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A Singular Poincaré Lemma

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1 Introduction

The classical Poincaré lemma asserts that a closed 1-form on a smooth manifold is locally exact. In other words, given m-functions g_i on an m-dimensional manifold for which $(\partial/\partial x_i)(g_j) = (\partial/\partial x_j)(g_i)$, there exists a smooth F in a neighborhood of each point such that $g_i = (\partial/\partial x_i)(F)$.

Now assume that we have a set of r functions g_i and a set of r vector fields X_i with a singularity at a point p and fulfilling a commutation relation of type $X_i(g_j) = X_j(g_i)$. We want to know if a similar expression for g_i exists in a neighborhood of p.

In case g_i are n functions on the symplectic manifold $(\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i)$ and X_i form a basis of a Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{R})$, a Poincaré-like lemma exists. This result was stated by Eliasson in [4]. In [5] Eliasson provided a proof of this statement in the completely elliptic case. As far as the nonelliptic cases are concerned, no complete proof of this result is known to the authors of this paper.

The analytical counterpart of this result dates back to the seventies and was proved by Vey [12]. The transition from the analytical case to the smooth case in cases other than elliptic entails a nontrivial work with flat functions along certain submanifolds and, in our opinion, cannot be neglected.

The aim of this paper is to prove a more general singular Poincaré lemma: the one that would correspond to a set of r functions on a $2n$-dimensional manifold with $r \leq n$ fulfilling similar commutation relations determined by a basis of a Cartan subalgebra of

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$\mathfrak{sp}(2r,\mathbb{R})$. In particular, in this way we obtain a complete proof also when $r=n$ in the non-completely elliptic cases which was missing in the literature. This result has a natural interpretation in terms of the cohomology associated to the infinitesimal deformation of completely integrable foliations (see Section 6).

This result has applications in establishing normal forms for completely integrable systems. The statement for $r=n$ was used by Eliasson in [4, 5] to give a symplectic normal form for nondegenerate singularities of completely integrable systems. The more general result we prove here could be useful to establish normal forms for more general singularities of completely integrable systems.

2 The result

All the objects considered in this paper will be $\mathcal{C}^\infty$. We are interested in germ-like objects attached to a point p of a smooth manifold M^{2n}.

We denote by $(x_1,y_1,\ldots,x_n,y_n)$ a set of coordinates centered at the origin. Consider the standard symplectic form $\omega=\sum_{i=1}^n dx_i\wedge dy_i$ in a neighborhood of the origin. Take $r\leq n$ and consider the embedding $i_r\colon\mathbb{R}^{2r}\to\mathbb{R}^{2n}$ defined by $i_r(x_1,y_1,\ldots,x_r,y_r)=(x_1,y_1,\ldots,x_r,y_r,0,\ldots,0)$. Consider $\omega_r=\sum_{i=1}^r dx_i\wedge dy_i$, then $i_r^*(\omega)=\omega_r$; in other words, this embedding induces an inclusion of Lie groups $Sp(2r,\mathbb{R})\subset Sp(2n,\mathbb{R})$. In this way $\mathfrak{sp}(2r,\mathbb{R})$ is realized as a subalgebra of $\mathfrak{sp}(2n,\mathbb{R})$. This particular choice of subalgebra is implicit throughout the paper.

In this paper we consider singular vector fields which constitute a basis of a Cartan subalgebra of the Lie algebra $\mathfrak{sp}(2r,\mathbb{R})$ with $r\leq n$. Recall that $\mathfrak{sp}(2m,\mathbb{R})$ is isomorphic to the algebra of quadratic forms in $2m$ variables, $Q(2m,\mathbb{R})$, via symplectic duality. Thus the above-chosen immersion induces, in turn, an inclusion of subalgebras $Q(2r,\mathbb{R})\subset Q(2n,\mathbb{R})$.

Cartan subalgebras of $Q(2r,\mathbb{R})$ were classified by Williamson in [17].

Theorem 2.1 (Williamson)**.** For any Cartan subalgebra $\mathcal{C}$ of $Q(2r,\mathbb{R})$, there are a symplectic system of coordinates $(x_1,y_1,\ldots,x_r,y_r)$ in $\mathbb{R}^{2r}$ and a basis $q_1,\ldots,q_r$ of $\mathcal{C}$ such that each q_i is one of the following:

$$
\begin{aligned}
&q_i=x_i^2+y_i^2 \quad \text{for } 1\leq i\leq k_e, \ \text{(elliptic)}\\
&q_i=x_iy_i \quad \text{for } k_e+1\leq i\leq k_e+k_h, \ \text{(hyperbolic)}\\
&\begin{bmatrix} q_i=x_iy_i+x_{i+1}y_{i+1}\\ q_{i+1}=x_iy_{i+1}-x_{i+1}y_i\end{bmatrix} \quad \text{for } i=k_e+k_h+2j-1,\ 1\leq j\leq k_f \ \text{(focus-focus pair).}
\end{aligned}
\tag{2.1}
$$

□

Observe that the number of elliptic components k_e, hyperbolic components k_h, and focus-focus components k_h is therefore an invariant of the algebra $\mathcal{C}$. The triple (k_e, k_h, k_f) is called the Williamson type of $\mathcal{C}$. Observe that $r = k_e + k_h + 2k_f$. Let $q_1, \ldots, q_r$ be a Williamson basis of this Cartan subalgebra. We denote by X_i the Hamiltonian vector field of q_i with respect to ω. Those vector fields are a basis of the corresponding Cartan subalgebra of $\mathfrak{sp}(2r, \mathbb{R})$. We say that a vector field X_i is hyperbolic (resp., elliptic) if the corresponding function q_i is so. We say that a pair of vector fields X_i, X_{i+1} is a focus-focus pair if X_i and X_{i+1} are the Hamiltonian vector fields associated to functions q_i and q_{i+1} in a focus-focus pair.

