbb{R}) \longrightarrow L(\mathbb{R}^*)$ that is the identity on the reals and ordinals.*

*Proof.* Since $\omega_1^{L(\mathbb{R})} = \omega_1^{L(\mathbb{R}^*)}$, $Coll(\omega, <\omega_1^{L(\mathbb{R})}) = Coll(\omega, <\omega_1^{L(\mathbb{R}^*)})$. Let $G, G^* \subseteq Coll(\omega, <\omega_1^{L(\mathbb{R})})$ be generic filters over $V$ such that $L(\mathbb{R}) = L(\mathbb{R})^{V[G]}$ and $L(\mathbb{R}^*) = L(\mathbb{R})^{V[G^*]}$. In order to prove that the identity map on reals and ordinals yields an elementary embedding of $L(\mathbb{R})$ into $L(\mathbb{R}^*)$, we only need to show that for every formula $\varphi(y, z)$, every ordinal $\alpha$, and every real $a \in \mathbb{R}$,

$$V[G] \vDash \varphi(\alpha, a)^{L(\mathbb{R})} \text{ iff } V[G^*] \vDash \varphi(\alpha, a)^{L(\mathbb{R})}.$$

By the Factor Lemma for the Levy-collapse (see Proposition 3.1), we may assume that $a$ belongs to the ground model. But by homogeneity of the Levy-collapse, we have

$$V[G] \models \varphi(\alpha, a)^{L(\mathbb{R})} \text{ iff } \Vdash_{Coll\left(\omega, <\omega_1^{L(\mathbb{R})}\right)} \varphi(\check{\alpha}, \check{a})^{L(\mathbb{R})} \text{ iff } V[G^*] \models \varphi(\alpha, a)^{L(\mathbb{R}^*)}.$$

$\square$

We are ready now to prove Theorem 3.1.

*Proof of Theorem 3.1.* Fix $A \in V_\kappa$. To simplify the argument, let us suppose $G$ is $Coll(\omega, <\kappa)$-generic over $L[A]$, and work in $L[A][G]$. Note that in $L[A][G]$, $A$ has countable transitive closure.

Let $\sigma$ be a $\underset{\sim}{\Sigma}_n^1$ subset of $\boldsymbol{B_1}$, and suppose that $\tau = \boldsymbol{b}(\sigma)$ is defined by a $\Sigma_n^1$ formula $\varphi(x, a_1, ..., a_m)$ where $a_1, ..., a_m$ are reals and $\boldsymbol{b}$ is the fixed Borel isomorphism between $\boldsymbol{B_1}$ and $\mathcal{N}$. For simplicity of notation, suppose that $m = 1$ and write $a$

for $a_1$. Let $\overline{a}$ be a real that codes both $A$ and $a$. So, by the Factor Lemma, let $H$ be $Coll(\omega, <\kappa)$-generic over $L[\overline{a}]$ such that $L[\overline{a}][H] = L[A][G]$. $\kappa$ is Mahlo in $L[\overline{a}]$. So, by Lemma 3.2, let $S \subseteq \omega_1$ be the stationary set of inaccessible cardinals in $L[\overline{a}]$ such that for every $\alpha \in S$,

$$L[\overline{a}]^{Coll(\omega,<\alpha)} \preccurlyeq_\omega L[\overline{a}]^{Coll(\omega,<\kappa)}.$$

Note that $S$ is definable in the parameters $\overline{a}$ and $\kappa$.

For each $\alpha \in S$ define a set of reals $A_\alpha$ as follows:

$$x \in A_\alpha \text{ iff there exists } g \text{ such that } g \text{ is } Coll(\omega, <\alpha)\text{-generic over } L[\overline{a}] \text{ and } x \in L[\overline{a}][g] \text{ and } L[\overline{a}][g] \models \varphi(x, a).$$

Thus, $A_\alpha$ is definable with $\alpha$ and $\overline{a}$ as parameters.

If $g$ is $Coll(\omega, <\alpha)$-generic ($\alpha \in S$) over $L[\overline{a}]$, then $\omega_1^{L[\overline{a}][g]} = \alpha$. So, we have

$$x \in A_\alpha \text{ iff } \exists g (L_\alpha[\overline{a}][g] \models (g \texttt{ is } Coll(\omega, <\alpha)\texttt{-generic over } L[\overline{a}]) \text{ and } x \in L_\alpha[\overline{a}][g] \text{ and } L_\alpha[\overline{a}][g] \models \varphi(x, a)).$$

Notice that for a real $g$ and a countable ordinal $\alpha$, the map $\langle g, \overline{a}, \alpha\rangle \mapsto L_\alpha[\overline{a}][g]$ is arithmetical in the codes. Hence, $A_\alpha$ is $\utilde{\Sigma}^1_1$ in the codes for $\alpha$ and $\overline{a}$.

We claim that

$$\tau = \{x \in \mathbb{R} \mid \varphi(x, a)\} = \bigcup_{\alpha \in S} A_\alpha.$$

For suppose that $x \in \tau$. Let $\alpha \in S$ be such that $x, a \in L[\overline{a}][H_\alpha]$, where $H_\alpha = H \cap Coll(\omega, <\alpha)$. Then, since $\alpha \in S$, $L[\overline{a}][H_\alpha] \models x \in \tau$. Hence, $x \in A_\alpha$.

Conversely, suppose that $x \in A_\alpha$, for some $\alpha \in S$. Fix $g$ a $Coll(\omega, <\alpha)$-generic over $L[\overline{a}]$ with $x \in L[\overline{a}][g]$ and $L[\overline{a}][g] \models x \in \tau$. Then, since $\alpha \in S$, $L[\overline{a}][H] \models x \in \tau$.