In the local coordinates specified above, the vector fields X_i take the following forms:

(i) X_i is an elliptic vector field,

$$X_i = 2\left(- y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i} \right); \tag{2.2}$$

(ii) X_i is a hyperbolic vector field,

$$X_i = -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}; \tag{2.3}$$

(iii) X_i, X_{i+1} is a focus-focus pair,

$$\begin{aligned} X_i &= -x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} - x_{i+1} \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_{i+1}}, \\ X_{i+1} &= -x_i \frac{\partial}{\partial x_{i+1}} + y_{i+1} \frac{\partial}{\partial y_i} + x_{i+1} \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_{i+1}}. \end{aligned} \tag{2.4}$$

With all this notation at hand we can now state the result proven in this paper.

Theorem 2.2. Let $g_1, \ldots, g_r$ be a set of germs of smooth functions on $(\mathbb{R}^{2n}, 0)$ with $r \leq n$, fulfilling the following commutation relations:

$$X_i(g_j) = X_j(g_i), \quad \forall i, j \in \{1, \ldots, r\}, \tag{2.5}$$

where the X_i's are the vector fields defined above. Then there exists a germ of smooth function G and r germs of smooth functions f_i such that

(1) $X_j(f_i) = 0$, for all $i, j \in \{1, \dots, r\}$;
(2) $g_i = f_i + X_i(G)$, for all $i \in \{1, \dots, r\}$. □

3 Preliminaries

In this section, we recall some basic facts which are proved elsewhere and which will be used in the proof. Here and in the rest of the paper the symbol X_i always refers to the Hamiltonian vector field associated to the quadratic function q_i, as precised above.

3.1 A special decomposition for elliptic vector fields

Assume X_i is an elliptic vector field. That is, it is the vector field associated to an elliptic $q_i = x_i^2 + y_i^2$. The following result was proved by Eliasson in [4] when $n = 1$.

Proposition 3.1. Let g be a smooth function; then there exist differentiable functions g_1 and g_2 such that

$$g = g_1(x_1, y_1, \dots, x_i^2 + y_i^2, \dots, x_n, y_n) + X_i(g_2). \tag{3.1}$$

Moreover,

(1) g_1 is uniquely defined and satisfies $X_j(g_1) = 0$ whenever $X_j(g) = 0$;
(2) g_2 can be chosen such that $X_j(g_2) = 0$ whenever $X_j(g) = 0$. □

Remark 3.2. There are explicit formulas for the functions g_1 and g_2 claimed above. Let ϕ_t be the flow of the vector field X_i; we define

$$\begin{aligned} g_1(x_1, y_1, \dots, x_n, y_n) &= \frac{1}{\pi}\int_0^{\pi} g(\phi_t(x_1, y_1, \dots, x_n, y_n))\,dt, \\ g_2(x_1, y_1, \dots, x_n, y_n) &= \frac{1}{\pi}\int_0^{\pi} (tg(\phi_t(x_1, y_1, \dots, x_n, y_n)) \\ &\qquad - g_1(x_1, y_1, \dots, x_n, y_n))\,dt. \end{aligned} \tag{3.2}$$

3.2 A special decomposition for hyperbolic vector fields

In this section we assume the vector field X_i corresponds to a hyperbolic function $q_i = x_i y_i$. As a matter of notation, S_i stands for the set $S_i = \{x_i = 0,\ y_i = 0\} \subset \mathbb{R}^{2n}$. When we refer to an (x_i, y_i)-flat function f along S_i we mean that

$$\frac{\partial^{k+l} f}{\partial x_i^k \partial y_i^l}_{|S_i} = 0. \tag{3.3}$$

The first result is a decomposition result for smooth functions.

Proposition 3.3. Given a smooth function g there exist smooth functions g_1 and g_2 such that

$$g = g_1(x_1, y_1, \ldots, x_i y_i, \ldots, x_n, y_n) + X_i(g_2). \tag{3.4}$$

Moreover, g_1 and g_2 can be chosen such that $X_j(g_1) = X_j(g_2) = 0$ whenever $X_j(g) = 0$ for some $j \neq i$. □

This proposition was proven by the first author of this paper in [9] (Proposition 2.2.2).

The main strategy of the proof is first to find a decomposition of this type in terms of (x_i, y_i)-jets and then solve the similar problem for (x_i, y_i)-flat functions along S_i. Main ingredients in the proof of the proposition above are the following lemmas which will be also used in the proof of the theorem in this paper. The proof of the following two lemmas is also contained in [9] (resp., Lemmas 2.2.1 and 2.2.2).

Lemma 3.4. Let g be a smooth function; the equation $X_i(f) = g$ admits a formal solution along the subspace S_i if and only if

$$\frac{\partial^{2k} g}{\partial x_i^k \partial y_i^k}_{|S_i} = 0. \tag{3.5}$$

□

Lemma 3.5. Let g be an (x_i, y_i)-flat function along the subspace S_i; then there exists a smooth function f for which $X_i(f) = g$. □

Remark 3.6. (1) We point out that when $n = 1$ the decomposition claimed in Proposition 3.3 had been formerly given by Guillemin and Schaeffer [8], by Colin de Verdière and Vey in [3], and by Eliasson in [4].