We will show that $\sigma = \bigcup_{\alpha \in S} \sigma_\alpha$ is a good decomposition, where $\sigma_\alpha = \boldsymbol{b}^{-1}(A_\alpha)$. Let $\mathbb{P}$ be one of the relevant partial orderings. We claim that if $F$ is $\mathbb{P}$-generic over $L[\overline{a}][H]$, then in $L[\overline{a}][H][F]$ also

$$\tau = \bigcup_{\alpha \in S} A_\alpha.$$

But this follows from Lemmas 3.3 and 3.4, for since $\mathbb{P}$ preserves $\omega_1$, we have

$$L(\mathbb{R})^{L[\overline{a}][H]} \preccurlyeq L(\mathbb{R})^{L[\overline{a}][H][F]},$$

and $S$ is definable in the parameters $\overline{a}$ and $\kappa$.

Every $(\sigma_\alpha)_\Delta$ is an analytic subset of $\boldsymbol{B_1}$, hence is weakly Ramsey. Now, proceed as in the proof of Theorem 2.2 to show that $\sigma$ is weakly Ramsey in $L[\overline{a}][H]$, hence in $L[A][G]$. □

## 4. Projective determinacy

We will show that the Axiom of Projective Determinacy implies that every projective set is weakly Ramsey.

We recall the notion of *integer game*: Fix $A \subseteq \mathbb{N}^\omega$. $G_\mathbb{N}(A)$ denotes the following game: There are two players, $I$ and $II$. $I$ initially chooses an $\alpha(1) \in \mathbb{N}$; then $II$ chooses some $\alpha(2) \in \mathbb{N}$; then $I$ again chooses an $\alpha(3) \in \mathbb{N}$, to which $II$ replies with $\alpha(4) \in \mathbb{N}$, and so on. $II$ wins the game if $\alpha = (\alpha(n))_n$ is in $A$. Otherwise, $I$ wins. $A$ is determined if $G_\mathbb{N}(A)$ is, i.e., either $I$ or $II$ has a winning strategy.

**Definition 6.** The *Axiom of Determinacy (AD)* is the assertion that every set $A \subseteq \mathbb{N}^\omega$ is determined. The Axiom of *Projective Determinacy (PD)* asserts that every projective subset of $\mathbb{N}^\omega$ is determined.

Let us explain how to convert the property of being weakly Ramsey in $\boldsymbol{B_1}$ into a property on $\mathcal{N}$. Let $\mathcal{V}$ be the set of normalized block vectors. There is a natural inclusion map $\mathcal{V} \hookrightarrow c_{00}$ ($c_{00}$ is the subspace of $c_0$ consisting of the sequences that are eventually 0). Let $Q$ be the subset of $\mathcal{V}$ consisting of the block vectors with rational coordinates. $\mathcal{V}$ is the closed closure of $Q$ in $c_{00}$ (the reason being that for every finite set of integers $a$, the projection $p_a$ is continuous).

Now, fix any enumeration $Q = \{q_n \mid n \geq 1\}$. Let $B_1 \subseteq \mathcal{N}$ be the set of infinite sequences of integers $\alpha = (\alpha_n)_n$ such that $q_{\alpha_n} < q_{\alpha_{n+1}}$ (i.e., $\max\operatorname{supp} q_{\alpha_n} < \min\operatorname{supp} q_{\alpha_{n+1}}$). $B_1$ is a closed subset of $\mathcal{N}$. Define $f_{\mathcal{X}} : B_1 \to \boldsymbol{B_1}$ by $f_{\mathcal{X}}(\alpha) = (q_{\alpha_n})_n$. $f_{\mathcal{X}}$ is injective and continuous.

Next, we define the relation $\preceq_{\mathcal{N}}$ and the game in $B_1$.

**Definition 7.** For $X, Y \in B_1$, $\sigma \subseteq B_1$ and $\Delta > 0$, define the relations $X \preceq_{\mathcal{N}} Y$ iff $f_{\mathcal{X}}(X) \preceq f_{\mathcal{X}}(Y)$, and $d(X,Y) \leq \Delta$ iff $d(f_{\mathcal{X}}(X), f_{\mathcal{X}}(Y)) \leq \Delta$.

Let $\mathbb{N}\Game_\sigma[X]$ be the integer game associated to $\Game_{f_{\mathcal{X}}(\sigma)}[f_{\mathcal{X}}(X)]$, when both players always choose elements of $Q$.

Then, we have the natural notions of being large in $X$ and being weakly Ramsey. Let us remark that these notions are absolute between transitive models of set theory.

*Remark* 4.1. For $\sigma \subseteq \mathcal{N}$, and $\Delta > 0$, the sentence $\sigma$ `is` $\Delta$`-weakly Ramsey` is projective in $\sigma$. In fact, it is a $\Sigma^1_3(\sigma, d, \Delta)$ sentence:

$$\sigma \texttt{ is } \Delta\texttt{-weakly Ramsey} \text{ iff } \exists\alpha\forall\beta(\beta \in \sigma \Rightarrow \alpha \not\preceq_{\mathcal{N}} \beta) \texttt{ or}$$
$$\exists\alpha(\alpha \texttt{ codes a strategy } S \texttt{ for } II$$
$$\texttt{and}\, \forall\beta\exists\gamma \in \sigma(d(S * \beta, \gamma) \leq \Delta)).$$

Hence, if $\sigma$ is a $\underset{\sim}{\Sigma}^1_n$ ($\underset{\sim}{\Pi}^1_n$) subset of $B_1$, then the sentence $\sigma$ `is weakly Ramsey` is $\Sigma^1_{n+2}$ ($\Pi^1_{n+3}$) in the parameters of $\sigma$.