(2) The recipe for solving the equation specified in the lemma above in the case $n = 1$ was given by Eliasson in [4]. The recipe for the general case follows the same guidelines. It is given by the following formula:

$$f(x_1, y_1, \ldots, x_n, y_n) = -\int_0^{T_i(x_1, y_1, \ldots, x_n, y_n)} g(\phi_t(x_1, y_1, \ldots, x_n, y_n))\, dt, \tag{3.6}$$

where T_i is the function

$$T_i(x_1, y_1, \dots, x_n, y_n) = \begin{cases} \frac{1}{2} \ln \frac{x_i}{y_i} & x_i y_i > 0, \\ \frac{1}{2} \ln \frac{-x_i}{y_i} & x_i y_i < 0, \end{cases} \tag{3.7}$$

and $\phi_t(x_1, y_1, \dots, x_n, y_n)$ is the flow of the vector field X_i. Observe that f is defined outside the set $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are the sets $\Omega_1 = \{(x_1, y_1, \dots, x_n, y_n),\ x_i = 0\}$ and $\Omega_2 = \{(x_1, y_1, \dots, x_n, y_n),\ y_i = 0\}$. In [9] it is proven that f admits a smooth continuation in the whole neighborhood considered and that it is a solution of the equation $X_i(f) = g$.

(3) From the formula specified above one deduces that if $X_j(g) = 0$ for $j \neq i$ then $X_j(f) = 0$.

(4) In contrast to the uniqueness of the function g_1 in the decomposition obtained in Proposition 3.1 for elliptic vector fields, the function g_1 specified in the decomposition is not unique. In fact, if g_1 and h_1 are two functions fitting in the decomposition their difference is an (x_i, y_i)-flat function along S_i. In order to check this, observe $g_1 - h_1 = X_i(h_2 - g_2)$, where h_2 is a function such that $g = h_1 + X_i(h_2)$. Now, on the one hand, the Taylor expansion of $g_1 - h_1$ in the x_i, y_i variables has the form $\sum_j c_j(\check{z}_i)(x_i \cdot y_i)^j$ but, on the other hand, the Taylor expansion of $X_i(h_2 - g_2)$ has the form $\sum_{jk} c_{jk}(\check{z}_i) x_i^j y_i^k$ with $j \neq k$, and since the equality $g_1 - h_1 = X_i(h_2 - g_2)$ holds, we deduce that $g_1 - h_1$ is an (x_i, y_i)-flat function along S_i.

(5) We show the last point of the proposition. The first step in the proof of the proposition was to take care of the formal Taylor series in (x_i, y_i). Then it is easy to see that one can always choose Borel resummations of these formal expansions which are annihilated by X_j $(j \neq i)$ whenever g is.

Finally we integrate the flat function using formula (3.6), on which one can check directly that f is invariant by the flow of X_j $(j \neq i)$ whenever g is, at least in a neighborhood of any point where the formula is well defined. In other words $X_j(g) = 0$ implies $X_j(f) = 0$ at these points and hence everywhere by continuity.

4 A special decomposition for focus-focus vector fields

The aim of this section is to prove the analogue of Propositions 3.1 and 3.3 for a focus-focus pair. But before stating and proving this result we need some preliminary material concerning the integration of equations of type $X(f) = g$ in a neighborhood of a hyperbolic zero (in the sense of Sternberg) of the vector field X. As we will see, the resolution of such an equation is closely related to the problem of finding the desired decomposition for focus-focus pairs.

4.1 Digression: two theorems of Guillemin and Schaeffer

A point is called a hyperbolic zero of a vector field X if the vector field vanishes at this point and all the eigenvalues of the matrix associated to the linear part of X have nonzero real part.

According to Sternberg's linearization theorem a vector field can be linearized in a neighborhood of a hyperbolic zero.

The following two theorems are concerned with the integration of equations of type $X(f) = g$ in a neighborhood of a hyperbolic zero.

Theorem 4.1 (see [8, Section 4, Theorem 2]). Let V be a linear vector field on $\mathbb{R}^n$ with a hyperbolic zero at the origin and let c be a fixed constant. Then given a smooth function g flat at the origin, there exists a smooth function defined in a neighborhood of the origin which is flat at the origin and such that

$$V(f) + cf = g. \tag{4.1}$$

□

The theorem that follows is used in the proof of Theorem 4.1. We recall it here because we will need it in order to show the smoothness of some constructions used in the next subsection. This theorem uses a trick previously used by Nelson [10] in his proof of Sternberg's linearization theorem.

Theorem 4.2 (see [8, Section 4, Theorem 4]). Let $U(t)$ be a group of linear transformations acting on $\mathbb{R}^n$. Let N be a subspace of $\mathbb{R}^n$ invariant under $U(t)$ and let E be the subspace of $\mathbb{R}^n$ consisting of all x in $\mathbb{R}^n$ such that

$$\lim_{t\to\infty} \big\|U(t)(x) - N\big\| = 0. \tag{4.2}$$

Let g be a compactly supported function on $\mathbb{R}^n$ which is flat along N. Set

$$f(x, s) = -\int_0^s e^{ct} g\big(U(t)(x)\big)\, dt. \tag{4.3}$$

Then, for all multi-indices α, $\lim_{s\to\infty} D^\alpha f(x, s)$ converges absolutely for all $x \in E$ and is a smooth function of x. Moreover, this limit is flat along N. □

Observe that the vector field X_i in a focus-focus pair X_i, X_{i+1} has a hyperbolic zero (in the sense of Sternberg) on the set $\{x_j = c_j,\ y_j = d_j,\ j \neq i,\ j \neq i+1\}$ for fixed constants c_j and d_j.

4.2 Our proposition for focus-focus pairs

When i is the index of a focus-focus component, we denote by S_i the set $S_i = \{x_i = 0,\ y_i = 0,\ x_{i+1} = 0,\ y_{i+1} = 0\}$. We state and prove the decomposition result for focus-focus pairs.