**Proposition 4.1.** *Suppose that $X = (x_n)_n \in f_{\mathcal{X}}(B_1) \subseteq \boldsymbol{B_1}$ (i.e., for every $n$, $x_n$ has rational coefficients over $(\mathbf{e}_n)_n$).*

1. *If I has a winning strategy for the game $\Game_{\sigma_\Delta}[X]$ (resp. $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\sigma_\Delta)}[f_{\mathcal{X}}^{-1}(X)]$), for some $\Delta > 0$, then I has a winning strategy for the game $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\sigma)}[f_{\mathcal{X}}^{-1}(X)]$ (resp. $\Game_\sigma[X]$).*
2. *If II has a winning strategy for the game $\Game_\sigma[X]$ (resp. $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\sigma)}[f_{\mathcal{X}}^{-1}(X)]$), then for every $\Delta > 0$, II has a winning strategy for $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\sigma_\Delta)}[f_{\mathcal{X}}^{-1}(X)]$ (resp. $\Game_{\sigma_\Delta}[X]$).*

*Proof.* Use that every block vector can be approximated by rational block vectors to pass from one game to the other. □

**Proposition 4.2.** *Fix $\sigma \subseteq \boldsymbol{B_1}$, $\Delta_1, \Delta_2, \Delta_3 > 0$.*

1. *If $\sigma_{\Delta_1}$ is $\Delta_2$-weakly Ramsey, then $f_{\mathcal{X}}^{-1}(\sigma) \subseteq B_1$ is $(\Delta_1 + \Delta_2 + \Delta_3)$-weakly Ramsey.*
2. *If $f_{\mathcal{X}}^{-1}(\sigma_{\Delta_1}) \subseteq B_1$ is $\Delta_2$-weakly Ramsey, then $\sigma \subseteq \boldsymbol{B_1}$ is $(\Delta_1 + \Delta_2 + \Delta_3)$-weakly Ramsey.*

*Proof.* Fix $\Delta_1, \Delta_2, \Delta_3 > 0$. For 1: Suppose that $f_{\mathcal{X}}^{-1}(\sigma)$ is large in $B_1$. Then $\sigma_{\Delta_1}$ is large in $\boldsymbol{B_1}$: For suppose that $(x_n)_n \in \boldsymbol{B_1}$, and choose a rational block sequence $(y_n)_n$ close enough to $(x_n)_n$ (for example, let $\Gamma$ be as in Fact 1.1 for $\Delta_1$, and choose $(y_n)_n$ such that $d((x_n)_n, (y_n)_n) \leq \Gamma$), and let $\alpha \in B_1$ be such that $f_{\mathcal{X}}(\alpha) = (y_n)_n$. Let $(\beta_n)_n \preceq_{\mathcal{N}} (\alpha_n)_n$, $(\beta_n)_n \in f_{\mathcal{X}}^{-1}(\sigma)$. Define $Z \preceq X$ as $f((\beta)_n) \preceq Y$. Then, $d(Z, f(\beta_n)_n) \leq \Delta_1$, and hence, $Z \in [X] \cap \sigma_{\Delta_1}$.

As $\sigma_{\Delta_1}$ is large and $\Delta_2$-weakly Ramsey, there is some $X$ such that $II$ has a winning strategy for the game $\eth_{\sigma_{\Delta_1+\Delta_2}}[X]$. Let $Y \in f_{\mathcal{X}}(B_1)$ be close enough to $X$. This implies that $II$ also has a winning strategy for the game $\eth_{\sigma_{\Delta_1+\Delta_2+\Delta_3/2}}[Y]$. By Proposition 4.1, we are done.

For 2: Suppose that $\sigma$ is large. Then it can be shown that $\tau = f_{\mathcal{X}}^{-1}(\sigma_{\Delta_1})$ is large in $B_1$. Let $\alpha \in B_1$ be such that $II$ has a winning strategy for the game $\mathbb{N}\eth_{\tau_{\Delta_2}}[\alpha]$. But $\tau_{\Delta_2} \subseteq f_{\mathcal{X}}^{-1}(\sigma_{\Delta_1+\Delta_2})$, and therefore $II$ also has a winning strategy for the game $\mathbb{N}\eth_{f_{\mathcal{X}}^{-1}(\sigma_{\Delta_1+\Delta_2})}[\alpha]$. By Proposition 4.1, $II$ has a winning strategy for the game $\eth_{\sigma_{\Delta_1+\Delta_2+\Delta_3}}[f(\alpha)]$. $\square$

**Definition 8.** Let $M \subseteq N$ be transitive models of (a fragment) of ZF. We say that $M$ is $\boldsymbol{\Sigma^1_n}$*-correct in* $N$ if for any $\Sigma^1_n$ formula $\varphi(x_1, ..., x_k)$ and reals $a_1, ..., a_k \in M$,

$$M \models \varphi(a_1, ..., a_k) \quad \text{iff} \quad N \models \varphi(a_1, ..., a_k).$$

We say $M$ is $\Sigma^1_n$*-correct* if $M$ is $\Sigma^1_n$-correct in $V$.

Given a partial ordering $\mathbb{P}$, we say that $M$ is $\Sigma^1_n$*-absolute* for $\mathbb{P}$ if $M$ is $\Sigma^1_n$-correct in $M^{\mathbb{P}}$. We say that $M$ is $\Sigma^1_n$*-absolute* if $M$ is $\Sigma^1_n$-absolute for every (set) partial ordering $\mathbb{P}$.

Woodin [23] shows that under certain assumptions, which hold under PD, one can build for every $n \geq 1$ a countable transitive model of ZF such that all its (set) forcing extensions are $\Sigma^1_{n+1}$-correct.

We present Woodin's construction under the following assumptions:

1. Every $\Pi^1_n$ set (i.e., a $\underset{\sim}{\Pi}^1_n$ set which is definable without parameters) can be uniformized by a $\Gamma^1_m$-function, where $\Gamma^1_m$ is some projective pointclass.
2. $V$ is $\Sigma^1_k$-absolute for *Cohen* forcing, where $k = \max\{n, m\}$.

Consistency-wise 1 is a rather weak assumption, since it holds in $L$, and, in fact, in any model with a projective well-ordering of the reals. However, for 1 and 2 to hold simultaneously large cardinals are needed, and they both hold under PD (see [17]).

Choose a $\Pi^1_n$ set $U \subseteq \mathcal{N} \times \mathcal{N}$ that is universal for $\Pi^1_n$ subsets of $\mathcal{N}$. Let $\varphi^*(x, y)$ be the formula that defines $U$. Let $f$ be a $\Gamma^1_m$-function that uniformizes $U$.

Given any $\Pi^1_n$ formula $\varphi(x, x_1, ..., x_i)$, $f$ effectively induces a Skolem function $f_\varphi$ for $\varphi$, i.e., for all reals $a_1, ..., a_i$, $f_\varphi(a_1, ..., a_i) = b$ iff $\varphi(f(b), a_1, ..., a_i)$. So, for all $a_1, ..., a_i$,

$$\exists x \varphi(x, a_1, ..., a_i) \rightarrow \varphi(f_\varphi(a_1, ..., a_i), a_1, ..., a_i).$$

Given $A \in HC$, we can code $A$ by a real $x$ (for example, $x$ can code the countable structure $(TC(A), \in)$). So, Suppose $\mathbb{Q}$ is a (atomless and separative) partial ordering, $\mathbb{Q} \in HC$. Let $\tau_1, ..., \tau_i$ be $\mathbb{Q}$-terms for reals. Then, we may assume both $\mathbb{Q}$ and $\tau_1, ..., \tau_i$ are coded by reals so that, for every $q \in \mathbb{Q}$, and every $\Sigma^1_n$ $(\Pi^1_n)$ formula $\varphi$, $n \geq 2$, the formula $q \Vdash_{\mathbb{Q}} \varphi(\tau_1, ..., \tau_i)$ is also $\Sigma^1_n$ $(\Pi^1_n)$.

Now suppose $\varphi(x, x_1, ..., x_i)$ is a $\Pi^1_n$ formula. Let $g$ be $\mathbb{Q}$-generic and suppose $V[g] \models \exists x \varphi(x, \tau_1[g], ..., \tau_i[g])$. Since $\mathbb{Q}$ is countable, $V[g] = V[c]$ for some real $c$ *Cohen*-generic. By $\Sigma^1_k$-absoluteness for *Cohen*,

$$V[g] \models \varphi(f_\varphi(\tau_1[g], ..., \tau_i[g]), \tau_1[g], ..., \tau_i[g]).$$

Let $\tau$ be the canonical term for $f_\varphi(\tau_1[g], ..., \tau_i[g])$. Define $F_\varphi(\mathbb{Q}, \tau_1, ..., \tau_i) = \tau$, i.e.,

$$F_\varphi(\mathbb{Q}, \tau_1, ..., \tau_i) = \{(q, s) : q \in \mathbb{Q}, s \in \omega^{<\omega} \text{ and } q \Vdash f_\varphi(\tau_1[\dot{g}], ..., \tau_i[\dot{g}]) \text{ extends } s\}.$$

If $\tau$ does not exist we view $F_\varphi$ as undefined. So, $F_\varphi$ is a partial function from $HC^{i+1}$ into $HC$. We can reinterpret $F_\varphi$ as a function from $HC$ into $HC$. Let $F = F_{\varphi^*}$, where $\varphi^*$ is the $\Pi^1_n$ formula that defines the universal set $U$ that $f$ uniformizes.

**Proposition 4.3.** *If $M \subseteq HC$ is any transitive model of a sufficiently large fragment of set theory, $M$ closed under $F$, and $\mathbb{Q}$ is a class-partial ordering in $M$, $\mathbb{Q} \in HC$, then for every $G \in V$, if $G$ is $\mathbb{Q}$-generic over $M$, then $M[G]$ is $\Sigma^1_{n+1}$ correct.*

*Proof.* Suppose $\varphi(x, x_1, ..., x_i)$ is $\Pi^1_n$, $\tau_1, ..., \tau_i \in M^{\mathbb{Q}}$ are terms for reals and $V^{\mathbb{Q}} \models \exists x \varphi(x, \tau_1, ..., \tau_i)$. Then, $V^{\mathbb{Q}} \models \varphi(F(\mathbb{Q}, \tau_1, ..., \tau_i), \tau_1, ..., \tau_i)$. But since $M$ is closed under $F$, $M^{\mathbb{Q}} \models \varphi(F(\mathbb{Q}, \tau_1, ..., \tau_i), \tau_1, ..., \tau_i)$.

This shows $M^{\mathbb{Q}}$ is correct in $V^{\mathbb{Q}}$, but since $V$ is absolute for Cohen, $M^{\mathbb{Q}}$ is correct in $V$. □

We can now prove the following.