Proposition 4.3. Let q_i, q_{i+1} be a focus-focus pair,

$$\begin{aligned} q_i &= x_i y_i + x_{i+1} y_{i+1}, \\ q_{i+1} &= x_i y_{i+1} - x_{i+1} y_i, \end{aligned} \tag{4.4}$$

and let g_1 and g_2 be two functions satisfying the commutation relation

$$X_i(g_2) = X_{i+1}(g_1). \tag{4.5}$$

Then there exist smooth functions f_1, f_2, and F such that

$$X_j(f_k) = 0, \quad j \in \{i, i+1\},\ k \in \{1, 2\}, \tag{4.6}$$

such that

$$\begin{aligned} g_1 &= f_1 + X_i(F), \\ g_2 &= f_2 + X_{i+1}(F). \end{aligned} \tag{4.7}$$

Moreover,

(1) f_2 is uniquely defined and satisfies $X_j(f_2) = 0$ whenever $X_j(g_2) = 0$ for some j;

(2) f_1 is uniquely defined modulo functions that are z_j-flat along S_j and satisfy (4.6);

(3) F and f_1 can be chosen such that $X_j(F) = X_j(f_1) = 0$ whenever $X_j(g_1) = X_j(g_2) = 0$ for some $j \neq i$. □

Remark 4.4. In the case $n = 2$ the proposition above was proven by Eliasson [4].

Proof. Here again the proof is a mild extension of Eliasson's. Without loss of generality, one can assume that $i = 1$. The flow of X_2 defines an S^1-action which will be used in the proof. We can visualize this S^1-action easily using complex coordinates $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$, so that $q_1 + iq_2 = \overline{z_1} z_2$. The flow of q_2 is the S^1-action given by $(z_1, z_2) \mapsto e^{-it}(z_1, z_2)$ whereas the flow of q_1 is the hyperbolic dynamics given by $(z_1, z_2) \mapsto (e^{-t} z_1, e^t z_2)$ (both flows act trivially on the remaining coordinates). When we say that a function H is S^1-invariant for this action we mean that $X_2(H) = 0$.

As in the proof of Eliasson, we will first integrate along this S^1-action and then along the hyperbolic flow in an S^1-invariant way. Instead of using the formula of Eliasson (which consists in integrating from a transversal hyperplane through the origin), we will embed everything in $\mathbb{R}^{2n}$ in order to apply the parametric versions of Theorems 4.1 and 4.2.

The proof consists of three steps.

(1) *Integrating along the* S^1*-action.* Let $\varphi_{2,t}$ be the flow of q_2. As in the elliptic case (Proposition 3.1) we define

$$F_2 = \frac{1}{2\pi}\int_0^{2\pi} (\theta - 1) g_2 \circ \varphi_{2,\theta}\, d\theta \tag{4.8}$$

and one obtains easily, by differentiating $F_2 \circ \varphi_{2,t}$ at $t = 0$, that

$$X_2(F_2) = g_2 - f_2, \tag{4.9}$$

where

$$f_2 = \frac{1}{2\pi}\int_0^{2\pi} g_2(\varphi_{2,\theta})\, d\theta, \tag{4.10}$$

which is obviously S^1-invariant. Notice that if f_2 is any S^1-invariant function satisfying equation (4.9) then by integrating along the S^1 flow f_2 is necessarily of the form given by (4.10). Hence such an f_2 is indeed unique.

If we check that $X_1(f_2) = 0$, then we can write $g_2 = f_2 + X_2(F_2)$, with f_2 satisfying $X_1(f_2) = 0$ and $X_2(f_2) = 0$. That is to say, these functions g_2 and f_2 solve the second equation stated in the proposition.

One can check this directly on formula (4.10), using the commutation relation $X_1(g_2) = X_2(g_1)$ and the fact that the flows of X_1 and X_2 commute; from equation (4.9) one can also write

$$0 = X_1(f_2) + X_2(X_1(F_2) - g_1), \tag{4.11}$$

where $X_2(X_1(f_2)) = 0$. This equation can be seen as a decomposition for the zero function. Using the uniqueness of the S^1-invariant function in this decomposition we obtain

$$X_1(f_2) = 0, \qquad X_2(X_1(F_2) - g_1) = 0. \tag{4.12}$$

In particular, this also yields that the function $\tilde{g}_1 = g_1 - X_1(F_2)$ is S^1-invariant.

(2) *Formal resolution of the system.* In order to solve the initial system we need to find a smooth function f_1 such that $X_1(f_1) = 0$ and $X_2(f_1) = 0$ and a smooth function F_1 solving the system

$$\begin{aligned} X_1(F_1) &= \tilde{g}_1 - f_1, \\ X_2(F_1) &= 0. \end{aligned} \tag{4.13}$$

Once this system has been solved, the desired function F solving the initial system can be written as $F = F_1 + F_2$.

In order to solve this system we will first find a formal solution using formal power series and in a further step we will take care of the remaining flat functions along S_1.

We first solve the system in formal power series in (z_1, z_2), which is fairly easy. It amounts to solving the first equation assuming that all terms in the series commute with q_2 (we can do this because $X_2(\tilde{g}_1) = 0$). As in the hyperbolic case, the formal series for f_1 is unique and is of the form $\sum c_{k,\ell}(\check{z}) q_1^k q_2^\ell$, where $\check{z} = (x_3, y_3, \dots, x_n, y_n)$. Now we can use a Borel resummation in the variables (q_1, q_2) for f_1 and an S^1-invariant Borel resummation for F_1, which ensures that the system is reduced to the situation where the right-hand side of the first equation of (4.13) is a function g_1 which is S^1-invariant and flat at $\{z_1 = z_2 = 0\}$. These Borel resummations can be chosen uniform in the $\check{z}$ variables.