**Theorem 4.1.** *If PD holds, then all projective sets are weakly Ramsey.*

*Proof.* Let $\sigma$ be projective (say $\underset{\sim}{\Sigma}^1_n$), and fix $\Delta > 0$. $\tau = f_{\mathcal{X}}^{-1}(\sigma_{\Delta/3})$ is a projective set of $B_1$ (in fact, also $\underset{\sim}{\Sigma}^1_n$). By Remark 4.1, the sentence $\tau$ `is` $\Delta/3$`-weakly Ramsey` is $\Sigma^1_{n+2}$. Let $M$ be the transitive collapse of some countable elementary substructure $N$ of $H(\lambda)$, some $\lambda$ inaccessible, with $F, \boldsymbol{B_1}, \sigma, \Delta \in N$. Notice that since reals collapse always to themselves, $M$ is closed under $F$ and contains the parameters of $\boldsymbol{B_1}$ and $\sigma$.

We may assume that $F$ is actually a function from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$. In $M$, let $R_F$ be the following relation:

$$\langle x, n \rangle \in R_F \text{ iff } x \in \operatorname{dom} F \text{ and } n \in F(x).$$

Let $\alpha = \sup(ORD \cap M)$, and let $a$ be a real that codes $\Delta$ and the parameters of $\boldsymbol{B_1}$ and $\sigma$. Then, $L_\alpha[a, R_F]$ is the least inner model of $M$ closed under $F$ which contains $a$. Hence, by Proposition 4.3, every generic extension of $L_\alpha[a, R_F]$, by a partial ordering which is countable in $V$, is $\underset{\sim}{\Sigma}^1_{n+2}$-correct in $V$.

Since large cardinals exist in $V$ (a consequence of assumptions 1 and 2, see [23]), they also exist in $L_\alpha[a, R_F]$. Let $\kappa$ be a Mahlo cardinal in $L_\alpha[a, R_F]$. Since $L_\alpha[a, R_F]$ is countable, there exists $G$ which is $Coll(\omega, <\kappa)$-generic over $L_\alpha[a, R_F]$. From Theorem 3.1, in $L_\alpha[a, R_F][G]$ all projective sets are weakly Ramsey. In particular, in $L_\alpha[a, R_F]$, $(\sigma_{\Delta/3})_{\Delta/9}$ is $\Delta/9$-weakly Ramsey. By Proposition 4.2,

$$L_\alpha[a, R_F][G] \models f_{\mathcal{X}}^{-1}(\sigma_{\Delta/3}) \texttt{ is } \Delta/3\texttt{-weakly Ramsey}.$$

By correctness of $L_\alpha[a, R_F][G]$,

$$V \models f_{\mathcal{X}}^{-1}(\sigma_{\Delta/3}) \texttt{ is } \Delta/3\texttt{-weakly Ramsey}.$$

And again, by Proposition 4.2, in $V$ $\sigma$ is $\Delta$-weakly Ramsey. □

It can be observed that the countable axiom of choice is enough to define the partial orderings $\mathbb{P}(\Delta, X)$. Further, PD (in fact, determinacy for analytic sets) plus the Axiom of dependent choices is enough to show $\Sigma^1_3$-absoluteness for $\mathbb{P}(\Delta, X)$. Therefore, by Theorem 2.2, if all $\underset{\sim}{\Sigma}^1_1$ sets are determined and the Axiom of Dependent Choices holds, then every $\underset{\sim}{\Sigma}^1_2$ set of block sequences has a good decomposition, and hence it is weakly Ramsey.

## 5. Hereditarily indecomposable spaces and determinacy

Assume that $\mathcal{X}$ is a hereditarily indecomposable space. We will give a proof that the Axiom of Determinacy plus the Principle of Dependent Choices implies that every subset of $\boldsymbol{B_1} = \boldsymbol{B_1}(\mathcal{X})$ is weakly Ramsey.

**Definition 9.** The *Axiom of Dependent Choices (DC)* (P. Bernays, see [10]):

For every set $X$ and every relation $R \subseteq X \times X$, if for every $x \in X$ there is some $y \in X$ such that $(x, y) \in R$, then there is a map $f : \mathbb{N} \to X$ such that for every $n \in \mathbb{N}$, $(f(n), f(n+1)) \in R$.

Recall the notion of hereditarily indecomposable space:

**Definition 10.** $\mathcal{X}$ is *hereditarily indecomposable (HI)* iff for every $\Delta > 0$ and every pair $X, Y$ of normalized block sequences of $\mathcal{X}$, there are $\widetilde{X} \in [X]$ and $\widetilde{Y} \in [Y]$ such that $d(\widetilde{X}, \widetilde{Y}) \leq \Delta$.

**Proposition 5.1.** *Suppose that for some $X$, $II$ has a winning strategy for the game $\eth_\sigma[X]$. Then, for every $\Delta$, $I$ has no winning strategy for the game $\eth_{\sigma_\Delta}[\mathcal{X}]$.*

*Proof.* Fix $S$ a winning strategy for $II$ in the game $\eth_\sigma[X]$. Let $\Gamma$ be for $\Delta$ as in Fact 1.1. Towards a contradiction, let $S'$ be any strategy for $I$ in the game $\eth_{\sigma_\Delta}[\mathcal{X}]$. Let $Y_1 = S' * (0, 0, ...)$, and let $Z_1 \preceq Y_1$ and $X_1 = (x^{(1)}_k)_k \preceq X$ be such that $d(Z_1, X_1) \leq \Gamma$ (this is possible since $\mathcal{X}$ is HI). Let $v_1$ be the first non-zero move of $II$ following $S$, if $I$ plays $x^{(1)}_1$,...,$x^{(1)}_k$ in the game $\eth_\sigma[X]$, and let $n_1$ be the corresponding $k$. Then, in the game $\eth_{\sigma_\Delta}[\mathcal{X}]$, $II$ plays 0 until she can play $w_1 \in Z_1$, defined as $v_1 \in X_1$. Restart the game, and let $Y_2$ be such that $(Y_1 \restriction n_1)^\frown Y_2 = S' * (\overset{(n_1)}{0, ..., 0}, w_1, 0, 0, ...)$. Let $Z_2 \preceq Y_2$ and $X_2 = (x^{(2)}_k)_k \preceq X \setminus n_1$ be such that $d(Z_2, X_2) \leq \Gamma \setminus n_1$. Let $v_2 > v_1$ be the second non-zero move of $II$ following $S$, if $I$ plays $x^{(1)}_1, ..., x^{(1)}_{n_1}, x^{(2)}_1, ..., x^{(2)}_k$ in the game $\eth_\sigma[X]$, and $n_2$ its corresponding $k$. Again, in the game $\eth_\sigma[\mathcal{X}]$ $II$ plays 0 until she can play $w_2 \in Z_2$ defined as $v_2 \in X_2$. And so on.

At the end of the game, $d(Z_1 \restriction n_1^\frown Z_2 \restriction n_2^\frown \cdots,\ X_1 \restriction n_1^\frown X_2 \restriction n_2^\frown \cdots) \leq \Gamma$ and $(w_k) \preceq [Z_1 \restriction n_1^\frown Z_2 \restriction n_2^\frown \cdots]$ is defined as $(v_k) \preceq [X_1 \restriction n_1^\frown X_2 \restriction n_2^\frown \cdots]$. By Fact 1.1, we have that $d((w_k)_k, (v_k)_k) \leq \Delta$, and since $(v_k)_k \in \sigma$, $(w_k)_k \in \sigma_\Delta$. So, $S'$ is not a winning strategy for $I$ in the game $\eth_{\sigma_\Delta}[\mathcal{X}]$. $\square$

As a consequence, we have

**Proposition 5.2** (Reflection Principle)**.** *Assume AD and suppose that $\sigma \subseteq \boldsymbol{B_1}$. If $II$ has a winning strategy for the game $\eth_\sigma[X]$, for some $X$, then she also has a winning strategy for the game $\eth_{\sigma_\Delta}[\mathcal{X}]$, for every $\Delta > 0$.*

*Proof.* Suppose that $II$ has a winning strategy for the game $\eth_\sigma[X]$. By Proposition 5.1, $I$ does not have a winning strategy for the game $\eth_{\sigma_{\Delta/3}}[\mathcal{X}]$. By Proposition 4.1,

$I$ does not have a winning strategy for the integer game $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\sigma_{2\Delta/3})}[f_{\mathcal{X}}^{-1}(\mathcal{X})]$. AD implies that $II$ has a winning strategy for the game $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\sigma_{2\Delta/3})}[f_{\mathcal{X}}^{-1}(\mathcal{X})]$, and therefore, again by Proposition 4.1, also for the game $\Game_{\sigma_\Delta}[\mathcal{X}]$. □

**Theorem 5.1.** *AD plus DC implies that every subset of $\boldsymbol{B_1}$ is weakly Ramsey.*

*Proof.* Suppose that $\sigma \subseteq \boldsymbol{B_1}$ is large and fix $\Delta > 0$. Suppose that $II$ does not have a winning strategy for the game $\Game_{\sigma_\Delta}[\mathcal{X}]$. Let $w = f_{\mathcal{X}}^{-1}(\sigma_{\Delta/2}) \subseteq B_1$. Then $II$ has no winning strategy for the integer game $\mathbb{N}\Game_w[f_{\mathcal{X}}^{-1}(\mathcal{X})]$. By AD, $I$ has a winning strategy for the game $\mathbb{N}\Game_w[f_{\mathcal{X}}^{-1}(\mathcal{X})]$. Let us call it $S$. Let $A$ be the set of integer sequences corresponding to non-trivial runs of $II$ against $S$, and let $\tau_1 = f_{\mathcal{X}}(A)$. $A$ is closed, and hence $\tau_1$ is analytic. Note that $\tau_1 \subseteq (\sigma_{\Delta/2})^c$ (it is easy to show that $\sigma \cap (\tau_1)_{\Delta/2} = \emptyset$ iff $\sigma_{\Delta/2} \cap \tau_1 = \emptyset$, and this is true because $S$ is a winning strategy for $I$ in the game $\mathbb{N}\Game_w[f_{\mathcal{X}}(\mathcal{X})]$). Then $\sigma \subseteq ((\tau_1)_{\Delta/2})^c$ (as before, $\sigma \cap (\tau_1)_{\Delta/2} = \emptyset$ iff $\sigma_{\Delta/2} \cap \tau_1 = \emptyset$). $(\tau_1)_{\Delta/2}$ is analytic, and hence $\tau = ((\tau_1)_{\Delta/2})^c$ is co-analytic and large, since it contains $\sigma$. So, using that AD+DC implies $\Sigma^1_3$-absoluteness (see the remarks at the end of section 4), every $\underset{\sim}{\Sigma}^1_2$ set is weakly Ramsey, and hence $II$ has a winning strategy for the game $\Game_{\tau_{\Delta/4}}[X]$, for some $X$. Using the reflection principle, we can find some winning strategy for the game $\Game_{\tau_{\Delta/3}}[\mathcal{X}]$. Let $S'$ be the corresponding winning strategy for the integer game $\mathbb{N}\Game_{f_{\mathcal{X}}^{-1}(\tau_{\Delta/2})}[f_{\mathcal{X}}^{-1}(\mathcal{X})]$. If $I$ plays according to $S$ and $II$ according to $S'$, $II$ produces an integer sequence $\alpha$ such that $f_{\mathcal{X}}(\alpha) \in \tau_{\Delta/2}$, and $I$ forces that $f_{\mathcal{X}}(\alpha) \in \tau_1$. And this is impossible because $\tau_{\Delta/2} \cap \tau_1 = \emptyset$ (this is equivalent to $\tau \cap (\tau_1)_{\Delta/2} = \emptyset$, and this is true because $\tau = ((\tau_1)_{\Delta/2})^c$). □