(3) *Solving the equation* $X_1(F_1) = g_1$ *for an* S^1*-invariant function which is flat along* S_1. We could finish the proof by invoking a similar formula as for the hyperbolic case (Lemma 3.5). But checking the smoothness in all variables is not so obvious; we present here a small variant which uses Theorems 4.1 and 4.2 stated in the preceding subsection and which are contained in [8].

The strategy is exactly the same as in [8], with the additional requirement of keeping track of the S^1-symmetry. We give below the arguments for the sake of completeness.

First of all, using an S^1-invariant cutoff function in $\mathbb{R}^{2n}$, one can assume that g_1 is compactly supported while still commuting with X_2. Again, we denote this new function by g_1. It is clear that if one solves the corresponding system (4.13) in $\mathbb{R}^{2n}$, the associated germs for F_1 and f_1 will solve the initial local problem. Let $\varphi_{1,t}$ be the flow of q_1. The matrix associated to the linear vector fields X_1 has two positive and two negative eigenvalues.

We first apply Theorem 4.2 with parameters x_j, y_j, $j \neq 1$ and $j \neq 2$ with $N = S_1$, $E = E^+ = \{z_1 = 0\}$, and $U(t) = \varphi_{1,-t}$. As explained in the proof of [8, Section 4, Theorem 2] this allows to solve the equation to infinite order on the $(2n-2)$-dimensional invariant subspace $E^+ = \{z_1 = 0\}$. Observe that the formula provided in the statement of

Theorem 4.2 shows that if the function g depends smoothly on the parameters x_j and y_j for $j \neq 1$ and $j \neq 2$ then the function f does also depend smoothly on these parameters because $\varphi_{1,-t}$ leaves the set S_1 fixed.

Therefore, using an S^1-invariant Borel resummation, we are then reduced to the case where g_1 is flat on E^+ and S^1-invariant, and we terminate by a second application of Theorem 4.2 with parameters $x_j, y_j, j \neq 1$ and $j \neq 2$ with $N = E^+$ and $E = \mathbb{R}^{2n}$. That is, the function F_1 is given by the formula

$$F_1 = -\int_0^\infty g_1 \circ \varphi_{1,t}\, dt. \tag{4.14}$$

Again this function F_1 is smooth in all the variables since g_1 is smooth in all the variables. Using this formula we see that $X_2(F_1) = 0$ because $\varphi_{1,t}$ and $\varphi_{2,\theta}$ commute.

The justification of the last claim of the proposition goes as before, by examining the explicit formulae and the Borel resummations. The claimed uniqueness of f_1 modulo z_j-flat functions along S_j is a direct consequence of the uniqueness of the formal solution in the z_j variables. Of course, one can also check it by an a posteriori argument as we did in Remark 3.6. ■

5 The proof of Theorem 2.2

Consider $s = k_e + k_h + k_f$. As we observed in Section 2, we have $r = k_e + k_h + 2k_f$. Observe also that $r = s$ if there are no focus-focus components. We prove the theorem using induction on s for a fixed n.

In order to simplify the statements involving focus-focus pairs, we introduce some more notation. Let the vector fields $Y_1, Y_2, \ldots, Y_s$ be such that $Y_j = X_j$ for elliptic or hyperbolic cases (i.e., for $j \leq k_e + k_h$) while $Y_j = X_{\sigma(j)} + \sqrt{-1}X_{\sigma(j)+1}$ for focus-focus pairs (i.e., $j > k_e + k_h$ and $\sigma(j) := 2j - k_e - k_h - 1$). Similarly we define γ_j to be g_j for elliptic or hyperbolic indices, and $\gamma_j = g_{\sigma(j)} + \sqrt{-1}g_{\sigma(j)+1}$ for focus-focus indices.

For any $j \leq s$, let $\mathcal{C}_j$ be the space of all germs of complex functions $f \in \mathcal{C}^\infty(\mathbb{R}^{2n}, 0)$ such that $Y_j(f) = \overline{Y_j}(f) = 0$, and $\mathcal{F}_s = \bigcap_{j \leq s} \mathcal{C}_j$.

With these notations, the system we wish to solve has the form $\gamma_j = f_j + Y_j(G)$ (for all $j \in \{1, \ldots, s\}$) for germs of smooth functions G and f_j, where $f_j \in \mathcal{F}_s$ and G and f_j, $j \leq k_h + k_e$, are real-valued. The commutation relations are $\overline{Y_i}(\gamma_j) = Y_j(\overline{\gamma_i})$ and $Y_i(\gamma_j) = Y_j(\gamma_i)$ (of course the second one is redundant except when both Y_i and Y_j are complex).

Suppose throughout the rest of the proof that $r < n$. For any subindex i corresponding to an elliptic or hyperbolic vector field Y_i, we denote $z_i = (x_i, y_i)$ and $\check{z}_i = (z_1, \ldots, \check{z}_i, \ldots, z_n)$. For any subindex j corresponding to a focus-focus pair Y_j, we denote

$z_j = (x_i, y_i, x_{i+1}, y_{i+1})$ and $\check{z}_j = (z_1, \dots, \check{z}_j, \dots, z_n)$ (with $i = \sigma(j)$). We denote by S_j the set $S_j = \{z_j = 0\}$.

This being said, one notices that there is no more need to keep the vector fields Y_j in a particular order, which is of course most convenient for the induction process.

Sublemma 5.1. Let Z be a (real or complex) vector field on $\mathbb{R}^{2n}$ acting trivially on the variables $(z_1, \dots, z_s)$. Let $j \leq s$. Let f be a smooth real-valued function on $\mathbb{R}^{2n}$ such that

(1) $f \in \mathcal{F}_s$;
(2) $Z(f)$ is flat along S_j.