## 6. Generalizations in $c_0$

It is shown in [8] and [15] that if $\sigma$ is a large analytic subset of $\boldsymbol{B_1}(c_0)$, then for every $\Delta$, $\sigma_\Delta$ contains a cube. For this section, assume that $\mathcal{X} = c_0$.

We recall from [8] and [15] that a set $\sigma \subseteq \boldsymbol{B_1}$ is *almost-Ramsey* iff either there is some $X$ such that $[X] \cap \sigma = \emptyset$ or for every $\Delta$, there is some $X$ such that $[X] \subseteq \sigma_\Delta$. We will use *very good decompositions* (see definition below) to prove that large cardinals and determinacy hypotheses imply that many sets are almost-Ramsey, for which we need a bit more than good decompositions.

**Definition 11.** $\sigma = \bigcup_{i \in I} \sigma_i$ is a *very good decomposition* iff for every $\Delta > 0$ the following holds:

1. For every $(s, A)$, if $\sigma$ is large in $[s; A]$, then there is some $i \in I$ and $(t, B) \leq (s, A)$ such that $(\sigma_i)_\Delta$ is large in $[t; B]$, and
2. for every $i \in I$, $s$ and $X$, $((\sigma_i)_\Delta)^s$ is almost-Ramsey in $X$.

**Theorem 6.1.** *Every set with a very good decomposition is almost-Ramsey.*

As in the proof of Theorem 2.1 we need some lemmas.

First, the analogue of Lemma 2.1:

**Lemma 6.1.** *There is $Y$ such that for every $(s, A) \in \mathbb{P}(Y)$, if $[s; A] \subseteq \sigma_{\Delta/2}$, then also $[s; Y] \subseteq \sigma_\Delta$.*

*Proof.* Consider the following subsets of $\mathbb{P}(c_0)$:

$$D_m := \{(s, A) \in \mathbb{P}(\mathcal{X}) \mid |s| \geq m \text{ and for every } t \in [s] \text{ if for some } B \preceq A, \\ [s; B] \subseteq \sigma_{\Delta/2}, \text{ then also } [s; A] \subseteq \sigma_{\Delta}\}.$$

Note that if $(s, A) \in D_m$, then for every $t \in [s]$, $(t, A) \in D_m$. Repeating the proof of Lemma 2.1, we can prove that each $D_m$ is dense. Choose $(s_n, A_n) \in D_m$, such that $|s_n| = n$, and so that $(s_n, A_n) \leq (s_{n-1}, A_{n-1})$. Let $Y = \bigcup_n s_n$, that satisfies what we want: For suppose that $(s, A) \in \mathbb{P}(Y)$ is such that $[s; A] \subseteq \sigma_{\Delta/2}$. Choose $m$ minimum such that $s \in [s_m]$. Note that this implies that $A \preceq A_m$. Then $[s; A] \subseteq \sigma_{\Delta}$, but $[s; Y] \subseteq [s; A]$, and we are done. □

The analogue of Lemma 2.2:

**Lemma 6.2.** *Let $\sigma = \bigcup_{i \in I} \sigma_i$ be a very good decomposition. If $\sigma$ is large in $[s; X]$, then there is some $Y \preceq X$ such that $[Y] \subseteq \tilde{\sigma}(\Delta/4, s, Y)$, where*

$$\tilde{\sigma}(\Delta, s, Z) = \{(y_m)_m \preceq Z \mid \text{ there exist } i \in I,\ k \geq 1 \text{ and } A \preceq Z \text{ such that} \\ (\sigma_i)_{\Delta} \text{ is large in } [s^\frown(y_1, \ldots, y_k); A]\}.$$

*Proof.* Use that $\tilde{\sigma}(\Delta, s, X)$ is an open set of $[X]$, and large because $\bigcup_{i \in I} \sigma_i$ is a very good decomposition. But every open set is almost-Ramsey, and we are done. □

*Proof of Theorem 6.1.* Suppose that $\sigma$ is large. Let $Y$ satisfy Lemma 6.1 and let $Z \preceq Y$ satisfy Lemma 6.2. We check that $[Z] \subseteq \sigma_{\Delta}$: For suppose that $W \preceq Z$. Fix $i \in I$ and $k$ such that $(\sigma_i)_{\Delta/4}$ is large in $[W{\upharpoonright}k; A]$, for some $A \preceq Z \preceq Y$, i.e., $(\sigma_i)_{\Delta/4}^{W{\upharpoonright}k}$ is large in $[A]$. We have a very good decomposition and, therefore, $(\sigma_i)_{\Delta/4}^{W{\upharpoonright}k}$ is almost-Ramsey. Choose any $B \preceq A$ such that $[B] \subseteq ((\sigma_i)_{\Delta/4}^{W{\upharpoonright}k})_{\Delta/4}$. Then, $[B] \subseteq ((\sigma_i)_{\Delta/2})^{W{\upharpoonright}k}$, i.e., $[W{\upharpoonright}k; B] \subseteq (\sigma_i)_{\Delta/2} \subseteq \sigma_{\Delta/2}$. $(W{\upharpoonright}k; B) \in \mathbb{P}(Y)$, and so $[W{\upharpoonright}k; Y] \subseteq \sigma_{\Delta}$, and this implies that $W \in \sigma_{\Delta}$. □

**Theorem 6.2.** *Suppose that $V$ is $\Sigma^1_3$-absolute for $\mathbb{P}$. Then, every $\underset{\sim}{\Sigma}^1_2$ subset of $\boldsymbol{B}_1(c_0)$ has a very good decomposition.*

*Proof.* The decomposition of every $\underset{\sim}{\Sigma}^1_2$ of $\boldsymbol{B}_1(c_0)$ in the proof of Theorem 2.2 is, in the case of $\boldsymbol{B}_1(c_0)$, a very good decomposition (every piece is analytic, and hence, almost-Ramsey). □

**Corollary 6.2.1.** *Suppose that $V$ is $\Sigma^1_3$-absolute for $\mathbb{P}$. Then, every $\underset{\sim}{\Sigma}^1_2$ subset of $\boldsymbol{B}_1(c_0)$ is almost-Ramsey.* □

**Theorem 6.3.** *Suppose that $\kappa$ is a Mahlo cardinal, and let $A \in V_\kappa$. Then in $L[A]^{Coll(\omega,<\kappa)}$, every projective subset of $\boldsymbol{B}_1(c_0)$ is almost-Ramsey.*

*Proof.* The good decomposition given in the proof of Theorem 3.1 is indeed very good. □

**Theorem 6.4.** *If PD holds, then all projective subsets of $\boldsymbol{B}_1(c_0)$ are almost-Ramsey.* □

## REFERENCES


[1] J. Bagaria, Bounded forcing axioms as principles of generic absoluteness. *Archiv for Mathematical Logic* **39** (2000), pp. 393–401. MR **2001i:**03103

[2] J. Bagaria and J. López-Abad, Weakly Ramsey sets in Banach spaces. *Advances in Mathematics*, **160** (2001), no. 2, pp. 133–174. CMP 2001:14

[3] J. Bagaria and W. H. Woodin, $\underset{\sim}{\Delta}^1_n$ sets of reals, *The Journal of Symbolic Logic*, **62** (1997), no. 4, pp. 1379–1428. MR **98k:**03108

[4] J. Baumgartner, Iterated forcing, in "Surveys in Set Theory", Cambridge University Press, Cambridge 1983, pp. 1–59. MR **87c:**03099

[5] W. T. Gowers, Lipschitz functions on classical spaces, *European Journal of Combinatorics* **13** (1992), pp. 141–151. MR **93g:**05142

[6] W. T. Gowers, Analytic sets and games in Banach spaces. Preprint IHES M/94/42.

[7] W. T. Gowers, A new dichotomy for Banach spaces, *Geometric and Functional Analysis* **6** (1996), pp. 1083-1093. MR **97m:**46017

[8] W. T. Gowers, An infinite Ramsey theorem and some Banach-space dichotomies. Preprint.

[9] T. Jech, Set Theory, Pure and Applied Mathematics, Academic Press, New York, 1978. MR **80a:**03062

[10] A. Kanamori, The Higher Infinite, Springer, Berlin, 1994. MR **96k:**03125

[11] A. S. Kechris, Classical Descriptive Set Theory, Springer-Verlag, 1995. MR **96e:**03057

[12] K. Kunen, Set theory. An introduction to independence proofs. Reprint of the 1980 original. Studies in Logic and the Foundations of Mathematics, 102. North-Holland Publishing Co., Amsterdam-New York, 1983. xvi+313 pp. MR **85e:**03003

[13] A. Levy and R. M. Solovay, On the decomposition of sets of reals to Borel sets, *Annals of Mathematical Logic* **5** (1972/73), pp. 1–19. MR **47:**38

[14] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I, Springer-Verlag, 1977. MR **58:**17766

[15] J. López-Abad, *Weakly Ramsey sets in Banach Spaces*, Ph.D. Thesis. Universitat de Barcelona. 2000.

[16] A. R. D. Mathias, Happy families, *Annals of Mathematical Logic* **12** (1977), no. 1, pp. 59–111. MR **58:**10462

[17] Y. N. Moschovakis, Descriptive Set Theory, North-Holland, 1980. MR **82e:**03002

[18] S. Shelah, Proper and improper forcing, Second edition. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998. MR **98m:**03002

[19] J. Silver, Every analytic set is Ramsey, *The Journal of Symbolic Logic* **35** (1970), pp. 60–64. MR **48:**10807

[20] R. Solovay, A model of set theory in which every set of reals is Lebesgue measurable, *Annals of Mathematics* **92** (1970), pp. 1–56. MR **42:**64

[21] B.S. Tsirelson, Not every Banach space contains $\ell_p$ or $c_0$, *Functional Analysis and Applications* **8** (1974), pp. 138–141 (translated from Russian).

[22] B. Velickovic, Forcing axioms and stationary sets, *Advances in Mathematics* **94** (1992), pp. 256–284. MR **93k:**03045

[23] W. H. Woodin, On the consistency strength of projective uniformization, in *Proceedings of Herbrand Symposium. Logic Colloquium '81*, North-Holland, 1982. MR **86f:**03079



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