Then there exists a smooth real-valued function $\tilde{f} \in \mathcal{F}_r$ such that

(1) $Z(\tilde{f}) = 0$;
(2) $f - \tilde{f}$ is flat along S_j. □

Proof. Consider the Taylor expansion of f in z_j. Because $Y_j(f) = 0$ this expansion is a formal series in q_j (in the case of an elliptic or hyperbolic Y_j) or in q_i, q_{i+1} (in the case of a focus-focus Y_j, with $i = \sigma(j)$). Moreover, the coefficients of this expansion are functions of $\check{z}_j$ that are annihilated by X_j, $j \leq r$, $j \neq i$, and Z. Hence, using a suitable Borel resummation, one can come up with a smooth $\tilde{f}$ satisfying the requirements of our statement. ■

5.1 Case $s = 1$

(1) The Cartan subalgebra has Williamson type $(1, 0, 0)$ or $(0, 1, 0)$. In this case there is only one function. Propositions 3.1 (in case X_i is elliptic) and 3.3 (in case X_i is hyperbolic) guarantee that the theorem holds.

(2) The Cartan subalgebra has Williamson type $(0, 0, 1)$. In this case there are two functions g_1 and g_2 fulfilling the conditions specified in Proposition 4.3, and the proposition guarantees that the theorem holds.

5.2 Passing from s to $s + 1$

By hypothesis we can construct G and $f_1, \dots, f_s$ such that

$$\forall j \leq s, \quad \gamma_j = f_j + Y_j(G), \tag{5.1}$$

with $f_j \in \mathcal{F}_r$, for all $j \leq r$. Observe that when we pass from s to $s + 1$ we are adding a real vector field if the Williamson type changes from (k_e, k_h, k_f) to $(k_e + 1, k_h, k_f)$ or from (k_e, k_h, k_f) to $(k_e, k_h + 1, k_f)$. In the case where we increase by one the number of focus-focus components we are adding a complex vector field. The proof will go in two steps. First we modify the existing f_j and G in such a way that the new f_j's, $j \leq s$, are in $\mathcal{F}_{s+1}$. The final step is to look for a new G of the form $\tilde{G} = G + K$ which leads to the system

$$Y_1(K) = \cdots = Y_s(K) = 0, \qquad \tilde{\gamma}_{s+1} = f_{s+1} + Y_{s+1}(K), \tag{5.2}$$

with $Y_j(\tilde{\gamma}_{s+1}) = \overline{Y_j}(\tilde{\gamma}_{s+1}) = 0$, for all $j \leq s$.

(1) We consider the commutation relations

$$\overline{Y_{s+1}}(\gamma_j) = Y_j(\overline{\gamma_{s+1}}), \qquad Y_{s+1}(\gamma_j) = Y_j(\gamma_{s+1}). \tag{5.3}$$

We distinguish three subcases.

(a) The vector field Y_j is elliptic: from the uniqueness of the function g_1 of the decomposition in **Proposition 3.1** (possibly applied to the real and imaginary parts of Y_{s+1}) this condition tells us that $Y_{s+1}(f_j) = 0$. Therefore, in this case, no modification of f_j is required and $f_j \in \mathcal{F}_{s+1}$.

(b) The vector field Y_j is hyperbolic: by applying **Lemma 3.4** we deduce that the z_j-jet of $Y_{s+1}(f_j)$ is zero. We can write $Y_{s+1}(f_j) = \alpha_j$, where α_j is a z_j-flat function along S_j. We can now apply **Sublemma 5.1** to obtain the decomposition $f_j = \tilde{f}_j + \phi_j$, where $\tilde{f}_j \in \mathcal{F}_{s+1}$ and $\phi_j \in \mathcal{F}_s$ is a z_j-flat function. We may apply **Lemma 3.5** to the function ϕ_j to find a function φ_j satisfying $Y_j(\varphi_j) = \phi_j$. According to **Proposition 3.3**, this function φ_j can be chosen such that $Y_j(\varphi_j) = 0$ for $j \neq i$ and $j \leq s$. Hence, for this γ_j, we can write

$$\gamma_j = \tilde{f}_j + Y_j(\varphi_j + G). \tag{5.4}$$

(c) The vector field Y_j is a focus-focus complex vector field. The commutation conditions also read as follows:

$$\begin{aligned} Y_{s+1}(\Re\gamma_j) &= \Re(Y_j)(\gamma_{s+1}), \\ Y_{s+1}(\Im\gamma_j) &= \Im(Y_j)(\gamma_{s+1}). \end{aligned} \tag{5.5}$$

From the second equation and the uniqueness of the function f_2 obtained in **Proposition 4.3** we obtain $Y_{s+1}(\Im f_j) = 0$, so we only need to modify $\Re f_j$. Now, since $\Im(Y_j)(\Re f_j) = 0$ and

$\Re(Y_j)(\Re f_j) = 0$, we can invoke the uniqueness up to a flat function of the function f_1 in the decomposition claimed in Proposition 4.3 applied to the first equality to deduce that $Y_{s+1}(\Re f_j)$ is z_j-flat along S_j. Hence, by Sublemma 5.1 applied to $Z = Y_{s+1}$, we can write $\Re f_j = h_j + \phi_j$, where h_j is a real function in $\mathcal{F}_{s+1}$ and $\phi_j \in \mathcal{F}_s$ is a real z_j-flat function along S_j; therefore, as in the proof of Proposition 4.3, we can integrate ϕ_j to a function φ_j satisfying $\Re Y_j(\varphi_j) = \phi_j$. Hence

$$\gamma_j = \tilde{f}_j + Y_j\big(G + \varphi_j\big), \tag{5.6}$$

with $\tilde{f}_j = f_j - \phi_j \in \mathcal{F}_{s+1}$.

(2) After considering all these cases we may write

$$g_j = \tilde{f}_j + Y_j\big(\varphi_j + G\big), \quad \forall j \leq s, \tag{5.7}$$

where $\varphi_j \in \mathcal{F}_s$ is a real function equal to the zero function for subindices corresponding to elliptic Y_j. Now define $\tilde{G} = \sum_i \varphi_i + G$. This function satisfies

$$Y_j(\tilde{G}) = Y_j\big(\varphi_j + G\big), \quad \forall j \leq s. \tag{5.8}$$

Finally, to prove the theorem, it suffices to find a real function K and $f_{s+1} \in \mathcal{F}_{s+1}$ such that

$$\begin{aligned} \gamma_j &= \tilde{f}_j + Y_j\big(\tilde{G} + K\big), \quad \text{for } j \leq s, \\ \gamma_{s+1} &= f_{s+1} + Y_{s+1}\big(\tilde{G} + K\big). \end{aligned} \tag{5.9}$$

But consider $\tilde{\gamma}_{s+1} := \gamma_{s+1} - Y_{s+1}(\tilde{G})$. The commutation relations yield

$$Y_j\big(\tilde{\gamma}_{s+1}\big) = \overline{Y_j}\big(\tilde{\gamma}_{s+1}\big) = 0 \tag{5.10}$$

for $j \leq s$, and we still have (in case $s+1$ is a focus-focus index)

$$Y_{s+1}\big(\overline{\tilde{\gamma}_{s+1}}\big) = \overline{Y_{s+1}}\big(\tilde{\gamma}_{s+1}\big). \tag{5.11}$$

Thus our system becomes

$$\begin{aligned} 0 &= Y_j(K), \quad \text{for } j \leq s, \\ \tilde{\gamma}_{s+1} &= f_{s+1} + Y_{s+1}(K), \end{aligned} \tag{5.12}$$

and since $\tilde{\gamma}_{s+1} \in \mathcal{F}_s$ (equation (5.10)), it is solved by an application of Proposition 3.1, 3.3, or 4.3, depending on the type of Y_{s+1} (notice that the relation (5.11) is precisely the commutation relation required in the focus-focus case). This ends the proof of the theorem.

6 Deformations of completely integrable systems

Theorem 2.2 has a natural interpretation in terms of infinitesimal deformations of integrable systems near nondegenerate singularities. This was stated without proof in [16]. We recall briefly the appropriate setting.

A completely integrable system on a symplectic manifold M of dimension $2n$ is the data of n functions $f_1, \dots, f_n$ which commute pairwise for the symplectic Poisson bracket $\{f_i, f_j\} = 0$ and whose differentials are almost everywhere linearly independent.

When we are interested in geometric properties of such systems, the main object under consideration is the (singular) Lagrangian foliation given by the level sets of the momentum map $f = (f_1, \dots, f_n)$. We introduce the notation $\mathfrak{f}$ for the linear span (over $\mathbb{R}$) of $f_1, \dots, f_n$. It is an n-dimensional vector space. It is also an abelian Poisson subalgebra of the Poisson algebra $X = (\mathcal{C}^\infty, \{\cdot, \cdot\})$. Let $\mathcal{C}_\mathfrak{f} = \{h \in X, \{\mathfrak{f}, h\} = 0\}$ be the set of functions that commute with all f_i. By the Jacobi identity, $\mathcal{C}_\mathfrak{f}$ is a Lie subalgebra of X. The fact that $df_1 \wedge \cdots \wedge df_n \neq 0$ almost everywhere implies that $\mathcal{C}_\mathfrak{f}$ is actually abelian. From now on, we are given a point $m \in M$ and everything is localized at m; in particular, X is the algebra of germs of smooth functions at m.

Definition 6.1. Two completely integrable systems $\mathfrak{f} = \langle f_1, \dots, f_n \rangle$ and $\mathfrak{g} = \langle g_1, \dots, g_n \rangle$ are *equivalent* (near m) if and only if

$$\mathcal{C}_\mathfrak{f} = \mathcal{C}_\mathfrak{g}. \tag{6.1}$$

Geometrically speaking, $\mathfrak{f}$ is equivalent to $\mathfrak{g}$ if and only if the functions f_i are constant along the leaves of the $\mathfrak{g}$-foliation (or vice versa).

We wish to describe infinitesimal deformations of integrable systems modulo this equivalence relation. For this we fix an integrable system $\mathfrak{f}$ and introduce a deformation complex as follows. Let $L_0 \simeq \mathbb{R}^n$ be the typical commutative Lie algebra of dimension n. L_0 acts on X by the adjoint representation

$$L_0 \times X \ni (\ell, g) \longmapsto \{f(\ell), g\} \in X. \tag{6.2}$$

Hence X is an L_0-module, in the Lie algebra sense, and we can introduce the corresponding Chevalley-Eilenberg complex [1]: for $q \in \mathbb{N}$, $C^q(L_0, X) = \mathrm{Hom}(L_0^{\wedge q}, X)$ is the space of alternating q-linear maps from L_0 to X (regarded merely as real vector spaces), with the convention $C^0(L_0, X) = X$. The associated differential is denoted by d_f. Following [1] for a 0-cochain $g \in X$, the 1-cochain $d_f(g)$ is $d_f(g)(l) = \{f(l), g\}$, $l \in L_0$, and for a k-cochain ϕ, the $(k+1)$-cochain $d_f(\phi)$ is

$$d_f(\phi)\big(l_1, \dots, l_{k+1}\big) = \frac{1}{k+1} \sum_{i=1}^{k+1} (-1)^{i+1} \big\{f\big(l_i\big), \phi\big(\check{l}_i\big)\big\}, \quad l_i \in L_0, \tag{6.3}$$

where $\check{l}_i = (l_1, \dots, \check{l}_i, \dots, l_{k+1})$.

Now, since L_0 acts trivially on $\mathcal{C}_f$, the quotient Lie algebra $X/\mathcal{C}_f$ is an L_0-module and we can define the corresponding Chevalley-Eilenberg complex: for $q \in \mathbb{N}$, $C^q(L_0, X/\mathcal{C}_f) = \mathrm{Hom}(L_0^{\wedge q}, X/\mathcal{C}_f)$, with differential denoted by $\bar{d}_f$.

Finally we define the *deformation complex* $C^\bullet(f)$ as follows:

$$0 \longrightarrow X/\mathcal{C}_f \xrightarrow{\bar{d}_f} C^1\big(L_0, X/\mathcal{C}_f\big) \xrightarrow{\partial_f} C^2\big(L_0, X\big) \xrightarrow{d_f} C^3\big(L_0, X\big) \xrightarrow{d_f} \cdots, \tag{6.4}$$

where ∂_f is defined by the following diagram, where all small triangles are commutative ($C^k(L_0, \mathcal{C}_f)$ is always in the kernel of d_f):

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "d_f"] \arrow[d, "\pi"'] & C^1(L_0, X) \arrow[r, "d_f"] \arrow[d, "\pi"'] & C^2(L_0, X) \arrow[r, "d_f"] \arrow[d, "\pi"'] & \cdots \\
0 \arrow[r] & X/\mathcal{C}_f \arrow[r, "\bar{d}_f"'] \arrow[ur, "\partial_f"] & C^1(L_0, X/\mathcal{C}_f) \arrow[r, "\bar{d}_f"'] \arrow[ur, "\partial_f"] & C^2(L_0, X/\mathcal{C}_f) \arrow[r, "\bar{d}_f"'] \arrow[ur, "\partial_f"] & \cdots
\end{tikzcd} \tag{6.5}$$

For all cochain complexes, cocycles and coboundaries are denoted the standard way: $Z^q(\cdot)$ and $B^q(\cdot)$. In the analytic category a similar deformation complex was introduced recently by Garay and van Straten (see [6, 7]) and (for the first degrees) by Stolovitch [11]. The equivalence used in the analytic category is much easier to handle due to the absence of flat functions.

Definition 6.2. $Z^1(\mathbf{f})$ is the space of infinitesimal deformations of $\mathbf{f}$ modulo equivalence.

If we fix a basis $(e_1, \dots, e_n)$ of L_0, a cocycle $\alpha \in Z^1(\mathbf{f})$ is just a set of functions $g_1 = \alpha(e_1), \dots, g_n = \alpha(e_n)$ (defined modulo $\mathcal{C}_{\mathbf{f}}$) such that

$$\forall i, j, \quad \{g_i, f_j\} = \{g_j, f_i\}. \tag{6.6}$$

It is an infinitesimal deformation of $\mathbf{f}$ in the sense that, modulo ϵ^2,

$$\{f_i + \epsilon g_i, f_j + \epsilon g_j\} \equiv 0. \tag{6.7}$$

A special type of infinitesimal deformations of $\mathbf{f}$ is obtained by the infinitesimal action of the group G of local symplectomorphisms: given a function $h \in X$, one can define the deformation cocycle $\alpha \in Z^1(\mathbf{f})$ by

$$L_0 \ni \ell \longmapsto \alpha(\ell) = \{h, \mathbf{f}(\ell)\} \bmod \mathcal{C}_{\mathbf{f}}. \tag{6.8}$$

In other words, the set of all such cocyles, with h varying in X, is the orbit of $\mathbf{f}$ under the adjoint action on $Z^1(\mathbf{f})$ of the Lie algebra of G. From equation (6.8) one immediately sees that this orbit is exactly $B^1(\mathbf{f})$.

In the particular case that ω is the Darboux symplectic form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ and $\mathbf{f} = (q_1, \dots, q_n)$ is a Williamson basis as specified in Theorem 2.1, we can reformulate the statement of Theorem 2.2 in cohomological terms.

Namely, in this case, since $\{f_i, f\} = X_i(f)$, we can write $C_q = \{f \in X,\ X_i(f) = 0,\ \forall i\}$. Let α be a 1-cocycle, the cocycle condition specified in formula (6.6) reads as $X_j(g_i) = X_i(g_j)$, where $g_i = \alpha(e_i)$. But this is nothing but the commutation hypothesis of Theorem 2.2; therefore, there exists a function G such that $g_i = f_i + X_i(G)$. Using formula (6.8) and the definition of g_i this shows that α is a coboundary. In other words, what Theorem 2.2 shows in cohomological terms is that any $\alpha \in Z^1(\mathbf{f})$ is indeed a coboundary. And this proves the following reformulation of Theorem 2.2.

Theorem 6.3. Let $q_1, \dots, q_n$ be a standard basis (in the sense of Williamson) of a Cartan subalgebra of $\mathcal{Q}(2n, \mathbb{R})$. Then the corresponding completely integrable system $\mathbf{q}$ in $\mathbb{R}^{2n}$ is $\mathcal{C}^\infty$-infinitesimally stable at $m = 0$, that is,

$$H^1(\mathbf{q}) = 0. \tag{6.9}$$

□

Remark 6.4. Our proof actually shows that the result is also true when we include a smooth dependence on parameters in the definition of the deformation complex.

This theorem should have important applications in semiclassical analysis, where we consider pseudodifferential operators with $\mathcal{C}^\infty$ symbols depending on a small parameter $\hbar$. One can define a similar deformation complex for pseudodifferential operators, where the deformation is understood with respect to the parameter $\hbar$. Then, in many situations, the vanishing of the classical H^1 implies the vanishing of the pseudodifferential H^1. See [16] for general remarks, and [2, 14] for applications in simple cases where the vanishing of the pseudodifferential H^1 was checked explicitly.

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